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Existence of spherically symmetric global solution to the semi-linear wave equation $u_{tt}-\Delta u=au_t^2+b(\nabla u)^2$ in five space dimensions

By

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§0. In this note we shall study the following initial-value problem for a semi-linear wave equation with spherical symmetry

$$
\begin{cases} u_{tt}-\Delta u=au_t^2+b(\nabla u)^2 & (x,t)\in \boldsymbol{R}^n\times \boldsymbol{R}_+ \\ u(x,0)=f(x) & x\in \boldsymbol{R}^n \\ u_t(x,0)=g(x) & x\in \boldsymbol{R}^n \end{cases} \tag{0.1}
$$

where a and b are real constants.

It seems to me very remarkable that nonlinear wave equations are likely to have global solutions rather in higher dimensions, if the initial data are sufficiently small and mild. S. Klainerman [3] proved that if the space dimension n is not smaller than 6, very general quasi-linear wave equations with quadratic nonlinearity have global solutions for sufficiently small and mild initial data. His proof is based on a version of Nash-Moser iteration scheme. Later G. Ponce [4] obtained the same result employing L_p-L_q decay property of the solution to the wave equation.

On the other hand, F. John [2] showed that this is not the case in $n=3$. He showed that certain quasi-linear (or semi-linear) wave equations, for example (0.1) with $a>0$ and $b\geqslant 0$, have no global C^2-solution in $n=3$ for initial data with compact support, however small and mild the data may be. In [2], he also studied the spherically symmetric solutions for

$$
u_{tt}-\Delta u=u_t^2 \qquad (x,t)\in \boldsymbol{R}^3\times \boldsymbol{R}_+. \tag{0.2}
$$

He showed that there exists a numerical constant c such that the solution exists at least for $0\leqslant t\leqslant (1/2)e^{c/\eta}$, if the magnitude η of (certain derivatives of) the initial data is sufficiently small.

In this note, we shall confine ourselves to the case $n=5$. We shall show that if f, g depend only on $r=|x|$ and the magnitude of the initial data is sufficiently small, then there exists a unique spherically symmetric global C^2-solution of (0.1).

Finally we must mention that G. Ponce [4] proved for more general quasi-linear wave equations in $n=5$ that the solution exists up to the time $T_0=O(e^{1/\eta})$ if the magnitude η of the initial data is sufficiently small.

§ **1.** We shall review the representation of the spherically symmetric solution of the wave equation.

Let $x=(x_1, x_2, \dots, x_n) \in \boldsymbol{R}^n$ and $\varphi \in C^2(\boldsymbol{R}^n)$. We set

$$v(r, x)=\frac{1}{\omega_n}\int_{|\beta|=1} \varphi(x+r\beta)d\omega, \tag{1.1}$$

where $\omega_n=2\pi^{n/2}/\Gamma(n/2)$ denotes the surface area of the unit sphere in $\boldsymbol{R}^n$ and $d\omega$ its surface element. Then v is an even function of r and satisfies the Darboux equation

$$v_{rr}+\frac{n-1}{r}v_r-\Delta_x v=0. \tag{1.2}$$

We also find for the initial-values

$$v(0, x)=\varphi(x), \quad v_r(0, x)=0.$$

Especially when $\varphi(x)$ depends only on x_1, say $\varphi(x)=\varphi(x_1)$, v is written as

$$v(r, x_1)=\frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1} \varphi(x_1+r\mu)(1-\mu^2)^{(n-3)/2}d\mu \tag{1.3}$$

and v satisfies

$$v_{rr}+\frac{n-1}{r}v_r-v_{x_1x_1}=0. \tag{1.4}$$

Furthermore, if φ is an even function, then $v(r, x_1)$ is even in each variable.

We set

$$u(r, t)=\int_{-1}^{1} \varphi(t+r\mu)(1-\mu^2)^{(n-3)/2}d\mu. \tag{1.5}$$

Then by (1.3), (1.4), we can see that $u(r, t)$ is a spherically symmetric solution of the wave equation.

We choose φ to satisfy

$$\begin{aligned} u(r, 0)&=\int_{-1}^{1} \varphi(r\mu)(1-\mu^2)^{(n-3)/2}d\mu \\ &=f(r). \end{aligned}$$

When $n=$odd, differentiating $r^{n-2}f(r)$ $(n-1)/2$-times with respect to $\sqrt{r}$, we have

$$\varphi(r)=\frac{1}{\Gamma\left(\frac{n-1}{2}\right)}r\left(\frac{1}{2r}\frac{d}{dr}\right)^{(n-1)/2}(r^{n-2}f(r)) \tag{1.6}$$

(see Courant-Hilbert [1], Chap. VI, §13).

Changing the variable of the integration, we obtain

Proposition 1.1. *Let* $n=\text{odd}$, $f\in C^{(n+1)/2}(\boldsymbol{R})$ *and* $f(-r)=f(r)$. *Then the solution of the initial-value problem*

$$(1.7)\qquad \begin{cases} u_{tt}-u_{rr}-\dfrac{n-1}{r}u_r=0 & (r,t)\in \boldsymbol{R}^2 \\ u(r,0)=f(r) & r\in \boldsymbol{R} \\ u_t(r,0)=0 & r\in \boldsymbol{R} \end{cases}$$

is represented as

$$(1.8)\qquad u(r,t)=\frac{1}{2\Gamma\left(\dfrac{n-1}{2}\right)r^{n-2}}\frac{\partial}{\partial t}\int_{t-r}^{t+r}\Phi_f(\rho)\{r^2-(\rho-t)^2\}^{(n-3)/2}d\rho$$

where $\Phi_f(r)=\left(\dfrac{1}{2r}\dfrac{d}{dr}\right)^{(n-3)/2}(r^{n-2}f)$.

In the similar manner, we have

Proposition 1.2. *Let* $n=\text{odd}$, $g\in C^{(n-1)/2}(R)$ *and* $g(-r)=g(r)$. *The solution of the initial-value problem*

$$(1.9)\qquad \begin{cases} u_{tt}-u_{rr}-\dfrac{n-1}{r}u_r=0 & (r,t)\in \boldsymbol{R}^2 \\ u(r,0)=0 & r\in \boldsymbol{R} \\ u_t(r,0)=g(r) & r\in \boldsymbol{R} \end{cases}$$

is represented as

$$(1.10)\qquad u(r,t)=\frac{1}{2\Gamma\left(\dfrac{n-1}{2}\right)r^{n-2}}\int_{t-r}^{t+r}\Phi_g(\rho)\{r^2-(\rho-t)^2\}^{(n-3)/2}d\rho$$

where Φ_g *has the same meaning as above.*

In the case $n=5$, we find

$$\begin{aligned}\Phi_f(\rho)&=\frac{1}{2\rho}\frac{d}{d\rho}(\rho^3 f(\rho))\\ &=\frac{1}{2}\{3\rho f(\rho)+\rho^2 f'(\rho)\}.\end{aligned}$$

Then integrating by parts (1.8), (1.9), we obtain

Theorem 1.3. *Let* $f\in C^2(\boldsymbol{R})$, $g\in C^1(\boldsymbol{R})$, $f(-r)=f(r)$, $g(-r)=g(r)$. *Then the solution of the initial-value problem*

$$(1.11)\qquad \begin{cases} u_{tt}-u_{rr}-\dfrac{4}{r}u_r=0 & (r,t)\in \boldsymbol{R}^2 \\ u(r,0)=f(r) & r\in \boldsymbol{R} \\ u_t(r,0)=g(r) & r\in \boldsymbol{R} \end{cases}$$

is represented as

$$(1.12)\qquad u(r, t)=\frac{1}{2r^2}\{(t+r)^2f(t+r)+(t-r)^2f(t-r)\}$$

$$-\frac{t}{2r^3}\int_{t-r}^{t+r}\rho f(\rho)d\rho+\frac{1}{4r^3}\int_{t-r}^{t+r}\rho g(\rho)(r^2-t^2+\rho^2)d\rho.$$

Remark 1.4. According to the previous propositions, we must assume $f\in C^3(\boldsymbol{R})$, $g\in C^2(\boldsymbol{R})$ to get the representation. But it is easy to verify that u, represented as (1.12), is the C^2-solution of (1.11), if we merely assume $f\in C^2(\boldsymbol{R})$, $g\in C^1(\boldsymbol{R})$.

Employing Duhamel's principle, we have

Theorem 1.5. *Let $w(r, t)\in C^1(\boldsymbol{R}^2)$ and $w(-r, t)=w(r, t)$. Then the solution of the inhomogeneous initial-value problem*

$$(1.13)\qquad \begin{cases} u_{tt}-u_{rr}-\dfrac{4}{r}u_r=w(r, t) & (r, t)\in \boldsymbol{R}^2 \\ u(r, 0)=u_t(r, 0)=0 & r\in \boldsymbol{R} \end{cases}$$

is represented as

$$(1.14)\qquad u(r, t)=\frac{1}{4r^3}\int_0^t\int_{t-\tau-r}^{t-\tau+r}\rho w(\rho, \tau)\{r^2-(t-\tau)^2+\rho^2\}d\rho d\tau.$$

§2. If there exists the spherically symmetric C^2-solution of (0.1), then $u(x, t)=u(r, t)$ is the solution of the following initial-value problem

$$(2.1)\qquad \begin{cases} u_{tt}-u_{rr}-\dfrac{4}{r}u_r=au_t^2+bu_r^2 & (r, t)\in \boldsymbol{R}\times\boldsymbol{R}_+ \\ u(r, 0)=f(r) & r\in \boldsymbol{R} \\ u_t(r, 0)=g(r) & r\in \boldsymbol{R}. \end{cases}$$

We shall denote sometimes $C(u_t, u_r)=au_t^2+bu_r^2$ for abbreviation.

By Theorem 1.3 and 1.5, we find that u is represented as

$$(2.2)\qquad u(r, t)=u^0(r, t)+\frac{1}{4r^3}\int_0^t\int_{t-\tau-r}^{t-\tau+r}\rho C(u_t, u_r)(\rho, \tau)\{r^2-(t-\tau)^2+\rho^2\}d\rho d\tau$$

where

$$(2.3)\qquad u^0(r, t)=\frac{1}{2r^2}\{(t+r)^2f(t+r)+(t-r)^2f(t-r)\}$$

$$-\frac{t}{2r^3}\int_{t-r}^{t+r}\rho f(\rho)d\rho+\frac{1}{4r^3}\int_{t-r}^{t+r}\rho g(\rho)(r^2-t^2+\rho^2)d\rho.$$

Differentiating (2.2) with respect to t, we have

$$(2.4)\qquad u_t(r, t)=u_t^0(r, t)$$

$$+\frac{1}{2r^2}\int_0^t [(t-\tau+r)^2 C(u_t, u_r)(t-\tau+r, \tau)$$

$$+(t-\tau-r)^2 C(u_t, u_r)(t-\tau-r, \tau)]d\tau$$

$$-\frac{1}{2r^3}\int_0^t (t-\tau)\int_{t-\tau-r}^{t-\tau+r} \rho C(u_t, u_r)(\rho, \tau)d\rho d\tau$$

(2.5) $$u_t(-r, t)=u_t(r, t).$$

where

(2.6) $$u_t^0(r, t)=\frac{1}{r^2}\{(t+r)f(t+r)+(t-r)f(t-r)\}$$

$$+\frac{1}{2r^2}[(t+r)^2\{f'(t+r)+g(t+r)\}$$

$$+(t-r)^2\{f'(t-r)+g(t-r)\}]$$

$$-\frac{t}{2r^3}\{(t+r)f(t+r)+(t-r)f(t-r)\}$$

$$-\frac{1}{2r^3}\int_{t-r}^{t+r}\rho f(\rho)d\rho-\frac{t}{2r^3}\int_{t-r}^{t+r}\rho g(\rho)d\rho.$$

In a similar manner, differentiating with respect to r, we have

(2.7) $$u_r(r, t)=u_r^0(r, t)$$

$$+\frac{1}{2r^2}\int_0^t [(t-\tau+r)^2 C(u_t, u_r)(t-\tau+r, \tau)$$

$$-(t-\tau-r)^2 C(u_t, u_r)(t-\tau-r, \tau)]d\tau$$

$$-\frac{3}{4r^4}\int_0^t\int_{t-\tau-r}^{t-\tau+r} \rho C(u_t, u_r)(\rho, \tau)\{r^2-(t-\tau)^2+\rho^2\}d\rho d\tau$$

$$+\frac{1}{2r^2}\int_0^t\int_{t-\tau-r}^{t-\tau+r} \rho C(u_t, u_r)(\rho, \tau)d\rho d\tau,$$

(2.8) $$u_r(-r, t)=-u_r(r, t)$$

where

(2.9) $$u_r^0(r, t)=\frac{1}{r^2}\{(t+r)f(t+r)-(t-r)f(t-r)\}$$

$$-\frac{1}{r^3}\{(t+r)^2 f(t+r)+(t-r)^2 f(t-r)\}$$

$$+\frac{1}{2r^2}[(t+r)^2\{f'(t+r)+g(t+r)\}$$

$$-(t-r)^2\{f'(t-r)+g(t-r)\}]$$

$$-\frac{t}{2r^3}\{(t+r)f(t+r)+(t-r)f(t-r)\}$$

$$+\frac{3t}{2r^4}\int_{t-r}^{t+r}\rho f(\rho)d\rho$$

$$-\frac{3}{4r^3}\int_{t-r}^{t+r}\rho g(\rho)(r^2-t^2+\rho^2)d\rho+\frac{1}{2r^2}\int_{t-r}^{t+r}\rho g(\rho)d\rho.$$

Conversely, if C^2-function $u_1=u_t$, $u_2=u_r$ satisfy the integral identity (2.4), (2.7), we can see easily

$$u(r,\ t)=f(r)+\int_0^t u_1(r,\ \tau)d\tau \tag{2.10}$$

is the solution of the original equation (2.1). In this way, in order to get the global solution of (2.1), we have only to show that there exists a solution of the integral equations (2.4), (2.7) for all $t\geqslant 0$, if we assign sufficiently mild initial data.

We imagine that the magnitude of u_t^0, u_r^0 may increase as t tends to ∞, unless we impose certain conditions on $f(r)$, $g(r)$.

Lemma 2.1. *If a continuous function $h(\rho)$ satisfies*

$$|h(\rho)|\leqslant\frac{H}{1+\rho^2}, \qquad \text{then we find} \tag{2.11}$$

$$\left|\frac{t}{r}\int_{t-r}^{t+r}h(\rho)d\rho\right|\leqslant CH$$

where C does not depend on any of r, t, $h(\rho)$.

Proof. We set $\quad I(r,\ t)=\dfrac{t}{r}\displaystyle\int_{t-r}^{t+r}h(\rho)d\rho.$

When $t\leqslant 2r$, we find

$$|I(r,\ t)|\leqslant 2\int_{t-r}^{t+r}|h(\rho)|d\rho\leqslant 2H\int_{-\infty}^{\infty}\frac{d\rho}{1+\rho^2}=2\pi H.$$

when $t\geqslant 2r$, we find

$$|I(r,\ t)|\leqslant\frac{t}{r}\times 2r\times\frac{H}{1+(t-r)^2}$$

$$=\frac{2tH}{1+(t-r)^2}.$$

Since $t\geqslant 2r$, $t-r\geqslant t/2$ holds in this case. Then it follows

$$|I(r,\ t)|\leqslant\frac{2tH}{1+t^2/2}\leqslant CH.$$

With the aid of the lemma above, we have

Proposition 2.2. *If f, g satisfy*

$$\sup_{r\in \boldsymbol{R}}(1+r)^3(|f(r)|+|f'(r)|+|g(r)|)<\infty, \tag{2.12}$$

then $r^2u_t^0$, $r^2u_r^0$ are uniformly bounded for all $(r,\ t)\in \boldsymbol{R}^2$.

§3. We introduce the characteristic coordinates

$$\text{(3.1)}\qquad \begin{cases} x=t+r \\ y=t-r \end{cases} \quad \text{and} \quad \begin{cases} \xi=\tau+\rho \\ \eta=\tau-\rho. \end{cases}$$

Under these transformations, $\boldsymbol{R}^2_+=\{(r, t)\in \boldsymbol{R}^2 \mid t\geqslant 0\}$ is mapped to

$$\Lambda=\{(x, y)\in \boldsymbol{R}^2 \mid x+y\geqslant 0\}.$$

We set
$$U(x, y)=r^2u_t(r, t)=\left(\frac{x-y}{2}\right)^2 u_t\left(\frac{x-y}{2}, \frac{x+y}{2}\right)$$
$$V(x, y)=r^2u_r(r, t)=\left(\frac{x-y}{2}\right)^2 u_r\left(\frac{x-y}{2}, \frac{x+y}{2}\right).$$

Then we can easily verify the following.

$$\text{(3.2)}\qquad U(y, x)=U(x, y), \qquad U(x, x)=0,$$

$$\text{(3.3)}\qquad V(y, x)=-V(x, y), \quad (D_1-D_2)V(x, x)=0.$$

Here D_jV denotes the derivative in the j-th variable of V. Multiplying r^2 to the both sides of (2.4), we have

$$\begin{aligned}
\text{(3.4)}\quad U(x, y)&=U^0(x, y)\\
&+\frac{1}{2}\int_0^{\frac{x+y}{2}}\Big[\frac{1}{(x-\tau)^2}\{aU(x, 2\tau-x)^2+bV(x, 2\tau-x)^2\}\\
&+\frac{1}{(\tau-y)^2}\{aU(2\tau-y, y)^2+bV(2\tau-y, y)^2\}\Big]d\tau\\
&-\frac{2}{x-y}\int_{-y}^{x}\int_{-\xi}^{y}(x-\xi+y-\eta)\frac{aU(\xi, \eta)^2+bV(\xi, \eta)^2}{(\xi-\eta)^3}d\eta d\xi\\
&=U^0(x, y)\\
&+\int_{-x}^{y}\frac{aU(x, \sigma)^2+bV(x, \sigma)^2}{(x-\sigma)^2}d\sigma+\int_{-y}^{x}\frac{aU(\sigma, y)^2+bV(\sigma, y)^2}{(\sigma-y)^2}d\sigma\\
&-\frac{2}{x-y}\int_{-y}^{x}\int_{-\xi}^{y}(x-\xi+y-\eta)\frac{aU(\xi, \eta)^2+bV(\xi, \eta)^2}{(\xi-\eta)^3}d\eta d\xi
\end{aligned}$$

where

$$\begin{aligned}
\text{(3.5)}\quad U^0(x, y)&=xf(x)+yf(y)+\frac{1}{2}\{x^2(f'(x)+g(x))+y^2(f'(y)+g(y))\}\\
&-\frac{x+y}{2(x-y)}(xf(x)-yf(y))\\
&-\frac{1}{x-y}\int_y^x \rho f(\rho)d\rho-\frac{x+y}{2(x-y)}\int_y^x \rho g(\rho)d\rho.
\end{aligned}$$

Here we used $\dfrac{X(\xi, \eta)^2}{(\eta-\xi)^3}=-\dfrac{X(\xi, \eta)^2}{(\xi-\eta)^3}$ for $X=U, V$.

In a similar manner V satisfies

(3.6) $$V(x, y)=V^0(x, y) + \int_{-x}^{y} \frac{aU(x, \sigma)^2+bV(x, \sigma)^2}{(x-\sigma)^2} d\sigma - \int_{-y}^{x} \frac{aU(\sigma, y)^2+bV(\sigma, y)^2}{(\sigma-y)^2} d\sigma$$
$$+\frac{6}{(x-y)^2}\int_y^x\int_{-\xi}^{y} \frac{aU(\xi, \eta)^2+bV(\xi, \eta)^2}{(\xi-\eta)^3} \cdot \{(x-\xi)(x-\eta)+(y-\xi)(y-\eta)\} d\eta d\xi$$
$$-4\int_y^x\int_{-\xi}^{y} \frac{aU(\xi, \eta)^2+bV(\xi, \eta)^2}{(\xi-\eta)^3} d\eta d\xi,$$

where

(3.7) $$V^0(x, y)=xf(x)-yf(y)+\frac{1}{2}\{x^2(f'(x)+g(x))-y^2(f'(y)+g(y))\}$$
$$-\frac{1}{2(x-y)}\{(5x^2+xy)f(x)-(5y^2+xy)f(y)\}$$
$$+\frac{3(x+y)}{(x-y)^2}\int_y^x \rho f(\rho)d\rho$$
$$-\frac{3}{x-y}\int_y^x \rho g(\rho)\frac{x(\rho-y)+y(\rho-y)}{x-y} d\rho.$$

We introduce certain integral operators Θ, Φ, Ψ, Ω as the following.

(3.8) $$(\Theta X)(x, y)=\int_{-x}^{y} \frac{X(x, \sigma)^2}{(x-\sigma)^2} d\sigma$$

(3.9) $$(\Phi X)(x, y)=\frac{1}{x-y}\int_y^x\int_{-\xi}^{y} (x-\xi+y-\eta)\frac{X(\xi, \eta)^2}{(\xi-\eta)^3} d\eta d\xi$$

(3.10) $$(\Psi X)(x, y)=\frac{1}{(x-y)^2}\int_y^x\int_{-\xi}^{y} (x-\xi)(x-\eta)\frac{X(\xi, \eta)^2}{(\xi-\eta)^3} d\eta d\xi$$

(3.11) $$(\Omega X)(x, y)=\int_y^x\int_{-\xi}^{y} \frac{X(\xi, \eta)^2}{(\xi-\eta)^3} d\eta d\xi.$$

We can readily verify

Proposition 3.1. *Let $X=U$ or V, then it follows*

(1) $$\int_{-y}^{x} \frac{X(\sigma, y)^2}{(\sigma-y)^2} d\sigma=(\Theta X)(y, x),$$

(2) $$(\Phi X)(y, x)=(\Phi X)(x, y),$$

(3) $$\frac{1}{(x-y)^2}\int_y^x\int_{-\xi}^{y} (y-\xi)(y-\eta)\frac{X(\xi, \eta)^2}{(\xi-\eta)^3} d\eta d\xi=-(\Psi X)(y, x),$$

(4) $$(\Omega X)(y, x)=-(\Omega X)(x, y).$$

Using these operators, the integral equations are written as

$$(3.12)\qquad U(x, y)=U^0(x, y)+a(\Theta U)(x, y)+b(\Theta V)(x, y)$$
$$+a(\Theta U)(y, x)+b(\Theta V)(y, x)$$
$$-2a(\Phi U)(x, y)-2b(\Phi V)(x, y),$$

$$(3.13)\qquad U(y, x)=U(x, y),\quad U(x, x)=0,$$

$$(3.14)\qquad V(x, y)=V^0(x, y)+a(\Theta U)(x, y)+b(\Theta V)(x, y)$$
$$-a(\Theta U)(y, x)-b(\Theta V)(y, x)$$
$$+6a(\Psi U)(x, y)+6b(\Psi V)(x, y)$$
$$-6a(\Psi U)(y, x)-6b(\Psi V)(y, x)$$
$$-4a(\Omega U)(x, y)-4b(\Omega V)(x, y),$$

$$(3.15)\qquad V(y, x)=-V(x, y),\quad (D_1-D_2)V(x, x)=0.$$

We sometimes denote for simplicity (3.12), (3.14) by

$$U(x, y)=U^0(x, y)+\Xi_1(U, V)(x, y)$$
$$V(x, y)=V^0(x, y)+\Xi_2(U, V)(x, y).$$

§**4.** Let $C^j(\Lambda)$ denote the class of functions which have continuous derivatives up to order j. $C^j(\Lambda)$ is considered as a Banach space with respect to the norm

$$\|X\|_j=\max_{|\alpha|<j}\sup_{\Lambda}|D^\alpha X|.$$

We shall seek the solution of the equations (3.12), (3.14) in the following classes of functions

$$\Gamma_1=\{U\in C^4(\Lambda)\mid U(y, x)=U(x, y),\ U(x, x)=0\}$$
$$\Gamma_2=\{V\in C^4(\Lambda)\mid V(y, x)=-V(x, y),\ (D_1-D_2)V(x, x)=0\}.$$

We denote $A=O(B)$ when there exists a numerical constant C such that $|A|\leqslant C|B|$ holds for all A, B in question. Following [2], we have

Proposition 4.1. *For U, $U^*\in\Gamma_1$, there exist M, $M^*\in C^3(\Lambda)$ satisfying*

$$(4.1)\ (1)\qquad U(x, y)=(x-y)M(x, y),\ U^*(x, y)=(x-y)M^*(x, y),$$

$$(4.2)\ (2)\qquad D^\alpha(M-M^*)(x, y)=O(\|U-U^*\|_4)\quad \textit{for}\quad |\alpha|\leqslant 3.$$

Proof. Since $U(x, x)=0$, we find

$$U(x, y)=U(x, y)-U\Big(\frac{x+y}{2}, \frac{x+y}{2}\Big)$$
$$=\int_0^1\frac{d}{dt}U\Big(\frac{x+y+t(x-y)}{2}, \frac{x+y-t(x-y)}{2}\Big)dt$$

$$=\frac{x-y}{2}\int_0^1(D_1-D_2)U\Big(\frac{x+y+t(x-y)}{2},\ \frac{x+y-t(x-y)}{2}\Big)dt.$$

If we set

$$M(x, y)=\frac{1}{2}\int_0^1(D_1-D_2)U\Big(\frac{x+y+t(x-y)}{2},\ \frac{x+y-t(x-y)}{2}\Big)dt,$$

we can easily verify (1), (2).

Proposition 4.2. *For U, $U^*\in\Gamma_1$, there exist P, $P^*\in C^2(\Lambda)$ satisfying*

(4.3) (1) $U(x, y)=(x-y)^2P(x, y)$, $U^*(x, y)=(x-y)^2P^*(x, y)$,

(4.4) (2) $D^\alpha(P-P^*)(x, y)=O(\|U-U^*\|_4)$ *for* $|\alpha|\leqslant 2$.

Proof. As before, we find

$$U(x, y)=\frac{x-y}{2}\int_0^1(D_1-D_2)U\Big(\frac{x+y+t(x-y)}{2},\ \frac{x+y-t(x-y)}{2}\Big)dt.$$

Since $U(x, y)=U(y, x)$, it follows $(D_1-D_2)U(x, x)=0$. Then integrating by parts we have

$$U(x, y)=\Big(\frac{x-y}{2}\Big)^2\int_0^1(1-t)(D_1-D_2)^2U\Big(\frac{x+y+t(x-y)}{2},\ \frac{x+y-t(x-y)}{2}\Big)dt$$

To verify (1), (2), we merely set

$$P(x, y)=\frac{1}{4}\int_0^1(1-t)(D_1-D_2)^2U\Big(\frac{x+y+t(x-y)}{2},\ \frac{x+y-t(x-y)}{2}\Big)dt.$$

Finally, we have

Proposition 4.3. *For U, $U^*\in\Gamma_1$, there exist R_1, R_2, R_1^*, $R_2^*\in C^2(\Lambda)$ satisfying*

(4.5) (1) $D_jU(x, y)=(x-y)R_j(x, y)$, $D_jU^*(x, y)=(x-y)R_j^*(x, y)$

for $j=1, 2$,

(4.6) (2) $D^\alpha(R_j-R_j^*)(x, y)=O(\|U-U^*\|_4)$ *for* $|\alpha|\leqslant 2$, $j=1, 2$.

Proof. Since $U(x, y)=(x-y)^2P(x, y)$, $D_jU(x, x)=0$ hold for $j=1, 2$. Then it follows

$$D_jU(x, y)=\frac{x-y}{2}\int_0^1(D_1-D_2)D_jU\Big(\frac{x+y+t(x-y)}{2},\ \frac{x+y-t(x-y)}{2}\Big)dt.$$

If we set

$$R_j(x, y)=\frac{1}{2}\int_0^1(D_1-D_2)D_jU\Big(\frac{x+y+t(x-y)}{2},\ \frac{x+y-t(x-y)}{2}\Big)dt,$$

We can see easily that (1), (2) hold.

In a similar fashion, for V, $V^*\in\Gamma_2$, we can prove the followings.

Proposition 4.4. *There exist N, $N^* \in C^3(\Lambda)$ satisfying*

(4.7) (1) $V(x, y) = (x-y)N(x, y),\ V^*(x, y) = (x-y)N^*(x, y),$

(4.8) (2) $D^\alpha(N-N^*)(x, y) = O(\|V-V^*\|_4)$ *for* $|\alpha| \leqslant 3.$

Proposition 4.5. *There exist Q, $Q^* \in C^2(\Lambda)$ satisfying*

(4.9) (1) $V(x, y) = (x-y)^2 Q(x, y),\ V^*(x, y) = (x-y)^2 Q^*(x, y),$

(4.10) (2) $D^\alpha(Q-Q^*)(x, y) = O(\|V-V^*\|_4)$ *for* $|\alpha| \leqslant 2.$

Proposition 4.6. *There exist S_1, S_2, S_1^*, $S_2^* \in C^2(\Lambda)$ satisfying*

(4.11) (1) $D_j V(x, y) = (x-y)S_j(x, y),\ D_j V^*(x, y) = (x-y)S_j^*(x, y)$ *for* $j = 1, 2,$

(4.12) (2) $D^\alpha(S_j - S_j^*)(x, y) = O(\|V-V^*\|_4)$ *for* $|\alpha| \leqslant 2, j = 1, 2.$

By the considerations above, if we find $U \in \Gamma_1$, $v \in \Gamma_2$ satisfying the equations (3.12), (3.14), then

$$u_1(x, y) = \frac{1}{(x-y)^2} U(x, y)$$

is C^2 and u, defined as (2.10), is the desired solution.
As in [2], let us introduce the function

$$z(x, y) = \min\left(1, \frac{1}{|x-y|}\right).$$

We shall show that M, N,... etc. have the following improved estimates.

Lemma 4.7. *Let M, N, M^*, N^* be the same as in the previous propositions. For $|\alpha| \leqslant 3$, we have*

(1) $$D^\alpha(M-M^*)(x, y) = O(z(x, y)\|U-U^*\|_4),$$

(2) $$D^\alpha(N-N^*)(x, y) = O(z(x, y)\|V-V^*\|_4),$$

Especially we have

(3) $$D^\alpha M(x, y) = O(z(x, y)\|U\|_4),$$

(4) $$D^\alpha N(x, y) = O(z(x, y)\|V\|_4).$$

Proof. Since $(x-y)M(x, y) = U(x, y)$, we find

$$(M-M^*)(x, y) = O\left(\frac{1}{|x-y|}\|U-U^*\|_0\right) \quad \text{for} \quad x \neq y.$$

Differentiating (4.1) in x or y, we have

(4.13) $$(x-y)D_i M + (-1)^{i-1} M = D_i U.$$

Then we find

$$D_i(M-M^*)(x, y)=C\Big(\frac{1}{|x-y|}\|U-U^*\|_4\Big).$$

Differentiating (4.13) with D_j and then D_k, we have

(4.14) $$(x-y)D_{ij}M+(-1)^{i-1}D_jM+(-1)^{j-1}D_iM=D_{ij}U,$$

(4.15) $$(x-y)D_{ijk}M+(-1)^{i-1}D_{jk}M+(-1)^{j-1}D_{ki}M+(-1)^{k-1}D_{ij}M=D_{ijk}M.$$

Here we abbreviate D_iD_jU to $D_{ij}U$ and $D_iD_jD_kU$ to $D_{ijk}U$.
Then we find as before

$$D_{ij}(M-M^*)(x, y)=O\Big(\frac{1}{|x-y|}\|U-U^*\|_4\Big),$$

$$D_{ijk}(M-M^*)(x, y)=O\Big(\frac{1}{|x-y|}\|U-U^*\|_4\Big)$$

In this way, together with Proposition 4.1, we obtain

$$D^\alpha(M-M^*)(x, y)=O(z(x, y)\|U-U^*\|_4) \qquad \text{for} \quad |\alpha|\leqslant 3.$$

In the same manner, we obtain (2), (3), (4).

Differentiating the both sides of

$$(x-y)^2P(x, y)=U(x, y),\ (x-y)^2Q(x, y)=V(x, y),$$

we obtain

Lemma 4.8. *For $|\alpha|\leqslant 2$, we have*

(1) $$D^\alpha(P-P^*)(x, y)=O(z(x, y)^2\|U-U^*\|_4),$$

(2) $$D^\alpha(Q-Q^*)(x, y)=O(z(x, y)^2\|V-V^*\|_4)$$

(3) $$D^\alpha P(x, y)=O(z(x, y)^2\|U\|_4),$$

(4) $$D^\alpha Q(x, y)=O(z(x, y)^2\|V\|_4).$$

For R_j, S_j $(j=1, 2)$, we obtain

Lemma 4.9. *For $|\alpha|\leqslant 2$, $j=1, 2$, we have*

(1) $$D^\alpha(R_j-R_j^*)(x, y)=O(z(x, y)\|U-U^*\|_4),$$

(2) $$D^\alpha(S_j-S_j^*)(x, y)=O(z(x, y)\|V-V^*\|_4),$$

(3) $$D^\alpha R_j(x, y)=O(z(x, y)\|U\|_4),$$

(4) $$D^\alpha S_j(x, y)=O(z(x, y)\|V\|_4).$$

Later, to establish the global existence of the solution, we shall make essential use of

(4.16) $$\int_{-\infty}^{\infty} z(x, y)^2dx=\int_{-\infty}^{\infty} z(x, y)^2dy<\infty.$$

§5. In this section we shall carry out the estimates of $\|\Theta X\|_4$, $\|\Phi X\|_4$, $\|\Psi X\|_4$, $\|\Omega X\|_4$. ΘX can be treated just in the same manner as [2]. Firstly we set

$$W(x, y) = \frac{U(x, y)^2}{(x-y)^2} = PU. \tag{5.1}$$

Differentiating the both sides of (5.1) once and twice, we have

$$\begin{aligned} D_i W &= \frac{2UD_iU}{(x-y)^2} - (-1)^{i-1}\frac{2U^2}{(x-y)^3} \\ &= 2PD_iU - 2(-1)^{i-1}MP \end{aligned} \tag{5.2}$$

$$\begin{aligned} D_{ij} W &= \frac{2D_iUD_jU}{(x-y)^2} + \frac{2UD_{ij}U}{(x-y)^2} - (-1)^{i-1}\frac{4UD_jU}{(x-y)^3} \\ &\quad - (-1)^{j-1}\frac{4UD_iU}{(x-y)^3} - 6(-1)^{i+j}\frac{U^2}{(x-y)^4} \\ &= 2R_iR_j + 2PD_{ij}U - 4(-1)^{i-1}PR_j - 4(-1)^{j-1}PR_i - 6(-1)^{i+j}P^2. \end{aligned} \tag{5.3}$$

In this way, we know that, for $|\alpha|\leqslant 2$, $D^\alpha W$ can be expressed in terms of P, R_j, and $D^\alpha U$. Differentiating (5.3) still more, we find that the third and the forth derivatives of W can be expressed in terms of $D^\alpha P$, $D^\alpha R_j$ ($|\alpha|\leqslant 2$) and $D^\alpha U$ ($|\beta|\leqslant 4$). Then together with Lemma 4.7, 4.8 and 4.9, we obtain

Proposition 5.1. *For* $|\alpha|\leqslant 4$, *we have*

(1) $$D^\alpha W(x, y) = O(z(x, y)^2\|U\|_4^2),$$

(2) $$D^\alpha(W - W^*)(x, y) = O(z(x, y)^2(\|U\|_4 + \|U^*\|_4)\|U - U^*\|_4).$$

Since
$$\begin{aligned} (\Theta U)(x, y) &= \int_{-x}^{y} \frac{U(x, \sigma)^2}{(x-\sigma)^2} d\sigma \\ &= \int_{-x}^{y} W(x, \sigma) d\sigma, \end{aligned} \tag{5.4}$$

we find by differentiating (5.4)

$$D_x\Theta U(x, y) = W(x, -x) + \int_{-x}^{y} D_1 W(x, \sigma) d\sigma$$

$$D_{xx}\Theta U(x, y) = (D_1 - D_2)W(x, -x) + D_1 W(x, -x) + \int_{-x}^{y} D_1^2 W(x, \sigma) d\sigma$$

$$D_{xxx}\Theta U(x, y) = \cdots\text{etc.}$$

In this way, we can see easily that

$$\begin{aligned} &D^\alpha(\Theta U - \Theta U^*)(x, y) \\ &= O(\|W - W^*\|_4) + O\Big(\int_{-x}^{y} |D^\alpha(W - W^*)(x, \sigma)| d\sigma\Big) \\ &= O((\|U\|_4 + \|U^*\|_4)\|U - U^*\|_4) \end{aligned}$$

$$+O((\|U\|_4+\|U^*\|_4)\|U-U^*\|_4\int_{-x}^{y} z(x,\sigma)^2 d\sigma)$$

$$=O((\|U\|_4+\|U^*\|_4)\|U-U^*\|_4).$$

Then we obtain

Proposition 5.2. *For* $X=U, V$, *we have*

$$\|\Theta X-\Theta X^*\|_4=O((\|X\|_4+\|X^*\|_4)\|X-X^*\|_4.$$

Next we shall carry out the estimates of ΦX. Let us denote

$$\Phi U(x, y)=\int_y^x\int_{-\xi}^y \frac{x-\xi}{x-y}\frac{U(\xi,\eta)^2}{(\xi-\eta)^3}d\eta d\xi+\int_y^x\int_{-\xi}^y \frac{y-\eta}{x-y}\frac{U(\xi,\eta)^2}{(\xi-\eta)^3}d\eta d\xi \tag{5.5}$$

$$=\Phi_1 U(x,y)+\Phi_2 U(x,y).$$

Differentiating $\Phi_1 U$ in x, we have

$$D_x\Phi_1 U(x,y)=-\int_y^x\int_{-\xi}^y \frac{y-\xi}{(x-y)^2}\frac{U(\xi,\eta)^2}{(\eta-\xi)^3}d\eta d\xi.$$

We change the variable of the integration as $\xi=y+t(x-y)$. Then we find

$$D_x\Phi_1 U(x,y)=\int_0^1 t\int_{-y-t(x-y)}^y \frac{U(y+t(x-y),\eta)^2}{\{y+t(x-y)-\eta\}^3}d\eta dt \tag{5.6}$$

$$=\int_0^1 t\int_{-y-t(x-y)}^y M(y+t(x-y),\eta)\ P(y+t(x-y),\eta)d\eta dt.$$

Differentiating $\Phi_1 U$ in y, we have

$$D_y\Phi_1 U(x,y)=-\int_{-y}^y \frac{U(y,\eta)^2}{(y-\eta)^3}d\eta+\int_y^x \frac{x-\xi}{x-y}\frac{U(\xi,y)^2}{(\xi-y)^3}d\xi$$

$$+\int_y^x\int_{-\xi}^y \frac{x-\xi}{(x-y)^2}\frac{U(\xi,\eta)^2}{(\xi-\eta)^3}d\eta d\xi$$

$$=-\int_{-y}^y \frac{U(y,\eta)^2}{(y-\eta)^3}d\eta+\int_y^x \frac{U(\xi,y)^2}{(\xi-y)^3}d\xi$$

$$-\frac{1}{x-y}\int_y^x \frac{U(\xi,y)^2}{(\xi-y)^2}d\xi+\int_y^x\int_{-\xi}^y \frac{x-\xi}{(x-y)^2}\frac{U(\xi,\eta)^2}{(\xi-\eta)^3}d\eta d\xi.$$

Changing the variable of the integration in the same manner as above, we obtain

$$D_y\Phi_1 U(x,y)=-\int_{-y}^y M(y,\eta)d\eta+\int_y^x M(\xi,y)P(\xi,y)d\xi \tag{5.7}$$

$$-\int_0^1 M(y+t(x-y),y)^2 dt$$

$$+\int_0^1(1-t)\int_{-y+t(x-y)}^y M(y+t(x-y),\eta)P(y+t(x-y),\eta)d\eta dt.$$

Differentiating (5.6) in x, we have

(5.8) $D_{xx}\Phi_1 U(x, y)$

$$=\int_0^1 t^2 P(y+t(x-y),\ -y-t(x-y))M(y+t(x-y),\ -y-t(x-y))dt$$

$$+2\int_0^1 t^2 \int_{-y-t(x-y)}^{y} P(y+t(x-y),\ \eta)R_1(y+t(x-y),\ \eta)d\eta dt$$

$$-3\int_0^1 t^2 \int_{-y-t(x-y)}^{y} P(y+t(x-y),\ \eta)^2 d\eta dt.$$

Similar computations show

(5.9) $D_{xy}\Phi_1 U(x, y)$

$$=\int_0^1 tM(y+t(x-y),\ y)P(y+t(x-y),\ y)dt$$

$$+\int_0^1 t(1-t)M(y+t(x-y),\ -y-t(x-y))P(y+t(x-y),\ -y-t(x-y))dt$$

$$+2\int_0^1 t(1-t)\int_{-y-t(x-y)}^{y} P(y+t(x-y),\ \eta)R_1(y+t(x-y),\ \eta)d\eta dt$$

$$-3\int_0^1 t(1-t)\int_{-y-t(x-y)}^{y} P(y+t(x-y),\ \eta)^2 d\eta dt,$$

(5.10) $D_{yy}\Phi_1 U(x, y)$

$$=-M(y,\ -y)P(y,\ -y)-2\int_{-y}^{y} R_1(y,\ \eta)P(y,\ \eta)d\eta$$

$$+3\int_{-y}^{y} P(y,\ \eta)^2 d\eta-\int_y^x M(\xi,\ y)P(\xi,\ y)d\xi$$

$$+2\int_y^x R_1(\xi,\ y)P(\xi,\ y)d\xi-3\int_y^x P(\xi,\ y)^2 d\xi$$

$$+\int_0^1 M(y+t(x-y),\ y)P(y+t(x-y),\ y)dt$$

$$-2\int_0^1 R_1(y+t(x-y),\ y)P(y+t(x-y),\ y)dt$$

$$+3\int_0^1 P(y+t(x-y),\ y)^2 dt$$

$$+\int_0^1 (1-t)M(y+t(x-y),\ y)P(y+t(x-y),\ y)dt$$

$$-\int_0^1 (1-t)^2 M(y+t(x-y),\ -y-t(x-y))P(y+t(x-y),\ -y-t(x-y))dt$$

$$+2\int_0^1\int_{-y-t(x-y)}^{y} R_1(y+t(x-y),\ \eta)P(y+t(x-y),\ \eta)d\eta dt$$

$$-3\int_0^1 (1-t)^2\int_{-y-t(x-y)}^{y} P(y+t(x-y),\ \eta)^2 d\eta dt.$$

In a similar fashion, we obtain the expressions of the derivatives of $\Phi_2 U$.

Since, as we have shown, $(D^\alpha \Phi U)(x, y)$ $(|\alpha| \leqslant 2)$ are written in terms of M, P, R_j, the derivatives of ΦU up to the forth order can be expressed in terms of at most the second derivatives of M, P, R_j. Then together with Lemma 4.7, 4.8, 4.9, we obtain

Proposition 5.3. *For $X = U$, V, we have*

$$\|\Phi X - \Phi X^*\|_4 = O((\|X\|_4 + \|X^*\|_4)\|X - X^*\|_4).$$

Now we carry out the estimates of

$$(\Psi U)(x, y) = \frac{1}{(x-y)^2}\int_y^x \int_{-\xi}^y (x-\xi)(x-\eta)\frac{U(\xi, \eta)^2}{(\xi-\eta)^3} d\eta d\xi, \tag{5.11}$$

Differentiating (5.11) in x, we have

$$\begin{aligned}
D_x \Psi U(x, y) = &-\frac{2}{(x-y)^3}\int_y^x \int_{-\xi}^y (x-\xi)(x-\eta)\frac{U(\xi, \eta)^2}{(\xi-\eta)^3} d\eta d\xi \\
&+\frac{1}{(x-y)^2}\int_y^x \int_{-\xi}^y (x-\xi)\frac{U(\xi, \eta)^2}{(\xi-\eta)^3} d\eta d\xi \\
&+\frac{1}{(x-y)^2}\int_y^x \int_{-\xi}^y (x-\eta)\frac{U(\xi, \eta)^2}{(\xi-\eta)^3} d\eta d\xi \\
= &-\frac{2}{(x-y)^3}\int_y^x (x-\xi)^2 \int_{-\xi}^y \frac{U(\xi, \eta)^2}{(\xi-\eta)^3} d\eta d\xi \\
&-\frac{2}{(x-y)^3}\int_y^x (x-\xi) \int_{-\xi}^y \frac{U(\xi, \eta)^2}{(\xi-\eta)^2} d\eta d\xi \\
&+\frac{2}{(x-y)^2}\int_y^x (x-\xi) \int_{-\xi}^y \frac{U(\xi, \eta)^2}{(\xi-\eta)^3} d\eta d\xi \\
&-\frac{1}{(x-y)^2}\int_y^x \int_{-\xi}^y \frac{U(\xi, \eta)^2}{(\xi-\eta)^2} d\eta d\xi \\
= &-\frac{2}{(x-y)^3}\int_y^x (x-\xi)^2 \int_{-\xi}^y \frac{U(\xi, \eta)^2}{(\xi-\eta)^3} d\eta d\xi \\
&+\frac{1}{(x-y)^3}\int_y^x \frac{\partial}{\partial\xi}\{(x-\xi)^2\}(\Theta U)(\xi, y) d\xi \\
&+\frac{2}{(x-y)^2}\int_y^x (x-\xi) \int_{-\xi}^y \frac{U(\xi, \eta)^2}{(\xi-\eta)^3} d\eta d\xi \\
&-\frac{1}{(x-y)^2}\int_y^x \frac{\partial}{\partial\xi}(x-\xi)(\Theta U)(\xi, y) d\xi
\end{aligned}$$

Integrating by parts and changing the variable of the integration as $\xi = y + t(x-y)$, we have

(5.12) $D_x\Psi U(x, y)$

$$=2\int_0^1 t(1-t)\int_{-y-t(x-y)}^{y} M(y+t(x-y), \eta)P(y+t(x-y), \eta)d\eta dt$$

$$+\int_0^1 t(1-t)(D_1\Theta U)(y+t(x-y), y)dt.$$

In a similar manner, we have

(5.13) $D_y\Psi U(x, y)$

$$=2\int_0^1 (1-t)^2\int_{-y-t(x-y)}^{y} M(y+t(x-y), \eta)P(y+t(x-y), \eta)d\eta dt$$

$$+\int_0^1 (1-t)^2(D_1\Theta U)(y+t(x-y), y)dt$$

$$-\int_{-y}^{y} P(y, \eta)M(y, \eta)d\eta+\int_y^x M(\xi, y)P(\xi, y)d\xi$$

$$-\int_0^1 P(y+t(x-y), y)U(y+t(x-y), y)dt$$

Differentiating (5.12) in x, we have

(5.14) $D_{xx}\Psi U(x, y)$

$$=2\int_0^1 t^2(1-t)M(y+t(x-y), -y-t(x-y))P(y+t(x-y), -y-t(x-y))dt$$

$$+4\int_0^1 t^2(1-t)\int_{-y-t(x-y)}^{y} P(y+t(x-y), \eta)R_1(y+t(x-y), \eta)d\eta dt$$

$$-6\int_0^1 t^2(1-t)\int_{-y-t(x-y)}^{y} P(y+t(x-y), \eta)^2 d\eta dt$$

$$+\int_0^1 t^2(1-t)(D_1^2\Theta U)(y+t(x-y), y)dt.$$

Similar computations show

(5.15) $D_{xy}\Psi U(x, y)$

$$=2\int_0^1 t(1-t)M(y+t(x-y), y)P(y+t(x-y), y)dt$$

$$+2\int_0^1 t(1-t)^2M(y+t(x-y), -y-t(x-y))P(y+t(x-y), -y-t(x-y))dt$$

$$+4\int_0^1 t(1-t)^2\int_{-y-t(x-y)}^{y} P(y+t(x-y), \eta)R_1(y+t(x-y), \eta)d\eta dt$$

$$-6\int_0^1 t(1-t)^2\int_{-y-t(x-y)}^{y} P(y+t(x-y), \eta)^2 d\eta dt$$

$$+\int_0^1 t(1-t)^2(D_1^2\Theta U)(y+t(x-y), y)dt$$

$$+\int_0^1 t(1-t)^2(D_1D_2\Theta U)(y+t(x-y), y)dt,$$

(5.16) $D_{yy}\Psi U(x, y)$

$$=2\int_0^1(1-t)^2M(y+t(x-y), y)P(y+t(x-y), y)dt$$

$$+2\int_0^1(1-t)^3M(y+t(x-y), -y-t(x-y))P(y+t(x-y), -y-t(x-y))dt$$

$$+8\int_0^1(1-t)^3\int_{-y-t(x-y)}^{y}P(y+t(x-y), \eta)R_1(y+t(x-y), \eta)d\eta dt$$

$$-6\int_0^1(1-t)^3\int_{-y-t(x-y)}^{y}P(y+t(x-y), \eta)^2d\eta dt$$

$$+\int_0^1(1-t)^3(D_1\Theta U)(y+t(x-y), y)dt$$

$$-M(y, -y)^2-2\int_{-y}^{y}M(y, \eta)R_1(y, \eta)d\eta-2\int_{-y}^{y}M(y, \eta)P(y, \eta)d\eta$$

$$+\int_0^1(1-t)M(y+t(x-y), y)P(y+t(x-y), y)dt$$

$$+4\int_y^x P(\xi, y)R_2(\xi, y)d\xi+4\int_0^1 M(y+t(x-y), y)R_2(y+t(x-y), y)dt$$

$$+3\int_y^x P(\xi, y)^2d\xi+3\int_0^1 M(y+t(x-y), y)P(y+t(x-y), y)dt.$$

In this way, we have shown that the second derivatives of ΨU are expressed in terms of M, P, R_j. By the same arguments as before, we obtain the estimates of the derivatives of ΨU up to the forth order. In a similar fashion, we carry out the estimates for ΨV. Thus we obtain

Proposition 5.4. *For $X=U, V$, we have*

$$\|\Psi X-\Psi X^*\|_4=O((\|X\|_4+\|X^*\|_4)\|X-X^*\|_4).$$

Since ΩX is handled with in a easier manner, we may omit the proof of the following estimate.

Proposition 5.5. *For $X=U, V$, we have*

$$\|\Omega X-\Omega X^*\|_4=O((\|X\|_4+\|X^*\|_4)\|X-X^*\|_4).$$

§6. Combining the preceeding estimates, we shall prove the existence theorem.

Theorem 6.1. *Let $f\in C^5(\boldsymbol{R}^4)$, $g\in C^4(\boldsymbol{R})$ be even functions satisfying*

$$\sup_{r\in\boldsymbol{R}}[(1+r)^3(\sum_{j=0}^{5}|D_r^jf(r)|+\sum_{j=0}^{4}|D_r^jg(r)|)]=\eta<\infty.$$

If η is sufficiently small, then there exists a unique global C^2-solution of the initial-value problem (2.1).

Proof. We set $W=(U, V)$, $\Xi=(\Xi_1, \Xi_2)$. Then the equations (3.12), (3.14) are expressed as

$$W=W^0+\Xi(W). \tag{6.1}$$

We seek the solution of (6.1) in the class of functions

$$\Gamma=\{W=(U, V) \mid U\in\Gamma_1,\ V\in\Gamma_2\}.$$

Firstly we must show that $\Xi(W)\in\Gamma$ for $W\in\Gamma$. It is clear that

$$\Xi_1(W),\quad \Xi_2(W)\in C^4(\Lambda),\quad U(y, x)=U(x, y),\quad V(y, x)=-V(x, y).$$

By (3.12), we find

$$\Xi_1(W)(x, x)=2a(\Theta U)(x, x)+2b(\Theta V)(x, x)-2a(\Phi U)(x. x)-2b(\Phi V)(x, x).$$

Since

$$(\Phi U)(x, x)=\int_{-x}^{x}\frac{U(x, \eta)^2}{(x-\eta)^2}d\eta=(\Theta U)(x, x),$$

$$(\Phi V)(x, x)=(\Theta V)(x, x),$$

then we find $\Xi_1(W)(x, x)=0$.

Similarly by (5.6), (5.7), (5.12), (5.13), we can see

$$D_1\Xi_2(x, x)=D_2\Xi_2(x, x)=0.$$

In this way we have shown $\Xi(W)\in\Gamma$ for $W\in\Gamma$.

The estimates in the previous section show that there exists a numerical onstant L such that

$$\|\Xi W-\Xi W^*\|_4\leqslant L(\|W\|_4+\|W^*\|_4)\|W-W^*\|_4 \qquad \text{for} \quad W,\ W^*\in\Gamma.$$

Let us define a sequence of functions W_n as

$$W_0=W^0,\ W_n=W^0+\Xi(W_{n-1}) \qquad \text{for} \quad n\geqslant 1.$$

If we take sufficiently small ε such that

$$\|W^0\|_4\leqslant\varepsilon\leqslant\frac{1}{8L},$$

then we find by induction

$$\|W_n\|_4\leqslant 2\varepsilon,$$

$$\|W_{n+1}-W_n\|_4\leqslant 4\varepsilon L\|W_n-W_{n-1}\|_4$$

$$\leqslant\frac{1}{2}\|W_n-W_{n-1}\|_4 \quad \text{hold for all} \quad n\geqslant 1.$$

Then we can see W_n converges to certain $W\in\Gamma\subset C^4(\Lambda)$ which is the solution of (6.1).

Thus we have obtained the theorem.

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