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On unitary representations and factor sets of covering groups of the real symplectic groups

Dedicated to Professor Hisaaki Yoshizawa on his 60th birthday

By

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Introduction.

In an attempt to get the metaplectic groups of "higher degree", Kubota presented a Weil type representation for $SL(2, \boldsymbol{C})$ in the papers [7]-[10]. A similar construction of the covering groups of $SL(2, \boldsymbol{R})$ was obtained by Yamazaki [16]. Briefly speaking, they replaced the role of the Fourier transformation in the construction of so-called Weil representation [14] by that of the Fourier-Bessel transformation. In the present paper we treat the case of the real symplectic group $Sp(m, \boldsymbol{R})$, using the Bessel functions of matrix argument defined by Herz [5]. We start from a certain family of unitary operators defined on an open dense subset of $Sp(m, \boldsymbol{R})$. Then this family determines a projective unitary representation of $Sp(m, \boldsymbol{R})$. For a closer investigation of matters, we introduce a factor set for the universal covering group of $Sp(m, \boldsymbol{R})$, which can be computed explicitly. The purpose of the present paper is to study such a family of unitary operators in connection with the factor set.

Let us explain our results in more detail. Let $S_m(\boldsymbol{R})$ be the space of all $m\times m$ real symmetric matrices and P_m the space of all $m\times m$ positive definite real symmetric matrices. For $\delta>-1$, we denote by $L^2_\delta(P_m)$ the Hilbert space of square integrable functions on P_m with respect to the measure $(\det x)^\delta dx$, where dx is the restriction of usual Lebesgue measure on $S_m(\boldsymbol{R})$. We denote three types of elements in $Sp(m, \boldsymbol{R})$ by $d(a)=\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1}\end{pmatrix}$, $t(b)=\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}$, $d'(c)=\begin{pmatrix} 0 & -{}^t c^{-1} \\ c & 0\end{pmatrix}$ for a, $c\in GL(m, \boldsymbol{R})$ and $b\in S_m(\boldsymbol{R})$. Corresponding to these elements, we define three types of unitary operators on $L^2_\delta(P_m)$ as follows. For $\varphi\in L^2_\delta(P_m)$,

$$\boldsymbol{d}_\delta(a)\varphi(x)=\varphi({}^t a x a)|\det a|^{\delta+p} \qquad (a\in GL(m, \boldsymbol{R})),$$

$$\boldsymbol{t}_\delta(b)\varphi(x)=\varphi(x)\,\mathrm{etr}(\sqrt{-1}\,bx) \qquad (b\in S_m(\boldsymbol{R})),$$

$$\boldsymbol{d}'_\delta(c)\varphi(x)=\varphi^*(c^{-1}x\,{}^t c^{-1})|\det c|^{-\delta-p} \qquad (c\in GL(m, \boldsymbol{R})).$$

Here $p=(m+1)/2$, $\mathrm{etr}(a)=\exp(\mathrm{tr}(a))$, and φ^* is the Hankel transform of φ defined by

$$\varphi^*(x)=\int_{P_m}\varphi(y)A_\delta(xy)(\det y)^\delta dy$$

with the Bessel function A_δ of Herz [5]. On the other hand, put

$$\Omega=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in Sp(m, \boldsymbol{R})\,;\ \det c\neq 0\right\}.$$

Then any element σ in Ω is uniquely decomposed in the form $\sigma=t(b_1)d'(c)t(b_2)$. Using this decomposition for $\sigma\in\Omega$, we define a unitary operator $\boldsymbol{r}_\delta(\sigma)$ on $L^2_\delta(P_m)$ by $\boldsymbol{r}_\delta(\sigma)=\boldsymbol{t}_\delta(b_1)\boldsymbol{d}'_\delta(c)\boldsymbol{t}_\delta(b_2)$. Let us now state our first theorem:

Theorem 3.2. *Let* $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\sigma'=\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, $\sigma''=\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$ *be three elements in* Ω *such that* $\sigma''=\sigma\sigma'$. *Then it holds that*

$$\boldsymbol{r}_\delta(\sigma)\boldsymbol{r}_\delta(\sigma')=\boldsymbol{r}_\delta(\sigma'')\boldsymbol{e}_\delta(\operatorname{sgn}(c^{-1}c''c'^{-1}))\,,$$

where $\boldsymbol{e}_\delta(\zeta)=\exp\left(\sqrt{-1}\,\frac{\pi}{2}(\delta+p)\zeta\right)$ *and* $\operatorname{sgn} b$ $(b\in S_m(\boldsymbol{R}))$ *is the index of inertia of* b.

From this theorem, we see that $\boldsymbol{r}_\delta$ determines a projective unitary representation of $Sp(m, \boldsymbol{R})$, so that we obtain a unitary representation of the universal covering group of $Sp(m, \boldsymbol{R})$. To investigate this representation, we describe the universal covering group of $Sp(m, \boldsymbol{R})$ using an explicit factor set, which we denote by $A(\sigma, \sigma')$, $(\sigma, \sigma'\in Sp(m, \boldsymbol{R}))$. For example, we have an expression

$$A(\sigma, \sigma')=\frac{1}{4}\{\operatorname{Sgn}(c)-\operatorname{Sgn}(c'')+\operatorname{Sgn}(c')-\operatorname{Sgn}(c^{-1}c''c'^{-1})\} \tag{5.1}$$

for σ, σ', σ'' in Theorem 3.2. (For the definition of Sgn, see § 5). Now, for a positive integer q, we consider the central extension G_q of $Sp(m, \boldsymbol{R})$ by $\boldsymbol{Z}$ with the factor set $qA(\sigma, \sigma')$. Here G_q is a group with the underlying set $Sp(m, \boldsymbol{R})\times\boldsymbol{Z}$ and the group operation $(\sigma, n)(\sigma', n')=(\sigma\sigma', n+n'+qA(\sigma, \sigma'))$. Then G_1 for $q=1$ is by definition the universal covering group of $Sp(m, \boldsymbol{R})$. For the structure of G_q, we see in Proposition 6.1 that G_q is a semidirect product of G_1 and $\boldsymbol{Z}/q\boldsymbol{Z}$. Further in Proposition 6.2, we determine the normal subgroups of G_q. For the representation of G_q, from Theorem 3.2 and (5.1), we obtain the following

Theorem 6.3. *For* $\delta>-1$, *there exists an irreducible unitary representation* $\boldsymbol{U}_{q,\delta}$ *of* G_q *on the Hilbert space* $L^2_\delta(P_m)$ *such that for* $(\sigma, n)\in G_q$ *with* $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Omega$,

$$\boldsymbol{U}_{q,\delta}((\sigma, n))=\boldsymbol{r}_\delta(\sigma)\boldsymbol{e}_\delta\left(-\frac{4}{q}n-\operatorname{Sgn}(c)\right).$$

By virtue of the explicit factor set, we can specify the group of operators on $L^2_\delta(P_m)$ generated by the set $\{\boldsymbol{r}_\delta(\sigma)\,;\ \sigma\in\Omega\}$ as $\boldsymbol{U}_{4,\delta}(G_4)$ for m odd, and $\boldsymbol{U}_{2,\delta}(G_2)$ for m even (Proposition 6.6).

The equivalence of the representation $\boldsymbol{U}_{1,\delta}$ to relative holomorphic discrete series representation of the universal covering group of $Sp(m, \boldsymbol{R})$ is given by the Laplace transformation. Therefore $\boldsymbol{U}_{1,\delta}$ is found to be essentially the same as that obtained in Yamada [15, Th. 3.5].

The contents of each section are as follows. § 1 is a preliminary and § 2 is a summary of the necessary facts about the Bessel functions of Herz. In § 3, we compute the factor associated with the family of operators $\{r_\delta(\sigma)\,;\,\sigma\in\Omega\}$. In § 4, we define and compute an explicit factor set $A(\sigma, \sigma')$, and describe the universal covering group of $Sp(m, \boldsymbol{R})$ by it. Gathering these results in §§ 3-4, we obtain unitary representations of the universal covering group of $Sp(m, \boldsymbol{R})$ in § 5. § 6 is devoted to study of the group G_q. § 7 is a remark on the relation to relative holomorphic discrete series representations. In Appendix, we give a sufficient condition that the commutant of a certain set of operators on $L^2(X)$ is the algebra of multiplication operators.

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§ 1. Notations and preliminaries.

1.1. We denote by $\boldsymbol{Z}$, $\boldsymbol{R}$, and $\boldsymbol{C}$, respectively, the ring of integers, the real number field, and the complex number field. Also we use the notation $M_m(F)$ and $GL(m, F)$ for the total matrix algebra and the general linear group of degree m with entries in F, where $F=\boldsymbol{R}$ or $\boldsymbol{C}$. For a matrix a, ${}^t a$ is the transposed of a. We denote by 1_m or 0_m the unit matrix or the zero matrix of degree m. For $z\in M_m(\boldsymbol{C})$, $\operatorname{Re} z$ or $\operatorname{Im} z$ denotes the real or the imaginary part of z: $\operatorname{Re} z$, $\operatorname{Im} z\in M_m(\boldsymbol{R})$, $z=\operatorname{Re} z+\sqrt{-1}\operatorname{Im} z$. The group of real or complex orthogonal matrices and the group of unitary matrices of degree m are denoted by $O(m, \boldsymbol{R})$, $O(m, \boldsymbol{C})$, and $U(m)$ respectively. Moreover we use the following spaces of matrices:

$S_m(\boldsymbol{R})$: the space of all $m\times m$ real symmetric matrices,
$S_m(\boldsymbol{C})$: the space of all $m\times m$ complex symmetric matrices,
P_m: the space of all $m\times m$ positive definite real symmetric matrices,
$\mathfrak{H}_m$: the Siegel upper half space of degree m, $\mathfrak{H}_m=\{z\in S_m(\boldsymbol{C})\,;\,\operatorname{Im} z\in P_m\}$.

For $a\in S_m(\boldsymbol{R})$, we write $a>0$ if a is positive definite. As usual $\det a$ or $\operatorname{tr} a$ means the determinant or the trace of a. Following Herz [5], we write $\operatorname{etr}(a)=\exp(\operatorname{tr} a)$.

1.2. Throughout this paper, we fix an integer $m>0$ and use the letter p for $(m+1)/2$ consistently, $p=(m+1)/2$.

On the vector space $S_m(\boldsymbol{R})$, we define the measure dx as $\prod_{i\leq j} dx_{ij}$. Here the coordinate x_{ij} is taken from the components of $x=(x_{ij})$, and dx_{ij} is the Lebesgue measure on $\boldsymbol{R}$.

Let $GL(m, \boldsymbol{R})$ act on $S_m(\boldsymbol{R})$ by $x\mapsto {}^t a x a=x^a$ ($x\in S_m(\boldsymbol{R})$, $a\in GL(m, \boldsymbol{R})$). Then P_m is an open orbit. It is easy to see that the module of the linear transformation $x\mapsto x^a$ with respect to the measure dx is $|\det a|^{2p}$, i.e., $dx^a=|\det a|^{2p}dx$. So we have a $GL(m, \boldsymbol{R})$-invaritnt measure $(\det x)^{-p}dx$ on P_m.

On the other hand, every element $x\in P_m$ is diagonalized by an element in $O(m, \boldsymbol{R})$. Using the eigenvalues of x, we can write the measure dx in the form

$$dx=\prod_{i<j}|t_i-t_j|\,dt_1\cdots dt_m du\,.$$

Here $x={}^tutu$, $t=\mathrm{diag}(t_1, \cdots, t_m)$, $u\in O(m, \boldsymbol{R})$, and du is a Haar measure on $O(m, \boldsymbol{R})$. Since $O(m, \boldsymbol{R})$ is compact, the absolute convergence of an integral with respect to dx depends only on the part $\prod_{i<j}|t_i-t_j|\,dt_1\cdots dt_m$. For example, the integral

$$\int_{P_m}(\det(x_0+\sqrt{-1}\,y))^{-\alpha}dy$$

is absolutely convergent if $\mathrm{Re}\,\alpha>m$ for a fixed $x_0\in P_m$.

1.3. As in Herz [5], we make the following convention.

A complex analytic function f on $S_m(\boldsymbol{C})$ is called *symmetric* if it satisfies $f({}^tuzu)=f(z)$ for all $u\in O(m, \boldsymbol{C})$. A symmetric function $f(z)$ is actually an analytic function of m elementary symmetric functions of z, $s_1=\mathrm{tr}\,z$, $s_2, \cdots, s_m=\det z$. Using this fact, for a symmetric function f, we extend its domain of definition from $S_m(\boldsymbol{C})$ to $M_m(\boldsymbol{C})$ naturally. Then we see $f({}^tz)=f(z)$ and $f(zw)=f(wz)$. Moreover, it is useful to note the following. Let $x\in P_m$ and $x^{1/2}$ be the positive definite square root of x. Then $x^{1/2}zx^{1/2}\in S_m(\boldsymbol{C})$ for $z\in S_m(\boldsymbol{C})$, and we have $f(xz)=f(zx)=f(x^{1/2}zx^{1/2})$.

§ 2. Bessel functions of matrix argument.

In this section we summarize some results of Herz [5].

2.1. Definition of the Bessel functions.

Let δ be a complex number with $\mathrm{Re}\,\delta>p-1$. The Bessel function $A_\delta(x)$ $(x\in S_m(\boldsymbol{C}))$ is defined as

$$A_\delta(x)=(2\pi\sqrt{-1})^{-mp}\int_{\substack{\mathrm{Re}\,z=x_0>0\\ z\in S_m(\boldsymbol{C})}}\mathrm{etr}(z-xz^{-1})(\det z)^{-\delta-p}dz\,. \tag{2.1}$$

Here the integral should be understood as

$$(2\pi)^{-mp}\int_{S_m(\boldsymbol{R})}\mathrm{etr}(z-xz^{-1})(\det z)^{-\delta-p}dy\,,$$

with the variable $z=x_0+\sqrt{-1}\,y$ for a fixed $x_0\in P_m$, and we take the branch of the function $(\det z)^{-\delta-p}$ for $\mathrm{Re}\,z>0$ determined by $(\det 1_m)^{-\delta-p}=1$. Since $\mathrm{etr}(z-xz^{-1})$ is bounded in $z=x_0+\sqrt{-1}\,y$, the integral (2.1) converges absolutely for $\mathrm{Re}\,\delta>p-1$. And by the Cauchy's theorem, (2.1) is independent of $x_0\in P_m$. Moreover we can see that for any fixed $x_0\in P_m$, $\mathrm{etr}(z-xz^{-1})$ is uniformly bounded in $z=x_0+\sqrt{-1}\,y$ whenever x varies in a compact subset of $S_m(\boldsymbol{C})$. Therefore $A_\delta(x)$ is an entire function in x and analytic in δ for $\mathrm{Re}\,\delta>p-1$. In addition, for $\mathrm{Re}\,\delta>p-1$, $A_\delta(x)$ is bounded in $x\in P_m$ and vanishes at infinity.

The analytic continuation in δ of $A_\delta(x)$ is carried out by the differential recurrence formula:

$$D((\det x)^\delta A_\delta(x))=(\det x)^{\delta-1}A_{\delta-1}(x)\,, \tag{2.2}$$

where $D=\det\left(\eta_{ij}\frac{\partial}{\partial x_{ij}}\right)$, $\eta_{ij}=1$ if $i=j$, and $=\frac{1}{2}$ if $i\neq j$. It can be shown that A_δ is analytically continued to all $\delta\in\boldsymbol{C}$, so that $A_\delta(x)$ is entire in δ and x simultaneously.

For $\operatorname{Re}\delta>p-1$, (2.1) shows that $A_\delta(x)=O(\operatorname{etr}(|x|))$, and the same estimate holds also for all derivatives of $A_\delta(x)$. Here $|x|$ is the positive definite hermitian matrix which satisfies $|x|^2={}^t\bar{x}x$.

From the definition (2.1), we see that $A_\delta(x)$ is symmetric. So we can extend the function $A_\delta(z)$ for all $z\in M_m(\boldsymbol{C})$.

The very important formula A_δ is the Laplace transform of (2.1):

$$\int_{P_m}\operatorname{etr}(-xz)A_\delta(xy)(\det x)^\delta dx=\operatorname{etr}(-yz^{-1})(\det z)^{-\delta-p}. \tag{2.3}$$

This converges absolutely for all $y\in P_m$, $\operatorname{Re}z>0$, and $\operatorname{Re}\delta>-1$. Formulae in the following subsections 2.2 and 2.3 are essentially based on (2.3).

Remark. For $m=1$, the relation of $A_\delta(x)$ to the ordinary Bessel function $J_\delta(x)$ is given by $J_\delta(x)=A_\delta\left(\frac{1}{4}x^2\right)\left(\frac{x}{2}\right)^\delta$. (c.f. Watson [13, 6.2])

2.2. The Hankel transform. Let δ be a real number greater than -1. We denote by $L_\delta^2(P_m)$ the Hilbert space of all square integrable functions on P_m with respect to the measure $(\det x)^\delta dx$. Let us consider the linear transformation with integral kernel $A_\delta(xy)$:

$$\varphi^*(x)=\int_{P_m}\varphi(y)A_\delta(xy)(\det y)^\delta dy\,. \tag{2.4}$$

Proposition 2.1 (C.f. Herz [5, Theorem 3.1]). *The transform $\varphi\mapsto\varphi^*$ on the space of continuous functions with compact supports can be extended on the whole $L_\delta^2(P_m)$ as a unitary operator, and $\varphi^{**}=\varphi$. The integral expression* (2.4) *is valid for $\varphi\in L_\delta^2(P_m)$ whenever it is absolutely convergent.*

2.3. Weber's second exponential integral. For $\operatorname{Re}\delta>-1$, $a, b\in P_m$, and $\operatorname{Re}z>0$, we have an integral formula which converges absolutely (Herz [5, (5.8)]).

$$\begin{aligned}&\int_{P_m}\operatorname{etr}(-xz)A_\delta(ax)A_\delta(bx)(\det x)^\delta dx\\ &\quad=\operatorname{etr}(-(a+b)z^{-1})A_\delta(-az^{-1}bz^{-1})(\det z)^{-\delta-p}.\end{aligned} \tag{2.5}$$

Here the branch of $(\det z)^{-\delta-p}$ for $\operatorname{Re}z>0$ is determined by $(\det 1_m)^{-\delta-p}=1$.

§3. Weil type factor of a family of unitary operators.

3.1. Let δ be a real number greater than -1. On the analogy of Weil [14], Kubota [9], and Yamazaki [16], we define the following three types of unitary operators on $L_\delta^2(P_m)$. For $\varphi\in L_\delta^2(P_m)$,

$$\boldsymbol{d}_{\delta}(a)\varphi(x)=\varphi({}^{t}axa)|\det a|^{\delta+p} \qquad (a\in GL(m, \boldsymbol{R})),$$

$$\boldsymbol{t}_{\delta}(b)\varphi(x)=\varphi(x)\operatorname{etr}(\sqrt{-1}\,bx) \qquad (b\in S_{m}(\boldsymbol{R})),$$

$$\boldsymbol{d}'_{\delta}(c)\varphi(x)=\varphi^{*}(c^{-1}x{}^{t}c^{-1})|\det c|^{-\delta-p} \qquad (c\in GL(m, \boldsymbol{R})).$$

Here φ^{*} is defined in 2.2.

In the following we often denote these operators as $\boldsymbol{d}(a)$, $\boldsymbol{t}(b)$, and $\boldsymbol{d}'(c)$ without the parameter δ in case there is no fear of confusion.

Proposition 3.1. *Let $b\in GL(m, \boldsymbol{R})$ be symmetric. Then we have*

$$(\boldsymbol{d}'_{\delta}(-b^{-1})\boldsymbol{t}_{\delta}(b))^{3}=\exp\left(\sqrt{-1}\,\frac{\pi}{2}(\delta+p)\operatorname{sgn} b\right).$$

Here sgn *b is the index of inertia of b, i.e., the dimension of positive eigenspace of b minus that of negative one.*

Proof. We show the equality

$$(\boldsymbol{d}'(-b^{-1})\boldsymbol{t}(b))^{2}=\boldsymbol{t}(-b)\boldsymbol{d}'(b^{-1})\exp\left(\sqrt{-1}\,\frac{\pi}{2}(\delta+p)\operatorname{sgn} b\right).$$

Let us compute $I=(\boldsymbol{d}'(-b^{-1})\boldsymbol{t}(b))^{2}\varphi(x)$ for a continuous function φ on P_{m} with compact support. Put

$$\Phi(x_{1}, x_{2})=\varphi(x_{1})\operatorname{etr}(\sqrt{-1}\,bx)A_{\delta}(x_{1}bx_{2}b)(\det x_{1})^{\delta}$$
$$\times\operatorname{etr}(\sqrt{-1}\,bx_{2})A_{\delta}(x_{2}bxb)(\det x_{2})^{\delta}.$$

Then we see formally

$$I=|\det b|^{2p+2\delta}\int_{P_{m}}\left(\int_{P_{m}}\Phi(x_{1}, x_{2})dx_{1}\right)dx_{2}.$$

To be precise, we consider the integral I_{ε} with convergence factor $\operatorname{etr}(-\varepsilon x_{2})$, $\varepsilon>0$,

$$I_{\varepsilon}=|\det b|^{2p+2\delta}\int_{P_{m}}\left(\int_{P_{m}}\operatorname{etr}(-\varepsilon x_{2})\Phi(x_{1}, x_{2})dx_{1}\right)dx_{2}.$$

Then by Fubini's theorem,

$$I_{\varepsilon}=|\det b|^{2p+2\delta}\int dx_{1}\int\operatorname{etr}(-\varepsilon x_{2})\Phi(x_{1}, x_{2})dx_{2}$$

$$=|\det b|^{2p+2\delta}\int\varphi(x_{1})\operatorname{etr}(\sqrt{-1}\,bx_{1})(\det x_{1})^{\delta}dx_{1}$$

$$\times\int\operatorname{etr}((-\varepsilon+\sqrt{-1}\,b)x_{2})A_{\delta}(x_{2}bxb)A_{\delta}(x_{1}bx_{2}b)(\det x_{2})^{\delta}dx_{2}.$$

By the Weber's second exponential integral (2.5), integral $\int\cdot dx_{2}$ is equal to

$$\operatorname{etr}(-b(x+x_{1})bz_{\varepsilon}^{-1})A_{\delta}(-bxbz_{\varepsilon}^{-1}bx_{1}bz_{\varepsilon}^{-1})(\det z_{\varepsilon})^{-\delta-p},$$

where $z_{\varepsilon}=\varepsilon-\sqrt{-1}\,b$. Therefore,

$$I_\varepsilon = |\det b|^{2p+2\delta} \operatorname{etr}(-bxbz_\varepsilon^{-1})(\det z_\varepsilon)^{-\delta-p}$$
$$\times \int \varphi(x_1) \operatorname{etr}(b(\sqrt{-1}-z_\varepsilon^{-1}b)x_1) A_\delta(-bz_\varepsilon^{-1}bxbz_\varepsilon^{-1}bx_1)(\det x_1)^\delta dx_1 .$$

Letting ε tend to zero, we have

$$I = |\det b|^{2p+2\delta} \varphi^*(bxb) \operatorname{etr}(-\sqrt{-1}\, bx) \times \lim_{\varepsilon \to 0} (\det z_\varepsilon)^{-\delta-p}$$
$$= \boldsymbol{t}(-b)\boldsymbol{d}'(b^{-1})\varphi(x) \times |\det b|^{\delta+p} \lim_{\varepsilon \to 0} (\det z_\varepsilon)^{-\delta-p}.$$

Recalling the choice of the branch for $(\det z)^{-\delta-p}$, we can easily compute the factor:

$$\lim_{\varepsilon \to 0} |\det z_\varepsilon|^{\delta+p} (\det z_\varepsilon)^{-\delta-p} = \exp\left(\sqrt{-1}\, \frac{\pi}{2}(\delta+p) \operatorname{sgn} b\right).$$

Thus we obtain the assertion. Q.E.D.

3.2. Let us consider the real symplectic group of degree m in the usual form,

$$Sp(m, \boldsymbol{R}) = \{\sigma \in GL(2m, \boldsymbol{R});\, {}^t\sigma J \sigma = J\}, \quad J = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix} \in M_{2m}(\boldsymbol{R}).$$

We write $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ using the $m \times m$ block components $a, b, c, d \in M_m(\boldsymbol{R})$, and denote $c = c(\sigma)$. We put

$$\Omega = \{\sigma \in Sp(m, \boldsymbol{R});\, \det c(\sigma) \neq 0\}.$$

Moreover we define three types of elements in $Sp(m, \boldsymbol{R})$ as follows:

$$d(a) = \begin{pmatrix} a & 0_m \\ 0_m & {}^t a^{-1} \end{pmatrix}, \quad t(b) = \begin{pmatrix} 1_m & b \\ 0_m & 1_m \end{pmatrix}, \quad d'(c) = \begin{pmatrix} 0_m & -{}^t c^{-1} \\ c & 0_m \end{pmatrix},$$

for $a, c \in GL(m, \boldsymbol{R})$ and $b \in S_m(\boldsymbol{R})$.

It is easy to see that every $\sigma \in \Omega$ can be written uniquely in the form

$$\sigma = t(b_1) d'(c) t(b_2), \quad b_1, b_2 \in S_m(\boldsymbol{R}), \quad c \in GL(m, \boldsymbol{R}). \tag{3.1}$$

In fact, for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $b_1 = ac^{-1}$ and $b_2 = c^{-1}d$. Using the decomposition (3.1) for $\sigma \in \Omega$, we define a unitary operator $\boldsymbol{r}(\sigma) = \boldsymbol{r}_\delta(\sigma)$ on $L_\delta^2(P_m)$ by

$$\boldsymbol{r}_\delta(\sigma) = \boldsymbol{t}_\delta(b_1) \boldsymbol{d}'_\delta(c) \boldsymbol{t}_\delta(b_2). \tag{3.2}$$

Let us put $\boldsymbol{e}_\delta(\zeta) = \exp\left(\sqrt{-1}\, \frac{\pi}{2}(\delta+p)\zeta\right)$ for $\zeta \in \boldsymbol{C}$.

Theorem 3.2. *Let $\sigma, \sigma', \sigma''$ be three elemeuts in Ω such that $\sigma'' = \sigma\sigma'$. Then it holds that*

$$\boldsymbol{r}_\delta(\sigma) \boldsymbol{r}_\delta(\sigma') = \boldsymbol{r}_\delta(\sigma'') \boldsymbol{e}_\delta(\operatorname{sgn}(c^{-1}c''c'^{-1})),$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\sigma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, $\sigma'' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$.

For the proof of Theorem, we prepare a computational lemma.

Lemma 3.3. *Let* ${}^tf=f$, $c, c'\in GL(m, \boldsymbol{R})$. *Then we have*

(3.3) $$d'(c)t(f)d'(c')=t(f_1)d'(cfc')t(f_2'),$$

(3.4) $$\boldsymbol{d}'(c)\boldsymbol{t}(f)\boldsymbol{d}'(c')=\boldsymbol{t}(f_1)\boldsymbol{d}'(cfc')\boldsymbol{t}(f_2')\boldsymbol{e}_\delta(\mathrm{sgn}\, f),$$

where $f_1=-{}^tc^{-1}f^{-1}c^{-1}$ *and* $f_2'=-c'^{-1}f^{-1}\,{}^tc'^{-1}$.

Proof. It is easy check the following equalities.

(1) $$d(a_1)d(a_2)=d(a_1a_2),\quad d(a)^{-1}=d(a^{-1}),$$

(2) $$t(b_1)t(b_2)=t(b_1+b_2),\quad t(b)^{-1}=t(-b),$$

(3) $$d'(c_1)d'(c_2)=d(-{}^tc_1^{-1}c_2),\quad d'(c)^{-1}=d'(-{}^tc),$$

(4) $$d(a)t(b)d(a)^{-1}=t(ab\,{}^ta),$$

(5) $$d(a)d'(c)=d'({}^ta^{-1}c),\quad d'(c)d(a)=d'(ca).$$

By a simple computation we see that there hold the equalities in which d, t, d' are substituted by $\boldsymbol{d}$, $\boldsymbol{t}$, $\boldsymbol{d}'$ respectively.

(1′) $$\boldsymbol{d}(a_1)\boldsymbol{d}(a_2)=\boldsymbol{d}(a_1a_2),\quad \boldsymbol{d}(a)^{-1}=\boldsymbol{d}(a^{-1}),$$

(2′) $$\boldsymbol{t}(b_1)\boldsymbol{t}(b_2)=\boldsymbol{t}(b_1+b_2),\quad \boldsymbol{t}(b)^{-1}=\boldsymbol{t}(-b),$$

(3′) $$\boldsymbol{d}'(c_1)\boldsymbol{d}'(c_2)=\boldsymbol{d}(-{}^tc_1^{-1}c_2),\quad \boldsymbol{d}'(c)^{-1}=\boldsymbol{d}'(-{}^tc),$$

(4′) $$\boldsymbol{d}(a)\boldsymbol{t}(b)\boldsymbol{d}(a)^{-1}=\boldsymbol{t}(ab\,{}^ta),$$

(5′) $$\boldsymbol{d}(a)\boldsymbol{d}'(c)=\boldsymbol{d}'({}^ta^{-1}c),\quad \boldsymbol{d}'(c)\boldsymbol{d}(a)=\boldsymbol{d}'(ca).$$

Moreover, recalling the equality $(d'(-f^{-1})t(f))^3=1$, or

$$t(f)=d'(f^{-1})t(-f)d'(f^{-1})t(-f)d'(f^{-1}),$$

we get by (1)$\sim$(5)

$$\begin{aligned}d'(c)t(f)d'(c')&=d'(c)d'(f^{-1})t(-f)d'(f^{-1})t(-f)d'(f^{-1})d'(c')\\&=d(-{}^tc^{-1}f^{-1})t(-f)d'(f^{-1})t(-f)d(-fc')\\&=t(-{}^tc^{-1}f^{-1}c^{-1})d(-{}^tc^{-1}f^{-1})d'(f^{-1})d(-fc')t(-c'^{-1}f^{-1}\,{}^tc'^{-1})\\&=t(f_1)d'(cfc')t(f_2').\end{aligned}$$

This proves the equality (3.3). The second equality (3.4) can be obtained in a parallel way. In fact, instead of $(d'(-f^{-1})t(f))^3=1$, we have only to use the equality $(\boldsymbol{d}'(-f^{-1})\boldsymbol{t}(f))^3=\boldsymbol{e}_\delta(\mathrm{sgn}\, f)$, which was shown in Proposition 3.1.

Q.E.D. for Lemma 3.3.

Proof of Theorem 3.2. Let the decomposition of σ, σ', σ'' be $\sigma=t(b_1)d'(c)t(b_2)$, $\sigma'=t(b_1')d'(c')t(b_2')$, $\sigma''=t(b_1'')d'(c'')t(b_2'')$. We have $b_2+b_1'=c^{-1}c''c'^{-1}$, because $c''=ca'+dc'$, $b_2=c^{-1}d$, $b_1'=a'c'^{-1}$. Put $f=b_2+b_1'=c^{-1}c''c'^{-1}$. Then f is symmetric and

$$\sigma''=\sigma\sigma'=t(b_1)d'(c)t(b_2)t(b_1')d'(c')t(b_2')$$
$$=t(b_1)d'(c)t(f)d'(c')t(b_2')\,.$$

So by (3.3), we get

$$\sigma''=t(b_1+f_1)d'(cfc')t(f_2'+b_2')\,,$$

where $f_1=-{}^tc^{-1}f^{-1}c^{-1}$ and $f_2'=-c'^{-1}f^{-1}\,{}^tc'^{-1}$. The uniqueness of the decomposition shows that $b_1''=b_1+f_1$, $b_2''=f_2'+b_2'$, and $c''=cfc'$. So by definition,

$$\boldsymbol{r}(\sigma'')=\boldsymbol{t}(b_1+f_1)\boldsymbol{d}'(bfc')\boldsymbol{t}(f_2'+b_2')\,.$$

On the other hand, we have

$$\boldsymbol{r}(\sigma)\boldsymbol{r}(\sigma')=\boldsymbol{t}(b_1)\boldsymbol{d}'(c)\boldsymbol{t}(b_2)\boldsymbol{t}(b_1')\boldsymbol{d}'(c')\boldsymbol{t}(b_2')$$
$$=\boldsymbol{t}(b_1)\boldsymbol{d}'(c)\boldsymbol{t}(f)\boldsymbol{d}'(c')\boldsymbol{t}(b_2')\,.$$

Then by (3.4), we get

$$\boldsymbol{r}(\sigma)\boldsymbol{r}(\sigma')=\boldsymbol{t}(b_1)\boldsymbol{t}(f_1)\boldsymbol{d}'(cfc')\boldsymbol{t}(f_2')\boldsymbol{t}(b_2')\boldsymbol{e}_\delta(\operatorname{sgn} f)$$
$$=\boldsymbol{t}(b_1+f_1)\boldsymbol{d}'(cfc')\boldsymbol{t}(f_2'+b_2')\boldsymbol{e}_\delta(\operatorname{sgn} f)$$
$$=\boldsymbol{r}(\sigma'')\boldsymbol{e}_\delta(\operatorname{sgn} f)\,.$$

Hence the theorem. Q. E. D.

§ 4. A factor set for the universal covering group of $Sp(m, \boldsymbol{R})$.

In this section we describe the univarsal covering group of $Sp(m, \boldsymbol{R})$ using a factor set, which is convenient for our purpose. We give some explicit computations for the factor set, too.

4.1. We introduce the following notations.

(1) For $\zeta\in\boldsymbol{C}$, $\zeta\neq 0$, we choose the principal value of its argument as $-\pi\leqq\operatorname{Arg}\zeta<\pi$.

(2) For $a\in M_m(\boldsymbol{C})$, we put

$$\operatorname{Arg}(a)=\sum_\mu \operatorname{Arg}\mu\,,$$

where the summation is taken over all non-zero eigenvalues μ of a with multiplicities.

Remark 1. For $a\in S_m(\boldsymbol{R})$, we have

$$\operatorname{Arg}(a)=\frac{\pi}{2}(\operatorname{sgn}(a)-\operatorname{rank}(a))\,.$$

2. If $a\in M_m(\boldsymbol{R})$, then $\operatorname{Arg}(a)\in\pi\boldsymbol{Z}$.

4.2. A decomposition for elements in $Sp(m, \boldsymbol{R})$.

Following Weil [14, Ch. V, n$^{\text{os}}$ 46–47, Prop. 6, Cor's 1 & 2], we explain a "normal" form for $\sigma\in Sp(m, \boldsymbol{R})$, which generalizes the expression (3.1) for elements of $\boldsymbol{\Omega}$.

Let $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d\in M_m(\boldsymbol{R})$, and $\operatorname{rank}(c)=r$. Let V be the space of row vectors of dimension m. Let $V_1\subset V$ be the range under right multiplication by c, and V_2 the orthogonal complement of V_1 in V. We choose orthonormal basis $u_1, \cdots, u_r$ of V_1 and $u_{r+1}, \cdots, u_m$ of V_2. And we make a matrix u by arranging $u_1, \cdots, u_m$ in m columns in this order. Then $u\in O(m, \boldsymbol{R})$. Moreover we put

$$(4.1)\qquad e_1=\begin{pmatrix} 1_r & \\ & 0_{m-r} \end{pmatrix},\quad e_2=1_m-e_1=\begin{pmatrix} 0_r & \\ & 1_{m-r} \end{pmatrix},\quad E_r=\begin{pmatrix} e_2 & -e_1 \\ e_1 & e_2 \end{pmatrix}.$$

In this situation, σ can be written in the form

$$(4.2)\qquad \sigma=d(u^{-1})t(g)d({}^t\lambda^{-1})E_r t(h)d(u),$$

where $g, h\in S_m(\boldsymbol{R})$, $e_1he_1=h$, $\lambda\in GL(m, \boldsymbol{R})$. Moreover for a fixed u, the decomposition is unique.

Next let us look over how g, h, λ change in case u is replaced. If we choose another orthonormal basis of V_1 and V_2 and make the matrix u' as above, then $v=u'u^{-1}$ is of the form

$$v=\begin{pmatrix} v_1 & \\ & v_2 \end{pmatrix},\quad v_1\in O(r, \boldsymbol{R}),\quad v_2\in O(m-r, \boldsymbol{R}).$$

So we see $v^{-1}e_1v=e_1$, $v^{-1}e_2v=e_2$, $d(v^{-1})E_rd(v)=E_r$. Using this, we get easily

$$(4.3)\qquad g=v^{-1}g'v,\quad h=v^{-1}h'v,\quad \lambda=v^{-1}\lambda'v,$$

where $\sigma=d(u'^{-1})t(g')d({}^t\lambda'^{-1})E_rt(h')d(u')$ is the decomposition corresponding to u'.

4.3. For $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in Sp(m, \boldsymbol{R})$ and $z\in\mathfrak{H}_m$, we put $J(\sigma, z)=cz+d$. It is well known that $J(\sigma, z)$ is invertible. It can be written in terms of the decomposition (4.2) as

$$J(\sigma, z)=u^{-1}\lambda(e_1(uz{}^tu+h)+e_2)u,$$

because $c=u^{-1}\lambda e_1u$, $d=u^{-1}\lambda(h+e_2)u$.

Now we define, using (4.2),

$$(4.4)\qquad AJ(\sigma, z)=\operatorname{Arg}(\lambda)+\operatorname{Arg}(e_1(uz{}^tu+h)+e_2)$$

Recalling (4.3), we see easily that this does not depend on the choise of u, so that $AJ(\sigma, z)$ is defined as a function of σ and z.

Let $\sigma\in\Omega$, i.e., $\operatorname{rank}(c)=m$. Then, from the decomposition (3.1), we can choose $u=1_m$, and get

$$(4.5)\qquad AJ(\sigma, z)=\operatorname{Arg}(c)+\operatorname{Arg}(z+c^{-1}d).$$

Let $c=0$, i.e., $\sigma=\begin{pmatrix} a & b \\ 0 & {}^ta^{-1} \end{pmatrix}$. Then we can also choose $u=1_m$ and get $AJ(\sigma, z)=\operatorname{Arg}({}^ta^{-1})=\operatorname{Arg}(a^{-1})$.

The important property of $AJ(\sigma, z)$ is the following.

Proposition 4.1. *For a fixed $\sigma\in Sp(m, \boldsymbol{R})$, the function $AJ(\sigma, z)$ of z is continuous on $\mathfrak{H}_m$.*

Proof. Let us write $uz^tu=\begin{pmatrix} z_1 & * \\ * & * \end{pmatrix}$, $z_1\in S_r(\boldsymbol{C})$. It is easy to see $z_1\in\mathfrak{H}_r$. Since $e_1he_1=h$, the matrix h is of the form $\begin{pmatrix} h_1 & \\ & 0_{m-r} \end{pmatrix}$, $h_1\in S_r(\boldsymbol{R})$. Then we have

$$e_1(uz^tu+h)+e_2=\begin{pmatrix} z_1+h_1 & * \\ 0 & 1_{m-r} \end{pmatrix}.$$

Therefore we have only to consider the eigenvalues of z_1+h_1. Since $z_1+h_1\in\mathfrak{H}_r$, its eigenvalues are in the complex upper half plane $\mathfrak{H}_1$. By definition, whenever μ is in $\mathfrak{H}_1$, the map $\mu\mapsto\operatorname{Arg}\mu$ is continuous. Thus the continuity of the map $z\mapsto\operatorname{Arg}(z_1+h_1)$ is verified, because the roots of a polynomial depend continuously on its coefficients. Q. E. D.

4.4. Let us put $j(\sigma, z)=\det J(\sigma, z)$. Then we see

$$j(\sigma, z)=|j(\sigma, z)|\exp(\sqrt{-1}\,AJ(\sigma, z)). \tag{4.6}$$

On the other hand, for $\sigma, \sigma'\in Sp(m, \boldsymbol{R})$, $z\in\mathfrak{H}_m$, we have

$$J(\sigma\sigma', z)=J(\sigma, \sigma'z)J(\sigma', z) \quad\text{and}\quad j(\sigma\sigma', z)=j(\sigma, \sigma'z)j(\sigma', z). \tag{4.7}$$

Here the action of $Sp(m, \boldsymbol{R})$ on $\mathfrak{H}_m$ is given by

$$\tau z=(az+b)(cz+d)^{-1}, \quad \tau=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in Sp(m, \boldsymbol{R}).$$

Now we consider

$$A(\sigma, \sigma'; z)=\frac{1}{2\pi}(AJ(\sigma, \sigma'z)-AJ(\sigma\sigma', z)+AJ(\sigma', z)). \tag{4.8}$$

From (4.6) and (4.7), we see that $\exp(2\pi\sqrt{-1}\,A(\sigma, \sigma'; z))=1$, so that $A(\sigma, \sigma'; z)\in\boldsymbol{Z}$. On the other hand, by Proposition 4.1, $A(\sigma, \sigma'; z)$ is continuous in z. Therefore we find that $A(\sigma, \sigma'; z)$ does not depend on $z\in\mathfrak{H}_m$. So we write it by $A(\sigma, \sigma')$ instead.

By a simple computation, the following cocycle condition for $A(\sigma, \sigma')$ is verified:

$$A(\sigma\sigma', \sigma'')+A(\sigma, \sigma')=A(\sigma, \sigma'\sigma'')+A(\sigma', \sigma'').$$

Now using this factor set $A(\sigma, \sigma')$, we construct a central extension G_1 of $Sp(m, \boldsymbol{R})$ as follows. As an underlying set, we take $G_1=Sp(m, \boldsymbol{R})\times\boldsymbol{Z}$. The group operation in G_1 is given by

$$(\sigma, n)(\sigma', n')=(\sigma\sigma', n+n'+A(\sigma, \sigma')).$$

Proposition 4.2. *The group G_1 is the universal covering group of $Sp(m, \boldsymbol{R})$.*

Proof. This can be seen by restricing the factor set on the maximal compact subgroup $K=\left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\in GL(2m, \boldsymbol{R})\,;\, a+\sqrt{-1}\,b\in U(m)\right\}$. But we give here another proof in order to mention the topology on the covering group.

Let us realize the universal covering group $Sp(m, \boldsymbol{R})^{\sim}$ of $Sp(m, \boldsymbol{R})$ as in Kashiwara-Vergne [6, (3.5)]: $Sp(m, \boldsymbol{R})^{\sim}=\{(\sigma, L_\sigma); \sigma\in Sp(m, \boldsymbol{R}), L_\sigma\in\mathcal{L}_\sigma\}$ endowed with the group operation $(\sigma, L_\sigma)(\sigma', L_{\sigma'})=(\sigma'', L_{\sigma''})$ with $\sigma''=\sigma\sigma'$ and $L_{\sigma''}(z)=L_\sigma(\sigma'z)+L_{\sigma'}(z)$. Here L_σ is an element of a family $\mathcal{L}_\sigma$ of functions on $\mathfrak{H}_m$ given as follows. For any fixed $\sigma\in Sp(m, \boldsymbol{R})$, $L_\sigma(z)=\log j(\sigma, z)$, where the values of logarithm are taken in such a way that we get a univalent continuous function on the simply connected domain $\mathfrak{H}_m$. Note that L_σ is determined by its value at $z=\sqrt{-1}$, and that the topology on $Sp(m, \boldsymbol{R})^{\sim}$ is given as the induced topology from $Sp(m, \boldsymbol{R})\times\boldsymbol{C}$ to $\{(\sigma, L_\sigma(\sqrt{-1}))\}$. Now, we put for $\sigma\in Sp(m, \boldsymbol{R})$, $s(\sigma)=(\sigma, \log|j(\sigma, z)|+\sqrt{-1}\,AJ(\sigma, z))$. Then Proposition 4.1 shows that $s(\sigma)\in Sp(m, \boldsymbol{R})^{\sim}$, so that s gives a cross section from $Sp(m, \boldsymbol{R})$ to $Sp(m, \boldsymbol{R})^{\sim}$. It is easy to see that $s(\sigma\sigma')^{-1}s(\sigma)s(\sigma')=(1, 2\pi\sqrt{-1}\,A(\sigma, \sigma'))$. Thus the factor set $A(\sigma, \sigma')$ determines the universal covering group of $Sp(m, \boldsymbol{R})$.

Let Ω' be the set of $\sigma\in\Omega$ such that $c(\sigma)$ has no negative eigenvalues. Then we see from (4.5) that the cross section s is continuous on Ω', so that the topology on the subset $\{(\sigma, n)\in G_1; \sigma\in\Omega', n\in\boldsymbol{Z}\}$ is the direct product topology of Ω' and $\boldsymbol{Z}$. Q. E. D.

4.5. A computation of $A(\sigma, \sigma')$ for a generic case.

Here we compute $A(\sigma, \sigma')$ for the case that $\sigma, \sigma', \sigma''=\sigma\sigma'\in\Omega$. The idea is simple. In the definition (4.8), we specialize z as $\sqrt{-1}\infty$.

As before, we write $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, $\sigma'=\begin{pmatrix} a' & b' \\ c' & d'\end{pmatrix}$, $\sigma''=\begin{pmatrix} a'' & b'' \\ c'' & d''\end{pmatrix}$. In the formula (4.5) for $\sigma\in\Omega$, put $z=\sqrt{-1}\,t$ (t: positive real number) and let t tend to infinity. Then we get $AJ(\sigma, \sqrt{-1}\infty)=\mathrm{Arg}(c)+\frac{\pi}{2}m$, whence

$$-AJ(\sigma'', \sqrt{-1}\infty)+AJ(\sigma', \sqrt{-1}\infty)=-\mathrm{Arg}(c'')+\mathrm{Arg}(c'). \tag{4.9}$$

There remains to compute $AJ(\sigma, \sigma'(\sqrt{-1}\infty))$. We see that $\sigma'(\sqrt{-1}\,t)=a'c'^{-1}+\varepsilon(t)$ with $\varepsilon(t)\in\mathfrak{H}_m$ and $\lim_{t\to+\infty}\varepsilon(t)=0$. Therefore by (4.5),

$$\begin{aligned} AJ(\sigma, \sigma'(\sqrt{-1}\,t)) &= \mathrm{Arg}(c)+\mathrm{Arg}(a'c'^{-1}+\varepsilon(t)+c^{-1}d) \\ &= \mathrm{Arg}(c)+\mathrm{Arg}(c^{-1}c''c'^{-1}+\varepsilon(t)). \end{aligned}$$

The following lemma leads us to the conclusion.

Lemma 4.3. *Let* ${}^t h=h\in GL(m, \boldsymbol{R})$. *For* $\varepsilon(t)\in\mathfrak{H}_m$ *with* $\lim_{t\to+\infty}\varepsilon(t)=0$, *we have*

$$\lim_{t\to+\infty}\mathrm{Arg}(h+\varepsilon(t))=-\mathrm{Arg}(h).$$

Proof. Since $h+\varepsilon(t)\in\mathfrak{H}_m$, its eigenvalues are in the complex upper half plane. On the other hand, the eigenvalues of h are non-zero real numbers, to which the eigenvalues of $h+\varepsilon(t)$ tend as $t\to+\infty$. According as the eigenvalue of $h+\varepsilon(t)$ tends to a positive or negative real number, its Arg tends to 0 or π respectively. By definition, Arg of a positive real number is 0 and that of a negative one is $-\pi$. Therefore we see that $\mathrm{Arg}(h+\varepsilon(t))$ tends to $-\mathrm{Arg}(h)$.

Q. E. D.

From this lemma, putting $h=c^{-1}c''c'^{-1}$, we see that $AJ(\sigma, \sigma'(\sqrt{-1}\,t))$ tends to $\mathrm{Arg}(c)-\mathrm{Arg}(c^{-1}c''c'^{-1})$ as t tends to infinity. Gathering this and (4.9), we get the following.

Proposition 4.4. *Let* $\sigma, \sigma', \sigma''=\sigma\sigma' \in \Omega$. *Then*

$$A(\sigma, \sigma')=\frac{1}{2\pi}\{\mathrm{Arg}(c)-\mathrm{Arg}(c'')+\mathrm{Arg}(c')-\mathrm{Arg}(c^{-1}c''c'^{-1})\}.$$

4.6. In the remainder of this section, we make preparations for §§5-6. First we give a computation of $AJ(\sigma, z)$ by reduction to lower dimensional case, when σ and z are written in the form of a direct sums of lower dimensional ones. Next we compute $A(\sigma, \sigma')$ for σ, σ' in the maximal compact subgroup of $Sp(1, \boldsymbol{R})$.

Let $\sigma^{(i)}=\begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} \in Sp(m^{(i)}, \boldsymbol{R})$, and $z^{(i)} \in \mathfrak{H}_{m^{(i)}}$ $(i=1, 2)$. We put $\sigma^{(1)} \hat{+} \sigma^{(2)} =\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a=a^{(1)} \oplus a^{(2)}$, $b=b^{(1)} \oplus b^{(2)}$, $c=c^{(1)} \oplus c^{(2)}$, $d=d^{(1)} \oplus d^{(2)}$. Here $a^{(1)} \oplus a^{(2)}$ means the direct sum of matrices $\begin{pmatrix} a^{(1)} & 0 \\ 0 & a^{(2)} \end{pmatrix}$.

Proposition 4.5. *It holds that*

$$AJ(\sigma^{(1)} \hat{+} \sigma^{(2)}, z^{(1)} \oplus z^{(2)})=AJ(\sigma^{(1)}, z^{(1)})+AJ(\sigma^{(2)}, z^{(2)}).$$

Corollary 4.6. *Let* $\sigma_j^{(i)} \in Sp(m^{(i)}, \boldsymbol{R})$ $(i, j=1, 2)$. *Then*

$$A(\sigma_1^{(1)} \hat{+} \sigma_1^{(2)}, \sigma_2^{(2)} \hat{+} \sigma_2^{(2)})=A(\sigma_1^{(1)}, \sigma_2^{(2)})+A(\sigma_1^{(2)}, \sigma_2^{(2)}).$$

Proof of Proposition 4.5. Let $r^{(i)}=\mathrm{rank}(c(\sigma^{(i)}))$ $(i=1, 2)$, and put $m= m^{(1)}+m^{(2)}$, $r=r^{(1)}+r^{(2)}$. We denote the matrices appearing in the decomposition (4.2) for $\sigma^{(i)}$ by the suffixed letters such as $u^{(i)}$, $g^{(i)}$, $h^{(i)}$, $\lambda^{(i)}$. Similarly we use the notations $e_j^{(i)}$ $(i, j=1, 2)$ for the matrices that define $E_{r^{(i)}}$ in (4.1). Then we have

$$\sigma^{(1)} \hat{+} \sigma^{(2)}=d(u_0^{-1})t(g_0)d({}^t\lambda_0^{-1})F\,t(h_0)d(u_0),$$

where $u_0=u^{(1)} \oplus u^{(2)}$, $g_0=g^{(1)} \oplus g^{(2)}$, $h_0=h^{(1)} \oplus h^{(2)}$, $\lambda_0=\lambda^{(1)} \oplus \lambda^{(2)}$, and $F= E_{r^{(1)}} \hat{+} E_{r^{(2)}}$. Let $v \in O(m, \boldsymbol{R})$ be a permutation matrix such that $v(e_1^{(1)}+e_1^{(2)})v^{-1} =e_1$, then we have $v(e_2^{(1)}+e_2^{(2)})v^{-1}=e_2$ and $d(v)F\,d(v^{-1})=E_r$. Here e_1, e_2 are as in (4.1). Hence

$$\sigma^{(1)} \hat{+} \sigma^{(2)}=d(u^{-1})t(g)d({}^t\lambda^{-1})E_r t(h)d(u)$$

with $u=vu_0$, $g=vg_0v^{-1}$, $h=vh_0v^{-1}$, $\lambda=v\lambda_0v^{-1}$. It is easy to verify that this gives the decomposition (4.2) for $\sigma^{(1)} \hat{+} \sigma^{(2)}$. So by definition (4.4), we get

$$AJ(\sigma^{(1)} \hat{+} \sigma^{(2)}, z^{(1)} \oplus z^{(2)})=\mathrm{Arg}(\lambda)+\mathrm{Arg}(e_1(u(z^{(1)} \oplus z^{(2)}){}^t u+e_2)),$$

and for the first term in the right-hand side

$$\mathrm{Arg}(\lambda)=\mathrm{Arg}(v(\lambda^{(1)} \oplus \lambda^{(2)})v^{-1})=\mathrm{Arg}(\lambda^{(1)})+\mathrm{Arg}(\lambda^{(2)}),$$

and similarly for the second term. Therefore

$$AJ(\sigma^{(1)}\hat{+}\sigma^{(2)},\ z^{(1)}\oplus z^{(2)})=AJ(\sigma^{(1)},\ z^{(1)})+AJ(\sigma^{(2)},\ z^{(2)})\,. \qquad \text{Q. E. D.}$$

Put $k(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ and

$$B(\theta,\ \theta')=\frac{1}{2\pi}(\operatorname{Arg} e^{\sqrt{-1}\,\theta}+\operatorname{Arg} e^{\sqrt{-1}\,\theta'}-\operatorname{Arg} e^{\sqrt{-1}(\theta+\theta')})\,.$$

Then we see for $-\pi\leqq\theta,\ \theta'<\pi$,

$$B(\theta,\ \theta')=\begin{cases}1 & \text{if}\quad \pi\leqq\theta+\theta'\\ 0 & \text{if}\quad -\pi\leqq\theta+\theta'<\pi\\ -1 & \text{if}\quad \theta+\theta'<-\pi.\end{cases}$$

Proposition 4.7. $A(k(\theta),\ k(\theta'))=B(\theta,\ \theta')$.

Proof. It suffices to show $AJ(k(\theta),\ \sqrt{-1})=\theta$ for $-\pi\leqq\theta<\pi$. For $\theta=0$ or $\theta=-\pi$, this equality is obvious. For the case $-\pi<\theta<\pi,\ \theta\neq 0$, we have the decomposition (4.2) for $k(\theta)$ with $r=1,\ u=1,\ g=\cot\theta,\ h=\cot\theta$, and $\lambda=\sin\theta$. Then by definition (4.4), we get $AJ(k(\theta),\ \sqrt{-1})=\operatorname{Arg}(\sin\theta)+\operatorname{Arg}(\sqrt{-1}+\cot\theta)$ $=\theta$. Hence the proposition. Q. E. D.

Put $k(\theta_1,\ \cdots,\ \theta_m)=k(\theta_1)\hat{+}\cdots\hat{+}k(\theta_m)$. Then from Corollary 4.6 and Proposition 4.7, we see

Proposition 4.8. *It holds that*

$$A(k(\theta_1,\ \cdots,\ \theta_m),\ k(\theta'_1,\ \cdots,\ \theta'_m))=\textstyle\sum_{i=1}^m B(\theta_i,\ \theta'_i)\,.$$

Corollary 4.9. *We have* $A(E_r,\ E_r)=r$, *especially for* $r=m$, $A(d'(1),\ d'(1))=m$. *Moreover we have* $A(-1_{2m},\ -1_{2m})=-m$.

§5. Representations of the universal covering group of $Sp(m, \boldsymbol{R})$.

First we recall a general proposition about generators of a group and their relations, which is found in Weil [14, Lemme 6].

Lemma 5.1. *Let G be a group, and U a subset of G such that the condition $U^{-1}\cap Ua\cap Ub\cap Uc\neq\emptyset$ holds for arbitrary elements a, b, c in G. Let G' be a group and η a map from U to G' satisfying the relation $\eta(uu')=\eta(u)\eta(u')$ for u, u', $uu'\in U$. Then η is uniquely extended as a gronp homomorphism from G to G'.*

Note that the condition for U is satisfied when U is an open dense subset of a topological group G.

Let us introduce a notation. For $a\in M_m(\boldsymbol{C})$, we put

$$\operatorname{Sgn}(a)=\frac{2}{\pi}\operatorname{Arg}(a)+\operatorname{rank}(a).$$

If a is real symmetric, $\mathrm{Sgn}(a)$ coincides with $\mathrm{Sgn}(a)$ as in Remark 1 in §4. Note that $\mathrm{Sgn}(a)\in \boldsymbol{Z}$ for $a\in M_m(\boldsymbol{R})$.

Using this notation Sgn, we can rewrite the formula in Proposition 4.4 as follows.

$$A(\sigma, \sigma')=\frac{1}{4}\{\mathrm{Sgn}(c)-\mathrm{Sgn}(c'')+\mathrm{Sgn}(c')-\mathrm{Sgn}(c^{-1}c''c'^{-1})\}. \tag{5.1}$$

We describe the universal covering group of $Sp(m, \boldsymbol{R})$ as the group G_1 defined by the factor set $A(\sigma, \sigma')$ as in Proposition 4.2.

Theorem 5.2. *For $\delta>-1$, there exists a unitary representation $\boldsymbol{U}_\delta$ of G_1 on the Hilbert space $L^2_\delta(P_m)$ such that for $(\sigma, n)\in G_1$ with $\sigma\in\Omega$,*

$$\boldsymbol{U}_\delta(\sigma, n)=\boldsymbol{r}_\delta(\sigma)\boldsymbol{e}_\delta(-4n-\mathrm{Sgn}(c(\sigma))).$$

Here $\boldsymbol{r}_\delta(\sigma)$ is defined in (3.2).

Proof. By Lemma 5.1, it is enough to see that $\boldsymbol{U}_\delta(\sigma, n)\boldsymbol{U}_\delta(\sigma', n')=\boldsymbol{U}_\delta(\sigma'', n'')$ for σ, σ', $\sigma''=\sigma\sigma'\in\Omega$, and $n''=n+n'+A(\sigma, \sigma')$. We write $c=c(\sigma)$, $c'=c(\sigma')$, and $c''=c(\sigma'')$. Now by definition,

$$\boldsymbol{U}_\delta(\sigma, n)\boldsymbol{U}_\delta(\sigma', n')=\boldsymbol{r}(\sigma)\boldsymbol{r}(\sigma')\boldsymbol{e}_\delta(-4n-4n'-\mathrm{Sgn}(c)-\mathrm{Sgn}(c')).$$

We have $\boldsymbol{r}(\sigma)\boldsymbol{r}(\sigma')=\boldsymbol{r}(\sigma'')\boldsymbol{e}_\delta(\mathrm{Sgn}(c^{-1}c''c'^{-1}))$ by Theorem 3.2, and $4A(\sigma, \sigma')=\mathrm{Sgn}(c)-\mathrm{Sgn}(c'')+\mathrm{Sgn}(c')-\mathrm{Sgn}(c^{-1}c''c'^{-1})$ by (5.1). So we get

$$\begin{aligned}\boldsymbol{U}_\delta(\sigma, n)\boldsymbol{U}_\delta(\sigma', n')&=\boldsymbol{r}(\sigma'')\boldsymbol{e}_\delta(-4n-4n'-4A(\sigma, \sigma')-\mathrm{Sgn}(c''))\\&=\boldsymbol{r}(\sigma'')\boldsymbol{e}_\delta(-4n''-\mathrm{Sgn}(c''))=\boldsymbol{U}_\delta(\sigma'', n'').\end{aligned}$$

This representation is strongly continuous, because it is continuous on the subset $\{(\sigma, n)\,;\,\sigma\in\Omega', n\in\boldsymbol{Z}\}$. Q. E. D.

Proposition 5.3. *The representation $\boldsymbol{U}_\delta$ $(\delta>-1)$ of G_1 is irreducible.*

Proof. We show that a bounded linear operator T on $L^2_\delta(P_m)$ commuting with every $\boldsymbol{U}_\delta(\sigma, n)$ is a scalar operator. Since T commutes with $\boldsymbol{t}(b)$ for all $b\in S_m(\boldsymbol{R})$, T is written in the form $T\varphi(x)=f(x)\varphi(x)$ $(\varphi\in L^2_\delta(P_m))$ for some essentially bounded function $f(x)$. (For its proof, see Appendix.) On the other hand, T commutes with $\boldsymbol{d}(a)$ for all $a\in GL(m, \boldsymbol{R})$. So the function $f(x)$ satisfies the condition that $f(x)=f({}^t a x a)$ for all $a\in GL(m, \boldsymbol{R})$. Since $GL(m, \boldsymbol{R})$ acts transitively on P_m by $x\mapsto {}^t a x a$, the function $f(x)$ must be a constant.

Q. E. D.

Let us determine the kernel $\mathrm{Ker}\,\boldsymbol{U}_\delta$ of this representation $\boldsymbol{U}_\delta$. Note that a normal subgroup of $G_1\cong Sp(m, \boldsymbol{R})^\sim$ is either equal to G_1 itself or contained in the centre, and that the centre of G_1 is $\{(\pm 1, n)\,;\,n\in\boldsymbol{Z}\}$. Since $\mathrm{Ker}\,\boldsymbol{U}_\delta$ is normal, we have only to compute $\boldsymbol{U}_\delta(1, n)$ and $\boldsymbol{U}_\delta(-1, n)$. From Corollary 4.9, we see $(-1, n)=(d'(1), n-m)(d'(1), 0)$, so that

$$\begin{aligned}\boldsymbol{U}_\delta(-1,\ n)&=\boldsymbol{U}_\delta(d'(1),\ n-m)\boldsymbol{U}_\delta(d'(1),\ 0)\\&=\boldsymbol{d}'(1)\boldsymbol{d}'(1)\boldsymbol{e}_\delta(-4n+4m-m)\boldsymbol{e}_\delta(-m)=\boldsymbol{e}_\delta(-4n+2m)\,.\end{aligned}$$

Moreover, we see also from Corollary 4.9, $(1,\ n)=(-1,\ n+m)(-1,\ 0)$, so that

$$\begin{aligned}\boldsymbol{U}_\delta(1,\ n)&=\boldsymbol{U}_\delta(-1,\ n+m)\boldsymbol{U}_\delta(-1,\ 0)\\&=\boldsymbol{e}_\delta(-4n-4m+2m)\boldsymbol{e}_\delta(2m)=\boldsymbol{e}_\delta(-4n)\,.\end{aligned}$$

Thus we have the following.

Proposition 5.4. *The kernel of the representation $\boldsymbol{U}_\delta$ of G_1 is given as*

$$\operatorname{Ker}\boldsymbol{U}_\delta=\{(1,\ n)\,;\,(\delta+p)n\in\boldsymbol{Z}\}\cup\left\{(-1,\ n)\,;\,(\delta+p)\left(n-\frac{m}{2}\right)\in\boldsymbol{Z}\right\}.$$

Let $\boldsymbol{G}_\delta$ be the image of G_1 under $\boldsymbol{U}_\delta$. Since the representation $\boldsymbol{U}_\delta$ is irreducible, the image of the centre of G_1 under $\boldsymbol{U}_\delta$ coincides with the set of all scalar operators in $\boldsymbol{G}_\delta$. We see that $\boldsymbol{G}_\delta$ is generated by the set of operators $\{\boldsymbol{r}_\delta(\sigma)\boldsymbol{e}_\delta(-\mathrm{Sgn}(c(\sigma)))\,;\,\sigma\in\Omega\}$, because G_1 is generated by the set $\{(\sigma,\ 0)\,;\,\sigma\in\Omega\}$. In the next section we determine the group generated by the set of operators $\{\boldsymbol{r}_\delta(\sigma)\,;\,\sigma\in\Omega\}$.

§ 6. Certain central extensions of $Sp(m,\ \boldsymbol{R})$ and their representations.

6.1. Let q be a positive integer. In this section we study the central extension G_q of $Sp(m,\ \boldsymbol{R})$ by $\boldsymbol{Z}$ with the factor set $qA(\sigma,\ \sigma')$ $(\sigma,\ \sigma'\in Sp(m,\ \boldsymbol{R}))$. Here G_q is by definition a group with the underlying set $Sp(m,\ \boldsymbol{R})\times\boldsymbol{Z}$ and the group operation $(\sigma,\ n)(\sigma',\ n')=(\sigma\sigma',\ n+n'+qA(\sigma,\ \sigma'))$. As we see in Proposition 4.2, G_1 is equal to the universal covering group of $Sp(m,\ \boldsymbol{R})$. To avoid any confusion, we denote an element in G_q by $(\sigma,\ n)_q$ throughout this section.

In case q devides q', consider the natural injection $j_{q,q'}$ from G_q to $G_{q'}$ defined by $j_{q,q'}(\sigma,\ n)_q=\left(\sigma,\ \frac{q'}{q}n\right)_{q'}$. Then through $j_{q,q'}$, we can (and do) identify G_q with a normal subgroup of $G_{q'}$ of index q'/q.

Proposition 6.1. *The group G_q is isomorphic to a semidirect product of G_1 and $\boldsymbol{Z}/q\boldsymbol{Z}$.*

Proof. It is enough to find a subgroup H_q of G_q such that $H_q\cong\boldsymbol{Z}/q\boldsymbol{Z}$ and $G_1\cap H_q=\{(1,\ 0)_q\}$. Put $k_q=k\left(\frac{2\pi}{q},\ 0,\ \cdots,\ 0\right)$. (See 4.6 for notation.) Then from Proposition 4.8 we see

(i) $A(k_2,\ k_2)=-1$,

(ii) for $q\geqq 3$ and $1\leqq l\leqq q-1$,

$$A(k_q,\ k_q^l)=\begin{cases}1 & \text{if}\quad l=q_0\\ 0 & \text{if}\quad l\neq q_0.\end{cases}$$

Here we put $q_0=\frac{q}{2}-1$ for q even, and $q_0=\frac{q-1}{2}$ for q odd. Now, let H_q be the subgroup of G_q generated by κ_q with $\kappa_q=(k_q, 1)_q$ for $q=2$ and $\kappa_q=(k_q, -1)_q$ for $q\geqq 3$. Then from (i) and (ii) we see that H_q is of order q, and that κ_q^l is of the form $(k_q^l, l')_q$ with $l'\not\equiv 0 \mod q$ for $1\leqq l\leqq q-1$. Therefore H_q satisfies the required conditions, and we have proved the proposition. Q. E. D.

We give here some remarks. If m is odd, then G_2 is isomorphic to the direct product group $G_1\times \boldsymbol{Z}/2\boldsymbol{Z}$. In fact, put $H=\{(1, 0)_2, (-1, m)_2\}$. Then by Corollary 4.9, H is a subgooup of G_2 of order 2. Clearly H is contained in the centre of G_2. In case m is odd, we have $G_1\cap H=\{(1, 0)_2\}$, whence $G_2=G_1H\cong G_1\times H$.

In general, the centralizer of G_1 in G_q is the centre of G_q, which is given as $\{(\pm 1, n)_q; n\in \boldsymbol{Z}\}$. It is easy to see that the centre of G_q contains a non-trivial element of finite order if and only if qm is even. And when qm is even, $(-1, qm/2)_q$ is the only non-trivial element of finite order contained in it. Therefore G_q is expressed as a direct product of G_1 and a subgroup of G_q if and only if m is odd and $q=2$.

6.2. Normal subgroups of G_q. Here we determine the normal subgroups of G_q.

Proposition 6.2. *Let N be a normal subgroup of G_q. Then we have the following two cases:* (i) *$N=G_l$ for some divisor l of q, or* (ii) *N is contained in the centre of G_q.*

Proof. Put $N_1=N\cap G_1$. Then N/N_1 is canonically isomorphic to the image of N under the projection of G_q to $G_q/G_1=\boldsymbol{Z}/q\boldsymbol{Z}$. Therefore N/N_1 is a cyclic group. Let l be the order of N/N_1. Take a $\xi\in G_q$ such that ξN_1 is a generator of N/N_1. Then we have $N=\bigcup_{i=0}^{l-1}\xi^i N_1$. On the other hand, since N_1 is a normal subgroup of $G_1\cong Sp(m, \boldsymbol{R})^\sim$, we have two cases: (i) $N_1=G_1$, or (ii) N_1 is contained in the centre of G_1. In case (i), we have $N=G_l$. In fact, since N/N_1 and G_l/G_1 have the same order in the cyclic group G_q/G_1, they coincide with each other. It follows from this that $N=G_l$, because N and G_l contain $N_1=G_1$.

Let us consider the case (ii). It suffices to show that ξ is in the centre of G_q. As N is normal, we see $\alpha\xi\alpha^{-1}\in N=\bigcup_{i=0}^{l-1}\xi^i N_1$ for $\alpha\in G_1$. So we can write it as $\alpha\xi\alpha^{-1}=\xi^{i(\alpha)}\nu(\alpha)$ with $i(\alpha)\in \boldsymbol{Z}$, $0\leqq i(\alpha)<l$ and $\nu(\alpha)\in N_1$. Consider $\beta\alpha\xi\alpha^{-1}\beta^{-1}$ for $\beta\in G_1$. Then we obtain $\xi^{i(\beta\alpha)-i(\beta)i(\alpha)}=\nu(\beta)^{i(\alpha)}\nu(\alpha)\nu(\beta\alpha)^{-1}\in N_1$, so that $i(\beta\alpha)=i(\beta)i(\alpha) \mod l$. Therefore the map $\alpha\mapsto i(\alpha) \mod l$ is a group homomorphism of G_1 to $(\boldsymbol{Z}/l\boldsymbol{Z})^\times$. On the other hand G_1 is equal to its commutator group. So we see $i(\alpha)\equiv 1 \mod l$ for all $\alpha\in G_1$. From this and $0\leqq i(\alpha)<l$, we find that $i(\alpha)=1$ for all $\alpha\in G_1$. At the same time we have proved that $\nu(\beta\alpha)=\nu(\beta)\nu(\alpha)$ for $\alpha, \beta\in G_1$. Now, let us write $\nu(\alpha)=(\nu_0(\alpha), n(\alpha))_q$. Since $\nu(\alpha)$ is in the centre of G_q, $\nu_0(\alpha)=\pm 1$. Then we have $\nu_0(\alpha)=1$ for all $\alpha\in G_1$ by the same reason above. Gathering these, we see $\alpha\xi\alpha^{-1}=\xi\nu(\alpha)$ with $\nu(\alpha)=(1, n(\alpha))_q$. Consider the l-th power of this equality. Then noting that ξ^l is in the centre of G_q, we get

$(1, 0)_q=\nu(\alpha)^l=(1, ln(\alpha))_q$, whence $n(\alpha)=0$. Consequently $\nu(\alpha)=(1, 0)_q$ for all $\alpha\in G_1$. Thus we see that ξ commutes with all elements in G_1, so that ξ is in the centre of G_q. Q. E. D.

6.3. Representations $U_{q,\delta}$. Now, we consider representations of G_q similarly as in Theorem 5.2.

Theorem 6.3. *For $\delta>-1$, there exists an irreducible unitary representation $U_{q,\delta}$ of G_q on the Hilbert space $L_\delta^2(P_m)$ such that for $(\sigma, n)_q\in G_q$ with $\sigma\in\Omega$,*

$$U_{q,\delta}((\sigma, n)_q)=r_\delta(\sigma)e_\delta\left(-\frac{4}{q}n-\mathrm{Sgn}(c(\sigma))\right).$$

This is proved quite similarly as Theorem 5.2. Moreover similarly as computations for Proposition 5.4, we can show $U_{q,\delta}((-1, n)_q)=e_\delta\left(-\frac{4}{q}n+2m\right)$ and $U_{q,\delta}((1, n)_q)=e_\delta\left(-\frac{4}{q}n\right)$, because $(-1, n)_q=(d'(1), n-qm)_q(d'(1), 0)_q$ and $(1, n)_q=(-1, n+qm)_q(-1, 0)_q$. So we have

Proposition 6.4. *The hernel of the representation $U_{q,\delta}$ of G_q is given as*

$$\mathrm{Ker}\, U_{q,\delta}=\left\{(1, n)_q\,;\,(\delta+p)\frac{n}{q}\in \boldsymbol{Z}\right\}\cup\left\{(-1, n)_q\,;\,(\delta+p)\left(\frac{n}{q}-\frac{m}{2}\right)\in \boldsymbol{Z}\right\}.$$

Remark. The representations $U_{q,\delta}$ are compatible with the inclusion $j_{q,q'}$: $G_q\rightarrow G_{q'}$, namely $U_{q,\delta}=U_{q',\delta}\circ j_{q,q'}$.

6.4. In the following, we determine the group generated by the operators $r_\delta(\sigma)$, $\sigma\in\Omega$. Note that $r_\delta(\sigma)=U_{4,\delta}((\sigma, -\mathrm{Sgn}(c))_4)$ with $c=c(\sigma)$ for $\sigma\in\Omega$. So we determine the subgroup of G_4 generated by the set $\{(\sigma, -\mathrm{Sgn}(c(\sigma)))_4\,;\,\sigma\in\Omega\}$.

Proposition 6.5. *The subgroup of G_4 generated by the set $\{(\sigma, -\mathrm{Sgn}(c(\sigma)))_4\,;\,\sigma\in\Omega\}$ is equal to G_4 if m is odd, and equal to G_2 if m is even.*

Proof. Let G be the subgroup of G_4 generated by the set $\{(\sigma, -\mathrm{Sgn}(c(\sigma)))_4\,;\,\sigma\in\Omega\}$. Put $u_r=k(\theta_1, \cdots, \theta_m)$ (see 4.6) with $\theta_1=\cdots=\theta_r=2\pi/3$, $\theta_{r+1}=\cdots=\theta_m=-2\pi/3$. Then $u_r\in\Omega$ and $\mathrm{Sgn}(c(u_r))=2r-m$, whence $(u_r, m-2r)_4\in G$. On the other hand, using Proposition 4.8, we get $(u_r, m-2r)_4^3=(1, 2r-m)_4$. Therefore G contains $(1, 2r-m)_4$. In case m is odd, taking r with $2r-m=1$, we have $(1, 1)_4\in G$. This shows that $G\ni(1, n)_4$ for $n\in\boldsymbol{Z}$, so that G contains all elements of the form $(\sigma, n)_4$ with $\sigma\in\Omega$ and $n\in\boldsymbol{Z}$. Therefore $G=G_4$ for m odd.

In case m is even, since $\mathrm{Sgn}(c(\sigma))$ for $\sigma\in\Omega$ is even, we have $G\subset G_2$. On the other hand, putting $r=\frac{m}{2}+1$, we see $(1, 2)_4\in G$. So G contains all elements of the form $(\sigma, 2n)_4$ with $\sigma\in\Omega$ and $n\in\boldsymbol{Z}$. Therefore $G=G_2$ for m even.

Q. E. D.

Let $G_{q,\delta}$ be the image of G_q under the representation $U_{q,\delta}$. Then we have

$G_{q,\delta}\cong G_q/\mathrm{Ker}\, U_{q,\delta}$. From Proposition 6.5, we obtain the following.

Proposition 6.6. *The group generated by the set of operators* $\{r_\delta(\sigma)\,;\,\sigma\in\Omega\}$ *on the Hilbert space* $L_\delta^2(P_m)$ *is equal to* $G_{4,\delta}$ *if* m *is odd, and equal to* $G_{2,\delta}$ *if* m *is even.*

§7. Relation to relative holomorphic discrete series representations.

For $\varphi\in L_\delta^2(P_m)$, we define the Laplace transform $\check{\varphi}$ of φ as

$$\check{\varphi}(z)=\int_{P_m}\varphi(x)\,\mathrm{etr}(\sqrt{-1}\,xz)(\det x)^\delta dx\,.$$

This integral converges absolutely for every $z\in\mathfrak{H}_m$, so that $\check{\varphi}$ is a holomorphic function on $\mathfrak{H}_m$. We denote by $\mathcal{H}_\delta$ the image of $L_\delta^2(P_m)$ under the Laplace transformation. By the isomorphism $\check{}: L_\delta^2(P_m)\to\mathcal{H}_\delta$, we transfer the operators $d_\delta(a)$, $t_\delta(b)$, $d'_\delta(c)$, and $r_\delta(\sigma)$ from $L_\delta^2(P_m)$ to $\mathcal{H}_\delta$, whicw we denote by $\check{d}_\delta(a)$, $\check{t}_\delta(b)$, $\check{d}'_\delta(c)$, and $\check{r}_\delta(\sigma)$ respectively.

It is easy to see that

$$\check{d}_\delta(a)\check{\varphi}(z)=|\det a|^{-\delta-p}\check{\varphi}(a^{-1}z\,{}^ta^{-1})\,,$$

$$\check{t}_\delta(b)\check{\varphi}(z)=\check{\varphi}(z+b)\,.$$

From the formula (2.3), we see

$$\check{d}'_\delta(1)\varphi(z)=\Big(\det\frac{z}{\sqrt{-1}}\Big)^{-\delta-p}\check{\varphi}(-z^{-1})\,.$$

So we have

$$\check{d}'_\delta(c)\check{\varphi}(z)=|\det c|^{-\delta-p}\Big(\det\frac{z}{\sqrt{-1}}\Big)^{-\delta-p}\check{\varphi}(-c^{-1}z^{-1}\,{}^tc^{-1})\,,$$

because $d'_\delta(c)=d_\delta({}^tc^{-1})\check{d}'_\delta(1)$.

Let us consider an anti-automorphism of $Sp(m, \boldsymbol{R})$ defind by $\sigma\mapsto{}^\circ\sigma=I\,{}^t\sigma I$, where $I=\begin{pmatrix}0_m & 1_m\\ 1_m & 0_m\end{pmatrix}$. We see ${}^\circ\sigma=\begin{pmatrix}{}^td & {}^tb\\ {}^tc & {}^ta\end{pmatrix}$ for $\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. Note also that ${}^\circ\sigma=J_1\sigma^{-1}J_1$, where $J_1=\begin{pmatrix}1_m & 0_m\\ 0_m & -1_m\end{pmatrix}$. Then we have for $\sigma\in\Omega$,

$$\check{r}_\delta(\sigma)\check{\varphi}(z)e_\delta(-\mathrm{Sgn}(c(\sigma)))=j({}^\circ\sigma, z)^{-\delta-p}\check{\varphi}({}^\circ\sigma, z)\,.$$

Here $j({}^\circ\sigma, z)^{-\delta-p}=|j({}^\circ\sigma, z)|^{-\delta-p}\exp(-\sqrt{-1}(\delta+p)AJ({}^\circ\sigma, z))$. It turns out from this that our representation U_δ of $G_1=Sp(m, \boldsymbol{R})^\sim$ in Theorem 5.2 is essentially identical with that constructed in the paper of Yamada [15, Th. 3.5], in which $Sp(m, \boldsymbol{R})^\sim$ is treated in more abstract manner than the present paper (see the proof of Proposition 4.2).

We note here that the formula (2.3) is a key in deducing the properties of the Bessel function A_δ. So we implicitly used the realization of the representation on the space $\mathcal{H}_\delta$. If we work on $\mathcal{H}_\delta$ not on $L_\delta^2(P_m)$, the proof of Theorem 3.2 is reduced to a computation on $j({}^\circ\sigma, z)^{-\delta-p}$, which follows from Proposition 4.4.

Appendix. On commutant of a certain set of operators on $L^2(X)$.

Let $(X, \mathfrak{B}, \mu)$ be a measure space, and put $\mathfrak{B}_0=\{B\in\mathfrak{B}\,;\,\mu(B)<\infty\}$. In this Appendix, we assume that the measure space is localizable. Here $(X, \mathfrak{B}, \mu)$ is said to be localizable if it satisfies the following. Let $\{\varphi_A\,;\,A\in\mathfrak{B}_0\}$ be a family of functions such that φ_A is a measurable function on A and that $\varphi_A(x)=\varphi_B(x)$ holds for almost all $x\in A\cap B$, then there exists a locally measurable function φ on X such that for every $A\in\mathfrak{B}_0$, $\varphi(x)=\varphi_A(x)$ holds for almost all $x\in A$. Note that any σ-finite measure space is localizable.

We denote by $L^\infty(X)$ the set of all locally measurable functions f on X such that ess. sup$|f(x)|$ is bounded for $A\in\mathfrak{B}_0$. For $f\in L^\infty(X)$, we denote by M_f the multiplication operator on $L^2(X)$ defined by $M_f\varphi=f\varphi$ $(\varphi\in L^2(X))$.

Theorem. *Let $\mathcal{A}$ be a linear subspace of $L^\infty(X)$ satisfying the following condition:*

(C) *every $f\in L^\infty(X)$ can be approximated on any $B\in\mathfrak{B}_0$ by elements in $\mathcal{A}$ in the sense of convergence in measure.*

If a bounded linear operator T on $L^2(X)$ commutes with M_φ for all $\varphi\in\mathcal{A}$, then T is a multiplication operator.

Proof. We devide the proof into the following two steps.
(1°) If T commutes with M_φ for all $\varphi\in\mathcal{A}$, then T commutes with M_f for all $f\in L^\infty(X)$.
(2°) If T commutes with M_f for all $f\in L^\infty(X)$, then T is of the form M_h for some $h\in L^\infty(X)$.

The step (2°) is a well-known fact that $\{M_f\,;\,f\in L^\infty(X)\}$ is a maximal abelian subalgebra in the algebra of all bounded linear operators on $L^2(X)$. So we prove here the step (1°) only.

Let us prove (1°) by contradiction. Suppose there exist an $f\in L^\infty(X)$ and a $\psi\in L^2(X)$ such that $\alpha=\|(TM_f-M_fT)\psi\|_{L^2(X)}$ is positive. Let ε be a positive number. Then there exists a $B\in\mathfrak{B}_0$ such that

$$\|(TM_f-M_fT)\psi\|_{L^2(B)}\geqq\alpha-\varepsilon\,,\quad \|\psi\|_{L^2(B^C)}\leqq\varepsilon\,,\quad\text{and}\quad \|T\psi\|_{L^2(B^C)}\leqq\varepsilon\,.$$

Here B^C denotes the complement of B in X. On the other hand, by the absolute continuity of indefinite integral, there exists a $\delta>0$ such that $\|(TM_f-M_fT)\psi\|_{L^2(e)}\leqq\varepsilon$, $\|\psi\|_{L^2(e)}\leqq\varepsilon$, and $\|T\psi\|_{L^2(e)}\leqq\varepsilon$ hold for arbitrary $e\in\mathfrak{B}_0$ with $\mu(e)\leqq\delta$. We fix these B and δ. By the condition (C) on $\mathcal{A}$, there exist $\varphi\in\mathcal{A}$ and $e\subset B$ such that $\sup_{x\in B\setminus e}|f(x)-\varphi(x)|\leqq\varepsilon$ and $\mu(e)\leqq\delta$. Put $B_1=B\setminus e$. Let χ_1 be the characteristic function of B_1. We put $\psi_1=\psi\chi_1$ and $\psi_2=\psi-\psi_1$. Then we have $\|\psi_2\|_{L^2(X)}\leqq2\varepsilon$. Hence on one hand,

$$\begin{aligned}\|(TM_f-M_fT)\psi\|_{L^2(B_1)}&\geqq\|(TM_f-M_fT)\psi\|_{L^2(B)}-\|(TM_f-M_fT)\psi\|_{L^2(e)}\\&\geqq\alpha-2\varepsilon\,.\end{aligned}$$

On the other hand, since $TM_\varphi=M_\varphi T$, we see $TM_f-M_fT=TM_{f-\varphi}-M_{f-\varphi}T$. Therefere we have

$$\begin{aligned}\|(TM_f-M_fT)\phi\|_{L^2(B_1)} &\leqq \|(TM_f-M_fT)\phi_1\|_{L^2(B_1)}+\|(TM_f-M_fT)\phi_2\|_{L^2(B_1)}\\ &\leqq \|(TM_{f-\varphi}-M_{f-\varphi}T)\phi_1\|_{L^2(B_1)}+2\varepsilon\|TM_f-M_fT\|\\ &\leqq \|TM_{f-\varphi}\phi_1\|_{L^2(B_1)}+\|M_{f-\varphi}T\phi_1\|_{L^2(B_1)}\\ &\qquad +2\varepsilon\|TM_f-M_fT\|.\end{aligned}$$

Note that $\|M_{f-\varphi}\phi_1\|_{L^2(X)}\leqq\varepsilon\|\phi_1\|_{L^2(X)}\leqq\varepsilon\|\phi\|_{L^2(X)}$, because ϕ_1 is zero outside B_1. Note also $\|M_{f-\varphi}T\phi_1\|_{L^2(B_1)}\leqq\varepsilon\|T\phi_1\|_{L^2(B_1)}\leqq\varepsilon\|T\|\|\phi\|_{L^2(X)}$. Consequently we get

$$\|(TM_f-M_fT)\phi\|_{L^2(B_1)}\leqq 2\varepsilon(\|T\|\|\phi\|_{L^2(X)}+\|TM_f-M_fT\|)\,.$$

Since $\varepsilon>0$ can be chosen small enough, this gives a contradiction. Q.E.D.

Corollary. *Let $\mathcal{A}$ be a linear subspace of $L^\infty(X)$ satisfying the condition* (C). *Let $\mathfrak{A}$ be the algebra generated by $\mathcal{A}$ and the complex conjugate of $\mathcal{A}$. Then $\mathfrak{A}$ is dense in $L^\infty(X)$ with respect to the weak* topology.*

Proof. Recall the theorem of Fuglede: if N and T are bounded operators on a Hilbert space and N is normal, then $TN=NT$ implies $TN^*=N^*T$ (see e.g. Strătilă-Zsidó [11, 2.31]). From this and $M_\varphi^*=M_{\bar\varphi}$, we see that a bounded operator T on $L^2(X)$ commutes with M_φ for all $\varphi\in\mathfrak{A}$ if T so does with M_φ for all $\varphi\in\mathcal{A}$. Then Corollary follows from von Neumann's double commutant theorem.

Q. E. D.

Remark. In the condition (C), the family $\mathfrak{B}_0$ can be replaced by a subfamliy $\mathcal{K}$ of $\mathfrak{B}_0$ satisfying the following (*).

(*) For any $B\in\mathfrak{B}_0$, there exist a countablly many $K_n\in\mathcal{K}$ and a locally null set N such that $B\subset N\cup(\bigcup_n K_n)$.

For example, in the case where μ is a Radon measure on a topological space X, the family $\mathcal{K}$ of all compact subsets of X satisfies (*).

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