J. Math. Kyoto Univ. (JMKYAZ)
22-4 (1983) 567-581

Classification of the double projections of Veronese varieties

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(Communicated by Prof. M. Nagata, June 30, 1981)

1. Introduction

Let r, s be arbitrary positive integers. The Veronese variety $V_{r,s}$ over a field k is the projective variety in $\boldsymbol{P}_k^{N-1}$, $N=\binom{r+s-1}{r-1}$, whose homogeneous coordinate ring is generated by the N monomials of degree s in r indeterminates. By a projection of $V_{r,s}$ we understand a projective variety in $\boldsymbol{P}_k^{N-d}$, $d>1$, whose homogeneous coordinate ring is generated by $N-d+1$ such monomials. In [5] Gröbner showed that the defining prime ideal of $V_{r,s}$ is perfect, i.e., the homogeneous coordinate ring of $V_{r,s}$ is Cohen-Macaulay, but certain projections of $V_{r,s}$ in $\boldsymbol{P}_k^{N-2}$ have imperfect defining prime ideals. From this phenomenon he then posed the problem of classifying projections of Veronese varieties.

There were first some efforts of Renschuch to solve this problem in [9], [10]. But the first important result is due to Schenzel, who showed in [12] which projections of $V_{r,s}$ in $\boldsymbol{P}_k^{N-2}$ have Cohen-Macaulay or Buchsbaum local rings at the vertex of their affine cones. Note that the homogeneous coordinate ring of a projective variety is Cohen-Macaulay if and only if the local ring at the vertex of its affine cone is Cohen-Macaulay (see e.g. [8, Proposition 4.10]), and that a local ring A is called Buchsbaum if for all ideals q generated by a system of parameters of A the difference $l(A/q)-e(q; A)$ between length and multiplicity is an invariant $i(A)$ of A; hence A is Cohen-Macaulay if and only if A is Buchsbaum with $i(A)=0$ (see [13], [14] for further informations). However, Schenzel's method doesn't work for the classification of the projections of $V_{r,s}$ in $\boldsymbol{P}_k^{N-d}$ $d>2$ [12, p. 396].

In this paper we shall give a complete classification of the projections of $V_{r,s}$ in $\boldsymbol{P}_k^{N-3}$ (double projections) under the same aspect by using some results on the Cohen-Macaulay or Buchsbaum property of affine semigroup rings.

Let $t_1,\dots, t_r$ be r indeterminates over k. If H is an additive semigroup in $\boldsymbol{N}^r$, one can define the semigroup ring $k[H]$ of H over k to be the subring of $k[t_1,\dots, t_r]$ generated by all monomials $t_1^{\alpha_1}\cdots t_r^{\alpha_r}$ with $(\alpha_1,\dots, \alpha_r)\in H$. Thus, if H is a finitely generated additive semigroup with zero in $\boldsymbol{N}^r$, $k[H]$ is an affine ring, hence we call H an affine semigroup. Notice that $k[H\backslash\{0\}]$ may be considered as an ideal of

$k[H]$, which we shall denote by m_H. Then, by definition, the homogeneous coordinate ring of a projection of $V_{r,s}$ in $\boldsymbol{P}_k^{N-d}$ is isomorphic to the semigroup ring $k[H]$ of an affine semigroup H in $\boldsymbol{N}^r$ generated by a $N-d+1$ element subset of the set $\{(\alpha_1,\dots,\alpha_r)\in\boldsymbol{N}^r;\ \alpha_1+\cdots+\alpha_r=s\}$, and the local ring at the vertex of its affine cone is just $k[H]_{m_H}$.

In [3, Chapter III, §3] and [4, §3] Goto and Watanabe have already found general criteria for an affine semigroup H to have Cohen-Macaulay or Buchsbaum $k[H]_{m_H}$. But these criteria are rather complicated and hence not proper for the classification of the double projections of Veronese varieties. Hence, in Section 2, we will first prepare simpler criteria for some classes of affine semigroups which contain the associated affine semigroups of the double projections of $V_{r,s}$. In order to check whether a such affine semigroup satisfies the conditions of a criterion we shall study, in Section 3, sum representations of elements of the set $\{(\alpha_1,\dots,\alpha_r)\in\boldsymbol{N}^r;\ \alpha_1+\cdots+\alpha_r=s\}$. Finally, in Section 4, using results of the preceding sections we shall give a table from which one can see which double projections of $V_{r,s}$ have Cohen-Macaulay or Buchsbaum local rings at the vertex of their affine cones.

2. On affine semigroup rings

Let H be an affine semigroup in $\boldsymbol{N}^r$. We shall denote by $G(H)$ the additive subgroup of $\boldsymbol{Z}^r$ generated by H and by $\bar{H}$ the set $\{e\in G(H);\ me\in H$ for some positive integer $m\}$.

Further, if E and F are two sets in $\boldsymbol{N}^r$, we shall denote by $E+F$ and $E-F$ the sets $\{e+f;\ e\in E$ and $f\in F\}$ and $\{e-f;\ e\in E$ and $f\in F\}$, respectively. If $F=\{f\}$ for some $f\in\boldsymbol{N}^r$, we shall replace $E+F$ and $E-F$ by $E+f$ and $E-f$.

Then the results on affine semigroup rings which we shall use in the classification of the double projections of Veronese varieties are the following lemmas:

Lemma 1. *$k[H]$ is Cohen-Macaulay if $\bar{H}=H$.*

Lemma 2. *Assume that there exist elements $e_1,\dots,e_n$ in H such that*
(i) *$e_1,\dots,e_n$ are linearly independent over $\boldsymbol{Q}$,*
(ii) *mH is contained in the affine semigroup in $\boldsymbol{N}^r$ generated by $e_1,\dots,e_n$ for some positive integer m.*

Set $H_1=\{e\in\bar{H};\ e+e_i\in H$ and $e+e_i\in H$ and $e+e_j\in H$ for some $i\neq j$, $1\leq i, j\leq n\}$. Then $k[H]$ is Cohen-Macaulay if and only if $H_1=H$.

Lemma 3. *Let H be of the type as in Lemma 2 with $n\geq 2$. Set $H_2=\{e\in\bar{H};\ e+2e_i\in H$ and $e+2e_j\in H$ for some $i\neq j$, $1\leq i, j\leq n\}$. Then $k[H]_{m_H}$ is a Buchsbaum ring if and only if $(H\backslash\{0\})+H_2\subset H$.*

Lemma 4. *Let $\operatorname{rank}_{\boldsymbol{Z}} G(H)\geq 2$. Assme that there exists an element $f\in\boldsymbol{N}^r\backslash H$ with $2f\in H$ such that $k[K]$ and $k[H]/k[(H-f)\cap H]$ are Cohen-Macaulay rings, where K denotes the affine semigroup in $\boldsymbol{N}^r$; generated by H and f. Then $k[H]_{m_H}$ is a Cohen-Macaulay or non-Cohen-Macaulay Buchsbaum ring if and only if $\dim k[H]/k(H-f)\cap H]\geq\operatorname{rank}_{\boldsymbol{Z}} G(H)-1$ or $(H\backslash\{0\})+f\subset H$, respectively.*

Lemma 1 is a result of Hochster [7, Theorem 1]. As a consequence, since $\bar{H}$ is also an affine semigroup in $\boldsymbol{N}^r$; with $\bar{\bar{H}}=\bar{H}$ (cf. the proof of [7, Proposition 1]), $k[\bar{H}]$ is always a Cohen-Macaulay ring.

Lemma 2 is a modified version of a result of Goto, Suzuki, and Watanabe [2, Theorem 1]. It differs from [2, Theorem 1] only in using H_1 instead of the set $H_1':=\{e\in G(H);\ e+e_i\in H$ and $e+e_j\in H$ for some $i\neq j,\ 1\leqq n\}$. But this difference is unessential. Indeed, since $\bar{H}$ is of the same type as H, we can apply [2, Theorem 1] to $\bar{H}$ and get $H_1'\subseteq\{e\in G(\bar{H});\ e+e_i\in H$ and $e+e_j\in H$ for some $i\neq j,\ 1\leqq i, j\leqq n\}=\bar{H}$. Now, regarding the definition of H_1 and H_1', it is easy to see that $H_1=H_1'$.

Lemma 3 and Lemma 4 are new results. For their proofs we shall need the following.

Lemma 5. *Let* $(R, \mathfrak{p})$ *be a local proper subring of a Cohen-Macaulay local ring* $(S, \mathfrak{q})$ *with* $\dim R=\dim S\geqq 2$ *such that*

(i) $\mathfrak{p}S$ *is a q-primary ideal in* S,

(ii) S/R *is a finitely generated Cohen-Macaulay R-module.*

Then R *is a Cohen-Macaulay or non-Cohen-Macaulay Buchsbaum ring if and only if* $\dim S/R\geqq\dim R-1$ *or* $\mathfrak{p}S\subset R$, *respectively.*

Proof. We shall use the notion of local cohomology groups (see e.g. [6]). Note first that a finitely generated module M over a local ring $(A, \mathfrak{m})$ is Cohen-Macaulay if and only if the ith local cohomology group $H_{\mathfrak{m}}^i(M)$ of M with respect to $\mathfrak{m}$ vanishes for $i=0,\dots,\dim M-1$. Then, using (i) we have $H_{\mathfrak{p}}^i(S)\cong H_{\mathfrak{q}}^i(S)=0$ for $i=0,\dots,\dim R-1$. Therefore, from the exact sequence of local cohomology groups of the sequence of R-modules

$$0\longrightarrow R\longrightarrow S\longrightarrow S/R\longrightarrow 0$$

with respect to p we get

$$H_{\mathfrak{p}}^i(R)\cong\begin{cases}0 & \text{for}\quad i=0,\\ H_{\mathfrak{p}}^{i-1}(S/R) & \text{for}\quad i=1,\dots,\dim R-1.\end{cases}$$

But by (ii) $H_{\mathfrak{p}}^{i-1};(S/R)=0$ for $i=1,\dots,\dim S/R$. Hence, since $H_{\mathfrak{p}}^{\dim S/R}(S/R)\neq 0$, R is a Cohen-Macaulay ring if and only if $\dim S/R\geqq\dim R-1$.

Now, if R is a non-Cohen-Macaulay Buchsbaum ring, $H_{\mathfrak{p}}^{\dim S/R}(S/R)\cong H_{\mathfrak{p}}^{\dim S/R+1}(R)$ is a vector space over $R/\mathfrak{p}$ [11, Lemma 3]. But that happens only if S/R is also a vector space over $R/\mathfrak{p}$ or, equivalently, $\mathfrak{p}S\subset R$.

Conversely, if $\mathfrak{p}S\subset R$, S/R is a vector space over $R/\mathfrak{p}$. Hence $H_{\mathfrak{p}}^i(R)=0$ for $i\neq 1, \dim R$ and $H_{\mathfrak{p}}^1(R)\cong S/R$. Thus, by [14, Corollary 1.1], R must be then a non-Cohen-Macaulay Buchsbaum ring, as required.

The proof of Lemma 5 is now complete.

Proof of Lemma 3. For all elements $a=(\alpha_1,\dots,\alpha_r)\in\boldsymbol{N}^r$, we shall denote by t^a the monomial $t_1^{\alpha_1}\cdots t_r^{\alpha_r}$. Assume that $k[H]_{m_H}$ is a Buchsbaum ring. Then, since $t^{2e_1},\dots,t^{2e_r}$ form a system of parameters for $k[H]_{m_H}$ by [2, (1.11)],

$$m_H(t^{2e_i}k[H]:t^{2e_j})\subseteq t^{2e_i}k[H]$$

for all $i\neq j$, $1\leqq i, j\leqq n$ (see [13, §2]). Let $f\in H_2$ arbitrary. Then for some $i\neq j$, $1\leqq i, j\leqq n$,

$$t^{f+2e_i}t^{2e_j}=t^{f+2e_j}t^{2e_i}$$

is a relation of elements of $k[H]$, hence $t^{f+2e_i}\in t^{2e_i}k[H]: t^{2e_j}$. Let $e\in H\backslash\{0\}$ arbitrary. Then $t^e\in m_H$, hence $t^e t^{f+2e_i}\in t^{2e_i}k[H]$. Thus, we can find an element $g\in H$ such that

$$t^e t^{f+2e_i}=t^g t^{2e_i},$$

hence $e+f=g\in H$. So we have proved the relation $(H\backslash\{0\})+H_2\subset H$.

Conversely, assume that $(H\backslash\{0\})+H_2\subset H$. Then, with similar arguments as above, we can first show that for $f\in H_2$ $t^{f+e_i}\in t^{e_i}k[H]$: m_H, and $f\in H$ only if $t^{f+e_i}\in t^{e_i}k[H]$. Hence, since $(t^{e_i}k[H]: m_H)\backslash t^{e_i}k[H]$ is a vector space over k with finite dimension, $H_2\backslash H$ is a finite set. Now it is easy to see that H_2 is an affine semigroup and that $k[H_2]$ satisfies the following conditions:

(i) $m_H k[H_2]\subset k[H]$ and it is a m_{H_2}-primary ideal in $k[H_2]$,

(ii) $k[H_2]/k[H]$ is a finite-dimensional vector space over k.

Further, since $G(H_2)=G(H)$, $\dim k[H_2]=\dim k[H]=\operatorname{rank}_{\boldsymbol{Z}} G(H)\geqq 2$. Therefore, by Lemma 5, to show that $k[H]_{m_H}$ is a Buchsbaum ring it suffices to show that $k[H_2]$ is a Cohen-Macaulay ring. Note first that H_2 is of the same type as H. Further, if $e\in\bar{H}=\bar{H}_2$ with $e+e_i$, $e+e_j\in H_2$ for some $i\neq j$, $1\leqq i, j\leqq n$, $e+2e_i$, $e+2e_j\in(H\backslash\{0\})+H_2\subset H$, hence by the definition of H_2 $e\in H_2$ again. Then, by Lemma 2, $k[H_2]$ is Cohen-Macaulay, as required.

Proof of Lemma 4. Since $f\notin H$, $k[H]$ is a proper subring of $k[K]$. Since $2f\in H$, $K=H\cup(H+f)$; hence we can conclude that $m_H k[K]$ is a m_K-primary ideal in $k[K]$ and that $k[K]/k[H]\cong k[H]/k[(H-f)\cap H]$. Further, since $G(K)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=G(H)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$, $\dim k[K]=\dim k[H]=\operatorname{rank}_{\boldsymbol{Q}} G(H)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=\operatorname{rank}_{\boldsymbol{Z}} G(H)\geqq 2$. Therefore, we can apply Lemma 5 to the rings $k[H]_{m_H}\subset k[K]_{m_K}$. Note that $m_H k[K]_{m_K}\subset k[H]_{m_H}$ if and only if $(H\backslash\{0\})+f=(H\backslash\{0\})+K\subset H$. Then the statement of Lemma 4 is immediate.

Remark. In Lemma 5, using [11, Satz 2] one can show that if R is a Buchsbaum ring, $i(R)=(\dim R-1)l(S/R)$. Thus, in Lemma 3 or Lemma 4, if $k[H]m_H$ is a Buchsbaum ring, $i(k[H]_{m_H})=(n-1)\#(H_2\backslash H)$ or $(\operatorname{rank}_{\boldsymbol{Z}} G(H)-1)\#(K\backslash H)$, respectively, (cf. [4, Theorem (3.1)].

3. On sum representations in $\boldsymbol{N}^r$

Let c, d be elements of $\boldsymbol{Z}^r$. By a representation of the sum $c+d$ we understand a sum $c_1+d_1=c+d$ with c_1, $d_1\in\boldsymbol{Z}^r$. Two representations c_1+d_1 and c_2+d_2 of $c+d$ are called to be identical if $\{c_1, d_1\}=\{c_2, d_2\}$; for the contrary we say that they are different.

The aim of this section is to study sum representations of the elements of the set $J:=\{(\alpha_1,\dots,\alpha_r)\in \boldsymbol{N}^r;\ \alpha_1+\cdots+\alpha_r=s\}$.

Set for some elements of J, if that is the case,

$$e_1=(s,\dots)$$

$$e_{12}=(s-1,\ 1,\dots)$$

$$e_{13}=(s-1,\ 0,\ 1,\dots)$$

$$f_1=(s-2,\ 2,\dots)$$

$$f_2=(s-3,\ 3,\dots)$$

$$f_3=(s-2,\ 1,\dots),$$

where the lined points mean that the rested components are zero.

Then we have the following result.

Lemma 6. *Let c, d be elements of such that by every permutation of the components of $\boldsymbol{N}^r$ $c+d$ doesn't have the form of e_1+e_1, e_1+e_{12}, e_1+f_1, e_1+f_2, e_1+f_3, $e_{12}+e_{12}$, $e_{12}+e_{13}$, $e_{12}+f_1$ (note that $e_1+f_1=e_{12}+e_{12}$, $e_1+f_2=e_{12}+f_1$, $e_1+f_3=e_{12}+e_{13}$). Then $c+d$ has at least two representations with summands in J which are different from $c+d$ and from each other. In other words, $c+d$ belongs to every additive semigroup in $\boldsymbol{N}^r$ generated by a set of the form $J\backslash\{c,d,a\}$, $a\in J$ arbitrary.*

Proof. We may assume that $r, s\geqq 2$. Let $c=(\gamma_1,\dots,\gamma_r)$ and $d=(\delta_1,\dots,\delta_r)$. Then $c+d$ has the following representations with summands in the set $\{(\alpha_1,\dots,\alpha_r)\in \boldsymbol{Z}^r;\ \alpha_1+\cdots+\alpha_r=s\}$:

$$(R_1)\quad (\gamma_1-,\ \gamma_2+1,\dots)+(\delta_1+1,\ \delta_2-1,\dots)$$

$$(R_2)\quad (\gamma_1-2,\ \gamma_2+2,\dots)+(\delta_1+2,\ \delta_2-2,\dots)$$

$$(R_3)\quad (\gamma_1-3,\ \gamma_2+3,\dots)+(\delta_1+3,\ \delta_2-3,\dots)$$

$$(R_4)\quad (\gamma_1+1,\ \gamma_2-1,\dots)+(\delta_1-1,\ \delta_2+1,\dots)$$

$$(R_5)\quad (\gamma_1+2,\ \gamma_2-2,\dots)+(\delta_1-2,\ \delta_2+2,\dots)$$

and if $r>2$,

$$(R_6)\quad (\gamma_1,\ \gamma_2-1,\ \gamma_3+1,\dots)+(\delta_1,\ \delta_2+1,\ \delta_3-1,\dots)$$

$$(R_7)\quad (\gamma_1,\ \gamma_2+1,\ \gamma_3-1,\dots)+(\delta_1,\ \delta_2-1,\ \delta_3+1,\dots)$$

$$(R_8)\quad (\gamma_1-1,\ \gamma_2+2,\ \gamma_3-1,\dots)+(\delta_1+1,\ \delta_2-2,\ \delta_3+1,\dots)$$

$$(R_9)\quad (\gamma_1-1,\ \gamma_2,\ \gamma_3+1,\dots)+(\delta_1+1,\ \delta_2,\ \delta_3-1,\dots)$$

$$(\mathrm{R}_{10})\quad (\gamma_1-2,\ \gamma_2+1,\ \gamma_3+1,\dots)+(\delta_1+2,\ \delta_2-1,\ \delta_3-1,\dots)$$

and, if $r>3$,

$$(R_{11})\quad (\gamma_1,\ \gamma_2+1,\ \gamma_3,\ \gamma_4-1,\dots)+(\delta_1,\ \delta_2-1,\ \delta_3,\ \delta_4+1,\dots).$$

Note that the summands of $(R_1),\dots,(R_{11})$ belongs to J if they belong to $\boldsymbol{N}^r$ and that since the first summands of $(R_1),\dots,(R_{11})$, $c+d$ are different from each other, from three arbitrary representations of $c+d$ among them one can always find two different ones. Then, to show that $c+d$ has at least two representations with summands in J which are different from $c+d$ and from each other it suffices also to show that among $(R_1),\dots,(R_{11})$ there are either three representations with summands in $\boldsymbol{N}^r$ which are different from $c+d$ or four representations with summands in $\boldsymbol{N}^r$ (because $c+d$ could be identical with only one of those four representations).

First we may assume that

$$\gamma_1=\max\{\gamma_1,\dots,\gamma_r,\ \delta_1,\dots,\delta_r\},$$

$$\delta_2=\max\{\gamma_2,\dots,\gamma_r,\ \delta_2,\dots,\delta_r\}.$$

Then $\gamma_1>0$ and, since $\delta_2=0$ would imply $c=d=e_1$, $\delta_2>0$ too.

(1) Case $\gamma_1=1$.

Then $\gamma_2,\dots,\gamma_r,\ \delta_1,\dots,\delta_r$ can take only the values 1,0. If $r=2$, $s=2$ and we get $c=d=e_{12}$. If $r=3$ and $s=2$ we can also conclude that by a permutation of the components of $\boldsymbol{N}^r$ $c=e_{12}$, $d=e_{12}$, e_{13}. Thus, by the assumption of the lemma, we may assume that $r>3$ or $s>2$. Now by a permutation of the components of $\boldsymbol{N}^r$ we always have one of the following cases:

$$\gamma_1=\gamma_3=\delta_2=1,\ \gamma_2=\delta_1=\delta_3=0, \tag{1.1}$$

$$\gamma_1=\gamma_2=\gamma_3=\delta_1=\delta_2=1. \tag{1.2}$$

In these cases, (R_1) and (R_7) have summands in $\boldsymbol{N}^r$ and one can check that (R_1) and (R_7) are different from $c+d$ and from each other.

(2) Case $\gamma_1=s$.

That means $c=e_1$. A straightforward check shows that $c+d$ is identical with (R_1), (R_2), (R_3), (R_9), (R_{10}) only if $d=e_{12}, f_1, f_2, e_{13}, f_3$, respectively. Therefore, by the assumption of the lemma, $c+d$ is different from (R_1), (R_2), (R_3), (R_9), (R_{10}). Now we distinguish two subcases:

(2.1) $\delta_2\leqq 2$. Then $\delta_1+\delta_2<s$ because $\delta_1+\delta_2=s$ would imply $d=e_{12}$ or f_1. Therefore, $r>2$ and we may assume that $\delta_3>0$. Now it is easily seen that (R_1), (R_9), (R_{10}) have summands in $\boldsymbol{N}^r$.

(2.2) $\delta_2>2$. Then $\gamma_1>2$ too. Hence (R_1), (R_2), (R_3) have summands in $\boldsymbol{N}^r$.

(3) Case $1<\gamma_1<s$.

We distinguish two subcases:

(3.1) $r=2$. Note that $\gamma_1+\gamma_2=s$, $\delta_1+\delta_2=s$, and $\gamma_1=\max\{\gamma_1,\ \gamma_2,\ \delta_1,\ \delta_2\}$. Then we can conclude that

$$\min\{\gamma_1,\ \gamma_2,\ \delta_1,\ \delta_2\}\geqq s-\gamma_1>0$$

and that $\delta_2\geqq 2$ because $\delta_2=1$ would imply $c=d=e_{12}$. If $\delta_2=2$, i.e. $d=f_1$, we have

$\gamma_1 < s-1$ because $\gamma_1 = s-1$ would imply $c = e_{12}$. Thus, $\min\{\gamma_1, \gamma_2, \delta_1, \delta_2\} \geqq 2$. In this case, (R_1), (R_2), (R_4), (R_5) have summands in $\boldsymbol{N}^r$. If $\delta_2 > 2$, then $\gamma_1 > 2$ and one can check that (R_1), (R_2), (R_3), (R_5) have summands in $\boldsymbol{N}^r$.

(3.2) $r > 2$. We may assume that

$$\delta_3 = \max\{\delta_3, \dots, \delta_r\}.$$

Suppose that $\delta_3 = 0$. Then we may further assume that $\gamma_3 = \max\{\gamma_3, \dots, \gamma_r\}$. If $\gamma_3 = 0$, we have a situation like (3.1). If $\gamma_3 > 0$, we have $\delta_2 > 1$ because $\delta_2 = 1$ would imply $d = e_{12}$, $c = e_{13}$. Now it is easily seen that (R_1), (R_2), (R_7), (R_8) have summands in $\boldsymbol{N}^r$.

It remains the case $\delta_3 > 0$. If $\gamma_2 > 0$, (R_1), (R_6), (R_9), (R_{10}) have summands in $\boldsymbol{N}^r$. If $\gamma_3 > 0$, (R_1), (R_7), (R_9), (R_{10}) have summands in $\boldsymbol{N}^r$. If $\gamma_2 = \gamma_3 = 0$, then $\gamma_1 + \gamma_2 + \gamma_3 = \gamma_1 < s$, hence $r > 3$ and we may assume that $\gamma_4 > 0$. Now it is easily seen that (R_1), (R_9), (R_{10}), (R_{11}) have summands in $\boldsymbol{N}^r$.

The proof of Lemma 6 is complete.

Lemma 6 has the following consequences, which we shall use later in the classification of the double projections of Veronese varieties.

Lemma 7. *Let c, d be elements of J such that by every permutation of the components of $\boldsymbol{N}^r$ $c+d$ doesn't have the form of $e_1 + e_1$, $e_1 + e_{12}$. Then $c+d$ has at least a different representation with summands in J. In other words, $c+d$ belongs to the affine semigroup in $\boldsymbol{N}^r$ generated by the set $J \backslash \{c, d\}$.*

Proof. Straightforward.

Lemma 8. *Let c, d be as in Lemma 7. Then $c+d$ belongs to the affine semigroup in $\boldsymbol{N}^r$ generated by the set $J \backslash \{e_1, c, d\}$ if by every permutation of the ith components of $\boldsymbol{N}^r$, $i = 2, \dots, r$, $c+d$ doesn't have the form of $e_{12} + e_{12}$, $e_{12} + f_1$, $e_{12} + e_{13}$.*

Proof. By Lemma 7, $c+d$ has a different representation $c_1 + d_1$ with $c_1, d_1 \in J$. If $c+d$ doesn't belong to the semigroup in $\boldsymbol{N}^r$ generated by $J \backslash \{e_1, c, d\}$, i.e. $c+d$ doesn't have a representation with summands in $J \backslash \{e_1, c, d\}$, then $e_1 \in \{c_1, d_1\}$ and $c_1 + d_1$ has only a different representation with summands in J, which is $c+d$. Assume that $c_1 = e_1$, $d_1 = (\delta_1, \dots, \delta_r)$ with $\delta_2 = \max\{\delta_2, \dots, \delta_r\}$ and, if $r > 2$, $\delta_3 = \max\{\delta_3, \dots, \delta_r\}$. Then from the proof of Lemma 6, Case (2) one can deduce that $d_1 \in \{f_1, f_2, f_3\}$. Thus, by Lemma 6, $c+d$ must be identical with one of the sums $e_{12} + e_{12}$, $e_{12} + f_1$, $e_{12} + e_{13}$.

Lemma 9. *Let c, d be as in Lemma 7 with $r = 2$ and $s > 4$. Then $c+d$ belongs to the affine semigroup in $\boldsymbol{N}^2$ generated by the set $J \backslash \{e_{12}, c, d\}$ if $c+d$ is not identical with $e_1 + f_1$, $e_1 + f_2$.*

Proof. By Lemma 7 $c+d$ has a different representation $c_1 + d_1$ with $c_1, d_1 \in J$. If $c+d$ doesn't belong to the semigroup in $\boldsymbol{N}^2$ generated by the set $J \backslash \{e_{12}, c, d\}$, i.e. $c+d$ doesn't have a representation with summands in $J \backslash \{e_{12}, c, d\}$, then $e_{12} \in$

$\{c_1, d_1\}$ and c_1+d_1 has only a different representation with summands in J, which is $c+d$. Thus, if $c+d$ is not identical with e_1+f_1, e_1+f_2, from Lemma 6 one can deduce that c_1+d_1 is identical with one of the sums $(0, s)+(2, s-2)$, $(0, s)+(3, s-3)$, $(1, s-1)+(1, s-1)$, $(1, s-1)+(2, s-2)$. Therefore, $e_{12}=(s-1, 1)$ must be one of the elements $(0, s)$, $(1, s-1)$, $(2, s-2)$, $(3, s-3)$, which then implies $s\leqq 4$, a contradiction.

4. Double projections of Veronese varieties

We shall adhere to the notations of the preceding sections.

Further, we call a projective variety arithmetically Cohen-Macaulay or Buchsbaum if the local ring at the vertex of its affine cone is Cohen-Macaulay or Buchsbaum, respectively.

We will first classify the projections of $V_{r,s}$ in $\boldsymbol{P}_k^{N-2}$ concerning the arithmetically Cohen-Macaulay and Buchsbaum property once more, cf. [12], because from our method one can see more clearly as in [12] (where different methods and results have been used) why there is a such classification.

Let U be an arbitrary projection of $V_{r,s}$ in $\boldsymbol{P}_k^{N-2}$ with $r, s>1$ (the cases $r=1$ and $s=1$ are trivial). Then the homogeneous coordinate ring of U is isomorphic to the semigroup ring over k of an affine semigroup in $\boldsymbol{N}^r$ generated by $N-1$ elements of the set $J:=\{(\alpha_1,\dots, \alpha_r)\in \boldsymbol{N}^r\ \alpha_1+\cdots+\alpha_r=s\}$. Note that J has N elements exactly. Then we denote by a the element of J deleted by the projection U. Of course, U is uniquely determined by this element a.

Set for some elements of J

$$e_i=(\dots, s,\dots),$$

where s stands at the ith place, $i=1,\dots, r$, and

$$e_{ij}=(\dots, s-1,\dots, 1,\dots),$$

where $s-1$ stands at the ith place and 1 at the jth place, $i, j=1,\dots, r$ with $i\neq j$.

Then, with the above notations, we have the following table which shows in terms of the element a when U is arithmetically Cohen-Macaulay or Buchsbaum.

Table I.

U is arithmetically Cohen-Macaulay	U is not arithmetically Cohen-Lacaulay	
	arith. Buchsbaum	not arith. Buchsbaum
$a=e_i$		
$a=e_{ij}$ with $r=2$ or $(r, s)=(3, 2)$		$a=e_{ij}$ with $r>2$ and $(r, s)\neq(3, 2)$
	$a\neq e_i, e_{ij}$	

Proof. Let H denote the affine semigroup in $\boldsymbol{N}^r$ generated by the set $J\setminus\{a\}$.

Then we have to study when $k[H]_{m_H}$ is a Cohen-Macaulay or Buchsbaum ring. Let I denote the affine semigroup in N^r generated by J and note that $\bar{I}=I$ (see [1, Theorem 5] or [7, §1]).

(1) Case $a=e_i$ (cf. [12, Proposition 2]).

We claim that $\bar{H}=H$, from which it then follows by Lemma 1 that $k[H]$ is a Cohen-Macaulay ring. Since $H\subseteq\bar{H}\subseteq\bar{I}=I$, it suffices to show that $(I\backslash H)\cap\bar{H}=\emptyset$. We may assume that $a=e_1$. Let f be an arbitrary element of $I\backslash H$. Then $f=me_1+g$ for some $m>0$ and $g\in H$. Note that by Lemma 7 $e_1+e\in H$ for all $e\in J\backslash\{e_1, e_{12},\dots, e_{1r}\}$. Then, choosing m as small as possible we must have $g=n_2e_{12}+\cdots+n_re_{1r}$ for some $n_2,\dots, n_r\geqq 0$. If $f\in\bar{H}$.

$$lf=\sum_{a_p\in\backslash I\{e_1\}} n_pa_p$$

for some $l>0$, $n_p\geqq 0$. Comparing the sums of all components of lf and $\sum n_pa_p$ we get $l(m+n_2+\cdots+n_r)s=\sum n_ps$, hence $l(m+n_2+\cdots+n_r)=\sum n_p$, so that $l[ms+(n_2+\cdots+n_r)(s-1)]=$the first component of $lf>1(m+n_2+\cdots+n_r)(s-1)=\sum n_p\cdot(s-1)\geqq$the first component of $\sum n_pa_p$, a contradiction. We have proved that $(I\backslash H)\cap\bar{H}=\emptyset$.

For the rested cases $(a\neq e_1,\dots, e_r)$ we always have $\text{rank}_{Z}\, G(H)=r\geqq 2$ and $2a\in H$ by Lemma 7. Note further that the affine semigroup in N^r generated by H and a is just I, which defines a Cohen-Macaulay ring $k[I]$ by Lemma 1. Then, by Lemma 4 it suffices to show below that $k[H]/k[(H-a)\cap H]$ is a Cohen-Macaulay ring of a suitable dimension or that $(H\backslash\{0\})+a\subset H$.

(2) Case $a=e_{ij}$ (cf. [5, §5] and [12, §4]).

(2.1) $s=2$. Then $e_{21}=e_{12}$, hence by Lemma 7 $e+e_{12}\in H$ for all $e\in J\backslash\{e_1, e_2\}$. It follows that $(H-e_{12})\cap H\supseteq H\backslash\{me_1+ne_{12};\ m, n\geqq 0\}$. If $me_1+ne_2+e_{12}\in H$ for some $m, n\geqq 0$,

$$me_1+ne_2+e_{12}=m'e_1+n'e_2+\sum_{a_p\in J\backslash\{e_1, e_2, e_{12}\}} n_pa_p$$

for some $m', n', n_p\geqq 0$. Since each $a_p\in J\backslash\{e_1, e_2, e_{12}\}$ always has a non-zero ith component for some $i\geqq 3$ but that of $me_1+ne_2+e_{12}$ is zero, $n_p=0$ for all $a_p\in J\backslash\{e_1, e_2, e_{12}\}$. Therefore, comparing the first components of the elements $me_1+ne_2+e_{12}$ and $m'e_1+n'e_2$ of N^2 we get $2m+1=2m'$, a contradiction. So we have proved that $(H-e_{12})\cap H=H\backslash\{me_1+ne_2;\ m, n\geqq 0\}$. From this it then follows that

$$k[H]/k[(H-e_{12})\cap H]=k[t_1^2, t_2^2]$$

is a Cohen-Macaulay ring of dimension 2. Thus, by Lemma 4, $k[H]_{m_H}$ is Cohen-Macaulay if $r\leqq 3$ and not Buchsbaum if $r>3$.

(2.2) $s>2$. Then $e_{21}\neq e_{12}$, hence by Lemma 7 $e+e_{12}\in H$ for all $e\in J\backslash\{e_1\}$. It follows that $(H-e_{12})\cap H\supseteq H\backslash\{me_1;\ m\geqq 0\}$. If $me_1+e_{12}\in H$ for some $m\geqq 0$,

$$me_1+e_{12}=m'e_1+\sum_{a_p\in J\backslash\{e_1, e_{12}\}} n_pa_p$$

for some $m', n_p\geqq 0$. Since for all $i\geqq 3$ the ith component of me_1+e_{12} is zero,

$n_p=0$ for all $a_p \in \{\alpha_1,\ldots,\alpha_r) \in J \backslash \{e_1, e_{12}\}; \alpha_i \neq 0$ for some $i \geqq 3\}$. Hence, comparing the second components of me_1+e_{12} and $m'e_1+\sum n_p a_p$ we get $n_p \neq 0$ for some $a_p \in \{(\alpha_1,\ldots,\alpha_r) \in J \backslash \{e_1, e_{12}\}; \alpha_i=0$ for all $i \geqq 3\}$ and $2 \leqq$ the second component of this $a_p \leqq 1 =$ the second component of me_1+e_{12}, a contradiction. So we have proved that $(H-e_{12}) \cap H = H \backslash \{me_1; m \geqq 0\}$. From this it then follows that

$$k[H]/k[(H-e_{12}) \cap H] \cong k[t_1^s]$$

is a Cohen-Macaulay ring of dimension 1. Thus, by Lemma 4, $k[H]_{m_H}$ is Cohen-Macaulay if $r=2$ and not Buchsbaum if $r>2$.

(3) Case $a \neq e_i, e_{ij}$ (cf. [12, Theorem 2]).

Then by Lemma 7 $e+a \in H$ for all $e \in J \backslash \{a\}$, hence $(H \backslash \{0\})+a \subset H$. Thus, by Lemma 4, $k[H]_{m_H}$ is a non-Cohen-Macaulay Buchsbaum ring.

Summarizing the above cases, we have just proved Table I.

Now we are going to formulate our main result on the classification of the double projections of Veronese varieties. For that we shall use the notations $f_1, f_2, f_3, e_i, e_{ij}$ of Section 3 and Table I.

Let V be an arbitrary projection of $V_{r,s}$ in $\boldsymbol{P}_k^{N-3}$ with $r, s>1$. (the cases $r=1$ and $s=1$ are trivial). Then the homogeneous coordinate ring of V is isomorphic to the semigroup ring over k of an affine semigroup in $\boldsymbol{N}^r$ generated by $N-2$ elements of the set $J:=\{(\alpha_1,\ldots,\alpha_r) \in \boldsymbol{N}^r; \alpha_1+\cdots+\alpha_r=s\}$. Let a and b denote the two elements of J deleted by the projection V, $a \neq b$. Then V is uniquely determined by the set $\{a, b\}$. Further, by a permutation of the components of $\boldsymbol{N}^r$ we always have one of the following situations:

(1) $a=e_1$, b arbitrary,

(2) $a=e_{12}$, $b \neq e_1,\ldots, e_r$,

(3) $a, b \neq e_i, e_{ij}$ for all $i, j=1,\ldots, i \neq j$.

For these situations we have the following table which shows in terms of $\{a, b\}$ when V is arithmetically Cohen-Macaulay or Buchsbaum, see Table II.

Note that "otherwise" denotes the cases which can not be reduced to one of the preceding cases of the corresponding situation by a permutation of the components of $\boldsymbol{N}^r$.

Proof. We will omit a detailed proof because it is rather technical. Let H denote the affine semigroup in $\boldsymbol{N}^r$ generated by the set $J \backslash \{a, b\}$. Then we have to study when $k[H]_{m_H}$ is a Cohen-Macaulay or Buchsbaum ring. Let K denote the affine semigroup in $\boldsymbol{N}^r$ generated by the set $J \backslash \{a\}$.

(1) Situation $a=e_1$, b arbitrary.

Then $\bar{K}=K$ by the proof of Table I, Case (1).

For $b=e_2, e_{12}$ we claim that $\bar{H}=H$, from which it then follows by Lemma 1 that $k[H]$ is a Cohen-Macaulay ring. Since $H \subseteq \bar{H} \subseteq \bar{K}=K$, it suffices to show that $(K \backslash H) \cap \bar{H}=\emptyset$.

(1.1) $b=e_2$.

Let f be an arbitrary element of $K \backslash H$. Then $f=me_2+g$ for some $m>0$ and $g \in H$. Since by Lemma 8 $e_2+e \in H$ for all $e \in J \backslash \{e_2, e_{21}, e_{23},\ldots, e_{2r}\}$, choosing m

Table II.

	V is arithmetically Cohen-Macaulay	V is not arithmetically Cohen-Macaulay: arith. Buchsbaum	V is not arithmetically Cohen-Macaulay: not arith. Buchsbaum
(1)	$b=e_2, e_{12}$ $b=f_1$ with $r=2$ or $(r, s)=(3, 3)$ $b=e_{21}$ with $r=2$ or $(r, s)=(3, 3)$ $b=e_{23}$ with $(r, s)=(3, 2)$	otherwise	$b=f_1$ with $r, s>2$, $(r, s)\neq(3, 3)$ $b=e_{21}$ with $r, s>2$, $(r, s)\neq(3, 3)$ $b=e_{23}$ with $(r, s)\neq(3, 2)$
(2)	$b=f_1, f_2, e_{21}$ with $r=2$ $b=e_{21}$ with $(r, s)=(3, 3), (3, 4)$ $b=e_{13}$ with $(r, s)=(3, 2), (4, 2)$	$b\neq f_1, f_2, e_{21}$ with $r=2$	otherwise
(3)		otherwise	$\{a, b\}=\{f_1, f_2\}$

as small as possible we must have $g=n_1e_{21}+n_3e_{23}+\cdots+n_re_{2r}$ for some $n_1, n_3,\dots, n_r\geqq 0$. Now, proceeding as in the proof of Table I, Case (1), we can show that $f\notin \bar{H}$; hence $(K\backslash H)\cap\bar{H}=\emptyset$, as required.

(1.2) $b=e_{12}$.

Let f be an arbitrary element of $K\backslash H$. Then $f=me_{12}+g$ for some $m>0$ and $g\in H$. Since by Lemma 8 $e_{12}+e\in H$ for all $e\in J\backslash\{e_1, e_{12},\dots, e_{1r}, f_1\}$, choosing m as small as possible we must have $g=n_3e_{13}+\cdots+n_re_{1r}+nf_1$ for some $n_3,\dots, n_r, n\geqq 0$. If $f\in\bar{H}$,

$$lf=n'_3e_{13}+\cdots+n'_re_{1r}+\sum_{a_p\in J\backslash\{e_1,e_{12},\dots,e_{1r}\}}n_pa_p$$

for some $l>0$, $n_p\geqq 0$. Comparing the components of lf and $n'_3e_{13}+\cdots+n'_re_{1r}+\sum n_pa_p$ we get $ln_i\geqq n'_i$ for $i=3,\dots, r$ and, as in the proof of Table I, Case (1), $l(m+n_3+\cdots+n_r+n)=n'_3+\cdots+n'_r+\sum n_p$, hence $l[(m+n_3+\cdots+n_r)(s-1)+n(s-2)]=$ the first component of $lf>n'_3+\cdots+n'_r+l(m+n_3+\cdots+n_r+n)(s-2)=(n'_3+\cdots+n'_r)(s-1)+\sum n_p(s-2)\geqq$ the first component of $n'_3e_{13}+\cdots+n'_re_{1r}+\sum n_pa_p$, a contradiction. Therefore, $(K\backslash H)\cap\bar{H}=\emptyset$, as required.

For $b\neq e_i, e_{1j}, i, j=2,\dots, r$, we always have $\mathrm{rank}_{\mathbf{Z}}\, G(H)=r\geqq 2$ and $2b\in H$ by Lemma 8. Note that the affine semigroup in $\mathbf{N}^r$ generated by H and b is just K, which defines a Cohen-Macaulay ring $k[K]$ by Lemma 1. Then, by Lemma 4 it suffices to show that $k[H]/k[(H-b)\cap H]$ is a Cohen-Macaulay ring of suitable dimension or that $(H\backslash\{0\})+b\subset H$.

(1.3) $b=f_1$.

If $s=2$, then $f_1=e_2$, hence see (1.1).

If $s=3$, then $f_1=e_{21}$. By Lemma 8 $e+f_1\in H$ for all $e\in J\backslash\{e_{12}, e_2\}$. From this we can show as in the proof of Table I, Case (2.1) that $(H-f_1)\cap H=H\backslash\{me_{12}+ne_2;\ m, n\geqq 0\}$. Hence

$$k[H]/K[(H-f_1)\cap H]\cong k[t_1^2t_2,\ t_2^3]$$

is a Cohen-Macaulay ring of dimension 2. Thus, by Lemma 4, $k[H]_{m_H}$ is Cohen-Macaulay if $r=2$, 3 and not Buchsbaum if $r>3$.

If $s>3$, then $f_1\neq e_2, e_{21}$. By Lemma 8 $e+f_1\in H$ for all $e\in J\backslash\{e_{12}\}$. From this we can show as in the proof of Table I, Case (2.2) that $(H-f_1)\cap H=H\backslash\{me_{12};\ m\geqq 0\}$. Hence

$$k[H]/k[(H-f_1)\cap H]\cong k[t_1^{s-1}t_2]$$

is a Cohen-Macaulay ring of dimension 1. Thus, by Lemma 4, $k[H]_{m_H}$ is Cohen-Macaulay if $r=2$ and not Buchsbaum if $r>2$.

(1.4) $b=e_{21}$.

If $s=2$, then $e_{21}=e_{12}$, hence see (1.2).

If $s=3$, then $e_{21}=f_1$, hence see (1.3).

If $s>3$, then $e_{21}\neq e_{12}, f_1$. By Lemma 8 $e+e_{21}\in H$ for all $e\in J\backslash\{e_2\}$. From this we can show as in the proof of Table I, Case (2.2) that $(H-e_{21})\cap H=H\backslash\{me_2;\ m\geqq 0\}$. Hence

$$k[H]/[(H-e_{21})\cap H)\cong k[t_2^s]$$

is a Cohen-Macaulay ring of dimension 1. Thus, by Lemma 4, $k[H]_{m_H}$ is Cohen-Macaulay if $r=2$ and not Buchsbaum if $r>2$.

(1.5) $b=e_{23}$.

If $s=2$, $e_{23}=e_{32}$. By Lemma 8 $e+e_{23}\in H$ for all $e\in J\backslash\{e_2, e_3\}$. From this we can show as in the proof of Table I, Case (2.1) that $(H-e_{23})\cap H=H\backslash\{me_2+ne_3;\ m, n\geqq 0\}$. Hence

$$k[H]/k[(H-e_{23})\cap H]\cong k[t_2^2,\ t_3^2]$$

is a Cohen-Macaulay ring of dimension 2. Thus, by Lemma 4, $k[H]_{m_H}$ is Cohen-Macaulay if $r=3$ and not Buchsbaum if $r>3$.

If $s>2$, then $e_{23}\neq e_{32}$. By Lemma $e+e_{23}\in H$ for all $e\in J\backslash\{e_2\}$. From this we can show as in the proof of Table I, Case (2.2) that $(H-e_{23})\cap H=H\backslash\{me_2;\ m\geqq 0\}$. Hence

$$\pi[H]/k[(H-e_{23})\cap H]\cong k[t_2^s]$$

is a Cohen-Macaulay ring of dimension 1. Thus, by Lemma 4, $k[H]_{m_H}$ is not Buchsbaum.

(1.6) Otherwise.

Then b doesn't have the form of $e_2, e_{12}, f_1, e_{21}, e_{23}$ by every permutation of the ith components of $\boldsymbol{N}^r$, $i=2,\dots, r$. By Lemma 8 one can see that $e+b\in H$ for all $e\in J$, hence $(H\backslash\{0\})+b\subset H$. Thus, by Lemma 4, $k[H]_{m_H}$ is a non-Cohen-Macaulay Buchsbaum ring.

For Situation (2) and Situation (3) we first observe that H is of the type as in Lemma 2 and Lemma 0.

(2) Situation $a=e_{12}$, $b\neq e_1,\dots, e_r$.

It is easily seen that $e_1+e_{12}\notin H$, so that by Lemma 3 a sufficient condition for $k[H]_{m_H}$ to be not Buchsbaum is $e_{12}\in\{e\in \bar{H};\ e+2e_i\in H$ and $e+2e_j\in H$ for some $i\neq j,\ 1\leqq i, j\leqq r\}$. Note that this condition may be reduced to $e_{12}+2e_i\in H$ and $e_{12}+2e_j\in H$ for some $i\neq j$, $2\leqq i, j\leqq r$, because $e_{12}\in\bar{H}$ is only a consequence of the relations $e_{12}\in H-2e_i\subseteq G(H)$ and $se_{12}=(s-1)e_1+e_2$. Then, using the following statements:

(i) $e_{12}+2e_2\in H$ if $s=3, 4$, $b\neq e_{21}$ or if $s>4$,

(ii) $e_{12}+2e_i\in H$ if $s=2$, $b\neq e_{1i}, e_{2i}$ or if $s>2$, $i=3,\dots, r$,

it is easy to verify that $k[H]_{m_H}$ is not Buchsbaum if $r\geqq 3$, $(r, s)\neq(3, 2)$, $b\neq e_{21}$ in the case $(r, s)=(3, 3), (3, 4)$, $b\neq e_{13}, e_{23}, e_{14}, e_{24}$ in the case $(r, s)=(4.2)$.

For the proof of (i) we may assume that $s\geqq 3$. Then $e_{12}\neq e_{21}\neq f_1$. Thus, if $b\neq f_1, e_{21}$; $e_{12}+2e_2=f_1+e_{21}+e_2\in H$, and, if $b=f_1$, $e_{12}+2e_2=f_2+2e_{21}\in H$. If $s>4$ and $b=e_{21}$, then $e_{12}=(2, s-2, 0)$ and $b\neq f_2$, hence $e_{12}+2e_2=f_2+(2, s-2, 0)+e_2\in H$. So we have proved (i).

For the proof of (ii) let $i=3,\dots, r$. If $s=2$ and $b\neq e_{1i}, e_{2i}$; $e_{12}+e_i=e_{1i}+e_{2i}\in H$. If $s>2$, $e_{12}+e_i$ satisfies the conditions of Lemma 6, hence belongs to H. Since $e_{12}+e_i\in H$ implies $e_{12}+2e_i\in H$, we have proved (ii).

Note that if $(r, s)=(3, 2)$ we may assume that $b=e_{13}$, and, if $(r, s)=(4, 2)$, the cases $b=e_{23}, e_{14}, e_{24}$ may be transformed into the case $b=e_{13}$. Then it remains to consider the cases $r=2$, $b=e_{21}$ with $(r, s)=(3, 3), (3, 4)$, $b=e_{13}$ with $(r, s)=(3, 2), (4, 2)$.

(2.1) $r=2$.

If $s=2$, we have no cases of Situation (2).

If $s=3$, we have $b=f_1$, and $k[H]=k[t_1^3, t_2^3]$ is a Cohen-Macaulay ring.

If $s=4$, we have $b=f_1, f_2$, and $k[H]=k[t_1^4, t_1t_2^3, t_2^4]$, $k[t_1^4, t_1^2t_2^2, t_2^4]$ is a Cohen-Macaulay ring.

If $s>4$, we apply Lemma 9 and get

$$e+b\in H \quad \text{for all} \quad \begin{cases} e\in J, & \text{if } b\neq f_1, f_2, e_{21}, \\ e\in J\backslash\{e_1\}, & \text{if } b=f_1, f_2, \\ e\in J\backslash\{e_2\}, & \text{if } b=e_{21}. \end{cases}$$

It follows that $(H\backslash\{0\})+b\subset H$ if $b\neq f_1, f_2, e_{21}$, and, as in the proof of Table I, Case (2.2), that

$$k[H]/k[(H-b)\cap H]\cong\begin{cases} k[t_1^s], & \text{if } b=f_1, f_2, \\ k[t_2^s], & \text{if } b=e_{21}, \end{cases}$$

is a Cohen-Macaulay ring of dimension 1. Note that $\operatorname{rank}_{\boldsymbol{Z}} G(H)=2$, $2b\in H$, and the affine semigroup in $\boldsymbol{N}^2$ generated by H and b is is just K, which defines a Cohen-Macaulay ring $k[K]$ by the proof of Table I, Case (2.1). Then, by Lemma 4, $k[H]_{m_H}$ is Cohen-Macaulay if $b=f_1, f_2, e_{21}$ and Buchsbaum if $b\neq f_1, f_2, e_{21}$.

For the cases $b=e_{21}$ with $(r, s)=(3, 3), (3, 4)$ and $b=e_{13}$ with $(r, s)=(3, 2), (4, 2)$ we set

$H_1=\{e\in \bar{H};\ e+e_i\in H$ and $e+e_j\in H$ for some $i\neq j,\ 1\leqq i, j\leqq r\}$ and claim that $H_1=H$, form which it then follows by Lemma 2 that $k[H]$ is a Cohen-Macaulay ring. Since $H\subseteq H_1\subseteq \bar{H}\subseteq \bar{I}=I$, where I denotes the affine semigroup in $\boldsymbol{N}^r$ generated by J, it suffices to show that $(I\backslash H)\cap H_1=\phi$.

(2.2) $b=e_{21}$ with $(r, s)=(3, 3), (4, 3)$.

Let f be an aribtrary element of $I\backslash H$. Then $f=me_{12}+ne_{21}+g$ for some $m, n\geqq 0$ and $g\in H$. Since $e_{12}+e\in H$, $e_{21}+e\in H$ for all $e\in\{(\alpha_1, \alpha_2, \alpha_3)\in J;\ \alpha_3\neq 0\}$ (which may be easily checked by Lemma 6) and $e_{12}+e_{21}=e_1+e_2$, $2e_{12}=e_1+f_1$, $2e_{21}=e_2+(2, s-2, 0)$, choosing $m+n$ as small as possible we must have $f=e_{12}+g$ or $e_{21}+g$ for some $g\in H'$, where H' denotes the affine semigroup in $\boldsymbol{N}^3$ generated by the set

$$\{(\alpha_1, \alpha_2, \alpha_3)\in J;\ \alpha_3=0\}\backslash\{e_{12}, e_{21}\}=\begin{cases}\{e_1, e_2\}, & \text{if } s=3,\\ \{e_1, f_1, e_2\}, & \text{if } s=4.\end{cases}$$

Without restriction we may assume that $f=e_{12}+g$ for some $g\in H'$. Then we can show as in the proof of Table I, Case (2.1) that $f+e_1\notin H, f+e_2\notin H$. Thus, $(I\backslash H)\cap H_1=\emptyset$, as required.

(2.3) $b=e_{13}$ with $(r, s)=(3, 2), (4, 2)$.

One can check directly that $e_{12}+e\in H$, $e_{13}+e\in H$ for all e $J\backslash\{e_1, e_2, e_3, e_{23}\}$. Therefore, as in (2.2), it is easy to see that every element $f\in I\backslash H$ has the form $e_{12}+le_1+me_2+ne_3+pe_{23}$ or $e_{13}+le_1+me_2+ne_3+pe_{23}$, $l, m, n, p\geqq 0$, so that we can show as in the proof of Table I, Case (2.1) that $f+e_1\notin H, f+e_2\notin H, f+e_3\notin H$. Thus, $(I\backslash H)\cap H_1=\emptyset$, as required.

(3) Situation $a, b\neq e_i, e_{ij}$ for all $i, j=1,\dots, r$, $i\neq j$.

(3.1) $\{a, b\}=\{f_1, f_2\}$.

It is easily seen that $f_1+2e_1=2e_{12}+e_1\in H, f_1+2e_2=(s-4, 4, 0)+2e_{21}\in H$ $f_1\in\bar{H}$ (because $f_1\in H-2e_1\subseteq G(H)$ and $2f_1=e_1+(s-4, 4, 0)\in H$), but $e_{12}+f_1\notin H$. Thus, by Lemma 3, $k[H]_{m_H}$ is not Buchsbaum.

(3.2) Otherwise.

Let $e\in J$ arbitrary. If $e+a\notin H$, $e+a$ hasn't a different representation with summands in $J\backslash\{b\}$. But by Lemma 6 that happens only if by a permutation of the components of $\boldsymbol{N}^r$ $\{a, b\}=\{f_1, f_2\}$. Thus, we must have $e+a\in H$, and similarly, $e+b\in H$. It follows that $(H\backslash\{0\})+I\subset H$. Note that $\bar{H}\subseteq\bar{I}=I$. Then, by Lemma 2 and Lemma 3, $k[H]_{m_H}$ is a non-Cohen-Macaulay Buchsbaum ring.

Summarizing the above cases, we have proved Table II.

Remark. From the proof of Table II one can also see that if $k[H]_{m_H}$ is a non-Cohen-Macaulay Buchsabum ring, then

$$i(k[H]_{m_H}) = \begin{cases} r-1 \text{ in Situation (1) and Situation (2)}, \\ 2(r-1) \text{ in Situation (3)}. \end{cases}$$

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