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The spectra of 1-forms on simply connected compact irreducible Riemannian symmetric spaces

By

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Introduction

Let G/K be a simply connected compact irreducible Riemannian symmetric space and let $\Lambda^p(G/K)$ be the space of complex continuous p-forms on G/K. Then it may be natural to ask: How does $\Lambda^p(G/K)$ decompose under the canonical action of G?

For several low rank G/K, such as the spheres, the complex projective spaces, the quarternion projective spaces and the complex quadrics, the answer to this question have been given (see Gallot-Meyer [5], Ikeda-Taniguchi [8], Levy-Bruhl-Laperrière [9], [10], Strese [11] and Tsukamoto [13]).

The purpose of this paper is to decompose $\Lambda^1(G/K)$ for all simply connected compact irreducible Riemannian symmetric spaces G/K. The method used in this paper is somewhat different from that used in the above papers.

Let $\Lambda^1(G)$ be the space of complex continuous 1-forms on G. We can regard $\Lambda^1(G)$ as a G-module under the action of G induced by left translations of G. Then in a natural way, $\Lambda^1(G/K)$ may be considered as a G-submodule of $\Lambda^1(G)$. Therefore to decompose $\Lambda^1(G/K)$, we have only to express $\Lambda^1(G)$ as a sum of irreducible G-submodules and find out all the factors of this decomposition that are contained in $\Lambda^1(G/K)$. Then our problem is to determine the function that assigns to each irreducible G-module the number of factors in $\Lambda^1(G/K)$ isomorphic to this G-module.

In §1, making use of the theorem of Peter-Weyl on the representative ring of G, we reduce our problem to the following problem: For each irreducible representation $\rho: G \to GL(V^\rho)$, determine the multiplicity of the eigenvalue -1 in $(V^\rho \otimes \mathfrak{g}^c)_K$ of the involutive automorphism $\tilde{\theta}$ of $(V^\rho \otimes \mathfrak{g}^c)_K$ induced by the canonical involution θ of $\mathfrak{g}^c$ associated to the symmetric pair (G, K) (see the definitions in §1). This problem can further be reduced to a problem of the complexified Lie algebra $\mathfrak{g}^c$. In §3 we define a map of $V^\rho \otimes \mathfrak{g}^c$ onto $\mathfrak{g}^c$ that sends $(V^\rho \otimes \mathfrak{g}^c)_K$ isomorphically onto a θ-invariant subspace $\mathfrak{p}$ of $\mathfrak{g}^c$. Then the problem stated above can be reduced to the problem of determination of the multiplicity of the eigenvalue -1 of θ in $\mathfrak{p}$. In order to solve this problem, we first clarify the relation between this multiplicity and the subset $B(\Lambda)$ of non-zero roots of $\mathfrak{g}^c$ determined by the highest weight Λ of ρ.

We next consider some conditions imposed on $B(\Lambda)$ and calculate the multiplicity by assuming that one of these conditions is satisfied. Thus the only problem that is left to us is to examine the conditions. In §4, we treat two examples: [*AIII*] $SU(p+q)/S(U(p)\times U(q))$; [*G*] $G_2/SU(2)\times SU(2)$ and show that in almost every case one of those conditions is satisfied. We treat in detail the only exceptional case that occurs in the case [*G*].

After the examinations of all simply connected compact irreducible Riemannian symmetric spaces G/K with G simple, we know that except the case [*G*] one of the conditions imposed on $B(\Lambda)$ is satisfied. The details are omitted in this paper. In the forthcoming paper [14], we exhibit the results obtained by the above arguments.

Finally we refer to the works [3], [4] of Dzjadyk. In [4], he obtained the decompositions of $\Lambda^1(G/K)$ by using his branching law established in [3]. Although there is left a question of whether his branching law is true or not his results are the same as ours.

§1. The G-modules $\Lambda^1(G/K)$ and $\Lambda^1(G, K)$

1.1. Let G be a compact connected semi-simple Lie group and $\mathfrak{g}$ the Lie algebra of G. Let B denote the Killing form of $\mathfrak{g}$. Naturally B can be extended to a complex symmetric bilinear form of $\mathfrak{g}^c$[1]; we also denote it by B. We define an $Ad(G)$-invariant hermitain inner product of $\mathfrak{g}^c$ by

$$(X, Y)=-B(X, \bar{Y}) \qquad X, Y\in\mathfrak{g}^c; \tag{1.1}$$

where $\bar{Y}$ means the complex conjugate of Y *w.r.t.* $\mathfrak{g}$.

Let $C(G)$ be the algebra of $\boldsymbol{C}$-valued continuous functions on G and let $\Lambda^1(G)$ be the vector space of complex continuous 1-forms on G. Let $\|\ \ \|_\infty$ be the maximal norm in $C(G)$, i.e.,

$$\|f\|_\infty=\max_{x\in G}|f(x)| \qquad f\in C(G).$$

Making use of $\|\ \ \|_\infty$ we introduce a norm $\|\ \ \|_\infty^{(1)}$ into $\Lambda^1(G)$ by

$$\|\omega\|_\infty^{(1)}=\max_{0\neq X\in\mathfrak{g}^c}\frac{\|\omega(X)\|_\infty}{(X, X)^{1/2}} \qquad \omega\in\Lambda^1(G). \tag{1.2}$$

Here $\Lambda^1(G)$ is identified with $C(G)\otimes(\mathfrak{g}^c)^*$ $(=\mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{g}^c, C(G)))$ in a natural manner.

We now define canonical actions of G on $C(G)$ and $\Lambda^1(G)$ by setting

$$(L_gf)(x)=f(g^{-1}x), \quad (R_gf)(x)=f(xg); \tag{1.3}$$

$$(\tilde{L}_g\omega)(X)=L_g(\omega(X)), (\tilde{R}_g\omega)(X)=R_g(\omega(Adg^{-1}X)); \tag{1.4}$$

where $f\in C(G)$, $\omega\in\Lambda^1(G)$, $X\in\mathfrak{g}^c$, $g, x\in G$. Then we can easily observe:

$$L_g\cdot R_h=R_h\cdot L_g;\ \tilde{L}_g\cdot\tilde{R}_h=\tilde{R}_h\cdot\tilde{L}_g \qquad g, h\in G.$$

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1) In the following we mean by V^c the complexification of a real vector space V.

We note that by the action L or R (resp. $\tilde{L}$ or $\tilde{R}$) G acts continuously on each finite dimensional invariant subspace of $C(G)$ (resp. $A^1(G)$) *w.r.t.* $\| \ \|_\infty$ (resp. $\| \ \|_\infty^{(1)}$).

In what follows we consider $C(G)$ (resp. $A^1(G)$) as a G-module under the action L (resp. $\tilde{L}$).

Let $\mathfrak{o}(G)$ (resp. $\mathfrak{o}(A^1(G))$) be the G-submodule of $C(G)$ (resp. $A^1(G)$) composed of all $f \in C(G)$ (resp. $\omega \in A^1(G)$) such that the G-orbit passing through f(resp. ω) is contained in a finite dimensional subspace. Clearly $\mathfrak{o}(G)$ (resp. $\mathfrak{o}(A^1(G))$ is invariant under the action R (resp. $\tilde{R}$).

Let $\rho: G \to GL(V^\rho)$ be an irreducible representation of G[2]. We denote by $\mathfrak{o}_{[\rho]}(G)$ (resp. $\mathfrak{o}_{[\rho]}(A^1(G))$ the sum of G-submodules of $\mathfrak{o}(G)$ (resp. $\mathfrak{o}(A^1(G))$ equivalent to V^ρ as G-modules. Then we can easily see that $\mathfrak{o}_{[\rho]}(G)$ (resp. $\mathfrak{o}_{[\rho]}(A^1(G))$) is a G-submodule of $\mathfrak{o}(G)$ (resp. $\mathfrak{o}(A^1(G))$) and is also invariant under the action R (resp. $\tilde{R}$).

Let us denote by $\mathscr{D}(G)$ the set of all equivalence classes of irreducible representations of G. Then we have:

Proposition 1.1. (1) *The G-submodule $\mathfrak{o}(A^1(G))$ is dense in $A^1(G)$ w.r.t.* $\| \ \|_\infty^{(1)}$. *Moreover*:

$$\mathfrak{o}(A^1(G)) = \sum_{[\rho] \in \mathscr{D}(G)} \mathfrak{o}_{[\rho]}(A^1(G)) \ (\textit{direct sum}). \tag{1.5}$$

(2) *For each $[\rho] \in \mathscr{D}(G)$ there exists a linear isomorphism $\tilde{\Phi}_\rho$: $V^\rho \otimes (V^\rho)^* \otimes (\mathfrak{g}^c)^* \to \mathfrak{o}_{[\rho]}(A^1(G))$ such that*

$$\tilde{L}_g \cdot \tilde{\Phi}_\rho(v \otimes \xi \otimes X^*) = \tilde{\Phi}_\rho(\rho(g)v \otimes \xi \otimes X^*), \tag{1.6}$$

$$\tilde{R}_g \cdot \tilde{\Phi}_\rho(v \otimes \xi \otimes X^*) = \tilde{\Phi}_\rho(v \otimes \rho^*(g)\xi \otimes Ad^*g X^*); \tag{1.7}$$

where $g \in G$, $v \in V^\rho$, $\xi \in (V^\rho)^*$ and $X^* \in (\mathfrak{g}^c)^*$.

Proof. The theorem of Peter-Weyl on $\mathfrak{o}(G)$ tells us that $\mathfrak{o}(G)$ is a dense subspace of $C(G)$ *w.r.t.* $\| \ \|_\infty$ and

$$\mathfrak{o}(G) = \sum_{[\rho] \in \mathscr{D}(G)} \mathfrak{o}_{[\rho]}(G) \qquad (\text{direct sum}).$$

Then if we note:

$$\mathfrak{o}(A^1(G)) = \mathfrak{o}(G) \otimes (\mathfrak{g}^c)^*, \ \mathfrak{o}_{[\rho]}(A^1(G)) = \mathfrak{o}_{[\rho]}(G) \otimes (\mathfrak{g}^c)^*,$$

then the assertion (1) follows immediately. To show the assertion (2), we set

$$\tilde{\Phi}_\rho(v \otimes \xi \otimes X^*) = \Phi_\rho(v \otimes \xi) \otimes X^* \quad v \in V^\rho, \ \xi \in (V^\rho)^*, \ X^* \in (\mathfrak{g}^c)^*; \tag{1.8}$$

where Φ_ρ implies the map $V^\rho \otimes (V^\rho)^* \to C(G)$ defined by

$$\Phi_\rho(v \otimes \xi)(x) = \xi(\rho(x^{-1})v) \qquad v \in V^\rho, \ \xi \in (V^\rho)^*, \ x \in G.$$

It is known that Φ_ρ gives an isomorphism between $V^\rho \otimes (V^\rho)^*$ and $\mathfrak{o}_{[\rho]}(G)$ and

2) Any representation ρ of G is understood to be unitary, i.e., V^ρ is a complex vector space with a $\rho(G)$-invariant hermitian inner product. We always regard V^ρ as a G-module (resp. $\mathfrak{g}^c$-module) under the action $g \mapsto \rho(g)$ (resp. $X \mapsto \rho(X)$).

$$L_g \cdot \Phi_\rho(v \otimes \xi) = \Phi_\rho(\rho(g)v \otimes \xi), \quad R_g \cdot \Phi_\rho(v \otimes \xi) = \Phi_\rho(v \otimes \rho^*(g)\xi).$$

Thus the assertion (2) can be easily verified. *Q.E.D.*

1.2. Let K be a closed subgroup of G such that the pair (G, K) is symmetric. Let θ be the canonical involution of $\mathfrak{g}$ associated with (G, K). Naturally θ can be extended to an involutive automorphism of the complexified Lie algebra $\mathfrak{g}^c$. We also denote it by θ.

Let $A^1(G/K)$ be the vector space of complex continuous 1-forms on G/K. We define a canonical action of T: $g \mapsto T_g$ of G on $A^1(G/K)$ by

$$(1.9) \qquad T_g\psi = (g^{-1})^*\psi \qquad \psi \in A^1(G/K), \quad g \in G$$

and regard $A^1(G/K)$ as a G-module under this action.

Let π: $G \to G/K$ be the canonical projection. Then it is clear that

$$(1.10) \qquad L_g\pi^*\psi = \pi^*T_g\psi \qquad \psi \in A^1(G/K), \quad g \in G.$$

Therefore we know that by the map

$$A^1(G/K) \ni \psi \longmapsto \pi^*\psi \in A^1(G),$$

$A^1(G/K)$ is mapped G-isomorphically into $A^1(G)$. In this meaning $A^1(G/K)$ may be considered as a G-submodule of $A^1(G)$. Then it can be easily checked that an element $\omega \in A^1(G)$ belongs to $A^1(G/K)$ if and only if

$$(1.11) \qquad \tilde{R}_k\omega = \omega \qquad \text{for any} \quad k \in K;$$

$$(1.12) \qquad \omega(X) = 0 \qquad \text{for any} \quad X \in \mathfrak{k}^c;$$

where $\mathfrak{k}$ denotes the Lie algebra of K. Let us define an involutive automorphism Θ of $A^1(G)$ by

$$(1.13) \qquad \Theta\omega(X) = \omega(\theta X) \qquad X \in \mathfrak{g}^c.$$

Then we clearly obtain

$$\tilde{L}_g \cdot \Theta = \Theta \cdot \tilde{L}_g, \ \tilde{R}_k \cdot \Theta = \Theta \cdot \tilde{R}_k \qquad g \in G, \ k \in K$$

and

$$\Theta \cdot \tilde{\Phi}_\rho(v \otimes \xi \otimes X^*) = \tilde{\Phi}_\rho(v \otimes \xi \otimes \theta^* X^*) \quad v \in V^\rho, \ \xi \in (V^\rho)^*, \ X^* \in (\mathfrak{g}^c)^*.$$

Since $\mathfrak{k}^c = \{X \in \mathfrak{g}^c \mid \theta X = X\}$, (1.12) is equivalent to

$$(1.14) \qquad \Theta\omega = -\omega.$$

Let us denote by $A^1(G, K)$ the G-submodule of $A^1(G)$ composed of all $\omega \in A^1(G)$ satisfying (1.11) and set

$$(1.15) \quad \mathfrak{o}(A^1(G, K)) = \mathfrak{o}(A^1(G)) \cap A^1(G, K), \quad \mathfrak{o}(A^1(G/K)) = \mathfrak{o}(A^1(G)) \cap A^1(G/K).$$

We further set for each $[\rho] \in \mathscr{D}(G)$, (1.16) $\mathfrak{o}_{[\rho]}(\Lambda^1(G, K))=\mathfrak{o}_{[\rho]}(\Lambda^1(G))\cap\Lambda^1(G, K)$, $\mathfrak{o}_{[\rho]}(\Lambda^1(G/K))=\mathfrak{o}_{[\rho]}(\Lambda^1(G))\cap\Lambda^1(G/K)$

Then by Proposition 1.1, we have

Proposition 1.2. *The G-submodule* $\mathfrak{o}(\Lambda^1(G, K))$ *(resp.* $\mathfrak{o}(\Lambda^1(G/K))$*) is dense in* $\Lambda^1(G, K)$ *(resp.* $\Lambda^1(G/K)$*) w.r.t.* $\|\ \|_\infty^{(1)}$. *Moreover*:

$$\mathfrak{o}(\Lambda^1(G, K))=\sum_{[\rho]\in\mathscr{D}(G)}\mathfrak{o}_{[\rho]}(\Lambda^1(G, K)) \qquad (\textit{direct sum}); \tag{1.17}$$

$$\mathfrak{o}(\Lambda^1(G/K))=\sum_{[\rho]\in\mathscr{D}(G)}\mathfrak{o}_{[\rho]}(\Lambda^1(G/K)) \qquad (\textit{direct sum}). \tag{1.18}$$

To obtain a more precise description of $\mathfrak{o}(\Lambda^1(G, K))$ and $\mathfrak{o}(\Lambda^1(G/K))$, we set for each $[\rho]\in\mathscr{D}(G)$

$$b([\rho])=\dim_{\boldsymbol{C}}\operatorname{Hom}_G(V^\rho, \mathfrak{o}_{[\rho]}(\Lambda^1(G, K))), \tag{1.19}$$

$$a([\rho])=\dim_{\boldsymbol{C}}\operatorname{Hom}_G(V^\rho, \mathfrak{o}_{[\rho]}(\Lambda^1(G/K))). \tag{1.20}$$

We call the function $a\colon\mathscr{D}(G)\ni[\rho]\mapsto a([\rho])\in\boldsymbol{Z}$ (resp. $b\colon\mathscr{D}(G)\ni[\rho]\mapsto b([\rho])\in\boldsymbol{Z}$) the spectrum of $\Lambda^1(G/K)$ (resp. $\Lambda^1(G, K)$). As is easily seen the spectrum a (resp. b) describes precisely how $\mathfrak{o}(\Lambda^1(G/K))$ (resp. $\mathfrak{o}(\Lambda^1(G, K))$ decomposes into irreducible G-submodules.

Let $[\rho]\in\mathscr{D}(G)$. We define a hermitian inner product of $V^\rho\otimes\mathfrak{g}^c$ by

$$(v\otimes X, v'\otimes X')=(v, v')(X, X') \quad v, v'\in V^\rho, X, X'\in\mathfrak{g}^c. \tag{1.21}$$

We consider $V^\rho\otimes\mathfrak{g}^c$ (resp. $(V^\rho\otimes\mathfrak{g}^c)^*$) as a G-module under the action $g\mapsto\rho(g)\otimes Ad\,g$ (resp. $g\mapsto\rho^*(g)\otimes Ad^*g$). We also define an involutive automorphism $\tilde\theta$ (resp. $\tilde\theta^*$) of $V^\rho\otimes\mathfrak{g}^c$ (resp. $(V^\rho\otimes\mathfrak{g}^c)^*$) by

$$\tilde\theta(v\otimes X)=v\otimes\theta X \qquad v\in V^\rho, X\in\mathfrak{g}^c, \tag{1.22}$$

$$\tilde\theta^*(\xi\otimes X^*)=\xi\otimes\theta^*X^* \quad \xi\in(V^\rho)^*, X^*\in(\mathfrak{g}^c)^*. \tag{1.23}$$

Then it can be easily observed that

$$\tilde\theta(k\cdot w)=k\cdot\tilde\theta(w),\ \tilde\theta^*(k\cdot\eta)=k\cdot\tilde\theta^*(\eta) \qquad w\in V^\rho\otimes\mathfrak{g}^c,\ \eta\in(V^\rho\otimes\mathfrak{g}^c)^*,\ k\in K.$$

Let us denote by $(V^\rho\otimes\mathfrak{g}^c)_K$ (resp. $(V^\rho\otimes\mathfrak{g}^c)^*_K$) the subspace of $V^\rho\otimes\mathfrak{g}^c$ (resp. $(V^\rho\otimes\mathfrak{g}^c)^*$) composed of all K-invariant vectors and denote by $(V^\rho\otimes\mathfrak{g}^c)^-_K$ (resp. $(V^\rho\otimes\mathfrak{g}^c)^{*-}_K$) the eigenspace in $(V^\rho\otimes\mathfrak{g}^c)_K$ (resp. $(V^\rho\otimes\mathfrak{g}^c)^*_K$) for the eigenvalue -1 of $\tilde\theta$ (resp. $\tilde\theta^*$). We remark that the semi-$\boldsymbol{C}$-linear isomorphism of $V^\rho\otimes\mathfrak{g}^c$ onto $(V^\rho\otimes\mathfrak{g}^c)^*$ induced from the hermitian inner product of $V^\rho\otimes\mathfrak{g}^c$ defined above, $(V^\rho\otimes\mathfrak{g}^c)_K$ (resp. $(V^\rho\otimes\mathfrak{g}^c)^-_K$) is mapped onto $(V^\rho\otimes\mathfrak{g}^c)^*_K$ (resp. $(V^\rho\otimes\mathfrak{g}^c)^*_K$). Then we have

Theorem 1.3. *For each* $[\rho]\in\mathscr{D}(G)$, *the following equalities hold*:

$$b([\rho])=\dim_{\boldsymbol{C}}(V^\rho\otimes\mathfrak{g}^c)_K; \tag{1.24}$$

$$a([\rho])=\dim_{\boldsymbol{C}}(V^\rho\otimes\mathfrak{g}^c)^-_K. \tag{1.25}$$

Proof. By Proposition 1.1, we know that the map $\tilde{\Phi}^\rho$ maps $V^\rho\otimes(V^\rho\otimes\mathfrak{g}^c)^*_K$ (resp. $V^\rho\otimes(V^\rho\otimes\mathfrak{g}^c)^{*-}_K$) G-isomorphically onto $\mathfrak{o}_{[\rho]}(A^1(G, K))$ (resp. $\mathfrak{o}_{[\rho]}(A^1(G/K))$). Hence we have:

$$\mathrm{Hom}_G(V^\rho, \mathfrak{o}_{[\rho]}(A^1(G, K)))\cong(V^\rho\otimes\mathfrak{g}^c)^*_K;$$

$$\mathrm{Hom}_G(V^\rho, \mathfrak{o}_{[\rho]}(A^1(G/K)))\cong(V^\rho\otimes\mathfrak{g}^c)^{*-}_K.$$

Therefore we obtain the equalities (1.24) and (1.25). *Q.E.D.*

1.3. Let us set $\tilde{G}=G\times G$ and set $\tilde{K}=\{(g, g)\mid g\in G\}$. As is well known G is expressed by the symmetric space $\tilde{G}/\tilde{K}$ where the involution is defined by $(g, h)\mapsto(h, g)$. We can consider that in a sense (1.5) gives a decomposition of $\mathfrak{o}(A^1(G/K))$. In this paragraph, however, as an application of Theorem 1.3 we determine the spectrum of $A^1(G/K)$ in the line of 1.2.

Let ρ: $G\to GL(V^\rho)$ and σ: $G\to GL(V^\sigma)$ be two representations of G. We define a representation $\rho\boxtimes\sigma$: $G\to GL(V^\rho\otimes V^\sigma)$ by

$$(\rho\boxtimes\sigma)(g, h)=\rho(g)\otimes\sigma(h) \qquad g, h\in G.$$

It is known that the set $\mathscr{D}(\tilde{G})$ of equivalence classes of irreducible representations of $\tilde{G}$ is given by

$$\mathscr{D}(\tilde{G})=\{[\rho\boxtimes\sigma^*]\mid[\rho], [\sigma]\in\mathscr{D}(G)\}.$$

Corollary to Theorem 1.3. *For each* $[\rho\boxtimes\sigma^*]\in\mathscr{D}(\tilde{G})$, *it holds that*

$$a([\rho\boxtimes\sigma^*])=\dim_C \mathrm{Hom}_G(V^\sigma, V^\rho\otimes\mathfrak{g}^c). \tag{1.26}$$

Proof. Let $\tilde{\mathfrak{g}}=\mathfrak{g}+\mathfrak{g}$ be the Lie algebra of G. Set

$$\tilde{\mathfrak{m}}=\{(X, -X)\mid X\in\mathfrak{g}\},$$

then $\tilde{\mathfrak{m}}$ is the eigenspace for the eigenvalue -1 of the canonical involution θ: $\tilde{\mathfrak{g}}\ni(X, Y)\mapsto(Y, X)\in\tilde{\mathfrak{g}}$. We regard $\tilde{\mathfrak{m}}^c$ as a G-module under the action $g\cdot(X, -X)=(AdgX, -AdgX)$; $g\in G$, $X\in\mathfrak{g}^c$. Since $\tilde{\mathfrak{m}}^c\cong\mathfrak{g}^c$ as G-modules, we have:

$$(V^{\rho\boxtimes\sigma^*}\otimes\tilde{\mathfrak{g}}^c)_{\tilde{K}}^-\cong \mathrm{Hom}_G(V^\sigma, V^\rho\otimes\tilde{\mathfrak{m}}^c)\cong \mathrm{Hom}_G(V^\sigma, V^\rho\otimes\mathfrak{g}^c).$$

This completes the proof. *Q.E.D.*

§2. The spectrum of $A^1(G/K)$

2.1. Let G be a compact connected simple Lie group and let ρ: $G\to GL(V^\rho)$ be an irreducible representation of G. The aim of this paragraph is to decompose the tensor product $\rho\otimes Ad$ as a sum of irreducible representations of G. As is easily seen this decomposition is obtained by determining the number $a([\rho\boxtimes\sigma^*])$ $(=\dim_C \mathrm{Hom}_G(V^\sigma, V^\rho\otimes\mathfrak{g}^c))$ for all $[\sigma]\in\mathscr{D}(G)$.

Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$. We denote by Δ the set of non-zero roots[3)] of $\mathfrak{g}^c$ with respect to $\mathfrak{t}^c$ and by $\Pi=\{\alpha_1,\dots,\alpha_n\}$ the set of simple roots with respect to a linear order in $\mathfrak{t}$. For each $\alpha\in\Delta$ we set $\alpha^*=\frac{2}{(\alpha,\alpha)}$ and denote by s_α the reflection of $\mathfrak{t}$ *w.r.t.* α, i.e., $s_\alpha(\lambda)=\lambda-(\lambda,\alpha^*)\alpha$ $(\lambda\in\mathfrak{t})$.

Let $Z(G)$ (resp. $D(G)$) be the set of integral forms (resp. dominant integral forms). Let Λ_0 be the highest weight of $[Ad]$. For each pair $\{\Lambda, \Lambda'\}$ $(\Lambda, \Lambda'\in D(G))$ we define a number $a(\Lambda, \Lambda')$ be setting

$$\chi(\Lambda)\cdot\chi(\Lambda_0)=\sum_{\Lambda'\in D(G)} a(\Lambda,\Lambda')\chi(\Lambda'); \tag{2.1}$$

where for each $\lambda\in Z(G)$, we mean by $\chi(\lambda)$ the formal character associated with λ. Let $[\rho], [\sigma]\in\mathscr{D}(G)$ and let Λ, Λ' be the highest weight of $[\rho], [\sigma]$ respectively. Then it can be easily seen that $a([\rho\boxtimes\sigma^*])=a(\Lambda,\Lambda')$[4)]. Hence the decomposition of $\rho\otimes Ad$ is given by the following

Theorem 2.1. *The number* $a(\Lambda, \Lambda')$ *is given as follows*:

(1) *The case* $\Lambda'=\Lambda$: $a(\Lambda,\Lambda)=\#\{\alpha_i\in\Pi\mid(\Lambda,\alpha_i^*)>0\}$.

(2) *The case* $\Lambda'=\Lambda+\alpha$ *for some* $\alpha\in\Delta$:

$$a(\Lambda,\Lambda+\alpha)=\begin{cases}0 & \textit{if the pair is contained in the following list } (\bigstar);\\ 1 & \textit{otherwise.}\end{cases}$$

(3) *The case* $\Lambda'\neq\Lambda, \Lambda+\alpha$ *for any* $\alpha\in\Delta$: $a(\Lambda,\Lambda')=0$.

		$\mathfrak{g}^c$	α	Λ
(★)	(1°)	$[B_n]$ ○—○⋯—○⇒○ (α_1, α_2, α_{n-1}, α_n)	$\pm(\sum_{k=i}^{n}\alpha_k)$ $(1\leqq i\leqq n-1)$	$(\Lambda,\alpha_n^*)=0$
	(2°)	$[C_n]$ ○—○⋯—○⇐○ (α_1, α_2, α_{n-1}, α_n)	$\pm\{\alpha_i+2(\sum_{k=i+1}^{n-1}\alpha_k)+\alpha_n\}$ $(1\leqq i\leqq n-1)$	$(\Lambda,\alpha_i^*)=0$
	(3°)	$[F_4]$ ○—○⇒○—○ (α_1, α_2, α_3, α_4)	(i) $\pm(\alpha_2+\alpha_3)$, $\pm(\alpha_1+\alpha_2+\alpha_3)$, $\pm(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)$	$(\Lambda,\alpha_3^*)=0$
			(ii) $\pm(\alpha_2+2\alpha_3+\alpha_4)$, $\pm(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)$, $\pm(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)$	$(\Lambda,\alpha_4^*)=0$
	(4°)	$[G_2]$ ○⇐○ (α_1, α_2)	(i) $2\alpha_1+\alpha_2$, $-(\alpha_1+\alpha_2)$	$(\Lambda,\alpha_1^*)=0$
			(ii) $\alpha_1+\alpha_2$, $-(2\alpha_1+\alpha_2)$	$(\Lambda,\alpha_1^*)=1$

3) By a root we mean a vector $\lambda\in\mathfrak{t}$ such that the subspace defined by

$$(\mathfrak{g}^c)_\lambda=\{X\in\mathfrak{g}^c\mid[H,X]=2\pi\sqrt{-1}(\lambda,H)X \text{ for any } H\in\mathfrak{t}\}$$

is not equal to $\{0\}$. Similarly we mean by a weight of a representation $\rho: G\to GL(V^\rho)$ (simply

(*Continued on next page*)

Before proceeding to the proof we show

Proposition 2.2. *Let* $\{\beta, \alpha_i\}$ $(\beta\in\Delta, \alpha_i\in\Pi)$ *be a pair satisfying* $(\beta, \alpha_i^*)\leqq -2$. *Then* $\{\beta, \alpha_i\}$ *coincides with one of the following*:

(☆)

		$\mathfrak{g}^c$	β	α_i	(β, α_i^*)
(I)			$-\alpha_i(1\leqq i\leqq n)$	α_i	-2
(II)	(1°)[B_n]		$\sum_{k=i}^{n-1}\alpha_k(1\leqq i\leqq n-1)$	α_n	-2
			$-(\sum_{k=i}^{n-1}\alpha_k+2\alpha_n)(1\leqq i\leqq n-1)$	α_n	-2
	(2°)[C_n]		$2(\sum_{k=i+1}^{n-1}\alpha_k)+\alpha_n(1\leqq i\leqq n-1)$	α_i	-2
			$-\{2(\sum_{k=i}^{n-1}\alpha_k)+\alpha_n\}(1\leqq i\leqq n-1)$	α_i	-2
	(3°)[F_4]		(i)$^+$ $\begin{cases}\alpha_2\\ \alpha_1+\alpha_2\\ \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4\end{cases}$	α_3	-2
			(i)$^-$ $\begin{cases}-(\alpha_2+2\alpha_3)\\ -(\alpha_1+\alpha_2+2\alpha_3)\\ -(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4)\end{cases}$	α_3	-2
			(ii)$^+$ $\begin{cases}\alpha_2+2\alpha_3\\ \alpha_1+\alpha_2+2\alpha_3\\ \alpha_1+2\alpha_2+2\alpha_3\end{cases}$	α_4	-2
			(ii)$^-$ $\begin{cases}-(\alpha_2+2\alpha_3+2\alpha_4)\\ -(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4)\\ -(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4)\end{cases}$	α_4	-2
	(4°)[G_2]		$\alpha_2,\ -(3\alpha_1+\alpha_2)$	α_1	-3

(*Continued*)

a weight of V^ρ) a vector $\lambda\in\mathfrak{t}$ such that the subspace defined by

$$(V^\rho)_\lambda=\{v\in V^\rho \mid \rho(H)v=2\pi\sqrt{-1}(\lambda, H)\, v \text{ for any } H\in\mathfrak{t}\}$$

is not equal to $\{0\}$.

4) This equality follows from

$$\chi_{[\rho\otimes Ad]}=\chi_{[\rho]}\cdot\chi_{[Ad]}=\sum_{[\sigma]\in\mathscr{D}(G)}a([\rho\boxtimes\sigma^*])\chi_{[\sigma]};$$

where for each $[\tau]\in\mathscr{D}(G)$, $\chi_{[\tau]}$ denotes the character of $[\tau]$.

Proof. We first note that if $\beta=-\alpha_i$ for some $\alpha_i\in\Pi$, then it holds $(\beta, \alpha_i^*)\leqq -2$ and $(\beta, \alpha_j^*)\geqq 0$ for any $\alpha_j\in\Pi$ such that $\alpha_j\neq\alpha_i$.

Now let us seek for all pairs $\{\beta, \alpha_i\}(\beta\in\Delta, \alpha_i\in\Pi)$ such that $(\beta, \alpha_i^*)\leqq -2$, $\beta\neq -\alpha_i$. We remark that under this assumption we have $\beta+2\alpha_i\in\Delta$ and $\beta+2\alpha_i\neq\alpha_i$. Thus we first seek for all pairs $\{\gamma, \alpha_i\}$ $(\gamma\in\Delta, \alpha_i\in\Pi)$ such that $\gamma-2\alpha_i\in\Delta$, $\gamma\neq\alpha_i$. If $\mathfrak{g}^c$ is not of the type $[E_i]$ ($i=6$, 7 or 8), we can easily find out all such pairs. We have:

(1°) $[B_n]$ $\{\sum_{k=i}^{n-1}\alpha_k+2\alpha_n, \alpha_n\}$ $(1\leqq i\leqq n-1)$,

$\{-(\sum_{k=i}^{n-1}\alpha_k), \alpha_n\}$ $(1\leqq i\leqq n-1)$.

(2°) $[C_n]$ $\{2(\sum_{k=i}^{n-1}\alpha_k)+\alpha_n, \alpha_i\}$ $(1\leqq i\leqq n-1)$,

$\{-\{2(\sum_{k=i+1}^{n-1}\alpha_k)+\alpha_n\}, \alpha_i\}(1\leqq i\leqq n-1)$.

(3°) $[F_4]$ (i) $\{\alpha_2+2\alpha_3, \alpha_3\}$, $\{\alpha_1+\alpha_2+2\alpha_3, \alpha_3\}$, $\{\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4, \alpha_3\}$,

$\{-\alpha_2, \alpha_3\}$, $\{-(\alpha_1+\alpha_2), \alpha_3\}$, $\{-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4), \alpha_3\}$.

(ii) $\{\alpha_2+2\alpha_3+2\alpha_4, \alpha_4\}$, $\{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4, \alpha_4\}$,

$\{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4, \alpha_4\}$, $\{-(\alpha_2+2\alpha_3), \alpha_4\}$,

$\{-(\alpha_1+\alpha_2+2\alpha_3), \alpha_4\}$, $\{-(\alpha_1+2\alpha_2+2\alpha_3), \alpha_4\}$.

(4°) $[G_2]$ (i) $\{2\alpha_1+\alpha_2, \alpha_1\}$, $\{-(\alpha_1+\alpha_2), \alpha_1\}$.

(ii) $\{3\alpha_1+\alpha_2, \alpha_1\}$, $\{-\alpha_2, \alpha_1\}$.

Setting $\beta=\gamma-2\alpha_i$ for the above pairs, we have $(\beta, \alpha_i^*)=-2$ except the case (4°). We have (4°) (i) $(\beta, \alpha_1^*)=-3$, (ii) $(\beta, \alpha_1^*)=-1$.

Finally we treat the case where $\mathfrak{g}^c$ is of the type $[E_i]$ ($i=6$, 7 or 8). We show directly: If $(\beta, \alpha_1^*)\leqq -2$, then $\beta=-\alpha_i$.

Since $(\beta, \beta)=(\alpha_i, \alpha_i)$, it holds that $(\alpha_i, \beta^*)=(\beta, \alpha_i^*)\leqq -2$. Hence $\alpha_i+2\beta\in\Delta$. Writing $\beta=\sum_{j=1}^{n} m_j\alpha_j$, we have $\alpha_i+2\beta=\sum_{j=1}^{n}(2m_j+\delta_{ij})\alpha_j$. Then it can be checked that such a root of $\mathfrak{g}^c$ is limited to $\pm\alpha_i$. Hence we have $\beta=-\alpha_i$, showing our assertion.

Thus we have obtained all the pairs satisfying our assumption. *Q.E.D.*

Proof of Theorem 2.1. Let us set

(2.2) $\Delta_0(\Lambda)=\{\alpha\in\Delta\mid\Lambda+\alpha\in D(G)\}$, $\Delta_1(\Lambda)=\{\alpha\in\Delta\mid\Lambda+\alpha\notin D(G), \chi(\Lambda+\alpha)\neq 0\}$.

Then the left hand side of (2.1) is computed as follows:

$$\chi(\Lambda)\cdot\chi(\Lambda_0)=n\chi(\Lambda)+\sum_{\alpha\in\Delta_0(\Lambda)}\chi(\Lambda+\alpha)+\sum_{\beta\in\Delta_1(\Lambda)}\chi(\Lambda+\beta). \tag{2.3}$$

We consider the third term of the right hand side of (2.3). Let $\beta\in\Delta_1(\Lambda)$. Since

$\Lambda+\beta\notin D(G)$, there is a simple root α_i such that $(\Lambda+\beta, \alpha_i^*)<0$. We first suppose that $(\Lambda+\beta, \alpha_i^*)=-1$. Then we have $(\Lambda+\beta+\delta, \alpha_i^*)=0$ where $\delta=\frac{1}{2}(\sum_{\substack{\alpha\in\Delta\\ \alpha>0}}\alpha)$. Hence we obtain $\chi(\Lambda+\beta)=0$, contradicting our assumption $\beta\in\Delta_1(\Lambda)$. Therefore we have $(\Lambda+\beta, \alpha_i^*)\leqq -2$. Since $(\Lambda, \alpha_i^*)\geqq 0$, it follows that $(\beta, \alpha_i^*)\leqq -2$. Thus the pair $\{\beta, \alpha_i\}$ coincides with one of the pairs listed in (☆) of Proposition 2.2. Consequently we have $(\Lambda, \alpha_i^*)=0$, $(\Lambda+\beta, \alpha_i^*)=-2$ in the cases (I), (II) (1°)~(3°) and have (4°) (i) $(\Lambda, \alpha_1^*)=0$, $(\Lambda+\beta, \alpha_1^*)=-3$ or (ii) $(\Lambda, \alpha_1^*)=1$, $(\Lambda+\beta, \alpha_1^*)=-2$. Now let us put $\alpha=s_{\alpha_i}(\Lambda+\beta+\delta)-(\Lambda+\delta)$. Then by a direct calculation, we have $\alpha=0$ in the case (I) and obtain the list (★) that exhibits all the pairs $\{\Lambda, \alpha\}$ corresponding to the cases (II) (1°)~(4°). Thus we can see that α is a root not contained in the list (☆) of Proposition 2.2. This implies $\alpha\notin\Delta_1(\Lambda)$. On the other hand, since $\Lambda+\alpha+\delta=s_{\alpha_i}(\Lambda+\beta+\delta)$, we have $\chi(\Lambda+\alpha)=-\chi(\Lambda+\beta)\neq 0$. Hence we know $\Lambda+\alpha\in D(G)$. Therefore if we set $n_0=\#\{\alpha_i\,|\,(\Lambda, \alpha_i^*)>0\}$ and denote by $\Delta_0'(\Lambda)$ the subset of $\Delta_0(\Lambda)$ composed of all α such that the pair $\{\Lambda, \alpha\}$ is not contained in the list (★), we have

$$\chi(\Lambda)\cdot\chi(\Lambda_0)=n_0\chi(\Lambda)+\sum_{\alpha\in\Delta_0'(\Lambda)}\chi(\Lambda+\alpha),$$

proving the theorem. *Q.E.D.*

2.2. Let G/K be a simply connected compact irreducible Riemannian symmetric space with G simple. In this paragraph we determine the spectrum of $\Lambda^1(G, K)$.

Let $\mathscr{D}(G, K)$ be the set of equivalence classes of spherical representations for the symmetric pair (G, K). Then we have

Propotion 2.3. *Let $V^\rho\otimes\mathfrak{g}^c=U_1+\dots+U_r$ be a G-irreducible decomposition of $V^\rho\otimes\mathfrak{g}^c$. Set $\rho_i=(\rho\otimes Ad)_{|U_i}$ for i $(1\leqq i\leqq r)$. Then:*

$$b([\rho])=\#\{\rho_i\,|\,[\rho_i]\in\mathscr{D}(G, K)\}. \tag{2.4}$$

Proof. Set $(U_i)_K=U_i\cap(V^\rho\otimes\mathfrak{g}^c)_K$ for i $(1\leqq i\leqq r)$. Then our assertion follows from the following facts: $(V^\rho\otimes\mathfrak{g}^c)_K=(U_1)_K+\dots+(U_r)_K$ (direct sum), $\dim_C(U_i)_K\leqq 1$ and $[\rho_i]\in\mathscr{D}(G, K)\Leftrightarrow\dim_C(U_i)_K=1$. *Q.E.D.*

Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ obtained by θ. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$ containing a maximal abelian subspace of $\mathfrak{m}$. Set $\mathfrak{a}=\mathfrak{t}\cap\mathfrak{m}$ and $\mathfrak{b}=\mathfrak{t}\cap\mathfrak{k}$. Then we have

$$\mathfrak{t}=\mathfrak{a}+\mathfrak{b}\ \text{(orthogonal direct sum)}.$$

Note that $\mathfrak{t}$ (and hence the set Δ) is invariant by θ. In the following we fix a system of vectors $\{X_\alpha\in(\mathfrak{g}^c)\,|\,\alpha\in\Delta\}$ of $\mathfrak{g}^c$ satisfying

$$\theta X_\alpha=X_{\theta\alpha};\ [X_\alpha, X_{-\alpha}]=\alpha^*.$$

Let us define a linear order "<" in $\mathfrak{t}$ such that

$$H>0,\ H\notin\mathfrak{b}\Longrightarrow\theta H<0 \qquad H\in\mathfrak{t}.$$

Let $\Pi=\{\alpha_1,\dots,\alpha_n\}$ be the set of simple roots with respect to this order. We set

$I=\{i \mid \alpha_i \notin \mathfrak{b}\}$ and denote by p the Satake involution of I.

Let $D(G, K)$ be the set of all the highest weights of spherical representations for the pair (G, K). Under our assumption that G/K is simply connected, the set $D(G, K)$ is identical with the additive semi-group generated by the following M_i's ($i \in I$, $p(i) \geqq i$):

$$(2.5) \qquad M_i = \begin{cases} 2\Lambda_i & p(i)=i, \quad (\alpha_i, \Pi \cap \mathfrak{b}) = \{0\}; \\ \Lambda_i & p(i)=i, \quad (\alpha_i, \Pi \cap \mathfrak{b}) \neq \{0\}; \\ \Lambda_i + \Lambda_{p(i)} & p(i) > i. \end{cases}$$

Here we mean by $\{\Lambda_1, \ldots, \Lambda_n\}$ the set of fundamental weights, i.e.,

$$(\Lambda_i, \alpha_j^*) = \delta_{ij} \quad 1 \leqq i, j \leqq n.$$

Let $\Delta^*(K)$ denote the subset of Δ composed of all $\alpha \in \Delta$ such that for any $\alpha_i \in \Pi \cap \mathfrak{b}$, $\alpha - \alpha_i$ is not a root of $\mathfrak{g}^c$. It is easily checked that $\Delta^*(K)$ is invariant by θ. Then we obtain the following

Proposition 2.4. *Let* $\rho\colon G \to GL(V^\rho)$ *be an irreducible representation of G with* Λ *as its highest weight and let* v_Λ *be a non-zero highest weight vector of* V^ρ. *Let* $M \in D(G, K)$. *Assume that there exists an irreducible G-submodule U of* $V^\rho \otimes \mathfrak{g}^c$ *with M as its highest weight. Then it holds either* $M=\Lambda$ *or* $M=\Lambda+\alpha_0$ *for some* $\alpha_0 \in \Delta^*(K)$ *and U contains, as its highest weight vector, a non-zero* u_M *written in the following form*:

(1) *The case* $M=\Lambda$:

$$u_M = v_\Lambda \otimes H + \sum_{\substack{\alpha \in \Delta \\ \alpha > 0}} v_{M-\alpha} \otimes X_\alpha;$$

where $v_{M-\alpha} \in (V^\rho)_{M-\alpha}$; $H \in \mathfrak{t}^c$, $H \neq 0$, $(H, \alpha_i^*)=0$ *for any* $\alpha_i \in \Pi$ *such that* $(M, \alpha_i^*)=0$.

(2) *The case* $M=\Lambda+\alpha_0$ *for some* $\alpha_0 \in \Delta^*(K)$:

$$u_M = v_\Lambda \otimes X_{\alpha_0} + \sum_{i=1}^n v_i \otimes \Lambda_i + \sum_{\substack{\alpha \in \Delta \\ \alpha > \alpha_0}} v_{M-\alpha} \otimes X_\alpha;$$

where $v_i \in (V^\rho)_M$; $v_{M-\alpha} \in (V^\rho)_{M-\alpha}$.

Proof. Let u_M be a non-zero highest weight vector of U. Since $u_M \in (V^\rho)_M \otimes \mathfrak{t}^c + \sum_{\alpha \in \Delta} (V^\rho)_{M-\alpha} \otimes (\mathfrak{g}^c)_\alpha$, u_M can be written in the form

$$u_M = \sum_{i=1}^n v_i \otimes \Lambda_i + \sum_{\alpha \in \Delta} v_{M-\alpha} \otimes X_\alpha;$$

where $v_i \in (V^\rho)_M$, $v_{M-\alpha} \in (V^\rho)_{M-\alpha}$. Now the following two cases are possible:

(1) It holds $v_{M-\alpha}=0$ for any $\alpha \in \Delta$, $\alpha<0$ and $v_i \neq 0$ for some i ($1 \leqq i \leqq n$).

(2) Otherwise.

The case (1). Let β be any positive root. Then we have

$$0 = X_\beta(u_M) = \sum_{i=1}^n (X_\beta v_i) \otimes \Lambda_i + \ldots;$$

and hence $X_\beta v_i=0$ $(1\leqq i\leqq n)$. This means $v_i\in(V^\rho)_\Lambda$ $(1\leqq i\leqq n)$, showing $M=\Lambda$. Let us take $H\in\mathfrak{t}^c$ such that $v\otimes H=\sum_{i=1}^n v_i\otimes\Lambda_i$, then we obtain

$$u_M=v_\Lambda\otimes H+\sum_{\substack{\alpha\in\Delta\\ \alpha>0}} v_{M-\alpha}\otimes X_\alpha;\ H\neq 0.$$

We now suppose $(M, \alpha_i^*)=0$. Since $M-\alpha_i$ is not a weight of U, we have

$$0=X_{-\alpha_i}(u_M)=2\pi\sqrt{-1}(H, \alpha_i)v_\Lambda\otimes X_{-\alpha_i}+\cdots;$$

hence $(H, \alpha_i^*)=0$.

The case (2). Let $\alpha_0\in\Delta$ be the minimum root such that $v_{M-\alpha_0}\neq 0$. Then for each positive root β, we have

$$0=X_\beta(u_M)=\cdots+(X_\beta v_{M-\alpha_0})\otimes X_{\alpha_0}+\cdots.$$

Hence $X_\beta v_{M-\alpha_0}=0$. This means $v_{M-\alpha_0}\in(V^\rho)_\Lambda$, proving $M=\Lambda+\alpha_0$. Multiplying a complex number if necessary, we obtain

$$u_M=\sum_{i=1}^n v_i\otimes\Lambda_i+\sum_{\substack{\alpha\in\Delta\\ \alpha>\alpha_0}} v_{M-\alpha}\otimes X_\alpha+v_\Lambda\otimes X_{\alpha_0}.$$

Finally we show $\alpha_0\in\Delta^*(K)$. Let $\alpha_i\in\Pi\cap\mathfrak{b}$. Since $(M, \alpha_i^*)=0$, $M-\alpha_i$ is not a weight of U (Note that $M\in D(G, K)\subset\mathfrak{a}$). Then we have

$$0=X_{-\alpha_i}(u_M)=v_\Lambda\otimes[X_{-\alpha_i}, X_{\alpha_0}]+\cdots.$$

Hence we know that $[X_{-\alpha_i}, X_{\alpha_0}]=0$. This means that $\alpha_0-\alpha_i$ is not a root. Therefore $\alpha_0\in\Delta^*(K)$. *Q. E. D.*

Lemma 2.5. *Let $\alpha\in\Delta^*(K)$. Then there exists a unique $M(\alpha)\in D(G, K)$ such that for any $M\in D(G, K)$ the following two conditions are equivalent*:

(1) $$-\alpha+M\in D(G).$$

(2) $$M-M(\alpha)\in D(G, K).$$

Proof. By the definition of $\Delta^*(K)$, we have $(\alpha, \alpha_i^*)\leqq 0$ for any $\alpha_i\in\Pi\cap\mathfrak{b}$. Hence if we write

$$-\alpha=\sum_{i=1}^n k_i^0\Lambda_i,$$

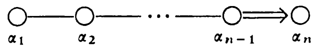
we have $k_i^0\geqq 0$ for any i such that $\alpha_i\in\Pi\cap\mathfrak{b}$. For each $i\in I$, $p(i)\geqq i$, let us set

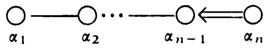
(2.6) $$k_i^1=\begin{cases}\max\left\{0,\ -\left[\frac{1}{2}k_i^0\right]\right\} & p(i)=i,\ (\alpha_i, \Pi\cap\mathfrak{b})=\{0\};\\ \max\{0,\ -k_i^0\} & p(i)=i,\ (\alpha_i, \Pi\cap\mathfrak{b})\neq\{0\};\\ \max\{0,\ -k_i^0,\ -k_{p(i)}^0\} & p(i)>i,\end{cases}$$

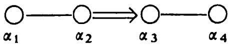
and put $M(\alpha)=\sum_{\substack{i\in I\\ p(i)\geqq i}} k_i^1 M_i$. Then it is easy to see that $M(\alpha)$ possesses the property stated in the lemma. The uniqueness is trivial. *Q.E.D.*

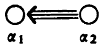
Lemma 2.6. *Any root listed in* (★) *of Theorem* 2.1 *is not contained in* $\Delta^*(K)$ *except the case where* G/K *is one of the following*:

[*BI*, *BII*] $SO(2p+1)/SO(p)\times SO(p+1)$:



[*CI*] $Sp(n)/U(n)$: 

[*FI*] $F_4/Sp(3)\cdot SU(2)$: 

[*G*] $G_2/SU(2)\times SU(2)$: 

Proof. It sufficies to consider the case where G/K is of type [*B*], [*C*], [*F*] or [*G*]. We note that in case the Satake diagram of G/K does not contain any black vertex (i.e., G/K coincides with one of the above), it holds that $\Delta^*(K)=\Delta$. Thus we have only to consider the following three cases:

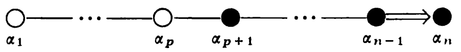
[*BI*, *BII*] $SO(p+q)/SO(p)\times SO(q)$ $(1\leqq p\leqq q-3,\ p+q\text{: odd})$;

[*CII*] $Sp(p+q)/Sp(p)\times Sp(q)$ $(1\leqq p\leqq q)$;

[*FII*] $F_4/Spin(9)$.

We assert that for each $\alpha\in\Delta$ listed in (★) there is a simple root $\alpha_i\in\Pi\cap\mathfrak{b}$ such that $\alpha-\alpha_i\in\Delta$.

[*BI*, *BII*] $SO(p+q)/SO(p)\times SO(q)$ $(1\leqq p\leqq q-3,\ p+q\text{: odd})$:



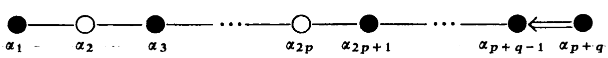
$\left(n=\frac{1}{2}(p+q-1)\right)$

We have: $\Pi\cap\mathfrak{b}=\{\alpha_{p+1},\dots,\alpha_n\}$.

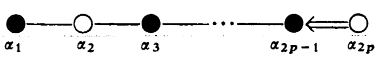
Hence: $\alpha=\sum_{k=i}^{n-1}\alpha_k+\alpha_n \overset{-\alpha_n}{\longmapsto} \sum_{k=i}^{n-1}\alpha_k\in\Delta\ (1\leqq i\leqq n-1)$;

$$\alpha=-\left(\sum_{k=i}^{n-1}\alpha_k+\alpha_n\right)\overset{-\alpha_n}{\longmapsto}-\left(\sum_{k=1}^{n-1}\alpha_k+2\alpha_n\right)\in\Delta\,(1\leqq i\leqq n-1).$$

[*CII*] $Sp(p+q)/Sp(p)\times Sp(q)$ $(1\leqq p\leqq q)$:



$(p<q)$



$(p=q)$

$$\text{We have: }\Pi\cap\mathfrak{b}=\begin{cases}\{\alpha_1,\alpha_3,\dots,\alpha_{2p-1},\alpha_{2p+1},\alpha_{2p+2},\dots,\alpha_{p+q}\} & (p<q);\\ \{\alpha_1,\alpha_3,\dots,\alpha_{2p-1}\} & (p=q).\end{cases}$$

Hence: $\alpha=\alpha_i+2(\sum_{k=i+1}^{n-1}\alpha_k)+\alpha_n \xmapsto{-\alpha_i} 2(\sum_{k=i+1}^{n-1}\alpha_k)+\alpha_n\in\Delta$ (i: odd),

$\xmapsto{-\alpha_{i+1}} \alpha_i+\alpha_{i+1}+2(\sum_{k=i+2}^{n-1}\alpha_k)+\alpha_n\in\Delta$ (i: even);

$\alpha=-\{\alpha_i+2(\sum_{k=i+1}^{n-1}\alpha_k)+\alpha_n\} \xmapsto{-\alpha_i} -\{2(\sum_{k=i}^{n-1}\alpha_k)+\alpha_n\}\in\Delta$ (i: odd),

$\xmapsto{-\alpha_{i-1}} -\{\alpha_{i-1}+\alpha_i+2(\sum_{k=i}^{n-1}\alpha_k)+\alpha_n\}\in\Delta$ (i: even).

Thus in the above cases our assertion is shown to be true. Similarly we can obtain the same result in the third case [*FII*] $F_4/\mathrm{Spin}(9)$: $\underset{\alpha_1}{\bullet}-\underset{\alpha_2}{\bullet}\Rightarrow\underset{\alpha_3}{\bullet}-\underset{\alpha_4}{\circ}$. The proof is left to the reader. *Q.E.D.*

We now prove

Theorem 2.7. *Let* $\alpha\in\Delta^*(K)$. *Then there exists a unique* $\Lambda(\alpha)\in D(G)$ *such that for any* $\Lambda\in D(G)$ *the following two conditions are equivalent*:

(1) $\Lambda+\alpha\in D(G, K)$ *and* $a(\Lambda, \Lambda+\alpha)=1$.

(2) $\Lambda-\Lambda(\alpha)\in D(G, K)$.

Precisely $\Lambda(\alpha)$ *is given as follows*:

	G/K	α	$\Lambda(\alpha)$
(I)	(1°) [*BI, BII*] ($p=q-1=n$)	$\pm(\sum_{k=i}^{n}\alpha_k)\ (1\leqq i\leqq n-1)$	$-\alpha+M(\alpha)+2\Lambda_n$
	(2°) [*CI*]	$\pm\{\alpha_i+2(\sum_{k=i+1}^{n-1}\alpha_k)+\alpha_n\}$ $(1\leqq i\leqq n-1)$	$-\alpha+M(\alpha)+2\Lambda_i$
	(3°) [*FI*]	(i) $\pm(\alpha_2+\alpha_3)$, $\pm(\alpha_1+\alpha_2+\alpha_3)$, $\pm(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)$	$-\alpha+M(\alpha)+2\Lambda_3$
		(ii) $\pm(\alpha_2+2\alpha_3+\alpha_4)$, $\pm(\alpha_1+\alpha_2+2\alpha_3+\alpha_4)$, $\pm(\alpha_1+2\alpha_2+2\alpha_3+\alpha_4)$	$-\alpha+M(\alpha)+2\Lambda_4$
	(4°) [*G*]	$\alpha_1+\alpha_2,\ -(2\alpha_1+\alpha_2)$	$-\alpha+M(\alpha)+2\Lambda_1$
(II)	otherwise		$-\alpha+M(\alpha)$

Proof. By Lemma 2.6, we know that the following two cases are possible:

(i) α is not contained in $(\star)$ of Theorem 2.1;

(ii) G/K coincides with one of the symmetric spaces listed in Lemma 2.6 and α is contained in $(\star)$ of Theorem 2.1.

The case (i). Let us set $\Lambda(\alpha)=-\alpha+M(\alpha)$. Then we have:

$$\Lambda-\Lambda(\alpha)=(\Lambda+\alpha)-M(\alpha).$$

By the above equality and by Lemma 2.5, we know that the two conditions $\Lambda+\alpha\in D(G, K)$ and $\Lambda-\Lambda(\alpha)\in D(G, K)$ are equivalent. This implies the equivalence of (1) and (2). (Note that $a(\Lambda, \Lambda+\alpha)=1$ follows immediately from $\Lambda+\alpha\in D(G, K)$.)

The case (ii). We first assume (1). By Lemma 2.5, we know that $M'=\Lambda-(-\alpha+M(\alpha))\in D(G, K)$. By a direct calculation, we have:

(1°) [BI, BII] $(p=q-1)(-\alpha+M(\alpha), \alpha_n^*)=0$; (2°) [$CI$] $(-\alpha+M(\alpha), \alpha_i^*)=0$;
(3°) [FI] (i) $(-\alpha+M(\alpha), \alpha_3^*)=0$, (ii) $(-\alpha+M(\alpha), \alpha_4^*)=0$;

(4°) [G] (i) $(-\alpha+M(\alpha), \alpha_1^*)=1$, (ii) $(-\alpha+M(\alpha), \alpha_1^*)=1$.

Hence by Theorem 2.1, we know that M' satisfy the following:

(1°) $(M', \alpha_n^*)>0$; (2°) $(M', \alpha_i^*)>0$;

(3°) (i) $(M', \alpha_3^*)>0$, (ii) $(M', \alpha_4^*)>0$; (4°) (ii) $(M', \alpha_1^*)>0$.

In the case (4°) (i), M' may be allowed to be any element of $D(G, K)$. Therefore if we define $\Lambda(\alpha)$ as the assertion of this theorem, we have $\Lambda-\Lambda(\alpha)\in D(G, K)$. (Note that since the Satake diagram of G/K does not contain any black vertex nor any arrow, the set $D(G, K)$ is given by

$$D(G, K)=\{\sum_{i=1}^{n}2k_i\Lambda_i \mid k_i\in \boldsymbol{Z},\ k_i\geqq 0\ (1\leqq i\leqq n)\}.)$$

Conversely if $\Lambda-\Lambda(\alpha)\in D(G, K)$, then it can be easily checked that $\Lambda+\alpha\in D(G, K)$ and $a(\Lambda, \Lambda+\alpha)=1$. The uniqueness of $\Lambda(\alpha)$ is straightforward. *Q.E.D.*

Let $\Lambda\in D(G)$. We set

$$(2.7)\quad I(\Lambda)=\begin{cases}\{i \mid 1\leqq i\leqq n,\ (\Lambda, \alpha_i^*)>0\} & \text{if } \Lambda\in D(G, K),\\ \emptyset & \text{if } \Lambda\notin D(G, K);\end{cases}$$

$$(2.8)\qquad B(\Lambda)=\{\alpha\in\Delta^*(K) \mid \Lambda-\Lambda(\alpha)\in D(G, K)\}.$$

The following theorem is an immediate consequence of Theorem 2.1, Proposition 2.3 and the above theorem.

Theorem 2.8. *Let* $[\rho]\in\mathscr{D}(G)$ *and let* $\Lambda\in D(G)$ *be the highest weight of* $[\rho]$. *Then the following equality holds*:

$$(2.9)\qquad b([\rho])=\#I(\Lambda)+\#B(\Lambda).$$

§ 3. The spectrum of $\Lambda^1(G/K)$

In this section we investigate the spectrum of $\Lambda^1(G/K)$. The notations used here are the same as in 2.2.

3.1. Let $\rho: G\to GL(V^\rho)$ be an irreducible representation of G with Λ as its highest weight and let v_Λ be a highest weight vector such that $(v_\Lambda, v_\Lambda)=1$. We define a $\boldsymbol{C}$-linear map Ψ: $V^\rho\otimes\mathfrak{g}^c\to\mathfrak{g}^c$ by

$$\Psi(v\otimes X)=(v, v_\Lambda)X \qquad v\in V^\rho,\ X\in\mathfrak{g}^c. \tag{3.1}$$

It is clear that

$$\Psi\cdot\tilde{\theta}=\theta\cdot\Psi. \tag{3.2}$$

Set $\mathfrak{p}=\Psi((V^\rho\otimes\mathfrak{g}^c)_K)$. Since $(V^\rho\otimes\mathfrak{g}^c)_K$ is invariant by $\tilde{\theta}$, $\mathfrak{p}$ is invariant by θ. For simplicity in the following we set $V=V^\rho$, $I=I(\Lambda)$, $B=B(\Lambda)$ and $C=\Delta\backslash B$.

Lemma 3.1. *There exists a basis* $\{Y_\alpha, Z_i\}_{\alpha\in B, i\in I}$ *of* $\mathfrak{p}$ *written in the form*:

$$Y_\alpha=X_\alpha+\sum_{\substack{\beta\in C\\ \beta<\alpha}}p_\alpha^\beta X_\beta+H_\alpha; \tag{3.3}$$

$$Z_i=\Lambda_i+\sum_{\substack{\beta\in C\\ \beta<0}}q_i^\beta X_\beta; \tag{3.4}$$

where p_α^β, $q_i^\beta\in\boldsymbol{C}$ *and* $H_\alpha\in\mathfrak{t}^c$ *such that* $(H_\alpha, \alpha_i^*)=0$ *for any* $i\in I$.

Proof. Let $\alpha\in B$ and let U be the irreducible G-submodule of $V\otimes\mathfrak{g}^c$ with M $(=\Lambda+\alpha)$ as its highest weight and u_M be the non-zero highest weight vector given in Proposition 2.4. We set

$$u_\alpha^0=\int_K k\cdot u_M dk,$$

where dk denotes a Haar measure of K. Then we have $u_\alpha^0\in(V\otimes\mathfrak{g}^c)_K$ and $(u_\alpha^0, u_M)\neq 0$. Let Ω be the set of weights of U. Then we can write

$$u_\alpha^0=c_\alpha u_M+\sum_{\substack{\mu\in\Omega\\ \mu<M}}u_\mu;\ c_\alpha\in\boldsymbol{C},\ u_\mu\in(U)_\mu.$$

Since $(u_\mu, u_M)=0$ for any μ $(<M)$, it follows that $c_\alpha\neq 0$. Hence we have

$$\Psi(u_\alpha^0)=c_\alpha X_\alpha+\sum_{\substack{\beta\in\Delta\\ \beta<\alpha}}c_\alpha^\beta X_\beta+H'_\alpha;\ c_\alpha^\beta\in\boldsymbol{C},\ H'_\alpha\in\mathfrak{t}^c.$$

(Note that $H'_\alpha=0$ in case $\alpha<0$.)

Assume now the case $\Lambda\in D(G, K)$. Then there exist linearly independent irreducible G-submodules U_i $(i\in I)$ whose highest weight vectors u_i $(i\in I)$ are of the form:

$$u_i=v_\Lambda\otimes\Lambda_i+\sum_{\substack{\alpha\in\Delta\\ \alpha>0}}v_{\Lambda-\alpha}\otimes X_\alpha;\ v_{\Lambda-\alpha}\in V_{\Lambda-\alpha}.$$

Let us set

$$u_i^0=\int_K k\cdot u_i dk,$$

then by the same argument as above we have

$$\Psi(u_i^0)=d_i A_i+\sum_{\substack{\beta\in\Delta\\ \beta<0}} d_i^\beta X_\beta;\ d_i\,(\neq 0),\ d_i^\beta\in \boldsymbol{C}.$$

Thus if we take linear combinations of $\Psi(u_\alpha^0)$'s $(\alpha\in B)$ and $\Psi(u_i^0)$'s $(i\in I)$ appropriately, we obtain a basis $\{Y_\alpha, Z_i\}$ of $\mathfrak{p}$ of the form stated in the lemma. *Q.E.D.*

By Lemma 3.1, it is known that the map Ψ gives a linear isomorphism between $(V\otimes\mathfrak{g}^c)_K$ and $\mathfrak{p}$. We set

$$\mathfrak{p}^-=\{Y\in\mathfrak{p}\mid \theta Y=-Y\}.$$

From (3.2) and Theorem 1.3, it follows the following

Lemma 3.2. *The map Ψ gives a linear isomorphism between $(V\otimes\mathfrak{g}^c)_K^-$ and $\mathfrak{p}^-$. Therefore*:

$$a([\rho])=\dim_{\boldsymbol{C}}\mathfrak{p}^-. \tag{3.5}$$

We now set: $I_1=\{i\in I\mid p(i)=i\}$, $I_2=\{i\in I\mid p(i)\neq i\}$;

$$B_0=\{\alpha\in B\mid \theta\alpha=\alpha\},\ B_1=\{\alpha\in B\mid \theta\alpha\neq\alpha,\ \theta\alpha\in B\},$$

$$B_2=\{\alpha\in B\mid \theta\alpha\notin B\}.$$

It is easily seen that $pI_2=I_2$ and $\theta B_1=B_1$.

Let us denote by $\mathfrak{p}_0$ the complex subspace of $\mathfrak{p}$ generated by the vectors $\{Y_\alpha\}_{\alpha\in B_2}$ stated in Lemma 3.1 and set

$$a_0([\rho])=\dim_{\boldsymbol{C}}(\mathfrak{p}_0\cap\mathfrak{p}^-). \tag{3.6}$$

Proposition 3.3. *The subspace $\mathfrak{p}_0$ is invariant by θ and the following equality holds*:

$$a([\rho])=a_0([\rho])+\#I_1+\frac{1}{2}\#I_2+\frac{1}{2}\#B_1. \tag{3.7}$$

Proof. Applying θ on both sides of (3.3) and (3.4), we easily have

$$\theta Y_\alpha=Y_{\theta\alpha}+\sum_{\substack{\beta\in B_2\\ \theta\beta<\alpha}} p_\alpha^{\theta\beta}Y_\beta \qquad (\alpha\in B_0 \text{ or } B_1); \tag{3.8}$$

$$\theta Y_\alpha=\sum_{\substack{\beta\in B_2\\ \theta\beta<\alpha}} p_\alpha^{\theta\beta}Y_\beta \qquad (\alpha\in B_2); \tag{3.9}$$

$$\theta Z_i=-Z_{p(i)}+\sum_{\substack{\beta\in B_2\\ \theta\beta<0}} q_i^{\theta\beta}Y_\beta \qquad (i\in I). \tag{3.10}$$

(Here we note $\theta\Lambda_i=-\Lambda_{p(i)}$.) Thus we can see that $\mathfrak{p}_0$ is invariant by θ. Let us set:

$$E_\alpha=2Y_\alpha+\sum_{\substack{\beta\in B_2\\ \theta\beta<\alpha}} p_\alpha^{\theta\beta}Y_\beta \qquad (\alpha\in B_0);$$

$$F_\alpha^\pm=Y_\alpha\pm(Y_{\theta\alpha}+\sum_{\substack{\beta\in B_2\\ \theta\beta<\alpha}} p_\alpha^{\theta\beta}Y_\beta) \qquad (\alpha\in B_1,\ \alpha<0);$$

$$P_i=2Z_i-\sum_{\substack{\beta\in B_2\\ \theta\beta<0}} q_i^{\theta\beta}Y_\beta \qquad (i\in I_1);$$

$$Q_i^\pm=Z_i\pm(-Z_{p(i)}+\sum_{\substack{\beta\in B_2\\ \theta\beta<0}} q_i^{\theta\beta}Y_\beta) \qquad (i\in I_2,\ p(i)>i).$$

Then we have: $\theta E_\alpha=E_\alpha$, $\theta F_\alpha^\pm=\pm F_\alpha^\pm$, $\theta P_i=-P_i$ and $\theta Q_i^\pm=\pm Q_i^\pm$. Since the vectors E_α $(\alpha\in B_0)$, $F_\alpha^\pm$ $(\alpha\in B_1,\ \alpha<0)$, Y_α $(\alpha\in B_2)$, P_i $(i\in I_1)$ and $Q_i^\pm$ $(i\in I_2,\ p(i)>i)$ form a basis of $\mathfrak{p}$, the equality (3.7) can be easily observed. *Q.E.D.*

3.2. In what follows we calculate the number $a_0([\rho])$ under some conditions. As is easily seen, $a_0([\rho])$ is closely related to the set B_2. We suppose that the set B_2 is composed of β_i $(1\leqq i\leqq m)$ and γ_s $(1\leqq s\leqq n)$ satisfying

$$\beta_1>\cdots>\beta_m>0>\gamma_n>\cdots>\gamma_1.$$

Proposition 3.4. *Assume that one of the following conditions* (1) *and* (2) *is satisfied. Then it holds* $a_0([\rho])=n$.

(1) $m=n,\ \beta_1>\theta\gamma_1>\beta_2>\theta\gamma_2>\cdots>\beta_n>\theta\gamma_n$.

(2) (i) $m=n+1,\ \beta_1>\theta\gamma_1>\beta_2>\theta\gamma_2>\cdots>\beta_n>\theta\gamma_n>\beta_{n+1}$;

(ii) $\theta\beta_{n+1}=-\beta_{n+1},\ (\Lambda,\ \beta_{n+1}^*)=0$.

Proof. We first note the following fact:

$$\beta,\ \gamma\in B_2,\ \beta>\gamma\Longrightarrow\theta\beta<\theta\gamma.$$

Therefore we know by (3.9) that there exists a non-zero complex number ε_1 such that $\theta Y_{\gamma_1}=\varepsilon_1 Y_{\beta_1}$. Hence we have $\theta Y_{\beta_1}=\frac{1}{\varepsilon_1}Y_{\gamma_1}$. This together with (3.9) means that both the terms Y_{β_1} and Y_{γ_1} vanish in θY_{β_i} $(2\leqq i\leqq m)$ and θY_{γ_s} $(2\leqq s\leqq n)$. Then we also know by (3.9) that there exist a non-zero complex number ε_2 such that $\theta Y_{\gamma_2}=\varepsilon_2 Y_{\beta_2}$. Hence we have $\theta Y_{\beta_2}=\frac{1}{\varepsilon_2}Y_{\gamma_2}$. This means that both the terms Y_{β_2} and Y_{γ_2} vanish in θY_{β_i} $(3\leqq i\leqq m)$ and θY_{γ_s} $(3\leqq s\leqq n)$. Applying the same argument successively we know that there exist non-zero complex numbers ε_i $(1\leqq i\leqq n)$ (and ε $(\varepsilon^2=1)$ in the case (2)) such that

$$\theta Y_{\gamma_i}=\varepsilon_i Y_{\beta i},\ \theta Y_{\beta_i}=\frac{1}{\varepsilon_i}Y_{\gamma_i} \qquad (1\leqq i\leqq n);$$

$$(\theta Y_{\beta n+1}=\varepsilon Y_{\beta n+1}).$$

Setting $R_i^\pm=Y_{\gamma_i}\pm\varepsilon_i Y_{\beta_i}$ for i $(1\leqq i\leqq n)$, we have $\theta R_i^\pm=\pm R_i^\pm$. Thus the proof of the proposition is completed by

Lemma 3.5. *Assume that an element $\beta \in B_2$ satisfies the following*:

(i) $\beta > 0$, $\theta\beta = -\beta$ *and* $(\Lambda, \beta^*) = 0$;

(ii) $\theta Y_\beta = \varepsilon Y_\beta \qquad (\varepsilon^2 = 1)$.

Then it holds $\varepsilon = 1$.

Proof. Take $w_\beta \in (V \otimes \mathfrak{g}^c)_K$ such that $\Psi(w_\beta) = Y_\beta$. Since $X_\beta + X_{-\beta} \in \mathfrak{k}^c$, it follows that

$$0 = (X_\beta + X_{-\beta}) w_\beta = v_\Lambda \otimes [X_\beta + X_{-\beta}, Y_\beta] + \cdots.$$

Hence we have $[X_\beta + X_{-\beta}, Y_\beta] = 0$. (Note that since $(\Lambda, \beta^*) = 0$, $\Lambda - \beta$ is not a weight of V.) On the other hand from the assumption $\theta Y_\beta = \varepsilon Y_\beta$, it follows

$$Y_\beta = X_\beta + \cdots + \varepsilon X_{-\beta} + \cdots.$$

Putting this into the above equality, we have

$$\varepsilon [X_\beta, X_{-\beta}] + [X_{-\beta}, X_\beta] = 0.$$

Therefore we have $\varepsilon = 1$. *Q.E.D.*

Proposition 3.6. *Assume that the set B_2 is composed of two positive roots β_1 and β_2 satisfying $\theta\beta_i = -\beta_i$ and $(\Lambda, \beta_i^*) = 0$ $(i = 1, 2)$. Then follows $\theta Y_{\beta i} = Y_{\beta_i}$ $(i = 1, 2)$. Therefore $a_0([\rho]) = 0$.*

Proof. If we express

$$Y_{\beta_1} = X_{\beta_1} + \cdots + q X_{-\beta_2} + \cdots + p X_{-\beta_1} + \cdots,$$

$$Y_{\beta_2} = X_{\beta_2} + \cdots + s X_{-\beta_2} + \cdots + r X_{-\beta_1} + \cdots,$$

we have

$$\theta Y_{\beta_1} = p Y_{\beta_1} + q Y_{\beta_2},\ \theta Y_{\beta_2} = r Y_{\beta_1} + s Y_{\beta_2}.$$

Take $w_{\beta_1}, w_{\beta_2} \in (V \otimes \mathfrak{g}^c)_K$ such that $\Psi(w_{\beta 1}) = Y_{\beta_1}$, $\Psi(w_{\beta_2}) = Y_{\beta_2}$. Applying a similar argument as in Lemma 3.5 to the relations

$$(X_{\beta_1} + X_{-\beta_1}) w_{\beta_1} = (X_{\beta_2} + X_{-\beta_2}) w_{\beta_2} = 0,$$

we obtain $p = s = 1$. From this it follows $q = r = 0$, implying $\theta Y_{\beta_i} = Y_{\beta_i}$ $(i = 1, 2)$. *Q.E.D.*

Remark. Let ρ: $G \to GL(V^\rho)$ be an irreducible representation of G with Λ as its highest weight. We put

$$a(\Lambda) = \#\{i \in I(\Lambda) \mid p(i) \geqq i\} + \#\{\alpha \in B(\Lambda) \mid \theta\alpha \neq \alpha,\ \alpha < 0\}.$$

Then it can be easily seen that if one of the assumptions (1) and (2) of Proposition 3.4

satisfied, then the equality $a([\rho])=a(\Lambda)$ holds. Similarly if the assumption of Proposition 3.6 is satisfied, the equality also holds. After the examinations of all simply connected compact irreducible Riemannian symmetric spaces G/K with G simple, we know that one of the assumptions (1) and (2) of Proposition 3.4 is satisfied except the following two cases: (1°) $[DI]$ $SO(2p)/SO(p)\times SO(p)$ $(p\geqq 3)$: $\Lambda=\Lambda_{p-2}+M_0$ $(M_0\in D(G, K)$, $(M_0, \alpha^*_{p-1})=(M_0, \alpha^*_p)=0)$; (2°) $[G]$ $G_2/SU(2)\times SU(2)$: $\Lambda=\Lambda_1$. In the case (1°) we can see that the assumption of Proposition 3.6 is satisfied. The details are ommited here. We treat the case (2°) in detail in the next section and show that the equality $a([\rho])=a(\Lambda)$ also holds in this case. Thus we have:

Theorem 3.7. *Let $[\rho]\in\mathscr{D}(G)$ and let $\Lambda\in D(G)$ be the highest weight of $[\rho]$. Then the following equality holds*:

$$a([\rho])=a(\Lambda). \tag{3.11}$$

In the forthcoming paper [14] we will exhibit the lists of the spectra of 1-forms on all simply connected compact irreducible Riemannian symmetric spaces G/K with G simple.

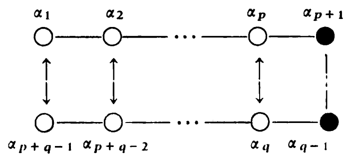
§ 4. Examples

In this section we calculate the spectra of $\Lambda^1(G/K)$ for $[AIII]$ $SU(p+q)/S(U(p)\times U(q))$ $(q\geqq p)$ and $[G]$ $G_2/SU(2)\times SU(2)$. The notations used here are the same as in the previous sections.

4.1. Example 1. $[AIII]$ $SU(p+q)/S(U(p)\times U(q))(q\geqq p)$.

Here we treat the cases $q\geqq p+2$ in detail. (As to the cases $q=p$, $p+1$, we only exhibit the results.)

$[AIII]$ $SU(p+q)/S(U(p)\times U(q))$ $\qquad (q\geqq p+2)$



Let us set $\hat{\alpha}_i=\alpha_{p+q-i}$ $(1\leqq i\leqq p)$, $\hat{\Lambda}_i=\Lambda_{p+q-i}$ $(1\leqq i\leqq p)$, $\beta=\sum_{i=p+1}^{q-1}\alpha_i$ and $\Lambda_0=\hat{\Lambda}_0=0$. Then we have:

(a) $\theta\alpha_i=-\hat{\alpha}_i$, $\theta\hat{\alpha}_i=-\alpha_i$ $\qquad (1\leqq i\leqq p-1)$;

$\theta\alpha_p=-(\hat{\alpha}_p+\beta)$, $\theta\hat{\alpha}_p=-(\alpha_p+\beta)$.

(b) $D(G, K)=\{\sum_{i=1}^{p} k_i(\Lambda_i+\hat{\Lambda}_i)\mid k_i\in\boldsymbol{Z},\ k_i\geqq 0,\ (1\leqq i\leqq p)\}$.

(c) $\Delta^*(K)$: (1°) $\sum_{k=i}^{j}\alpha_k\ (=\xi_{ij})$, $\sum_{k=i}^{j}\hat{\alpha}_k\ (=\hat{\xi}_{ij})$,

$$\theta\xi_{ij}\ (=-\hat{\xi}_{ij}),\ \theta\hat{\xi}_{ij}\ (=-\xi_{ij}) \qquad (1\leqq i\leqq j\leqq p-1);$$

(2°) $$\sum_{k=i}^{j}\alpha_k+\sum_{k=j+1}^{p}(\alpha_k+\hat{\alpha}_k)+\beta\ (=\eta_{ij}),$$

$$\sum_{k=i}^{j}\hat{\alpha}_k+\sum_{k=j+1}^{r}(\alpha_k+\hat{\alpha}_k)+\beta\ (=\hat{\eta}_{ij}),$$

$$\theta\eta_{ij}\ (=-\hat{\eta}_{ij}),\ \theta\hat{\eta}_{ij}\ (=-\eta_{ij}) \qquad (1\leqq i\leqq j\leqq p-1);$$

(3°) $$\sum_{k=i}^{p}\alpha_k\ (=\zeta_i),\ \sum_{k=i}^{p}\hat{\alpha}_k\ (=\hat{\zeta}_i),$$

$$\theta\zeta_i\ (=-(\hat{\zeta}_i+\beta)),\ \theta\hat{\zeta}_i\ (=-(\zeta_i+\beta)) \qquad (1\leqq i\leqq p);$$

(4°) $$\sum_{k=i}^{p}(\alpha_k+\hat{\alpha}_k)+\beta\ (=\beta_i),\ \theta\beta_i\ (=-\beta_i) \qquad (1\leqq i\leqq p);$$

(5°) $-\beta$.

Computing $\Lambda(\alpha)$ for each $\alpha\in\Delta^*(K)$ we obtain the following list of $\Lambda\in D(G)$ such that $B(\Lambda)\neq\phi$:

	Λ	$B(\Lambda)$
(I)	M_0	$\beta_i\ (i\in I_0^+),\ -\beta_j (j\in I_0)$
(II)	$\Lambda_{i-1}+\hat{\Lambda}_i+\hat{\Lambda}_j+\Lambda_{j+1}+M_0$ $(1\leqq i\leqq j\leqq p-1)$	$\xi_{ij},\ \theta\xi_{ij},\ \eta_{ij},\ \theta\eta_{ij}$
	$\hat{\Lambda}_{i-1}+\Lambda_i+\Lambda_j+\hat{\Lambda}_{j+1}+M_0$ $(1\leqq i\leqq j\leqq p-1)$	$\hat{\xi}_{ij},\ \theta\hat{\xi}_{ij},\ \hat{\eta}_{ij},\ \theta\hat{\eta}_{ij}$
(III)	$\Lambda_{i-1}+\hat{\Lambda}_i+\hat{\Lambda}_p+\Lambda_{p+1}+M_0$ $(1\leqq i\leqq p)$	$\zeta_i,\ \theta\zeta_i$
	$\hat{\Lambda}_{i-1}+\Lambda_i+\Lambda_p+\Lambda_{q-1}+M_0$ $(1\leqq i\leqq p)$	$\hat{\zeta}_i,\ \theta\hat{\zeta}_i$
(IV)	$\Lambda_{p+1}+\Lambda_{q-1}+M_0$	$-\beta$

Here we mean by M_0 an arbitrary element of $D(G, K)$ and set

$I_0=\{i\mid 1\leqq i\leqq p,\ i\in I(M_0)\}$, $I_0^+=\{1\}\cup\{i+1\mid i\in I_0,\ i\neq p\}$.

Thus we can easily observe $B_2=\phi$ except the case (I). We now assert that one of the assumptions (1) and (2) or Proposition 3.4 is satisfied in the case (I). We obtain by the definitions of B_2, I_0 and I_0^+:

$$B_2=\{\beta_i\mid i\in I_0^+\backslash I_0\}\cup\{-\beta_j\mid j\in I_0\backslash I_0^+\}.$$

Let us set

$$I_0^+\backslash I_0=\{i_1<i_2\ldots<i_m\};$$

$$I_0\backslash I_0^+=\{j_1<j_2\ldots<j_n\}.$$

Then we have:

(1) If $p\in I_0$, then we have $m=n$ and $i_1<j_1<i_2<j_2<\ldots<i_n<j_n$.

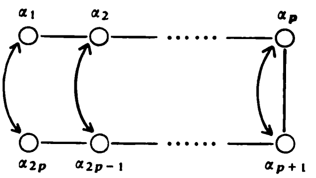
(2) If $p\notin I_0$, then we have $m=n+1$ and $i_1<j_1<i_2<j_2<\ldots<i_n<j_n<i_{n+1}$. Noting the fact $i<j\Leftrightarrow\beta_i>\beta_j$, we know that our assertion has been already shown except the equality $(M_0, \beta^*_{i_{n+1}})=0$ in the case (2). To show $(M_0, \beta^*_{i_{n+1}})=0$ we set $i_0=\max I_0$. Since $p\notin I_0$, we have $i_0\leqq p-1$ and hence $i_0+1=\max(I_0^+\backslash I_0)=i_{n+1}$. This means $(M_0, \alpha_i^*)=(M_0, \hat{\alpha}_i^*)=0$ for all i $(i_{n+1}\leqq i\leqq p)$. Since $M_0\in D(G, K)\subset$ $\mathfrak{a}$, we have $(M_0, \alpha_j^*)=0$ for all j $(p+1\leqq j\leqq q-1)$. Therefore we have $(M_0, \beta^*_{i_{n+1}})=$ 0, proving our assertion. Thus we know that the equality (3.11) holds (see Remark in §3).

In the same manner as above we can show that the equality also holds in the case $q=p$ or $p+1$. Consequently we obtain the following tables:

(1) $[AIII]$ $SU(p+q)/S(U(p)\times U(q))(q\geqq p+2)$

	Λ	$a(\Lambda)$
(I)	M_0	$\sharp I(M_0)$
(II)	$\Lambda_{i-1}+\hat{\Lambda}_i+\hat{\Lambda}_j+\Lambda_{j+1}+M_0$ $(1\leqq i\leqq j\leqq p-1)$	2
	$\hat{\Lambda}_{i-1}+\Lambda_i+\Lambda_j+\hat{\Lambda}_{j+1}+M_0$ $(1\leqq i\leqq j\leqq p-1)$	2
(III)	$\Lambda_{i-1}+\hat{\Lambda}_i+\hat{\Lambda}_p+\Lambda_{p+1}+M_0$ $(1\leqq i\leqq p)$	1
	$\hat{\Lambda}_{i-1}+\Lambda_i+\Lambda_p+\hat{\Lambda}_{p+1}+M_0$ $(1\leqq i\leqq p)$	1
(IV)	$\Lambda_{p+1}+\Lambda_{q-1}+M_0$	0

(2) $[AIII]$ $SU(2p+1)/S(U(p)\times U(p+1))$

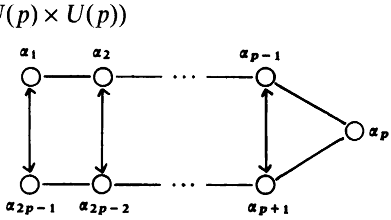


	Λ	$a(\Lambda)$
(I)	M_0	$\sharp I(M_0)$

(II)	$\Lambda_{i-1}+\hat{\Lambda}_i+\hat{\Lambda}_j+\Lambda_{j+1}+M_0$ $(1\leqq i\leqq j\leqq p-1)$	2
	$\hat{\Lambda}_{i-1}+\Lambda_i+\Lambda_j+\hat{\Lambda}_{j+1}+M_0$ $(1\leqq i\leqq j\leqq p-1)$	2
(III)	$\Lambda_{i-1}+\hat{\Lambda}_i+2\hat{\Lambda}_p+M_0$ $(1\leqq i\leqq p)$	1
	$\hat{\Lambda}_{i-1}+\Lambda_j+2\Lambda_p+M_0$ $(1\leqq i\leqq p)$	1

Notation: $\hat{\Lambda}_i=\Lambda_{2p-i+1}$ $(1\leqq i\leqq p)$

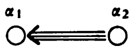
(3) [$AIII$] $SU(2p)/S(U(p)\times U(p))$



	Λ	$a(\Lambda)$
(I)	M_0	$\sharp I(M_0)$ $(p\notin I(M_0))$ $\sharp I(M_0)+1$ $(p\in I(M_0))$
(II)	$\Lambda_{i-1}+\hat{\Lambda}_i+\hat{\Lambda}_j+\Lambda_{j+1}+M_0$ $(1\leqq i\leqq j\leqq p-1)$	2
	$\hat{\Lambda}_{i-1}+\Lambda_i+\Lambda_j+\Lambda_{j+1}+M_0$ $(1\leqq i\leqq j\leqq p-1)$	2

Notation: $\Lambda_i=\Lambda_{2p-i}$ $(1\leqq i\leqq p)$

4.2. Example 2. [G] $G_2/SU(2)\times SU(2)$



We have

(a) $\theta\alpha_i=-\alpha_i$ $(i=1, 2)$.

(b) $D(G, K)=\{2k_1\Lambda_1+2k_2\Lambda_2 \mid k_1, k_2\in \boldsymbol{Z}, k_1, k_2\geqq 0\}$.

(c) $\Delta^*(K)=\Delta=\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2), \pm(3\alpha_1+\alpha_2), \pm(3\alpha_1+2\alpha_2)\}$.

Computing $\Lambda(\alpha)$ for each $\alpha\in\Delta^*(K)$, we have the following list of $\Lambda\in D(G, K)$ such that $B(\Lambda)\neq\phi$ or $I(\Lambda)\neq\phi$:

Λ			$B(\Lambda)$
(I) M_0			ϕ
(II) Λ_1+M_0	(M_0, α_1^*)	(M_0, α_2^*)	
	0	0	$2\alpha_1+\alpha_2$
	+	0	α_2, $\pm(2\alpha_1+\alpha_2)$
	0	+	$-\alpha_2$, $2\alpha_1+\alpha_2$
	+	+	$\pm\alpha_2$, $\pm(2\alpha_1+\alpha_2)$
(III) Λ_2+M_0	(M_0, α_1^*)		
	0		α_1, $\pm(3\alpha_1+2\alpha_2)$
	+		$\pm\alpha_1$, $\pm(3\alpha_1+2\alpha_2)$
(IV) $\Lambda_1+\Lambda_2+M_0$	(M_0, α_1^*)		
	0		$-(\alpha_1+\alpha_2)$, $3\alpha_1+\alpha_2$
	+		$\pm(\alpha_1+\alpha_2)$, $\pm(3\alpha_1+\alpha_2)$

Here we mean by M_0 an arbitrary element of $D(G, K)$. It is easy to see that except the case (II) $(M_0, \alpha_1^*)=(M_0, \alpha_2^*)=0$, i.e., $\Lambda=\Lambda_1$, one of the assumptions (1) and (2) of Proposition 3.4 is satisfied.

In the following we examine the case $\Lambda=\Lambda_1$. By the dimensional formula of Weyl we know the degree of $[\rho]$ (with Λ_1 as its highest weight) to be 7. Let Ω be the set of weights of $[\rho]$. Then we have:

$$\Omega=\{\pm\Lambda_1, \pm(-\Lambda_1+\Lambda_2), \pm(2\Lambda_1-\Lambda_2), 0\}.$$

This can be easily verified by considering the sequence of weights:

$$\Lambda_1 \xrightarrow{-\alpha_1} -\Lambda_1+\Lambda_2 \xrightarrow{-\alpha_2} 2\Lambda_1-\Lambda_2 \xrightarrow{-\alpha_1} 0$$

$$\xrightarrow{-\alpha_1} -2\Lambda_1+\Lambda_2 \xrightarrow{-\alpha_2} \Lambda_1-\Lambda_2 \xrightarrow{-\alpha_1} -\Lambda_1.$$

Consequently we know the multiplicity of each weight equals 1. We now select a basis $\{v_i\}_{-3\leqq i\leqq 3}$ of V such that

$$v_{\pm 3}\in V_{\pm\Lambda_1}, \quad v_{\pm 2}\in V_{\pm(-\Lambda_1+\Lambda_2)}, \quad v_{\pm 1}\in V_{\pm(2\Lambda_1-\Lambda_2)}, \quad v_0\in V_0;$$

$$(v_3, v_3)=1, \quad X_{-\alpha_1}v_3=v_2, \quad X_{\alpha_1}\cdot v_2=2\pi\sqrt{-1}v_3.$$

Let w be a non-zero vector of $(V\otimes\mathfrak{g}^c)_K$. Since $B(\Lambda_1)=B_2=\{2\alpha_1+\alpha_2\}$, we have $\theta w=\varepsilon w$ ($\varepsilon^2=1$). In order to show $\varepsilon=1$, we first suppose $\varepsilon=-1$. Then w can be written:

$$w=\sum_{i=-3}^{3} v_i\otimes Y_i,$$

where $Y_i\in\mathfrak{g}^c$ such that $\theta Y_i=-Y_i$. We remark that each Y_i can be expressed by a linear combination of the vectors $(X_\alpha-X_{-\alpha})$'s ($\alpha\in\Delta$) and Λ_i's ($i=1, 2$). Since $X_{\alpha_1}+X_{-\alpha_1}, X_{\alpha_2}+X_{-\alpha_2}\in\mathfrak{k}^c$, we have

$$0=(X_{\alpha_1}+X_{-\alpha_1})w=\cdots+(X_{\alpha_1}+X_{-\alpha_1})v_2\otimes Y_2+v_2\otimes[X_{\alpha_1}+X_{-\alpha_1}, Y_2]$$
$$+(X_{\alpha_1}+X_{-\alpha_1})v_3\otimes Y_3+v_3\otimes[X_{\alpha_1}+X_{-\alpha_1}, Y_3];$$

$$0=(X_{\alpha_2}+X_{-\alpha_2})w=\cdots+v_3\otimes[X_{\alpha_2}+X_{-\alpha_2}, Y_3].$$

Hence we obtain

(4.1) $$[X_{\alpha_1}+X_{-\alpha_1},\ Y_3]+2\pi\sqrt{-1}Y_2=0;$$

(4.2) $$[X_{\alpha_1}+X_{-\alpha_1},\ Y_2]+Y_3=0;$$

(4.3) $$[X_{\alpha_2}+X_{-\alpha_2},\ Y_3]=0.$$

From (4.1) and (4.2) it follows

(4.4) $$[X_{\alpha_1}+X_{-\alpha_1},\ [X_{\alpha_1}+X_{-\alpha_1},\ Y_3]]=2\pi\sqrt{-1}Y_3.$$

Now we define non-zero complex numbers a, b, c and d by the equalities

(4.5) $$\begin{aligned}&[X_{\alpha_1},\ X_{\alpha_2}]=aX_{\alpha_1+\alpha_2},\quad [X_{\alpha_1},\ X_{\alpha_1+\alpha_2}]=bX_{2\alpha_1+\alpha_2},\\ &[X_{\alpha_1},\ X_{2\alpha_1+\alpha_2}]=cX_{3\alpha_1+\alpha_2},\quad [X_{\alpha_2},\ X_{3\alpha_1+\alpha_2}]=dX_{3\alpha_1+2\alpha_2}.\end{aligned}$$

Then by simple calculations we obtain:

(4.6) $$\begin{aligned}&[X_{-\alpha_1},\ X_{\alpha_1}]=-\alpha_1^*,\quad [X_{-\alpha_1},\ X_{\alpha_2}]=0,\\ &[X_{-\alpha_1},\ X_{\alpha_1+\alpha_2}]=\frac{6\pi\sqrt{-1}}{a}X_{\alpha_2},\quad [X_{-\alpha_1},\ X_{2\alpha_1+\alpha_2}]=\frac{8\pi\sqrt{-1}}{b}X_{\alpha_1+\alpha_2},\\ &[X_{-\alpha_1},\ X_{3\alpha_1+\alpha_2}]=\frac{6\pi\sqrt{-1}}{c}X_{2\alpha_1+\alpha_2},\quad [X_{-\alpha_1},\ X_{3\alpha_1+2\alpha_2}]=0.\end{aligned}$$

(4.7) $$\begin{aligned}&[X_{-\alpha_2},\ X_{\alpha_1}]=0,\ [X_{-\alpha_2},\ X_{\alpha_2}]=-\alpha_2^*,\\ &[X_{-\alpha_2},\ X_{\alpha_1+\alpha_2}]=-\frac{2\pi\sqrt{-1}}{a}X_{\alpha_1},\quad [X_{-\alpha_2},\ X_{2\alpha_1+\alpha_2}]=0,\\ &[X_{-\alpha_2},\ X_{3\alpha_1+\alpha_2}]=0,\quad [X_{-\alpha_2},\ X_{3\alpha_1+2\alpha_2}]=-\frac{2\pi\sqrt{-1}}{d}X_{3\alpha_1+\alpha_2}.\end{aligned}$$

Hence we have:

(4.8) $$\begin{aligned}&[X_{\alpha_2}+X_{-\alpha_2},\ X_{\alpha_1}-X_{-\alpha_1}]=-a(X_{\alpha_1+\alpha_2}-X_{-(\alpha_1+\alpha_2)}),\\ &[X_{\alpha_2}+X_{-\alpha_2},\ X_{\alpha_2}-X_{-\alpha_2}]=-2\alpha_2^*,\\ &[X_{\alpha_2}+X_{-\alpha_2},\ X_{\alpha_1+\alpha_2}-X_{-(\alpha_1+\alpha_2)}]=-\frac{2\pi\sqrt{-1}}{a}(X_{\alpha_1}-X_{-\alpha_1}),\\ &[X_{\alpha_2}+X_{-\alpha_2},\ X_{2\alpha_1+\alpha_2}-X_{-(2\alpha_1+\alpha_2)}]=0,\\ &[X_{\alpha_2}+X_{-\alpha_2},\ X_{3\alpha_1+\alpha_2}-X_{-(3\alpha_1+\alpha_2)}]=d(X_{3\alpha_1+2\alpha_2}-X_{-(3\alpha_1+2\alpha_2)}),\\ &[X_{\alpha_2}+X_{-\alpha_2},\ X_{3\alpha_1+2\alpha_2}-X_{-(3\alpha_1+2\alpha_2)}]=-\frac{2\pi\sqrt{-1}}{d}(X_{3\alpha_1+\alpha_2}-X_{-(3\alpha_1+\alpha_2)}),\\ &[X_{\alpha_2}+X_{-\alpha_2},\ \Lambda_1]=0,\quad [X_{\alpha_2}+X_{-\alpha_2},\ \Lambda_2]=-2\pi\sqrt{-1}(\Lambda_2,\ \alpha_2)(X_{\alpha_2}-X_{-\alpha_2}).\end{aligned}$$

Therefore Y_3 can be written in the form

(4.9) $$Y_3=e(X_{2\alpha_1+\alpha_2}-X_{-(2\alpha_1+\alpha_2)})+fA_1;\ e,f\in \boldsymbol{C}.$$

Putting (4.9) into (4.4), we have

$$e\left\{14\pi\sqrt{-1}(X_{2\alpha_1+\alpha_2}-X_{-(2\alpha_1+\alpha_2)})-\frac{48\pi^2}{ab}(X_{\alpha_2}-X_{-\alpha_2})\right\}$$
$$+4\pi\sqrt{-1}(A_1,\ \alpha_1)f\alpha_1^*=2\pi\sqrt{-1}\{e(X_{2\alpha_1+\alpha_2}-X_{-(2\alpha_1+\alpha_2)})+fA_1\}.$$

Then it is easy to see that $e=f=0$. This implies $Y_3=0$, contradicting our assumption $w\neq 0$ (see Lemma 3.1). Therefore we have $\varepsilon=1$.

As a consequence of the above arguments we obtain the quality (3.11).

We resume the results in the following table:

[G] $G_2/SU(2)\times SU(2)$

	Λ			$a(\Lambda)$
(I)	M_0			$\sharp I(M_0)$
(II)	Λ_1+M_0	$(M_0,\ \alpha_1^*)$	$(M_0,\ \alpha_2^*)$	
		0	0	0
		+	0	1
		0	+	1
		+	+	2
(III)	Λ_2+M_0	$(M_0,\ \alpha_1^*)$		
		0		1
		+		2
(IV)	$\Lambda_1+\Lambda_2+M_0$	$(M_0,\ \alpha_1^*)$		
		0		1
		+		2

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