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On pseudo-poles of Abrikosov equation

By

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1. Introduction

In this paper we investigate the ordinary differential equation

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{\nu^2}{r^2}u=u^3-u \qquad (r>0). \tag{1}$$

Here ν is a positive parameter. This equation is called Abrikosov equation and describes vortex lines of a superconductor of type II in the theory of superconductivity. The existence and uniqueness of a global solution satisfying $0<u\leqq 1$ have been established by Y. Kametaka, [3]. Besides this global solution there are solutions with movable infinitudes, that is, solutions blowing up at $r=R$, R being an arbitrary finite positive number. We are interested in the nature of the movable infinitudes of the equation.

If $\nu=1/3$ the equation is equivalent to the second Painlevé equation

$$\frac{d^2w}{dz^2}=2w^3+zw$$

by the change of variables*

$$z=-\left(\frac{3}{2}r\right)^{\frac{2}{3}},\quad w=2^{-\frac{1}{2}}\left(\frac{3}{2}r\right)^{\frac{1}{3}}u.$$

Therefore if $\nu=1/3$ all possible infinitudes are simple poles. However if $\nu\neq 1/3$ there appear movable infinitudes which involve logarithmic terms in the expansions. In Section 2 of this paper such infinitudes are constructed. They can be called 'pseudo-poles' after E. Hille, who emphasized that such infinitudes appear in the Thomas-Fermi and Emden's equations ([2], Chapter IX).

On the other hand it is known that Abrikosov equation has a family of solutions which are asymptotically equivalent to a_0r^ν as $r\to+0$, a_0 being an arbitrary positive constant ([3], Theorem 5). The bounded solution mentioned above is asymptotic to $a_0(\nu)r^\nu$ with a certain value $a_0=a_0(\nu)$ of the constant. What happens if we continue the solution starting with $a_0\neq a_0(\nu)$ from $r=+0$ to the right? We will devote

* due to a private communication with Y. Kametaka.

Section 3 to a proof of the answer: If $a_0 > a_0(\nu)$ the solution in question is continued to an infinitude investigated in Section 2, that is, a pole if $\nu = 1/3$ or a pseudo-pole if $\nu \neq 1/3$ respectively.

2. Existence of movable infinitudes

This section is devoted to construction of local solutions having infinitudes of a certain type. We will prove the following theorem.

Theorem 1. *Abrikosov equation* (1) *has a solution* $u = u(r; R, C)$ *having the following series expansion in a neighborhood of* $r = R$:

$$(2)\qquad u = \frac{\sqrt{2}}{R-r}\Big[1 + \frac{1}{6}\frac{R-r}{R} + \Big(\frac{5}{36} - \frac{1}{6}\nu^2 + \frac{1}{6}R^2\Big)\Big(\frac{R-r}{R}\Big)^2 + $$
$$+ \Big(\frac{31}{216} - \frac{5}{12}\nu^2 - \frac{1}{12}R^2\Big)\Big(\frac{R-r}{R}\Big)^3 +$$
$$+ \frac{4}{15}\Big(\nu^2 - \frac{1}{9}\Big)\log\Big(\frac{R-r}{R}\Big)\Big(\frac{R-r}{R}\Big)^4 + C\Big(\frac{R-r}{R}\Big)^4 +$$
$$+ \sum_{5 \leq 2j_0 + j_1 + 4j_2} a_{j_0 j_1 j_2} R^{2j_0}\Big[\frac{4}{15}\Big(\nu^2 - \frac{1}{9}\Big)\log\Big(\frac{R-r}{R}\Big) + C\Big]^{j_2}\Big(\frac{R-r}{R}\Big)^{2j_0 + j_1 + 4j_2}\Big].$$

Here R, $R > 0$, *and* C *are arbitrary constants, the triple power series* $\sum a_{j_0 j_1 j_2} \cdot X_0^{j_0} X_1^{j_1} X_2^{j_2}$ *has positive radii of convergence and the coefficients* $a_{j_0 j_1 j_2}$*'s are polynomials in* ν^2 *whose coefficients are rational numbers independent of* R *and* C. *The series* (2) *converges and gives an actual real-valued solution if* $R - r$ *is sufficiently small and positive.*

Remark. We can substitute complex-valued r in the series provided that

$$|X_0| = |R - r|, \quad |X_1| = \frac{|R-r|}{R} \quad \text{and}$$

$$|X_2| = \Big|C + \Big(\frac{4}{15}\nu^2 - \frac{4}{135}\Big)\log\Big(\frac{R-r}{R}\Big)\Big|\,\Big|\frac{R-r}{R}\Big|^4$$

are smaller than the radii of convergence of the triple power series $\sum a_{j_0 j_1 j_2} X_0^{j_0} X_1^{j_1} X_2^{j_2}$.

If $\nu = 1/3$, that is, if $\frac{4}{15}\Big(\nu^2 - \frac{1}{9}\Big) = 0$, the expansion is reduced to that of a simple pole. However if $\nu \neq 1/3$ the infinitude is not a true pole but a pseudo-pole. Indeed the remainder of the expansion is $O(|\log(R-r)|(R-r)^5)$ as $r \uparrow R$. Therefore it is clear that the expansion cannot be that of a true pole.

If we replace $\frac{\sqrt{2}}{R-r}$ of the first term by $\frac{\sqrt{2}}{r-R}$ and $\log\frac{R-r}{R}$ by $\log\frac{r-R}{R}$ we get a real-valued solution for $R < r < R + \delta$, where C is supposed to be real and δ is sufficiently small. If $\nu = 1/3$ this solution coincides with the analytic continuation of $-u(r; R, C)$. However if $\nu \neq 1/3$ it is not the case. Indeed the continuation of

$-u(r; R, C)$ around $r=R$ along a small semi-circle in the complex r-plane will have the expansion in which $\frac{4}{15}\left(v^2-\frac{1}{9}\right)\log\frac{R-r}{R}+C$ is replaced by $\frac{4}{15}\left(v^2-\frac{1}{9}\right)$ $\log\frac{r-R}{R}+C+\frac{4}{15}\left(v^2-\frac{1}{9}\right)\pi\sqrt{-1}$ and will not be real-valued.

Suppose that $v\neq 1/3$ and consider the analytic continuation of the solution $u(r; R, C)$ along the curve

$$r=r(t)=R-\delta e^{\sqrt{-1}t}, \qquad 0\leqq t<+\infty,$$

δ being a sufficiently small positive number. Then although $r(t)$ remains on the small circle with center R, the absolute value of X_2,

$$|X_2|=\delta^4\left[\left(C+\left(\frac{4}{15}v^2-\frac{4}{135}\right)\log\delta\right)^2+\left(\frac{4}{15}v^2-\frac{4}{135}\right)^2t^2\right]^{1/2},$$

tends to infinity and may exceed the radius of convergence of the series if it is finite; the substitution of $r(t)$ in the series expansion may fail to be reasonable for large t. What happens for the further continuation? It is an open problem beyond the scope of this paper.

Proof of Theorem 1. The existence of a formal solution of the form (2) can be seen by substituting the series in the equation (1) and by equating the coefficients of $[\log(R-r)]^j(R-r)^k$'s. To prove the convergence we apply the scheme of R. A. Smith, [4].

Let us begin with transforming the equation (1) to a convenient system of first order equations. Put

$$f(u)\equiv u^3-u, \quad F(u)\equiv\int_1^u f(v)dv=\frac{1}{4}u^4-\frac{1}{2}u^2+\frac{1}{4}, \tag{3}$$

$$G(u)\equiv\frac{uf(u)}{F(u)}=\frac{4}{1-u^{-2}},$$

and write the equation (1) as

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{v^2}{r^2}u=f(u).$$

Then we see that the change of variables

$$x=\frac{u}{r}\frac{dr}{du}, \quad y=\left(\frac{dr}{du}\right)^2F(u) \tag{4}$$

transforms the equation (1) to the following two-dimensional nonautonomous system:

$$\begin{aligned} u\frac{dx}{du}&=x[1-G(u)y-v^2x^2], \\ u\frac{dy}{du}&=y[G(u)+2x-2G(u)y-2v^2x^2]. \end{aligned} \tag{5}$$

Given a solution $x=x(u)$, $y=y(u)$, $(1<)u_0\leqq u<+\infty$, of the system (5) such that $x(u)>0$ and $y(u)>0$, we will define a function $r=r(u)$ by

$$r(u)=uF(u)^{-\frac{1}{2}}x(u)^{-1}y(u)^{\frac{1}{2}}.$$

Then $r=r(u)$ is a strictly increasing function satisfying the equation

$$u\frac{dr}{du}=x(u)r, \tag{6}$$

and the inverse function $u=u(r)$ turns out to be a solution of the equation (1).

The right-hand sides of the system (5) vanish simultaneously for $x=0$, $y=1/2$ identically with respect to u. We will investigate solutions of (5) which converge to $x=0$, $y=1/2$ as u tends to infinity.

The change of variables

$$t=u^{-2},\ x=x,\ y=\frac{1}{2}+\frac{1}{3}x+Y$$

transforms the system (5) to an equivalent system of the following form:

$$t\frac{dx}{dt}=x\left[\frac{1}{2}+\frac{t}{1-t}+\frac{2}{3(1-t)}x+\frac{2}{1-t}Y+\frac{1}{2}v^2x^2\right],$$

$$t\frac{dY}{dt}=2Y+\frac{t}{3(1-t)}x+\frac{2t}{1-t}Y+\left(-\frac{1-3t}{9(1-t)}+\frac{1}{2}v^2\right)x^2+$$
$$+\frac{1+t}{1-t}xY+\frac{1}{6}v^2Y^2+\frac{1}{6}v^2x^3+v^2x^2Y.$$

Thus we can apply Picard's theorem concerning the equations of Briot-Bouquet type ([2], p. 50 and p. 81) to this system. As a result of solving a suitable recurrence formula, we can reduce (5) to a simplified system

$$t\frac{d\xi}{dt}=\frac{1}{2}\xi,\ t\frac{d\eta}{dt}=2\eta+\left(\frac{2}{3}v^2-\frac{2}{27}\right)\xi^4. \tag{7}$$

The reduction is given by a local analytic transformation of the form

$$x=\xi[1+\sum_{1\leqq j_0+j_1+j_2}P_{1;j_0j_1j_2}t^{j_0}\xi^{j_1}\eta^{j_2}], \tag{8}$$

$$y=\frac{1}{2}+\frac{1}{3}\xi+\eta+\sum_{2\leqq j_0+j_1+j_2}P_{2;j_0j_1j_2}t^{j_0}\xi^{j_1}\eta^{j_2}.$$

Here and hereafter $P_m(X_0, X_1, X_2)=\sum P_{m;j_0j_1j_2}X_0^{j_0}X_1^{j_1}X_2^{j_2}$, $m=1, 2,\dots, 7$, always denotes a triple power series with positive radii of convergence such that $P_m(0, 0, 0)=0$. The coefficients are polynomials in v^2 independent of the other parameters c_1, c_2, R and C introduced later.

The transformation $(\xi, \eta)\to(x, y)$ given by (8) sends a neighborhood of $\xi=\eta=0$ onto a neighborhood of $x=0$, $y=1/2$ provided that $t=u^{-2}$ is sufficiently small. Any solution of the reduced system (7), say

$$\xi = c_1 t^{\frac{1}{2}},\ \eta = \left[c_2 + \left(\frac{4}{3}\nu^2 - \frac{4}{27}\right) c_1^4 \log(t^{\frac{1}{2}})\right] t^2,$$

c_1 and c_2 being arbitrary constants, can be inserted into (8) provided that $t = u^{-2}$, $c_1 t = c_1 u^{-1}$ and $[c_2 + hc_1^4 \log(t^{1/2})]t^2 = [c_2 + hc_1^4 \log(u^{-1})]u^{-4}$ are sufficiently small. Here we write

$$h = \frac{4}{3}\nu^2 - \frac{4}{27}$$

for brevity. Then we obtain a solution $x = x(u;\ c_1,\ c_2)$, $y = y(u;\ c_1,\ c_2)$ of the system (5) of the following form.

$$x(u;\ c_1,\ c_2) = c_1 u^{-1}[1 + P_1(u^{-2},\ c_1 u^{-1},\ [c_2 + hc_1^4 \log(u^{-1})]u^{-4})],$$

(9) $$y(u;\ c_1,\ c_2) = \frac{1}{2} + \frac{1}{3} c_1 u^{-1} +$$

$$+ [c_2 + hc_1^4 \log(u^{-1})]u^{-4} + P_2(u^{-2},\ c_1 u^{-1},\ [c_2 + + hc_1^4 \log(u^{-1})]u^{-4}).$$

Supposing that $c_1 > 0$, we put

(10) $$r(u;\ c_1,\ c_2) = uF(u)^{-\frac{1}{2}} x(u;\ c_1,\ c_2)^{-1} y(u;\ c_1,\ c_2)^{\frac{1}{2}}.$$

Then it is clear that

(11) $$R \equiv \lim_{u\to\infty} r(u;\ c_1,\ c_2) = \frac{\sqrt{2}}{c_1}.$$

The inverse function $u = u(r)$ of $r = r(u;\ c_1,\ c_2)$ satisfies the equation (1). We are going to find a series expansion of the solution at $r = R$.

The expansion of $r(u;\ c_1,\ c_2)$ can be obtained by integrating the equation (6):

$$r = R \exp\left[-\frac{1}{2}\int_0^{u^{-2}} c_1 t^{\frac{1}{2}}\ [1 + P_1(t^{\frac{1}{2}},\ c_1 t,\ [c_2 + hc_1^4 \log(t^{\frac{1}{2}})]t^2)]t^{-1} dt\right] =$$

$$= R \exp[-c_1 u^{-1}[1 + P_3(u^{-2},\ c_1 u^{-1},\ [c_2 + hc_1^4 \log(u^{-1})]u^{-4})]].$$

Here the power series P_3 whose coefficients are independent of c_1 and c_2 is obtained by integrating term-by-term. The absolute convergence of P_3 can be verified easily by the majorant series argument.

Let us rewrite the above expression as

(12) $$s = u^{-1}[1 + P_3(u^{-2},\ c_1 u^{-1},\ [c_2 + hc_1^4 \log(u^{-1})]u^{-4})],$$

$$\text{where } s = \frac{R}{\sqrt{2}} \log\left[\frac{R}{r(u;\ c_1,\ c_2)}\right].$$

This relation can be regarded as an equation for the unknown function u^{-1} of s. A slight generalization of R. A. Smith' lemma ([4], p. 309, Lemma) implies that the equation (12) has a unique small solution u^{-1} provided that s^2, $c_1 s$ and $[c_2 + hc_1^4 \log(s)]s^4$ are sufficiently small, and that this small solution admits the following series expansion

$$u^{-1}=s[1+P_4(s^2, c_1s, [c_2+hc_1^4\log(s)]s^4)]. \tag{13}$$

This expansion is obtained by imbedding the equation (12) into the system of equations

$$s_0=t_0[1+P_3(t_0, t_1, t_2)]^2, \quad s_1=t_1[1+P_3(t_0, t_1, t_2)],$$
$$s_2=t_2[1+P_3]^4+ht_1^4[1+P_3]^4\log[1+P_3],$$

where

$$s_0=s^{-2}, \quad s_1=c_1s, \quad s_2=[c_2+hc_1^4\log(s)]s^4,$$
$$t_0=u^{-2}, \quad t_1=c_1u^{-1}, \quad t_2=[c_2+hc_1^4\log(u^{-1})]u^{-4}.$$

Since the Jacobian determinant of the analytic mapping $(t_0, t_1, t_2)\to(s_0, s_1, s_2)$ is equal to 1 for $t_0=t_1=t_2=0$, the analytic implicit function theorem implies the existence of its inverse mapping of the following form:

$$t_0=s_0[1+P_5(s_0, s_1, s_2)], \quad t_1=s_1[1+P_4(s_0, s_1, s_2)], \quad t_2=s_2+P_6(s_0, s_1, s_2),$$

where $P_{6;100}=P_{6;010}=P_{6;001}=0$. The second equation gives (13).

Finally let us substitute

$$s=\frac{R}{\sqrt{2}}\log\frac{R}{r}=\frac{R-r}{\sqrt{2}}\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{R-r}{R}\right)^k$$

into (13). Then we get the following expansion for the inverse function of $r(u; c_1, c_2)$:

$$u=\frac{\sqrt{2}}{R-r}\left[1+P_7\left((R-r)^2, \frac{R-r}{R}\left[c'+h\log\frac{R-r}{R}\right]\left(\frac{R-r}{R}\right)^4\right)\right]=$$
$$=\frac{\sqrt{2}}{R-r}\left[1+\sum P_{7;j_0j_1j_2}R^{2j_0}\left[c'+h\log\frac{R-r}{R}\right]^{j_2}\left(\frac{R-r}{R}\right)^{2j_0+j_1+4j_2}\right],$$

where

$$c'=\frac{R^4}{4}c_2+\left(\frac{4}{3}\nu^2-\frac{4}{27}\right)\log\frac{R}{\sqrt{2}}, \quad h=\frac{4}{3}\nu^2-\frac{4}{27}.$$

This gives the announced series expansion (2) through a suitable change of the parameter from c' to C.

The coefficients of the lower terms in the expansion (2) are determined by substituting it in the equation (1) formally and equating the coefficients of $[\log(R-r)]^k\cdot(R-r)^j$. The procedure is the same as that of E. Hille's textbook, [1], pp. 453–454, so we describe the outline.

Let us introduce the variables

$$u=\frac{R}{\sqrt{2}}z^{-1}U, \quad z=\frac{R-r}{R}$$

and rewrite the equation (1) as

$$\left(z\frac{d}{dz}\right)^2 U-\left(3+\frac{z}{1-z}\right)z\frac{dU}{dz}+\left(2+\frac{z}{1-z}-\nu^2\frac{z^2}{(1-z)^2}\right)U=$$
$$=2U^3-R^2z^2U.$$

It has been verified that there exists a solution of the form

$$U=\sum_{j=0}^{\infty}U_j(\log z;\ R,\ C,\ \nu)z^j,$$

where the U_j's are polynomials in $\log z=\log\frac{R-r}{R}$ of degree $\leqq j/4$. Substituting this in the above equation and equating the coefficient of z^j, we have an infinite system of second order equations for the successive determination of the U_j's. For $j=0$ we get

$$U_0''-3U_0'+2U_0=2U_0^3,\quad '=\frac{d}{d(\log z)},$$

which admits the expected solution $U_0\equiv 1$. Then for $j\geqq 1$ we get

$$U_j''+(2j-3)U_j'+(j-4)(j+1)U_j=H_j,$$

where H_j is a polynomial in U_k, $0\leqq k\leqq j-1$, ν^2 and R^2 as follows.

$$H_j=\sum_{k=0}^{j-1}[U_k'+(k-1)U_k]+\nu^2\sum_{k=0}^{j-2}(j-k-1)U_k+$$
$$+2\sum_{\substack{1\leqq k_1,k_2,k_3\\ k_1+k_2+k_3=j}}U_{k_1}U_{k_2}U_{k_3}-R^2U_{j-2}.$$

For $j=1, 2, 3$ the U_j's are constants. For $j=4$ the equation will take the form

$$U_4''+5U_4'=H_4\equiv\frac{4}{3}\nu^2-\frac{4}{27}.$$

The polynomial solution is

$$U_4=\frac{H_4}{5}\log z+C,$$

where C is an arbitrary constant. If we fix the constant C, then the equation for $j\geqq 5$ will determine the unique solution U_j which is a polynomial in $\log z$ successively. Thus we can determine the coefficients $a_{j_0j_1j_2}$'s from the coefficients of the U_j's as polynomials in $\log z$, R and C. This completes the proof of Theorem 1.

From the manner of construction of the solutions $u(r;\ R,\ C)$ we have the following proposition immediately, which will be used in the next section.

Proposition 1. *Let* $0<r_0<R\leqq+\infty$ *and let* $u=u(r)$, $r_0\leqq r<R$, *be a solution of the equation* (1) *such that*

(14) $$u\longrightarrow+\infty,\quad \frac{r}{u}\frac{du}{dr}\longrightarrow+\infty\ \text{and}\ \frac{F(u)}{\left(\frac{du}{dr}\right)^2}\longrightarrow\frac{1}{2}$$

as $r\uparrow R$. Then $R<+\infty$ and there exists a constant C such that $u(r)$ is identical with $u(r;R,C)$ for $R-\delta<r<R$, δ being a sufficiently small positive number.

Proof. Let $r=r(u)$ be the inverse function of $u=u(r)$ and put

$$x(u)=\frac{u}{r(u)u'(r(u))},\quad y(u)=\frac{F(u)}{u'(r(u))^2},\quad '=\frac{d}{dr}.$$

They are well-defined for sufficiently large u because $u'(r)$ is supposed to be positive when r is near R. Then $x=x(u)$, $y=y(u)$ is a solution of the system (5). The condition (14) means that $x(u)\to 0$, $y(u)\to 1/2$ as $u\to+\infty$. On the other hand the inverse transformation $(x,y)\to(\xi,\eta)$ of (8) can be applied provided that x, $y-\frac{1}{2}$ and u^{-2} are sufficiently small. Therefore the solution $x(u)$, $y(u)$ should be identical with $x(u;c_1,c_2)$, $y(u;c_1,c_2)$ for suitably chosen c_1, $c_1>0$, and c_2. Then $R=\sqrt{2}/c_1$, which is finite, and the inverse function $r(u)$ of the solution $u(r)$ in question turns out to be identical with $r(u;c_1,c_2)$ defined by (9). This completes the proof of Proposition 1.

3. Continuation to movable infinitudes

Let us recall the results of Y. Kametaka.

1) The exists a unique solution $u=w(r;\nu)$ of the equation (1) satisfying $0<u\leqq 1$ for any r, $r>0$ ([3], Theorem 6 and 7).

2) In a neighborhood of $r=0$ there exists a solution $u=w(r;\nu,a_0)$ such that

$$w(r;\nu,a_0)=a_0r^\nu+O(r^{\nu+2})$$

as $r\to 0$. Here a_0 is an arbitrary positive constant ([3], Theorem 5).

3) The unique solution $u=w(r;\nu)$ satisfies

$$w(r;\nu)=a_0(\nu)r^\nu+O(r^{\nu+2}),$$

where $a_0(\nu)$ is a suitably chosen positive constant ([3], Theorem 6).

In this section we discuss the behavior of the solution $w(r;\nu,a_0)$ continued from $r=+0$ to the right as long as possible along the positive r-axis. We will prove the following theorem.

Theorem 2. *If $a_0>a_0(\nu)$ then there exist constants $R=R(\nu,a_0)$, $0<R<+\infty$, and $C=C(\nu,a_0)$, $|C|<+\infty$, such that the solution $w(r;\nu,a_0)$ exists for $0<r<R$ and coincides with the local solution $u(r;R,C)$ having the series expansion (2) for $R-\delta<r<R$, δ being a sufficiently small positive number. This infinitude is a pole if $\nu=1/3$ or a pseudo-pole if $\nu\neq 1/3$ respectively.*

We need some preliminary propositions for the proof of this theorem. For brevity we will write

$$u(r)=w(r;\nu,a_0),$$

supposing that $a_0>a_0(\nu)$. Hereafter the symbol $'$ denotes differentiation.

Proposition 2. *There exists $r_0>0$ such that*

$$u(r_0)>1 \quad \text{and} \quad u'(r_0)>0. \tag{15}$$

Proof. Let $(0, R)$ be the maximal interval of existence of the solution $u(r)$ in question. The inequality

$$w(r;\, \nu)<u(r) \tag{16}$$

holds for sufficiently small r, since $u(r)=a_0r^\nu+O(r^{\nu+2})$, $w(r;\, \nu)=a_0(\nu)r^\nu+O(r^{\nu+2})$ as $r\downarrow 0$ and since $a_0>a_0(\nu)$. We shall show that the inequality (16) holds for any r, $0<r<R$. If it were not true there would exist a point $r=r^*$, $0<r^*<R$, such that $w(r^*;\, \nu)=u(r^*)\equiv u^*$ and $w(r;\, \nu)<u(r)$ for $0<r<r^*$. This leads a contradiction as follows. The equation (1) yields the equation

$$r^{-1}\frac{d}{dr}\left[r\frac{du}{dr}w-r\frac{dw}{dr}u\right]=\tilde{f}(r,\, u)w-\tilde{f}(r,\, w)u,$$

where $u=u(r)$, $w=w(r,\, \nu)$ and

$$\tilde{f}(r,\, u)=\frac{\nu^2}{r^2}u+u^3-u.$$

Integrating this equation yields

$$u'(r^*)-w'(r^*;\, \nu)=\frac{1}{r^*u^*}\int_0^{r^*}\left[\frac{\tilde{f}(r,\, u(r))}{u(r)}-\frac{\tilde{f}(r,\, w(r;\, \nu))}{w(r;\, \nu)}\right]u(r)w(r;\, \nu)r\,dr,$$

where $u^*=u(r^*)=w(r^*;\, \nu)>0$. Since $0<w(r;\, \nu)<u(r)$ for $0<r<r^*$ and since $\frac{\partial}{\partial u}\left[\frac{\tilde{f}(r,\, u)}{u}\right]=2u>0$ for $u>0$, the above relation yields

$$w'(r^*;\, \nu)<u'(r^*)$$

so that $u(r)<w(r;\, \nu)$ for $r^*-\delta<r<r^*$, δ being sufficiently small. This contradicts the definition of r^*. Therefore the inequality (16) holds for any r, $0<r<R$.

Suppose that $u(r)\leqq 1$ for any r, therefore that the inequality

$$0<w(r;\, \nu)<u(r)\leqq 1$$

holds for $0<r<R$. Then this a priori estimate implies the existence of $u(r)$ on the whole positive r-axis, that is, $R=+\infty$, for $\tilde{f}(r,\, u)$ is bounded and continuous in $0\leqq u\leqq 1$ and $r_0\leqq r<+\infty$ jointly, r_0 being an arbitrary positive number. However this contradicts the uniqueness of $w(r;\, \nu)$. Therefore there must exist $r_1>0$ such that $u(r_1)>1$. Since $u(r)$ vanishes as $r\to 0$ it is clear that there exists r_0 between 0 and r_1 for which $u(r_0)>1$ and $u'(r_0)>0$. This completes the proof of Proposition 2.

Let $(0, R)$ be the maximal interval on which $u(r)$ exists and satisfies $u>u(r_0)$. Then we have

Proposition 3. $u'(r)>0$ *for any* r, $r_0\leqq r<R$, *and*

(17) $$\lim_{r\to R} u(r)=+\infty.$$

Proof. Integrating the equation (1), we get

(18) $$u'(r)=r_0r^{-1}u'(r_0)+r^{-1}\int_{r_0}^{r}\tilde{f}(s,\ u(s))sds,$$

where $\tilde{f}(r,\ u)=\frac{\nu^2}{r^2}u+u^3-u$. Since $u(r_0)>1$, we can find a positive constant ε such that

$$\tilde{f}(r,\ u)\geqq\varepsilon>0$$

for any $r\geqq r_0$ and any $u\geqq u(r_0)$. Then, since $u'(r_0)>0$ and since $\tilde{f}(s,\ u(s))\geqq\varepsilon>0$ for $r_0\leqq s<r$, we see from (18) that $u'(r)>0$ for any $r<R$. Integrating (18), we get

$$u(r)\geqq u(r_0)+\left[r_0u'(r_0)-\frac{\varepsilon}{2}r_0^2\right]\log\frac{r}{r_0}-\frac{\varepsilon}{4}r_0^2+\frac{\varepsilon}{4}r^2.$$

If $R=+\infty$ the above inequality shows that $u(r)\to+\infty$ as $r\to R=+\infty$. If $R<+\infty$ and if $u(r)$ is bounded, then, as easily seen from (18), $u'(r)$ should have a finite limit as $r\to R$, as well as $u(r)$, so that $u(r)$ should have a continuation such that $u(r)>u(r_0)$ across $r=R$ to the right. It contradicts the definition of R. Therefore $u(r)$ cannot be bounded, whether R is finite or not. This completes the proof of Proposition 3.

Proposition 4. *We have*

(19) $$\lim_{r\to R}.\sup\frac{ru'(r)}{u(r)}=+\infty.$$

Proof. Put

$$v(r)=\frac{ru'(r)}{u(r)}.$$

Suppose that $v(r)$ were bounded.

Case A: Suppose that $R=+\infty$. The equation (1) yields

$$r\frac{dv}{dr}=\nu^2-v^2+r^2\frac{f(u(r))}{u(r)},$$

where $f(u)=u^3-u$. Since $f(u)/u\to+\infty$ as $u\to+\infty$ and since $v(r)$ is supposed to be bounded, we have the inequality

$$r\frac{dv}{dr}\geq -M+\delta r^2,$$

where $0<\delta$, $M<+\infty$. Then it is clear that $v(r)\to+\infty$ as $r\to R=+\infty$; this contradicts the boundedness of $v(r)$.

Case B: Suppose that $R<+\infty$. Then the boundedness of $v(r)$ implies that $u(r)$, which can be written as

$$u(r)=u(r_0)\exp\left[\int_{r_0}^{r} v(s)s^{-1}ds\right]$$

from the definition of $v(r)$, converges to a finite limit as $r\to R<+\infty$; this contradicts the unboundedness of $u(r)$.

Therefore in each case $v(r)$ cannot be bounded. This completes the proof of Proposition 4.

Thus we know that $u(r)$ is monotone increasing for $r_0\leqq r<R$ and tends to infinity as $r\uparrow R$. Hence the inverse function $r=r(u)$ of $u=u(r)$ is well-defined for $u_0\equiv u(r_0)\leqq u<+\infty$. Put

$$x(u)=\frac{u}{r(u)u'(r(u))}, \quad y(u)=\frac{F(u)}{u'(r(u))^2}.$$

Then we have a solution of the system

$$u\frac{dx}{du}=x[1-G(u)y-v^2x^2], \tag{5}$$

$$u\frac{dy}{du}=y[G(u)+2x-2G(u)y-2v^2x^2].$$

Here the notations

$$f(u)=u^3-u,\quad F(u)=\frac{1}{4}u^4-\frac{1}{2}u^2+\frac{1}{4}\quad \text{and} \tag{3}$$

$$G(u)=\frac{4}{1-u^{-2}}$$

are used samely as in Section 2. We want to prove that $\lim_{u\to+\infty} x(u)$ and $\lim_{u\to+\infty} y(u)$ exist and are equal to 0 and 1/2 respectively.

We will assume the already verified fact that the solution $(x(u), y(u))$, $u_0\leqq u<+\infty$, in question of the system (5) admists the following two properties:

$$0<x(u),\quad 0<y(u), \tag{20}$$

which follows from Proposition 3 and the definition; and

$$\lim_{u\to+\infty}\text{. inf}\, x(u)=0, \tag{21}$$

which is the conclusion of Proposition 4. We are going to replace "lim. inf" in (21) by "lim". In order to do it we use the auxiliary function

$$S(x, y)=1-4y-v^2x^2. \tag{22}$$

Thanks to the estimate

$$G(u)>4$$

we see that if $S(x(u), y(u))<0$ the first equation of (5) will imply $ud(\log x)/du=$

$S-(G-4)y<0$ so that we will be able to remove "inf" of (21).

Proposition 5. *Under the conditions* (20) *and* (21) *there exists* $u_1 \geqq u_0$ *such that the inequality*

$$S(x(u),\ y(u))<0 \tag{23}$$

holds for $u=u_1$.

Proof. Suppose on the contrary that

$$S(x(u),\ y(u)) \equiv 1-4y(u)-v^2x(u)^2 \geqq 0$$

holds for any $u \geqq u_0$. Then it is necessary that

$$0<y(u) \leqq \frac{1}{4}. \tag{24}$$

Since $\lim\inf_{u\to+\infty} x(u)=0$ and since $y(u)$ is confined to the compact interval $[0,\ 1/4]$ by (24), there exist a sequence $(u_n)_{n=1,2,\ldots}$ and accumulation value b of $y(u_n)$ such that

$$u_n \longrightarrow +\infty, \quad x(u_n) \longrightarrow 0 \quad \text{and} \quad y(u_n) \longrightarrow b,$$

as $n\to\infty$, where $0 \leqq b \leqq 1/4$. Let us prove that $b>0$ making use of the function

$$V(x,\ y)=\sqrt{x^2+y^2}$$

in a neighborhood of $x=y=0$. It is clear that the trajectory derivative of the function V

$$u\frac{dV}{du} \equiv x[1-Gy-v^2x^2]\frac{\partial V}{\partial x}+y[G-2Gy+2x-2v^2x^2]\frac{\partial V}{\partial y}=$$

$$=\frac{1}{V}[x^2+Gy^2-Gx^2y+2xy^2-2Gy^3-v^2x^4-2v^2x^2y^2]$$

turns out to be positive provided that $u_0 \leqq u<+\infty$ and $0<V(x,\ y) \leqq V_0$, V_0 being a sufficiently small positive number, because the estimate

$$4<G(u) \leqq \frac{4}{1-u_0^{-2}}$$

holds for any $u \geqq u_0$. Since $y(u_0)>0$ we can find a positive number $V_1 \leqq V_0$ such that

$$0<V_1<V(x(u_0),\ y(u_0)).$$

Then we have $V(x(u),\ y(u))>V_1$ for any $u \geqq u_0$, a fortiori

$$b=V(0,\ b)=\lim_{n\to\infty} V(x(u_n),\ y(u_n)) \geqq V_1>0.$$

This shows that $b \neq 0$, therefore

$$0<b \leqq \frac{1}{4}.$$

Let us fix a positive constant ε satisfying the condition

$$0<\varepsilon<2-2\varepsilon-2v^2\varepsilon^2. \tag{25}$$

Then we can find a sufficiently large n such that

$$x(u_n)<(2b)^{\frac{1}{\varepsilon}}\varepsilon \quad \text{and} \quad \frac{b}{2}<y(u_n).$$

The first equation of (5) yields

$$u\frac{dx}{du}=x[1-G(u)y-v^2x^2]$$
$$<x.$$

Integrating this differential inequality, we see that the estimate

$$0<x(u)\leqq x(u_n)\frac{u}{u_n}<\varepsilon$$

holds as long as $u_n\leqq u\leqq u^*\equiv(2b)^{-\frac{1}{\varepsilon}}u_n$. Since $G>4$, $0<y\leqq 1/4$, $|x|\leqq\varepsilon$ and since $2-2\varepsilon-2v^2\varepsilon^2>\varepsilon$, the second equation of (5) yields

$$u\frac{dy}{du}=y[G(u)(1-2y)+2x-2v^2x^2]$$
$$\geqq y\left[4\left(1-2\cdot\frac{1}{4}\right)-2\varepsilon-2v^2\varepsilon^2\right]$$
$$>\varepsilon y$$

as long as $u_n\leqq u\leqq u^*$. Integrating this differential inequality from $u=u_n$ to $u=u^*$ yields

$$y(u^*)\geqq y(u_n)\left(\frac{u^*}{u_n}\right)^{\varepsilon}=\frac{1}{2b}y(u_n).$$

The right-hand side exceeds $1/4$ because $y(u_n)>b/2$. This contradicts the assumed estimate (24); therefore $S\geqq 0$ cannot remain valid. This completes the proof of Proposition 5.

Proposition 6. *Under the conditions* (20) *and* (21) *the inequality* (23) *holds for any* $u\geqq u_1$ *if it holds for* $u=u_1$. *Moreover we have*

$$\lim_{u\to+\infty}x(u)=0. \tag{26}$$

Proof. Firstly we will prove that the inequality

$$S(x(u),\ y(u))\equiv 1-4y(u)-v^2x(u)^2<0 \tag{23}$$

holds for any $u\geqq u_1$. If this were not true there would exist u^*, $u^*\geqq u_1$, for which $S(x(u^*),\ y(u^*))=0$ and $S(x(u),\ y(u))<0$ for $u_1\leqq u<u^*$. Applying $S=0$ to the trajectory derivative of S at $u=u^*$, we have

$$u\frac{d}{du}S(x(u), y(u))|_{u=u^*} = -y(u^*)[2G(u^*)+8x(u^*)].$$

The right-hand side is negative since $y>0$, $G>4$ and $x>0$. Then we have $S(x(u), y(u))>0$ for $u^*-\delta<u<u^*$, δ being a sufficiently small positive number. This contradicts the definition of u^*. Therefore (23) holds for any $u\geqq u_1$. Now we are ready to prove $x'(u)<0$. The first equation of (5) can be written as

$$u\frac{d}{du}x(u)=x(u)[S(x(u), y(u))-(G(u)-4)y(u)].$$

This is strictly negative because $0<x$, $0<y$, $4<G$ and since $S<0$. Therefore $x(u)$ is monotone decreasing and $\lim_{u\to+\infty} x(u)$ exists and is equal to $\liminf_{u\to+\infty} x(u)=0$. This completes the proof of Proposition 5.

Proposition 7. *For any solution* $(x(u), y(u))$ *satisfying* $y(u)>0$ *the condition* (26) *implies*

$$\lim_{u\to+\infty} y(u)=\frac{1}{2}. \tag{27}$$

Proof. Let ε be a positive constant such that

$$0<\varepsilon(1+v^2)<1. \tag{28}$$

Fixed such an ε, we can find $u_2=u_2(\varepsilon)$ such that the inequality

$$4<G(u)\leqq 4(1+\varepsilon)$$

and the estimate

$$|x(u)|\leqq\varepsilon$$

holds for $u\geqq u_2$. Such a large u_2 can be found because $\lim_{u\to+\infty} G(u)=4$ and $\lim_{u\to+\infty} x(u)=0$, (26). Then the second equation of the system (5)

$$u\frac{dy}{du}=y[G(u)+2x-2v^2x^2-2G(u)y]$$

yields the following differential inequalities.

$$y[4-2\varepsilon(1+v^2)-8(1+\varepsilon)y]\leqq u\frac{dy}{du}\leqq$$

$$\leqq y[4(1+\varepsilon)+2\varepsilon(1+v^2)-8y].$$

Applying the comparison theorem ([2], p. 18, Theoreme 9), we have

$$\frac{cau^a}{1+cbu^a}\leqq y(u)\leqq\frac{CAu^A}{CBu^A-1}$$

for $u>u_2$, where

$$a=4-2\varepsilon(1+v^2),\ b=8(1+\varepsilon),\ c=a^{-1}u_2^{-A}y(u_2);$$

$$A=4(1+\varepsilon)+2\varepsilon(1+\nu^2),\ B=8 \quad \text{and} \quad C=B^{-1}u_2^{-A}.$$

Therefore

$$\frac{a}{b}=\frac{2-\varepsilon(1+\nu^2)}{4(1+\varepsilon)}\leqq \lim_{u\to+\infty}\inf y(u)\leqq$$

$$\leqq \lim_{u\to+\infty}\sup y(u)\leqq \frac{A}{B}=\frac{2(1+\varepsilon)+\varepsilon(1+\nu^2)}{4}.$$

Since ε can be chosen arbitrarily small see that $\lim_{u\to+\infty} y(u)$ exists and is equal to 1/2. This completes the proof of Proposition 7.

Combining the above propositions with Proposition 1 prepared in Section 2, we can give a proof of Theorem 2.

Proof of Theorem 2. We are concerned with the solution $u(r)=w(r;\ \nu,\ a_0)$ provided that $a_0>a_0(\nu)$. Let $(0,\ R)$ be the maximal interval of existence. From Propositions 2 and 3 we see that $u(r)$ is monotone increasing and $u(r)>1$ for $r\geqq r_0$, r_0 being a suitable positive number, and that $u(r)$ tends to infinity as $r\uparrow R$. Thus we can define the corresponding solution

$$x(u)=\frac{u}{r(u)u'(r(u))},\quad y(u)=\frac{F(u)}{u'(r(u))^2}$$

of the system (5). It follows from Propositions 6 and 7 that the solution $x=x(u)$, $y=y(u)$ converges to $x=0$, $y=1/2$ as $u\to+\infty$. Then Proposition 1 can be applied to the solution $u(r)$, for the required condition (14) follows from

$$\lim_{r\to R}\frac{ru'(r)}{u(r)}=\lim_{u\to+\infty}\frac{1}{x(u)}=+\infty$$

and

$$\lim_{r\to R}\frac{F(u(r))}{u'(r)^2}=\lim_{u\to+\infty}y(u)=\frac{1}{2}.$$

This completes the proof of Theorem 2.

4. Appendix

As for real solutions we can assert that there will not appear essentially different movable infinitudes other than those having the expansion (2). We shall show it in this appendix.

Let us consider a solution $x=x(u;\ c_1,\ c_2)$, $y=y(u;\ c_1,\ c_2)$, $u_0\leqq u<+\infty$, given by (9), of the system (5). Now we assume that $c_1<0$. Then we put

$$(10)^- \qquad r(u;\ c_1,\ c_2)=-uF(u)^{-\frac{1}{2}}x(u;\ c_1,\ c_2)^{-1}y(u;\ c_1,\ c_2)^{\frac{1}{2}}$$

and

$$(11)^- \qquad R \equiv \lim_{u \to +\infty} r(u; c_1, c_2) = -\frac{\sqrt{2}}{c_1}.$$

(Compare with (10) or (11) of Section 2, respectively.) Then the inverse function $u = u(r)$ is well-defined, monotone decreasing for $R < r \leqq r_0 \equiv r(u_0; c_1, c_2)$ and satisfies the equation (1). This solution admits a series expansion which is convergent for $R < r < R + \delta$, δ being a sufficiently small positive number, and is essentially the same as the expansion (2). That is, in parallel with Theorem 1 we have a solution $u = u^-(r; R, C)$ having a series expansion of the form

$$(2)^- \quad u = \frac{\sqrt{2}}{r-R}\left[1 + \sum a_{j_0 j_1 j_2} R^{2j_0}\left[\left(\frac{4}{15}v^2 - \frac{4}{135}\right)\log\left(\frac{r-R}{R}\right) + \right.\right.$$
$$\left.\left. + C\right]^{j_2}\left(\frac{R-r}{R}\right)^{2j_0+j_1+4j_2}\right].$$

Here the coefficients $a_{j_0 j_1 j_2}$'s are the same as those of (2). This is an actual solution for $R < r < R + \delta$ if C is real and δ is sufficiently small.

There exist no movable infinitudes for real-valued solutions of real variable other than those of $\pm u(r; R, C)$ and $\pm u^-(r; R, C)$'s. More precisely we have the following theorems.

Theorem 3. *Let* $0 < R < +\infty$ *and let* $u = u(r)$, $r_0 \leqq r < R$, *be a solution of* (1) *such that* $u(r) \to +\infty$ *as* $r \uparrow R$. *Then there exists a constant* C *such that* $u(r) = u(r; R, C)$ *for* $R - \delta < r < R$, δ *being a sufficiently small positive number.*

Theorem 3⁻. *Let* $0 < R < +\infty$ *and let* $u = u(r)$, $R < r \leqq r_0$, *be a solution of* (1) *such that* $u(r) \to +\infty$ *as* $r \downarrow R$. *Then there exists a constant* C *such that* $u(r) = u^-(r; R, C)$ *for* $R < r < R + \delta$, δ *being a sufficiently small positive number.*

Remark. It is clear that a negative movable infinitude of a real solution can be identified with $-u(r; R, C)$ or $-u^-(r; R, C)$, because the equation (1) is invariant under the change of the sign of u.

Proof of Theorem 3. Let $u = u(r)$, $r_0 \leqq r < R$, be a solution such that $\lim_{r \uparrow R} u(r) = +\infty$. We may assume that $u(r_0) > 1$ and $u'(r_0) > 0$. Then we can repeat Propositions 3 and 4. Thus the inverse function $r = r(u)$ is well-defined and we obtain a solution $(x(u), y(u))$, $u_0 \equiv u(r_0) \leqq u < +\infty$, of the system (5), which satisfies (20) and (21). Propositions 5, 6 and 7 can be repeated to deduce from (20) and (21) that $\lim_{u \to +\infty} x(u) = 0$ and $\lim_{u \to +\infty} y(u) = 1/2$. Then we are ready to apply Proposition 1. This completes the proof of Theorem 3.

Remark. The above argument does not require the hypothesis that $R < +\infty$. Therefore we see that any real-valued solution cannot approach a "fixed" infinitude at $r = +\infty$ along the real axis.

Proof of Theorem 3⁻. Let $0 < R$ and let $u = u(r)$, $R < r \leqq r_0$, be a solution such that $\lim_{r \downarrow R} u(r) = +\infty$. Then the integration of the equation (1) from r, $R < r \leqq r_0$, to r_0 shows that $u'(r) < 0$ for $R < r \leqq r_0$ (see (18)). Therefore the inverse function $r = r(u)$ of $u = u(r)$ is well-defined and monotone decreasing for $u_0 \equiv u(r_0) \leqq u < +\infty$. Moreover we have

$$(19)^- \qquad \lim_{r \to R}.\inf \frac{ru'(r)}{u(r)} = -\infty.$$

Indeed, otherwise

$$u(r) = u(r_0) \exp\left[-\int_r^{r_0} \frac{su'(s)}{u(s)} s^{-1} ds \right]$$

would tend to a finite limit as $r \to R > 0$. Putting

$$x(u) = \frac{u}{r(u)u'(r(u))}, \quad y(u) = \frac{F(u)}{u'(r(u))^2},$$

we get a solution $(x(u), y(u))$ of (5), which satisfies

$$(20)^- \qquad x(u) < 0 < y(u)$$

and

$$(21)^- \qquad \lim_{u \to +\infty}.\sup x(u) = 0.$$

Then we can verify the following propositions successively.

Proposition 5⁻. *Under the conditions* $(20)^-$ *and* $(21)^-$ *there exists* $u_1 \geqq u_0$ *such that the inequality*

$$(23)^- \qquad S(x(u), y(u)) \equiv 1 - 4y(u) - v^2 x(u)^2 < 0$$

and the estimate

$$(29) \qquad -1 < x(u) < 0$$

hold simultaneously for $u = u_1$.

Proposition 6⁻. *Under the conditions* $(20)^-$ *and* $(21)^-$ *the inequality* $(23)^-$ *and the estimate* (29) *hold for any* $u \geqq u_1$. *Moreover we have*

$$(26) \qquad \lim_{u \to +\infty} x(u) = 0.$$

Let us postpone proofs of these propositions, which are essentially the same as those of Propositions 5 and 6. Assuming these propositions at the instance, we can repeat the statement and the proof of Proposition 7 to derive $\lim_{u \to +\infty} y(u) = 1/2$. Then the analogue of Proposition 1 shows that $r(u)$ is identical with $r(u; c_1, c_2)$ if we choose suitable constants c_1, $c_1 < 0$, and c_2. This completes the proof of Theorem 3^-.

It remains to prove Propositions 5^- and 6^-.

Proof of Proposition 5^-. Suppose on the contrary that

$$-1<x(u)<0 \quad \text{implies} \quad S(x(u),\, y(u))\geqq 0,$$

a fortiori

$$-1<x(u)<0 \quad \text{implies} \quad 0<y(u)\leqq \frac{1}{4}. \tag{30}$$

Since $\limsup_{u\to+\infty} x(u)=0$, there exists a sequence $(u_n)_{n=1,2,\ldots}$ such that $u_n\to+\infty$ and $x(u_n)\to 0$. We may assume that $-1<x(u_n)<0$. Then the hypothesis (30) confines the values of $y(u_n)$ to the compact interval $[0, 1/4]$, therefore we may assume that $y(u_n)\to b$, $0\leqq b\leqq 1/4$. By using the function $V(x, y)=\sqrt{x^2+y^2}$ we see that b should be positive. Fixed a positive number ε satisfying the condition (25), we find a large number n such that

$$-(2b)^{\frac{1}{\varepsilon}}\varepsilon<x(u_n)<0 \quad \text{and} \quad \frac{b}{2}<y(u_n).$$

Then we have

$$-\varepsilon<x(u)<0$$

for any u, $u_n\leqq u\leqq u^*\equiv(2b)^{-1/\varepsilon}u_n$, by integrating the first equation of (5) in the same manner as in Proposition 5. Since we can take ε so that $\varepsilon<1$, we may assume that the above estimate of $x(u)$ requires $y(u)\leqq 1/4$ by the hypothesis (30). Then the second equation of (5) yields

$$u\frac{d}{du}y(u)>\varepsilon y(u)$$

as long as $u_n\leqq u\leqq u^*$ and it follows that $y(u^*)>1/4$ while $-1<-\varepsilon<x(u^*)<0$; this contradicts the hypothesis (30). This completes the proof of Proposition 5^-.

Proof of Proposition 6^-. Suppose that the inequality $(23)^-$ and the estimate (29) hold simultaneously for $u=u_1$. Then they hold for any $u\geqq u_1$. If this were not true there would exist $u^*>u_1$ such that $S(x(u), y(u))<0$, $-1<x(u)<0$ for $u_1\leqq u<u^*$ and either

$$A:\quad S(x(u^*),\, y(u^*))\leqq 0 \quad \text{and} \quad -1=x(u^*),$$

or

$$B:\quad S(x(u^*),\, y(u^*))=0 \quad \text{and} \quad -1\leqq x(u^*)$$

If A is the case, then the first equation of (5) yields

$$u\frac{d}{du}x(u)|_{u=u^*}=x(u^*)[S(x(u^*),\, y(u^*))-(G(u^*)-4)y(u^*)]>0,$$

therefore $x(u) < -1$ for $u^* - \delta < u < u^*$, δ being a sufficiently small positive number. This contradicts the definition of u^*. If B is the case, then applying $S=0$ to the trajectory derivative of S, we have

$$u \frac{d}{du} S(x(u), y(u))|_{u=u^*} = -y(u^*)[2G(u^*) + 8x(u^*)] < 0,$$

because $G > 4$ and $|x| \leqq 1$, therefore $S(x(u), y(u)) > 0$ for $u^* - \delta < u < u^*$. This contradicts the definition of u^*. Therefore $(23)^-$ and (29) hold for any $u \geqq u_1$.

Then it is clear from the first equation of (5) that $x'(u) > 0$ for any $u \geqq u_1$ and $\lim_{u \to +\infty} x(u) = \lim.\sup_{u \to +\infty} x(u) = 0$. This completes the proof of Proposition 6^-.

Remark. The above modification of Proposition 5 and 6 is unnecessary if we suppose that $\nu \geqq 1$ from the start. Indeed, in the case where $\nu \geqq 1$, the hypothesis $S=0$ implies $|x| < 1/\nu \leqq 1$ provided that $y > 0$. Then, avoiding the additional discussion about the estimate (29), we can assert that the trajectory derivative of S turns out to be negative everywhere as long as $S=0$ and $y > 0$.

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