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A note on the Segal-Becker type splittings

Dedicated to Professor Minoru NAKAOKA on his sixtieth birthday

By

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§1. Introduction

For a pointed space X, we define an infinite loop space $Q(X)$ by $Q(X)=\operatorname{Colim}_n \Omega^n \Sigma^n X$. If X is an infinite loop space, then there is an infinite loop map $\xi: Q(X)\to X$ called the structure map.

The natural inclusion $j: BU(1)=\boldsymbol{C}P^\infty\to BU$ and the structure map $\xi: Q(BU)\to BU$ of BU defined by the Bott periodicity theorem define an infinite loop map

$$\lambda: Q(\boldsymbol{C}P^\infty) \longrightarrow BU.$$

Quite similarly we can define $\lambda: Q(\boldsymbol{H}P^\infty)\to BSp$ and $Q(BO(2))\to BO$. In (7) Segal showed that λ has a splitting, that is there is a map $\varepsilon: BU\to Q(\boldsymbol{C}P^\infty)$ such that $\lambda\circ\varepsilon$ is a homotopy equivalence. On the other hand in (2) Becker constructed a splitting explicitly.

In this paper we give another construction of the splitting $\varepsilon_{\boldsymbol{C}}$ using the representation theory of compact Lie groups.

For the real and quaternionic cases, we can construct the splittings $\varepsilon_{\boldsymbol{R}}: BO\to Q(BO(2))$ and $\varepsilon_{\boldsymbol{H}}: BSp\to Q(HP^\infty)$ quite similarly.

The natural maps $BU\to BSp$ and $\boldsymbol{C}P^\infty\to \boldsymbol{H}P^\infty$ defined by the natural inclusion $\boldsymbol{C}\hookrightarrow\boldsymbol{H}$ are denoted by j' and the natural maps $BU\to BO$ and $BU(1)\to BO(2)$ defined by $\boldsymbol{C}\cong\boldsymbol{R}^2$ are denoted by r. Then the purpose of this paper is to show

Theorem. *The diagrams*

$$\begin{array}{ccc} BU & \xrightarrow{j'} & BSp \\ \downarrow{\scriptstyle \varepsilon_{\boldsymbol{C}}} & & \downarrow{\scriptstyle \varepsilon_{\boldsymbol{H}}} \\ Q(\boldsymbol{C}P^\infty) & \xrightarrow[Q(j')]{} & Q(\boldsymbol{H}P^\infty) \end{array}$$

$$\begin{array}{ccc} BU & \xrightarrow{r} & BO \\ \downarrow{\scriptstyle \varepsilon_{\boldsymbol{C}}} & & \downarrow{\scriptstyle \varepsilon_{\boldsymbol{R}}} \\ Q(\boldsymbol{C}P^\infty) & \xrightarrow[Q(r)]{} & Q(BO(2)) \end{array}$$

are homotopy commutative.

Let $\widetilde{HP}^\infty$ and $\widetilde{BO}(2)$ be the mapping cones of j': $CP^\infty \to HP^\infty$ and r: $CP^\infty \to BO(2)$. Then as a corollary of the above theorem we can easily show the following:

Corollary. *There are spaces F_0 and F'_0 such that*
(1) $\pi_*(F_0)$ *and* $\pi_*(F'_0)$ *are finite abelian groups for any* $*$, *and*
(2) $\Omega Q(\widetilde{HP}^\infty) \simeq (Sp/U) \times F_0$ *and* $\Omega Q(\widetilde{BO}(2)) \simeq (O/U) \times F'_0$.

§ 2. Construction of the splittings

Let G be a compact Lie group, H its closed subgroup and E a compact free G-space. A homomorphism α: $R(H) \to K(E/H)$ is defined by $M \to (E \times_H M \to E/H)$. The following is Proposition 5.4 of (5):

Lemma 2.1. *The following diagram is commutative:*

$$\begin{array}{ccc} R(H) & \xrightarrow{\alpha} & K(E/H) \\ \downarrow{\scriptstyle \mathrm{Ind}_H^G} & & \downarrow{\scriptstyle p_*} \\ R(G) & \xrightarrow[\alpha]{} & K(E/G), \end{array}$$

where Ind_H^G *is the induction homomorphism* (cf. (6)) *and* p_* *is the Becker-Gottlieb transfer.*

Note that the Becker-Gottlieb transfer for the fibre bundle p: $E \to B$ is defined by making use of a map $t(p)$: $B_+ \to Q(E_+)$. Consider the following homogeneous spaces

$$B_n = U(2n)/U(n) \times U(n),$$
$$E_n = U(2n)/U(1) \times U(n-1) \times U(n),$$
$$\bar{E}_n = U(2n)/U(1) \times U(2n-1) = CP^{2n-1},$$

and

$$\tilde{E}_n = U(2n)/\{1\} \times U(n).$$

The space $\tilde{E}_n$ is a compact free $U(n)$-space and there are natural projections p_n: $E_n \to B_n$ and q_n: $E_n \to \bar{E}_n$. Let $\iota_n \in R(U(n))$ be the identity representation and $\beta_n \in R(U(1) \times U(n-1))$ be the representation defined by the first projection. The following is Theorem 2.1 of (4):

Lemma 2.2. *$Ind_{U(1)\times U(n-1)}^{U(n)}(\beta_n) = \iota_n$.*

Consider the composition

$$\varepsilon_n\colon B_n \hookrightarrow B_{n+} \xrightarrow{t(p_n)} Q(E_{n+}) \xrightarrow{Q(p_n)} Q(\bar{E}_{n+}) \xrightarrow{r} Q(\bar{E}_n),$$

where r is the canonical projection. Note that $B = \mathrm{Colim}\, B_n$ is homotopy equivalent to BU. Define an $S^1 \times U(n)$ action on CP^{n+1} by

$$\alpha(z_0; z_1; \dots; z_{n+1}) = (z_0; z_1; \dots; \alpha z_{n+1}) \quad (\alpha \in S^1)$$

$$A(z_0; z_1; \dots; z_{n+1}) = (A(z_0, \dots, z_n); z_{n+1}) \quad (A \in U(n)).$$

The fixed point of S^1, $(\boldsymbol{CP}^{n+1})^{S^1} = \boldsymbol{CP}^n \amalg (pt)$ is clearly an $S^1 \times U(n)$-submanifold. Using the above fact we can construct ε_C: $B \to Q(\boldsymbol{CP}^\infty)$ by a similar method to that of (2). Applying Lemma 2.1, Lemma 2.2 and the fact that λ corresponds to the canonical line bundle, we can easily show $\lambda_*(\tilde{\varepsilon}_n) = p^*(\alpha(\beta_n)) = \alpha(\iota_n)$ and so ε_C is a splitting, where $\tilde{\varepsilon}_n$ is the composition $B_n \xrightarrow{\varepsilon_n} Q(E_n) \hookrightarrow Q(\boldsymbol{CP}^\infty)$ (cf. (2), (4)).

Next consider the quaternionic case. Put

$$B'_n = Sp(2n)/Sp(n) \times Sp(n),$$

$$E'_n = Sp(2n)/Sp(1) \times Sp(n-1) \times Sp(n),$$

and

$$\bar{E}'_n = Sp(2n)/Sp(1) \times Sp(2n-1) = \boldsymbol{HP}^{2n-1}.$$

Let p'_n: $E'_n \to B'_n$ and q'_n: $E'_n \to \bar{E}'_n$ be natural projections. Consider the following map

$$\varepsilon'_n: B'_n \hookrightarrow B'_{n+} \xrightarrow{t(p'_n)} Q(E'_{n+}) \xrightarrow{Q(p'_n)} Q(\bar{E}'_{n+}) \xrightarrow{r} Q(\bar{E}'_n).$$

Then we can define ε_H: $BSp \to Q(HP^\infty)$ similarly. To prove that ε_H is a splitting, we need the Bott periodicity theorem for KSp_G-theory (cf. §4 and §5 of (5)), which is proved in section 4.

The real case is similar.

§3. Proof of the main theorem

First we prove the following:

Lemma 3.1. *The diagram is homotopy commutative*:

$$\begin{array}{ccc} B_{n+} & \xrightarrow{t(p_n)} & Q(E_{n+}) \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle Q(j')} \\ B'_{n+} & \xrightarrow{t(p'_n)} & Q(E'_{n+}). \end{array}$$

Proof. Put $L_n = j'^*(E'_n)$. Then the structure group of this bundle $\boldsymbol{HP}^{n-1} \to L_n \to B_n$ can be reduced to $U(n)$. Note that this $U(n)$-action on $\boldsymbol{HP}^{n-1}$ can be extended to $S^1 \times U(n)$-action, since the center of $U(n)$ is S^1. Moreover $(\boldsymbol{HP}^{n-1})^{S^1} = \boldsymbol{CP}^{n-1}$ is an $S^1 \times U(n)$ submanifold and the associated $\boldsymbol{CP}^{n-1}$ bundle is p_n: $E_n \to B_n$. Therefore Lemma 3.1 follows from Lemma 1 of (2).

Let $\boldsymbol{PH}^*(\ )$ be the cohomology theory defined by the infinite loop space $Q(\boldsymbol{HP}^\infty)$ and $\tilde{\varepsilon}'_n$ be the composition $B'_n \xrightarrow{\varepsilon'_n} Q(E'_n) \hookrightarrow Q(\boldsymbol{HP}^\infty)$. By Lemma 3.1, $Q(j') \circ \tilde{\varepsilon}_n = \tilde{\varepsilon}'_n \circ j'$ in $\boldsymbol{PH}^0(B_n)$. Therefore to prove the main theorem we need only show the following:

Lemma 3.2. $\varprojlim_n^1 PH^{-1}(B_n)=0.$

Proof. We need only show that $PH^{-1}(B_n)$ is a finite abelian group for any n. Since $\lambda_*: PH^*(X)\to KSp^*(X)$ is split epic for any X, $\operatorname{Ker}\lambda_*$ defines a cohomology theory $F^*(\ )$. Note that $F^*(pt)$ is a finite abelian group for any $*$ and $PH^*(\ )=KS^*(\ )\oplus F^*(\ )$ as a cohomology theory. On the other hand $KSp^{-1}(B_n)$ and $F^{-1}(B_n)$ are finite abelian groups by the Atiyah-Hirzebruch spectral sequence and so Lemma 3.2 is proved.

The second case is proved similarly.

Proof of Corollary. First recall the fact that there are fiberings $Q(\widetilde{HP^\infty})\xrightarrow{f} Q(CP^\infty)\xrightarrow{Q(j')} Q(HP^\infty)$ and $Sp/U\xrightarrow{g} BU\xrightarrow{j'} BSp$. Using $j'\circ\lambda\simeq\lambda\circ Q(j')$ and $Q(j')\circ\varepsilon_C\simeq\varepsilon_H\circ j'$, we have two maps $\tilde{\lambda}: Q(\widetilde{HP^\infty})\to Sp/U$ and $\tilde{\varepsilon}: Sp/U\to Q(\widetilde{HP^\infty})$ satisfying $g\circ\tilde{\lambda}\simeq\lambda\circ f$ and $\varepsilon_C\circ g\simeq f\circ\tilde{\varepsilon}$. Then $\tilde{\lambda}\circ\tilde{\varepsilon}$ is a homotopy equivalence by the exact commutative diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \pi_*(Sp/U) & \xrightarrow{g_*} & \pi_*(BU) & \xrightarrow{j'_*} & \pi_*(BSp) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle(\tilde{\lambda}\circ\tilde{\varepsilon})_*} & & \downarrow{\scriptstyle(\lambda\circ\varepsilon_C)_*} & & \downarrow{\scriptstyle(\lambda\circ\varepsilon_H)_*} & \\
\cdots \longrightarrow & \pi_*(Sp/U) & \xrightarrow[g_*]{} & \pi_*(BU) & \xrightarrow[j'_*]{} & \pi_*(BSp) & \longrightarrow \cdots.
\end{array}$$

Now the corollary is obtained by a standard argument (cf. (2)).

§4. The Bott periodicity theorem for KSp_G-theory

Let G be a compact Lie group, V a real Spin G-module of dimension $8n$ and $u\in KO_G(V)$ the Bott class. For a compact G-space X, the multiplication by u defines a homomorphism

$$\beta': KSP_G(X)\longrightarrow KSp_G(V\times X).$$

On the other hand we can define a homomorphism

$$\alpha': KSp_G(V\times X)\longrightarrow KSp_G(X)$$

satisfying the following conditions by a similar method to that of (1):

(i) α' is functorial in X,

(ii) α' is a $KO_G(X)$-module homomorphism,

(iii) the diagram

$$\begin{array}{ccc}
KO_G(V)\otimes KSp_G(X) & \longrightarrow & KSp_G(V\times X) \\
\downarrow{\scriptstyle\alpha\otimes 1} & & \downarrow{\scriptstyle\alpha'} \\
KO_G(*)\otimes KSp_G(X) & \longrightarrow & KSp_G(X)
\end{array}$$

is commutative, where $\alpha(u)=1$.

Then we can show that $\alpha'\circ\beta'=1$ and $\beta'\circ\alpha'=1$ similarly (cf. (1)). Thus we have:

Theorem 4.1. *Let X be a compact G-space, V a real Spin G-module of dimension $8n$ and let $u \in KO_G(V)$ be the Bott class of V. Then multiplication by u induces an isomorphism*

$$KSp_G(X) \longrightarrow KSp_G(V \times X).$$

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References

[1] M. F. Atiyah, Bott periodicity and the index of elliptic operators, Quart. J. Math. Oxford (2), **19** (1968), 113–140.
[2] J. C. Becker, Characteristic classes and K-theory, Lecture Notes in Math., **428** (1974), 132–143.
[3] J. C. Becker and G. H. Gottlieb, The transfer map and fibre bundles, Topology, **14** (1975), 1–12.
[4] A. Kono, Segal-Becker thoerem for KR-theory, Japan. J. Math., **7** (1981), 195–199.
[5] G. Nishida, The transfer homomorphism in equivariant generalized cohomology theories, J. Math. Kyoto Univ., **18** (1978), 435–451.
[6] G. B. Segal, The representation ring of compact Lie groups, Publ. I. H. E. S., **34** (1968), 113–128.
[7] G. B. Segal, The stable homotopy of complex projective spaces, Quart. J. Math. Oxford (2), **24** (1973), 1–5.