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Existence and uniqueness of classical solutions for certain degenerated elliptic equations of the second order

By

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§ 0. Introduction

Let Ω be a bounded domain in $\boldsymbol{R}^n$ with smooth boundary. The operators $\mathscr{A}$ which we shall treat in this memoire are of second order, linear, elliptic in the interior of Ω and degenerated only in normal direction at each point of the boundary. Under some assumptions on $\mathscr{A}$, the existence and uniqueness of the classical solution u of the equation $\mathscr{A}u=f$ will be shown for any given function f with certain Hölder continuity up to the boundary. We impose no boundary condition because we assume the "entrance property" of the boundary with respect to $\mathscr{A}$.

There are many authors who have studied various types of degenerated elliptic equations. Baouendi [2] treated the equations degenerated at the boundary, but for which the boundary is non-characteristic. Baouendi-Goulaouic [3] studied the equations degenerated in all directions at the boundary. The main tool in these two works is the elliptic regularization initiated by Oleinik (see Oleinik-Radkevic [7]) and the theory of interpolation in L^2 framework.

Recently, Goulaouic-Shimakura [6] studied the same class of operators as in [3] in the Hölder spaces. And Graham [10] studied the Dirichlet problems for Bergman Laplacian also in some Hölder spaces. Our interest in this memoire is to study the same type of operators as in the Chapter V of Graham's article. But the Hölder spaces with which we work are not the same because of the difference of the boundary conditions. Our method is, as in [6] and [10], to make use of the elementary solution for the simplest model of our operators.

In §1, we consider the model L_α in the half-space, and explain the non-isotropic degeneracy at the boundary. In §2, we describe the general setting of our equations in a bounded domain, and state the main results. In this work, some a priori inequalities of Schauder type for solutions are essential. In §3, we reduce these inequalities to the case of the half-space. And the a priori inequalities in the half-space are finally established in §5. The §4 is devoted to introduce the elementary solutions of L_α and $L_\alpha+\lambda$. The results on the existence and uniqueness of the solution stated in §2 are proved in §6. In §§7 and 8, we establish some detailed estimates which are needed in §5.

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§ 1. Preliminaries

The operators $\mathscr{A}$ treated in this memoire have, roughly speaking, the following properties (see the hypotheses [H-1]~[H-6] in §2):

(a) Elliptic in the interior of the domain;

(b) Degenerated in the normal direction at each point of the boundary surface (supposed to be smooth);

(c) Not degenerated in tangential directions at the boundary.

They are approximated, near the boundary, by the following simple operator L_α in the half-space $\boldsymbol{R}^n_+$:

$$L_\alpha = -x_n\frac{\partial^2}{\partial x_n^2} - \sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j^2} + \alpha\frac{\partial}{\partial x_n},$$

where α is a complex parameter with negative real part.

As C. R. Graham [10] pointed out, $\boldsymbol{R}^n_+$ itself has a group structure G whose multiplication is defined by

$$g\times x = (g' + g_n x', g_n^2 x_n) \qquad \text{for} \quad g = (g', g_n) \in G.$$

It is easy to verify the identity

$$L_\alpha\left(x, \frac{\partial}{\partial x}\right)\{u(g\times x)\} = g_n^2 L_\alpha\left(y, \frac{\partial}{\partial y}\right)u(y)\,|_{y=g\times x}. \tag{1}$$

This identity explains very well the non-isotropic structure of L_α: For any fixed point y' of $\boldsymbol{R}^{n-1}$, L_α is invariant by the translation $x\to(x'+y', x_n)$ (we regard y' as an element $(y', 1)$ of G). And for any fixed positive number k, L_α is of degree 2 with respect to the homothety: $x\to(kx', k^2x_n)$ (we identify k with an element $(0, k)$ of G). Therefore, let us introduce a non-isotropic distance function d on $\bar{\boldsymbol{R}}^n_+$ as follows;

$$d(x, y) = \{(\sqrt{x_n} - \sqrt{y_n})^2 + |x' - y'|^2/4\}^{1/2}. \tag{2}$$

This gose well with the group structure G because $d(g\times x, g\times y) = g_n d(x, y)$ (The distance function adopted by Graham is left invariant one and is equivalent to $d(x, y)/\mathrm{Min}(\sqrt{x_n}, \sqrt{y_n})$ in our notation).

We shall introduce, in §4, an elementary solution $E_\alpha(x, y)$ of L_α. This kernel is homogeneous of degree $1-n$ with respect to left translations:

$$E_\alpha(g\times x, g\times y) = g_n^{1-n}E_\alpha(x, y)$$

(As the volume element in $\boldsymbol{R}^n_+$, we do not use the left invariant Haar measure of G but the ordinary Lebesgue measure $dx = dx_1\wedge, \dots, \wedge dx_n$).

It would be helpful for the readers to compare L_α with the model M_α due to Baouendi-Goulaouic [2]

$$M_\alpha = -x_n\Delta + \alpha \frac{\partial}{\partial x_n}$$

which is degenerated at the boundary in all directions. M_α is associated with another group structure $\tilde{G}$ of $\boldsymbol{R}_+^n$ whose multiplication is $g \circ x = (g' + g_n x', g_n x_n)$. M_α is invariant by the translations: $x \to (x' + y', x_n)$ and is of degree 1 with respect to the ordinary homotheties $x \to kx$ $(k > 0)$. And the distance function to be introduced to $\bar{\boldsymbol{R}}_+^n$ is ordinary Euclidian one $|x-y|$ (or left invariantly $|x-y|/\mathrm{Min}(x_n, y_n)$).

Let us return to our model L_α. To study the regularity near the boundary of the solutions u of the equation $L_\alpha u = f$, the ordinary distance $|x-y|$ is not appropriate but we must work with $d(x, y)$ introduced in (2) above, because of the non-isotropic property of L_α.

Let us introduce two basic function spaces $C_d^\mu(\bar{\boldsymbol{R}}_+^n)$ and $V_d^\mu(\bar{\boldsymbol{R}}_+^n)$ for $0 < \mu < 1$ (then L_α is a continuous map from the second space to the first space):

$$(3) \qquad C_d^\mu(\bar{\boldsymbol{R}}_+^n) = \left\{ u \in C^0(\bar{\boldsymbol{R}}_+^n);\ \sup_{\boldsymbol{R}_+^n} |u| + \sup_{\boldsymbol{R}_+^n \times \boldsymbol{R}_+^n} \frac{|u(x) - u(y)|}{d(x, y)^\mu} < +\infty \right\}$$

$$(4) \qquad V_d^\mu(\bar{\boldsymbol{R}}_+^n) = \left(u \in C^0(\bar{\boldsymbol{R}}_+^n) \;\middle|\; \begin{array}{l} \partial_j u \in C^0(\bar{\boldsymbol{R}}_+^n), \\ \partial_n u,\ \partial_j \partial_k u,\ \sqrt{x_n}\,\partial_j \partial_n u \text{ and } x_n \partial_n^2 u \in C_d^\mu(\bar{\boldsymbol{R}}_+^n) \\ \text{for } j, k = 1, \dots, n-1. \end{array} \right)$$

Where, we denote $\dfrac{\partial}{\partial x_j}$ by ∂_j for $j = 1, \dots, n$.

Let us set,

$$(5) \qquad \|u\|_\infty = \sup_{\boldsymbol{R}_+^n} |u| \quad \text{and} \quad |u|_{\mu,d} = \sup_{\boldsymbol{R}_+^n \times \boldsymbol{R}_+^n} \frac{|u(x) - u(y)|}{d(x, y)^\mu}.$$

With these semi-norms, we can define the related norms,

$$(6) \qquad \|u\|_{\mu,d} = \|u\|_\infty + |u|_{\mu,d} \quad \text{and}$$

$$\|u\|_{2+\mu,d} = \|u\|_\infty + \sum_{j=1}^{n-1} \|\partial_j u\|_\infty + \|\partial_n u\|_{\mu,d} + \sum_{j,k=1}^{n-1} \|\partial_j \partial_k u\|_{\mu,d}$$

$$+ \sum_{j=1}^{n-1} \|\sqrt{x_n}\,\partial_j \partial_n u\|_{\mu,d} + \|x_n \partial_n^2 u\|_{\mu,d}.$$

Clearly, $C_d^\mu(\bar{\boldsymbol{R}}_+^n)$ and $V_d^\mu(\bar{\boldsymbol{R}}_+^n)$ are Banach spaces with the norms $\|\ \|_{\mu,d}$ and $\|\ \|_{2+\mu,d}$ respectively.

Next, we consider the case in which Ω is a bounded domain of $\boldsymbol{R}^n$. Let ϕ be a given non-negative smooth function defined on $\bar{\Omega}$ and equivalent to a distance to the boundary, that is to say,

(7)
$$\Omega=\{x\in \boldsymbol{R}^n;\ \phi(x)>0\},$$
$$\partial\Omega=\{x\in \boldsymbol{R}^n;\ \phi(x)=0\},$$
$$d\phi\neq 0 \quad \text{on} \quad \partial\Omega,$$

where $\bar{\Omega}$ and ϕ are supposed to be smooth.
Similarly to $d(x, y)$ in $\bar{\boldsymbol{R}}^n_+$, we introduce a non-isotropic distance $d_\phi(x, y)$ to $\bar{\Omega}$.

(8)
$$d_\phi(x, y)=\{(\sqrt{\phi(x)}-\sqrt{\phi(y)})^2+|x-y|^2\}^{1/2},$$

for $(x, y)\in\bar{\Omega}\times\bar{\Omega}$.
For $0<\mu<1$, we define $C^\mu(\bar{\Omega};\ d_\phi)$ and $V^\mu(\bar{\Omega};\ d_\phi)$ by the following.

(9)
$$C^\mu(\bar{\Omega};\ d_\phi)=\left\{u\in C^0(\bar{\Omega});\ \sup_{\Omega}|u|+\sup_{\Omega\times\Omega}\frac{|u(x)-u(y)|}{d_\phi(x, y)^\mu}<+\infty\right\},$$

and set, $\|u\|_{\mu,d_\phi}=\|u\|_\infty+|u|_{\mu,d_\phi}$ as in (5) and (6). In order to define $V^\mu(\bar{\Omega};\ d_\phi)$ which is analogous to $V^\mu_d(\bar{\boldsymbol{R}}^n_+)$, we take a system of a finite number of smooth vector fields $\{X_j\}_{j=0}^N$ on $\bar{\Omega}$ as follows,

(10)
$$X_j=\sum_{k=1}^n g_{jk}(x)\partial_k,\quad \text{for } x\in\bar{\Omega} \text{ and } j=0, 1,\dots, N.$$

Here, all $g_{jk}(x)$'s are smooth functions and are chosen so as to satisfy the following conditions:

(A) There is a constant $C>0$ such that, for any $x\in\partial\Omega$, we have

$$X_0\phi(x)\geqq C \quad \text{and} \quad X_j\phi(x)=0, \qquad \text{for} \quad j=1,\dots, N.$$

(B) There is a constant $C>0$ such that, for any $x\in\partial\Omega$ and any tangent vector η at $x\in\partial\Omega$, we have

$$\sum_{j=1}^N\left|\sum_{k=1}^n g_{jk}(x)\eta_k\right|^2\geqq C|\eta|^2.$$

(C) For any compact set $K\subset\Omega$, there is a constant $C=C(K)>0$ such that, for any $x\in K$ and $\xi\in\boldsymbol{R}^n$, we have

$$\sum_{j=0}^N\left|\sum_{k=1}^n g_{jk}(x)\xi_k\right|^2\geqq C|\xi|^2.$$

The existence of $\{X_j\}_{j=0}^N$ as above is obvious for a large number $N>0$, and one can see that X_0 is corresponding to ∂_n, and $\sum_{j=1}^N {}^tX_jX_j$ to $\sum_{j=1}^{n-1}\partial_j^2$, in the case of half-space $\boldsymbol{R}^n_+$

Now we can define $V^\mu(\bar{\Omega};\ d_\phi)$.

(11)
$$V^\mu(\bar{\Omega};\ d_\phi)=\left\{u\in C^0(\bar{\Omega})\ \middle|\ \begin{array}{l} X_ju\in C^0(\bar{\Omega}), \\ X_0u,\ X_jX_ku,\ \sqrt{\phi}X_jX_0u \text{ and } \phi X_0^2u\in C^\mu(\bar{\Omega};\ d_\phi), \\ \text{for } j,\ k=1,\dots, N. \end{array}\right\}$$

And set,

$$\|u\|_{2+\mu,d_\phi}=\|u\|_\infty+\sum_{j=1}^{N}\|X_ju\|_\infty+\|X_0u\|_{\mu,d_\phi}+\|\phi X_0^2u\|_{\mu,d_\phi}$$
$$+\sum_{j,k=1}^{N}\|X_jX_ku\|_{\mu,d_\phi}+\sum_{j=1}^{N}\|\sqrt{\phi}X_jX_0u\|_{\mu,d_\phi}. \tag{12}$$

Remark 1, It is easy to see that the space $V^\mu(\bar{\Omega}; d_\phi)$ does not essentially depend on the choice of $\{X_j\}_{j=0}^N$.

§2. Hypotheses and principal results

Let Ω and ϕ be identical to the ones in (7), and for simplicity, we denote $\partial_j\phi$ and $\partial_j\partial_k\phi$ by ϕ_j and ϕ_{jk} respectively. We consider a class of differential operators on Ω of the form

$$\mathscr{A}=-\sum_{j,k=1}^{n}a_{jk}(x)\partial_j\partial_k+\sum_{j=1}^{n}a_j(x)\partial_j+a_0(x). \tag{13}$$

[H-1] (Regularity of the coefficients and the boundary)

$a_{jk}(x)$ and $a_j(x)$ are in $C^\mu(\bar{\Omega}; d_\phi)$. Moreover we suppose that all $a_{jk}(x)$'s are real valued, $a_{jk}(x)=a_{kj}(x)$ for $j, k=1,\dots, n$, $a_j(x)$'s are complex valued for $j=1,\dots, n$, and $\partial\Omega$ and ϕ are smooth.

Remark 1. In order to establish the theorem 1, it is sufficient to suppose that $\partial\Omega$ and ϕ are of $C^{2+\mu}$-class.

Moreover we suppose that $\mathscr{A}$ satisfies the following hypotheses.

[H-2] (Ellipticity in the interior)

For any compact set $K\subset\Omega$, there exists a constant $C=C(K)>0$ such that, for any $x\in K$ and $\xi\in \boldsymbol{R}^n$, we have

$$\sum_{j,k=1}^{n}a_{jk}(x)\xi_j\xi_k\geqq C(K)|\xi|^2.$$

[H-3] (Non degeneracy in tangential directions)

There exists a constant $C>0$ such that, for any $x\in\partial\Omega$ and any tangent vector η at x to $\partial\Omega$, we have

$$\sum_{j,k=1}^{n}a_{jk}(x)\eta_j\eta_k\geqq C|\eta|^2.$$

Let us set $g(x)=\sum_{j,k=1}^{n}a_{jk}(x)\phi_j\phi_k/\phi$. Then,

[H-4] (Degeneracy in normal direction)

$g(x)$ belongs to $C^\mu(\bar{\Omega}; d_\phi)$, and there exists a constant $0<C\leqq 1$ such that, for any $x\in\partial\Omega$, we have

$$C^{-1}\geqq g(x)>C.$$

The behavior near the boundary of the solution of "$\mathscr{A}u=f$" depends essentially on the values of the following function $\alpha(x)$,

$$(14) \qquad \alpha(x)=\{\sum_{j=1}^{n} a_j(x)\phi_j - \sum_{j,k=1}^{n} a_{jk}(x)\phi_{jk}\}/g(x), \qquad \text{for} \quad x\in\partial\Omega.$$

Remark 2. The definition of $\alpha(x)$ is independent of the choice of the function $\phi(x)$ satisfying (7).

We suppose in this paper that:

[H–5] (Entrance property of the boundary)

$$\alpha(x)<0, \quad \text{on} \quad \partial\Omega.$$

Remark 3. In order to establish the theorem 1, it sufficies to assume $\operatorname{Re}\alpha(x)<0$, on $\partial\Omega$.

Let us set

$$(15) \qquad d_j(x)=\sum_{k=1}^{n} a_{jk}(x)\phi_k, \qquad \text{for} \quad j=1,\dots, n.$$

Finally, we add:

[H–6] (A supplementary assumption)

There exists a constant $\delta>0$ such that, for any $x\in\bar{\Omega}$ and $j=1,\dots, n$, we have

$$d_j(x)=O(\phi^{\delta+1/2}) \quad \text{and} \quad d_j(x)/\phi^{1/2}\in C^{\mu}(\bar{\Omega};\, d_\phi).$$

We note that [H–6] is one of sufficient conditions to reduce $\mathscr{A}$ to the model operator L_α. If we assume that all $a_{jk}(x)$'s are smooth, [H–6] is automatically fulfilled. (Because, if all $a_{jk}(x)$'s are in $C^1(\bar{\Omega})$ and ϕ is in $C^2(\bar{\Omega})$, all $d_j(x)$'s are of $O(\phi)$.)

[H–3] and [H–4] imply that $\mathscr{A}$ has a Fuchsian principal part transversal to $\partial\Omega$ and $\alpha(x)+1$ is a characteristic root of this principal part, the other root being O. Therefore, [H–5] means that these characteristic roots at every point $x\in\partial\Omega$ are smaller than 1. By virtue of L^2-theory for such operators as $\mathscr{A}$, we know that the equation "$\mathscr{A}u=f$" can be treated without any boundary conditions.

Now we can state the principal results. First, the following a priori estimate holds.

Theorem 1. *Suppose that $\mathscr{A}$ satisfies the hypotheses* [H–1], [H–2], [H–3], [H–4], [H–5] *and* [H–6]. *Then, there exists a constant $C>0$ such that, for any $u\in V^{\mu}(\bar{\Omega};\, d_\phi)$, we have*

$$(16) \qquad \|u\|_{2+\mu,d_\phi}\leqq C\{\|\mathscr{A}u\|_{\mu,d_\phi}+\|u\|_{\mu,d_\phi}\}\,.$$

Here, the constant C depends only on Ω and $\mathscr{A}$.

To establish (16), we reduce it by localization, diffeomorphism and perturbation, to the case of the model operator in $\boldsymbol{R}^n_+$ of the form

$$L_\alpha = -x_n\partial_n^2 - \sum_{j=1}^{n-1} \partial_j^2 + \alpha\partial_n, \qquad \text{for} \quad x \in \boldsymbol{R}_+^n, \tag{17}$$

here α is a parameter corresponding to $\alpha(x)$ (see (14)). Then the proof of (16) relies on the study of the explicite kernel of Green's operator of L_α.

Next, let us suppose that all the coefficients are real and $a_0(x)>0$. Then we can show that $\mathscr{A}$ is injective from $V^\mu(\bar{\Omega}; d_\phi)$ into $C^\mu(\bar{\Omega}; d_\phi)$ by the maximum principle. And by virtue of L^2-theory in the case all the coefficients are smooth, we can show that $\mathscr{A}$ is surjective, that is to say,

Theorem 2. *Suppose that $\mathscr{A}$ satisfies the hypotheses* [H–1], [H–2], [H–3], [H–4], [H–5] *and* [H–6]. *Moreover, we suppose that all the coefficients of $\mathscr{A}$ are real and $a_0(x)>0$. Then, the operator $\mathscr{A}$ is an isomorphism from $V^\mu(\bar{\Omega}; d_\phi)$ onto $C^\mu(\bar{\Omega}; d_\phi)$.*

If the coefficients of lower order terms of $\mathscr{A}$ are complex valued, we can not use the maximum principle. But, from L^2-theory, we can prove the analogous result.

Theorem 3. *Suppose that $\mathscr{A}$ satisfies the hypotheses* [H–1], [H–2], [H–3] *and* [H–4]. *In place of* [H–5], *we suppose that $\alpha(x) \leqq -1$ and that $a_{jk}(x)$ $(j, k=1,\dots, n)$ belong to $C^2(\bar{\Omega})$ and $a_j(x)$ $(j=1,\dots, n)$ belong to $C^1(\bar{\Omega})$. Then, there exists a sufficiently large positive number C such that the operator $\mathscr{A}$ is an isomorphism from $V^\mu(\bar{\Omega}; d_\phi)$ onto $C^\mu(\bar{\Omega}; d_\phi)$, if* $\operatorname{Re} a_0(x) \geqq C$ *on $\bar{\Omega}$.*

Remark 4. In this case, [H–6] is automatically fulfilled.

At last, we consider the resolvent of the operator $\mathscr{A}$. Let us set,

$$\Lambda_\psi = \{\lambda \in \boldsymbol{C} - \{0\};\ |\arg \lambda| \leqq \pi - \psi\} \cup \{0\}, \tag{18}$$

where, $\psi \in (0, \pi)$ and $\boldsymbol{C}$ is the complex plane.
Then we have the following theorem which is sharper than the previous ones.

Theorem 4. *We fix a $\psi \in (0, \pi)$. Suppose that $\mathscr{A}$ satisfies the hypotheses* [H–1], [H–2], [H–3], [H–4], [H–5] *and* [H–6]. *Then, there exists a constant $\lambda_0>0$ such that, for any $\lambda \in \Lambda_\psi \cap \{\lambda;\ |\lambda| \geqq \lambda_0\}$, the operator $\mathscr{A}+\lambda$ is an isomorphism from $V^\mu(\bar{\Omega}; d_\phi)$ onto $C^\mu(\bar{\Omega}; d_\phi)$. Moreover, if all the coefficients of $\mathscr{A}$ are real, $a_0(x)$ is non-negative and if λ is positive, then we have*

$$\lambda\|u\|_\infty \leqq \|(\mathscr{A}+\lambda)u\|_\infty, \qquad \textit{for any} \quad u \in V^\mu(\bar{\Omega}; d_\phi). \tag{19}$$

Where, the constant λ_0 depends only on Ω, ψ and $\mathscr{A}$.

Remark 5. Under the hypotheses [H–1]~[H–6], we can show the following a priori estimate:

$$\begin{aligned} &\|\phi X_0^2 u\|_{\mu,d_\phi} + \sum_{j=1}^N \|\sqrt{\phi} X_j X_0 u\|_{\mu,d_\phi} + \sum_{j,k=1}^N \|X_j X_k u\|_{\mu,d_\phi} \\ &\quad + \|X_0 u\|_{\mu,d_\phi} + \sqrt{|\lambda|} \sum_{j=1}^N \|X_j u\|_{\mu,d_\phi} + (|\lambda| - \lambda_0)\|u\|_{\mu,d_\phi} \\ &\leqq C\|(\mathscr{A}+\lambda)u\|_{\mu,d_\phi}, \end{aligned} \tag{20}$$

where C is a positive number depending on Ω, ψ and $\mathscr{A}$ but independent of u. And this inequality (20) will be proved in parallel with the inequality (16).

Remark 6. By virtue of this theorem and the above Remark 5, we can easily show that the operator $-\mathscr{A}$ generates a holomorphic semi-group.

§3. A priori estimate in Ω (proof of the theorem 1)

In this section, we reduce the inequality (16) to the model case. To do this, we begin with a localization.

Let $\{(\Omega_j, \xi_j)\}_{j=0}^{M}$ be a partition of unity of $\bar{\Omega}$ such that:

$$\{\Omega_j\}_{j=0}^{M} \text{ is an open covering of } \bar{\Omega},$$

$$\bar{\Omega}_0 \subset \Omega, \quad \bar{\Omega} \subset \bigcup_{j=0}^{M} \Omega_j, \quad \xi_j \in C_0^\infty(\Omega_j) \quad \text{and} \quad \sum_{j=0}^{M} \xi_j = 1.$$

To establish the inequality (16) for $u \in V^u(\bar{\Omega}; d_\phi)$, it suffices to consider each $\xi_j u$ for $j=1,\dots, M$. For each Ω_j, there exists a diffeomorphism ω_j such as $(\phi \cdot \omega_j^{-1})(y) = y_n$, then Ω_j is mapped by ω_j onto W (a small neighborhood of the origin of $\bar{\boldsymbol{R}}_+^n$ whose coordinates are denoted by $y=(y_1,\dots, y_n)$). By virtue of ω_j for $j=1,\dots, M$, we can reduce the required inequality for each $\xi_j u$ to an analogous one in $\bar{\boldsymbol{R}}_+^n$.

Let us set, for $R>0$,

$$\begin{aligned} (21) \qquad & W_R = \{x=(x', x_n)\ \boldsymbol{R}_+^n;\ |x'|<R,\ 0<x_n<\boldsymbol{R}^2\} \quad \text{and} \\ & \underline{W}_R = W_R \cup \{\partial W_R \cap (x_n=0)\}. \end{aligned}$$

By a suitable ω_j, the operator $\mathscr{A}$ is reduced to the following one defined on $\overline{W}_R$ for sufficiently small R. For simplicity, we adopt the same coordinates $x=(x', x_n)$ in place of $y=(y', y_n)$. Then,

$$\begin{aligned} (22) \qquad \mathscr{B} = & -x_n b_{nn}(x)\partial_n^2 - x_n^{1/2} \sum_{k=1}^{n-1} b_{nk}(x)\partial_k\partial_n - \sum_{j,k=1}^{n-1} b_{jk}(x)\partial_j\partial_k \\ & + \sum_{j=1}^{n} b_j(x)\partial_j + b_0(x), \end{aligned}$$

where all the coefficients of $\mathscr{B}$ are in $C_d^u(\overline{W}_R)$:
Moreover:

$$\begin{aligned} (23) \qquad & b_{nn}(x)>0 \quad ([\text{H}-4]); \quad b_n(x)<0, \quad \text{for } x_n=0 \quad ([\text{H}-5]); \\ & b_{nk}(x) = O(x_n^\delta), \quad \text{for } k=1,\dots, n-1 \quad ([\text{H}-6]) \quad \text{and} \\ & \{b_{jk}(x)\}_{j,k=1}^{n-1} \text{ is a symmetric positive matrix } ([\text{H}-3]). \end{aligned}$$

The next proposition 1 is a local version of the theorem 1 and the latter is a consequence of the former.

Proposition 1. *Suppose that* $\mathscr{B}$ *is of the form* (22) *and satisfies* (23). *Then,* *for a sufficiently small $R>0$, there exists a constant $C=C(R)>0$ such that, for any $u\in V_d^\mu(\overline{W}_{2R})$, we have*

$$\|u\|_{2+\mu,d,W_R}\leqq C\{\|Bu\|_{\mu,d,W_{2R}}+\|u\|_{\mu,d,W_{2R}}\}+q_R(u), \tag{24}$$

where $q_R(u)$ is a semi-norm of $V_d^\mu(\overline{W}_{2R})$ defined by

$$q_R(u)=C\{\sum_{j=1}^{n-1}\|\partial_j u\|_{\mu,d,W_{2R}}+\|x_n\partial_n u\|_{\mu,d,W_{2R}}\}+o(R)\|\partial_n u\|_{\mu,d,W_{2R}}.$$

Let us set

$$\mathscr{B}_0=-x_n b_{nn}(0)\partial_n^2-\sum_{j,k=1}^{n-1}b_{jk}(0)\partial_j\partial_k+b_n(0)\partial_n. \tag{25}$$

From (23), there is a suitable linear transformation of coordinates in $\boldsymbol{R}_+^n$ which reduced $\mathscr{B}_0$ to L_α with $\alpha=b_n(0)/b_{nn}(0)$. Then, the proposition 1 is a consequence of the following proposition 2 which will be established in the section 5 (see the lemma 2). For $W\subset\boldsymbol{R}^n$, we denote by $\mathscr{E}'(W)$ the set of distributions having compact support in W.

Proposition 2. *For any $R>0$, there exists a constant $C=C(R)>0$ such that, for any $u\in V_d^\mu(\overline{W}_R)\cap\mathscr{E}'(\underline{W}_R)$, we have*

$$\|u\|_{2+\mu,d,W_R}\leqq C\{\|L_\alpha u\|_{\mu,d,W_R}+\|u\|_{\mu,d,W_R}\}. \tag{26}$$

Remark 1. Since $\mathscr{B}-\mathscr{B}_0$ can be regarded as perturbation, it is not difficult to see that this proposition assures the proposition 1.

§4. Elementary solutions for L_α and $L_\alpha+\lambda$

Let us set, for $(x, y)\in\boldsymbol{R}_+^n\times\boldsymbol{R}_+^n$ with $x\neq y$,

$$E_\alpha(x, y)=\gamma(n, \alpha)y_n^{-\alpha-1}\int_0^1 A(x, y, \theta)^{2\alpha-n+3}\{\theta(1-\theta)\}^{-\alpha-3/2}d\theta. \tag{27}$$

Where,
$$A(x, y, \theta)=\{d(x, y)^2\theta+\check{d}(x, y)^2(1-\theta)\}^{1/2},$$
$$\check{d}(x, y)=\{(\sqrt{x_n}+\sqrt{y_n})^2+|x'-y'|^2/4\}^{1/2},$$
$$\gamma(n, \alpha)=(4\pi)^{-n/2}2^{-2\alpha-1}\,\Gamma\Big(\frac{n-3-2\alpha}{2}\Big)\Big/\Gamma\Big(\frac{-2\alpha-1}{2}\Big),$$
and
$$\mathrm{Re}\,\alpha<-1/2 \quad (*).$$

Then, $E_\alpha(x, y)$ is one of elementary solutions of L_α in $\boldsymbol{R}_+^n$. Let Λ be the complement of the negative real axis in the complex plane, that is to say,

$$\Lambda=\{\lambda\in\boldsymbol{C}-\{0\};\ |\arg\lambda|\neq\pi\}\cup\{0\}. \tag{28}$$

And let us set, for $(x, y)\in\boldsymbol{R}_+^n\times\boldsymbol{R}_+^n$ with $x\neq y$ and $\lambda\in\Lambda$,

$$(29)\qquad E_{\lambda,\alpha}(x, y)=\delta(n, \alpha)\lambda^{\beta/2}y_n^{-\alpha-1}\int_0^1 K_\beta(2\sqrt{\lambda}A(x, y, \theta))\cdot A(x, y, \theta)^{-\beta}\{\theta(1-\theta)\}^{-\alpha-3/2}d\theta.$$

Where,

$$\beta=\frac{n-3}{2}-\alpha,\ \mathrm{Re}\,\alpha<-1/2\quad (*),$$

$$\delta(n, \alpha)=(4\pi)^{-n/2}2^{-2\alpha-1/2}/\Gamma\left(\frac{-2\alpha-1}{2}\right)$$

and $K_\beta(z)$ is the modified Bessel function of order β defined by

$$(30)\qquad K_\beta(z)=\left\{\Gamma\left(\frac{2\beta+1}{2}\right)(2z)^\beta/\sqrt{\pi}\right\}\int_0^\infty (x_0^2+z^2)^{-\beta-1/2}\cos x_0\, dx_0$$

$$=\left\{\sqrt{\pi}(z/2)^\beta/\Gamma\left(\frac{2\beta+1}{2}\right)\right\}\int_1^\infty \exp(-zx_0)\,(x_0^2-1)^{\beta-1/2}dx_0,$$

for $\mathrm{Re}\,\beta>-1/2$ and $z>0$.
Then, $E_{\lambda,\alpha}(x, y)$ is one of elementary solutions of $L_\alpha+\lambda$ in $\boldsymbol{R}_+^n$, and $E_{\lambda,\alpha}$ tends to E_α as $\lambda\to 0$.

(*) **Remark 1.** The integral of the right hand side of (27) and (29) are not convergent if $\mathrm{Re}\,\alpha\in[-1/2, 0)$, but if $n\geqq 3$, we see that E_α and $E_{\lambda,\alpha}$ can be continued analytically with respect to α in that case (see the Remark (**) of §7).
In particular for $\alpha=-1/2$, $n=3$ and $\lambda\in\Lambda$, we have

$$(31)\qquad E_{\lambda,-1/2}(x, y)=(8\pi)^{-1}y_n^{-1/2}\left\{\frac{\exp\{-2\sqrt{\lambda}d(x, y)\}}{d(x, y)}+\frac{\exp\{-2\sqrt{\lambda}\check{d}(x, y)\}}{\check{d}(x, y).}\right\}.$$

Remark 2. We also note that: For $\tilde{x}=(x_0, x', x_n)\in\boldsymbol{R}_+^{n+1}$ and $\tilde{y}=(y_0, y', y_n)\in\boldsymbol{R}_+^{n+1}$, let $\tilde{E}_\alpha(\tilde{x}, \tilde{y})$ be an elementary solution of $\tilde{L}_\alpha$ defined by

$$(32)\qquad \tilde{L}_\alpha=L_\alpha-\partial_0^2,$$

where ∂_0 is the derivative with respect to an auxiliary variable x_0. Then, by the definitions of $\tilde{E}_\alpha$ and $E_{\lambda,\alpha}$, we can show the relation

$$(33)\qquad E_{\lambda,\alpha}(x, y)=\int_{\Sigma(\sqrt{\lambda})}\exp\{-i\sqrt{\lambda}(x_0-y_0)\}\tilde{E}_\alpha(\tilde{x}, \tilde{y})\, dy_0,$$

where $\Sigma(\sqrt{\lambda})$ is the line $\{y_0=\tau/\sqrt{\lambda};\ -\infty\leqq\tau\leqq+\infty\}$ in the complex plane C for $\lambda\in\Lambda$ (if $\lambda=0$, $\Sigma(\sqrt{\lambda})$ is the entire real axis.). By virtue of (33), one can see that $E_{\lambda,\alpha}$ has the same singularity as E_α in a neighborhood of $x=y$. And by virtue of (29), (30) and the fact $A(x, y, \theta)\geqq d(x, y)$, $E_{\lambda,\alpha}$ and its derivatives with respect to x are exponentially decreasing as $\mathrm{Re}\sqrt{\lambda}d(x, y)$ tends to $+\infty$, for a fixed $y\in\boldsymbol{R}_+^n$ and a fixed $\lambda\in\Lambda-\{0\}$.

After all, we have the following lemma 1 (the proof is omitted).

Lemma 1. *For $\lambda\in\Lambda$ and* $\mathrm{Re}\,\alpha<0$ ($\mathrm{Re}\,\alpha<-1/2$, *if* $n=2$), $E_{\lambda,\alpha}(x, y)$ *is an ele-*

mentary solution of $L_\alpha+\lambda$ *in* $\boldsymbol{R}_+^n$.

Remark 3. The proof of this lemma and the construction of $E_{\lambda,\alpha}$ can be done by the same method in Goulaouic-Shimakura [6] (see also Graham [10]).

§5. A priori estimates in $\boldsymbol{R}_+^n$ (proof of the proposition 2)

The proposition 2 is the direct consequence of the following lemma 2.

Lemma 2. *Suppose that* $\operatorname{Re}\alpha$ *is contained in a compact set* K *of the interval* $(-\infty, 0)$. *Take any* $f\in C_d^\mu(\overline{\boldsymbol{R}}_+^n)$ *having support in* $\underline{W}_R=\{x\in\overline{\boldsymbol{R}}_+^n;\ |x'|<R$ *and* $0\leqq x_n<R^2\}$ *and put* $u=E_\alpha f$. *Then, each of the quantities listed in* (a)~(d) *below is majorated by*

$$C(K)\{\|f\|_\infty+R^\mu|f|_{\mu,d}\},$$

where $C(K)$ *is a positive number depending on* K *but neither on* R *nor on* f:

(a) $R^{-2}\|u\|_\infty$; (b) $R^{-1}\|\partial_j u\|_\infty$ $(1\leqq j\leqq n-1)$;

(c) $\|\partial_n u\|_\infty$ *and* $R^\mu|\partial_n u|_{\mu,d}$;

(d) $\|\partial_j\partial_k u\|_\infty$, $\|\sqrt{x_n}\partial_j\partial_n u\|_\infty$, $\|x_n\partial_n^2 u\|_\infty$, $R^\mu|\partial_j\partial_k u|_{\mu,d}$, $R^\mu|\sqrt{x_n}\partial_j\partial_n u|_{\mu,d}$ *and* $R^\mu|x_n\partial_n^2 u|_{\mu,d}$ $(1\leqq j,\ k\leqq n-1)$.

If $n=2$, the proposition 2 can be reduced to the case where $n\geqq 3$ as follows. Let $\tilde{L}_\alpha$ be an operator with 3 variables x_0, x_1 and x_2 defined by $L_\alpha-\partial_0^2$ (see (32)). Then, $u(x)$ can be regarded as a solution of $\tilde{L}_\alpha u=f\,(\equiv L_\alpha u)$ in $\boldsymbol{R}_+^3$ independent of x_0. Therefore the detailed proof of this lemma will be done only for $n\geqq 3$.

In order to prove the lemma 2, let us prepare the following properties [P–1], [P–2] and [P–3] of $E_{\lambda,\alpha}(x, y)$, which will be verified after the proof of this lemma. We denote any ∂_j $(1\leqq j\leqq n-1)$ by D, various multi-indexes by γ and $\operatorname{Re}\alpha$ by α'. And we fix a n $(n\geqq 3)$ and fix a compact set K of the interval $(-\infty, 0)$, and assume $\operatorname{Re}\alpha\in K$.

[P–1]: For any $\kappa\in(0, 1)$ and $\psi\in(0, \pi)$, there is a constant $C_1=C_1(K, \kappa, \psi, n)>0$ such that, for any $\lambda\in\Lambda_\psi$, $m+|\gamma|\leqq 3$ and $(x, y)\in\boldsymbol{R}_+^n\times\boldsymbol{R}_+^n$ with $x\neq y$, we have

$$|\partial_n^m D^\gamma E_{\lambda,\alpha}(x, y)|\leqq C_1 g_{\kappa,\lambda}(x, y)y_n^{-\alpha'-1}\check{d}(x, y)^{2\alpha'+1-m}d(x, y)^{2-n-m-|\gamma|},$$

Where, $g_{\kappa,\lambda}(x, y)=\exp\{-\kappa\operatorname{Re}\sqrt{\lambda}d(x, y)\}$.
In particular, these estimates are valid for $\lambda=0$.

[P–2]: Let us set $K_n(x, y)=\partial_n E_\alpha(x, y)$.
There is a constant $C_2>0$ and a non-negative function $\xi\in C_0^\infty(\boldsymbol{R}^n)$ whose value is 1 for $|x|\leqq 1$ and 0 for $|x|\geqq 2$ such that, for any $s\in\boldsymbol{R}_+^n$ and $\rho>0$, we have

$$\|K_n\xi_{\rho,s}\|_\infty\leqq C_2,\quad\text{where}\quad \xi_{\rho,s}(x)=\xi\Big(\frac{x'-s'}{\rho},\ \frac{\sqrt{x_n}-\sqrt{s_n}}{\rho}\Big).$$

[P–3]: Let us set $K_n(x, y) = \partial_n E_\alpha(x, y)$.
There is a constant $C_3 > 0$ and a non-negative function $\eta \in C_0^\infty(\bar{\boldsymbol{R}}_+^n)$ whose value is 1 in a neighborhood of the origin such that, for any $R > 0$, we have

$$|K_n \eta_R|_{\mu,d} \leqq C_3/R^\mu, \quad \text{where} \quad \eta_R(x) = \eta\left(\frac{x'}{R}, \frac{\sqrt{x_n}}{R}\right).$$

Proof of the lemma 2. By virtue of [P–1], one can see that, for a fixed $x \in \boldsymbol{R}_+^n$, $E_{\lambda,\alpha}(x, y)$ and $\partial_j E_{\lambda,\alpha}(x, y)$ $(1 \leqq j \leqq n-1)$ are locally summable if $\lambda = 0$ and summable if $\operatorname{Re}\sqrt{\lambda} > 0$ with respect to y on $\bar{\boldsymbol{R}}_+^n$. Therefore the estimates of the quantities in the group (a) and (b) are not difficult. It suffices to consider (c) and (d).
Group (d): By virtue only of [P–1], we can obtain the required inequality for each of the quantities of (d). Since the proof can be done by the same procedure as the estimation of derivatives of the Newtonian potential in bounded domains, the proof is omitted (cf. [5]).
Group (c): Fix a point $s \in \boldsymbol{R}_+^n$. If $d(0, s) \geqq 2R$, we have $|K_n f(s)| \leqq CC_1 \|f\|_\infty$. If $d(0, s) < 2R$, by subtraction, we have

$$f = (f - f(s))\xi_{3R,s} + f(s)\xi_{3R,s} \equiv f_1 + f_2.$$

Then we obtain

$$(34) \qquad |K_n f_1(s)| \leqq CC_1 R^\mu |f|_{\mu,d} \quad \text{and} \quad |K_n f_2(s)| \leqq C_2 \|f\|_\infty.$$

Next, we take another point $t \in \boldsymbol{R}_+^n$ with $s \neq t$ and write $\rho = 2d(s, t)$. Suppose that $R' > R + 2\rho$. Again, we write $f = f_1 + f_2 + f_3$, where

$$f_1 = (f - f(s))\xi_{\rho,s}, \quad f_2 = (f - f(s))(\eta_{R'} - \xi_{\rho,s})$$

and

$$f_3(x) = f(s)\eta_{R'}(x).$$

First, [P–3] implies

$$|K_n f_3(s) - K_n f_3(t)| \leqq C_3(\rho/R')^\mu \|f\|_\infty \leqq CC_3(\rho/R)^\mu \|f\|_\infty.$$

Secondly, we have

$$|K_n f_2(s)| \leqq CC_1 \rho^\mu |f|_{\mu,d} \quad \text{and} \quad |K_n f_1(t)| \leqq C(C_1 + C_2)\rho^\mu |f|_{\mu,d}.$$

Finally, we also obtain

$$|K_n f_2(s) - K_n f_2(t)| \leqq CC_1 \rho^\mu |f|_{\mu,d}.$$

Collecting these inequalities, we have

$$(35) \qquad |K_n f(s) - K_n f(t)| \leqq C\{C_3(\rho/R)^\mu \|f\|_\infty + (C_1 + C_2)\rho^\mu |f|_{\mu,d}\}.$$

By (34) and (35), (c) is established. Q.E.D.

Now we verify [P–2] and [P–3] by using the facts which will be proved in Appendix ([P–1] will also be proved there).

Proof of [P–3]. Set $\eta(x)=\tilde{\eta}(x_n)$, where $0\leqq\tilde{\eta}\leqq 1$, $\tilde{\eta}\in C_0^\infty(\bar{\boldsymbol{R}}_+)$ and $\tilde{\eta}=1$ for $0\leqq x_n\leqq 1$, $\tilde{\eta}=0$ for $x_n\geqq 2$. Then we have

$$(36)\qquad K_n\eta_R(x)=-\int_0^\infty \frac{\tau^{-\alpha-1}}{(1+\sqrt{\tau})^{-2\alpha-1}}M_\alpha\Big(\frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\Big)\tilde{\eta}\Big(\frac{\sqrt{x_n\tau}}{R}\Big)d\tau \quad \text{and}$$

$$(37)\qquad \int_0^\infty \frac{\tau^{-\alpha'-1+\mu/2}}{(1+\sqrt{\tau})^{-2\alpha'-1}}\Big|M_\alpha\Big(\frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\Big)\Big|d\tau<+\infty,$$

by virtue of the lemma A.3.
Hence, we obtain $|K_n\eta_R|_{\mu,d}\leqq C|\eta|_\mu/R^\mu=C_3/R^\mu$.

Proof of [P–2]. It follows immediately from the homogeneity of $K_n(x, y)$ that $K_n\xi_{\rho,s}(x)$ depends only on the parameter $s_n\geqq 0$. Therefore, we put $\rho=1$ and we take for ξ the following function,

$$\xi(x;\, s_n)=a(x')b(\sqrt{x_n}-\sqrt{s_n})\,.$$

Where, $a(x')\in C^\infty(\boldsymbol{R}^{n-1})$, $b(x_n)\in C^\infty(\boldsymbol{R})$ and having supports in $|x'|<2$ and $|x_n|<2$, their values are 1 for $|x'|\leqq 1$ and $|x_n|\leqq 1$, $0\leqq a\leqq 1$ and $0\leqq b\leqq 1$ in $\boldsymbol{R}^{n-1}$ and $\boldsymbol{R}$ respectively.
Let us set

$$V_n(x,\, s_n)=\{K_n\xi(\cdot\,,\, s_n)\}(x)\,.$$

By the lemma A.4, we have

$$V_n(x,\, s_n)=\int_0^\infty b(\sqrt{y_n}-\sqrt{s_n})U_n(x,\, y_n)dy_n\equiv V_n^1+V_n^2.$$

Where,

$$V_n^1(x,\, s_n)=\int_0^\infty a(x')b(\sqrt{\tau x_n}-\sqrt{s_n})\frac{\tau^{-\alpha-1}}{(1+\sqrt{\tau})^{-2\alpha-1}}M_\alpha\Big(\frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\Big)d\tau.$$

Then, $\|V_n^1\|_\infty\leqq C$, since the lemma A.3 in §8 below holds. And we also have

$$|V_n^2|\leqq C\int_0^\infty b(\sqrt{y_n}-\sqrt{s_n})\frac{y_n^{-1+\sigma}}{|\sqrt{x_n}-\sqrt{y_n}|^\sigma}dy_n\leqq C',$$

since $b(\tau-\sqrt{s_n})$ has its support in the interval $[(\sqrt{s_n}-2)_+,\, 2+\sqrt{s_n}]$. Q.E.D.

At the end of this section, we show the following lemma which provides the basic estimate to demonstrate the theorem 4.

Lemma 3. *We fix a* $\psi\in(0, \pi)$. *Suppose that* $\lambda\in\Lambda_\psi$ *and* $\operatorname{Re}\alpha$ *is contained in a compact set K of the interval* $(-\infty, 0)$. *Then, for any* $u\in V_d^\mu(\bar{\boldsymbol{R}}_+^n)$ *having support in* $\underline{W}_1$, *each of the quantities listed below is majorated by*

$$C(K,\, \psi)\|(L_\alpha+\lambda)\mu\|_{\mu,d},$$

where $C(K, \psi)$ *is a positive number depending on K and* ψ *but independent of u.*

(a) $\|x_n\partial_n^2 u\|_{\mu,d}$, $\|\sqrt{x_n}\partial_j\partial_n u\|_{\mu,d}$, $\|\partial_j\partial_k u\|_{\mu,d}$ *and* $\|\partial_n u\|_{\mu,d}$ $(1\leqq j,\ k\leqq n-1)$;

(b) $\sqrt{|\lambda|}\|\partial_j u\|_{\mu,d}$ $(1\leqq j\leqq n-1)$ *and* $|\lambda|\|u\|_{\mu,d}$.

Proof. For $v\in V_d^\mu(\bar{\boldsymbol{R}}_+^n)\cap\mathscr{E}'(\underline{W}_R)$, $\operatorname{Re}\alpha\in K$ and $\gamma\in\Lambda_\psi\cap\{|\gamma|=1\}$, if we put $g(x)=(L_\alpha+\gamma)v(x)$, we obtain $v(x)=E_{\gamma,\alpha}g(x)$. Then, it follows from the Remark 2 of §4 that we can apply the lemma 2 to g and v. (By virtue of (33), it is not difficult to see that [P–2] and [P–3] remain valid for this case, and we also note that $E_{\gamma,\alpha}(x,\ y)$ and $\partial_j E_{\gamma,\alpha}(x,\ y)$ $(1\leqq j\leqq n-1)$ are summable with respect to y on $\bar{\boldsymbol{R}}_+^n$ for a fixed $x\in\bar{\boldsymbol{R}}_+^n$.). Hence, in order to obtain the required estimations, we have only to put $R=\sqrt{|\lambda|}$, $\gamma=\lambda/|\lambda|$ and carry out a change of variables defined by

$$(x',\ x_n)\longrightarrow(\sqrt{|\lambda|}y',\ |\lambda|y_n). \qquad \text{Q.E.D.}$$

§6. Proofs of the theorem 2, 3 and 4

Proof of the theorem 2. First, we show that the operator $\mathscr{A}$ is injective from $V^\mu(\bar{\Omega};\ d_\phi)$ into $C^\mu(\bar{\Omega};\ d_\phi)$. Suppose $u\in V^\mu(\bar{\Omega};\ d_\phi)$ and $\mathscr{A}u=0$. We assume, contrary to the theorem, that u is not identically zero. Then, by the maximum principle, u attains the maximum only on $\partial\Omega$. Therefore, we may assume that u attains the maximum at a point $x^0\in\partial\Omega$ and $u(x^0)>0$ (if not, we consider $-u$). Without loss of generality, we suppose that $x^0=0$, $\{x_n=0\}$ is the tangent space at x^0 to $\partial\Omega$ and that $x_n>0$ is locally the side of Ω. Then we have $\mathscr{A}u(0)=0$. And evidently we have

$$-\sum_{j,k=1}^{n-1} a_{jk}(0)\partial_j\partial_k u(0)\geqq 0,\quad \partial_n u(0)<0 \quad\text{and}\quad \partial_j u(0)=0 \quad (1\leqq j\leqq n-1).$$

Since $\partial_j u(0)$ $(1\leqq j\leqq n)$ are bounded, we can show

$$a_{jn}(0)\partial_j\partial_n u(0)=0 \qquad (1\leqq j\leqq n).$$

After all, we have

$$-\sum_{j,k=1}^{n-1} a_{jk}(0)\partial_j\partial_k u(0)+a_n(0)\partial_n u(0)+a_0(0)u(0)=0$$

But, this contradicts to the fact $\alpha(0)=a_n(0)<0$.

In order to prove the surjectivity of the operator $\mathscr{A}$, we prepare some propositions. First, from the inequality of the theorem 1 and the above uniqueness result, we can derive, by a argument of functional analysis, the following a priori estimate which is very strong (the proof is omitted):

Prosposition 3. *Suppose that $\mathscr{A}$ satisfies the same hypotheses as in theorem* 2. *Then, there is a constant $C>0$ such that, for any $u\in V^\mu(\bar{\Omega};\ d_\phi)$, we have*

$$\|u\|_{2+\mu,d_\phi}\leqq C\|\mathscr{A}u\|_{\mu,d_\phi}, \tag{38}$$

where the constant C depends only on $\mathscr{A}$ and Ω.

Let us set

(39) $$\mathscr{A}^0 = -\operatorname{div}\phi\,\mathrm{grad} + \sum_{j=1}^{N} {}^tX_jX_j + 1,$$

where X_j is defined by (10) and ${}^tX_j$ is the formal transpose of X_j. It follows from L^2-theory (cf. [3]) that $\mathscr{A}^0$ is an isomorphism from $C^\infty(\bar{\Omega})$ onto $C^\infty(\bar{\Omega})$. Similarly we obtain:

Proposition 4. *$\mathscr{A}^0$ is an isomorphism from $V^\mu(\bar{\Omega}; d_\phi)$ onto $C^\mu(\bar{\Omega}; d_\phi)$.*

Proof. We use the following facts which are well-known in the Hölder theory (cf. [5]):

(a) For $f \in C^\mu(\bar{\Omega}; d_\phi)$, there is a sequence $\{f^m\}_{m\in N}$ such as $f^m \in C^\infty(\bar{\Omega})$, $\|f^m\|_{\mu,d_\phi} \leqq 2\|f\|_{\mu,d_\phi}$ and, for any $\nu \in [0, \mu)$, $\|f^m - f\|_{\nu,d_\phi} \to 0$ (as $m \to \infty$).

(b) If a sequence $\{u^m\}_{m\in N}$ is bounded in $C^\mu(\bar{\Omega}; d_\phi)$ and is a Cauchy sequence in $C^0(\bar{\Omega})$, then it is a Cauchy sequence in $C^\nu(\bar{\Omega}; d_\phi)$ for any $\nu \in [0, \mu)$ and its limit is contained in $C^\mu(\bar{\Omega}; d_\phi)$.

It suffices to prove that the operator $\mathscr{A}^0$ is surjective. Suppose $f \in C^\mu(\bar{\Omega}; d_\phi)$, then we can take a sequence $\{f^m\}_{m\in N}$ which satisfies (a). For each $m \in N$, there is a $u^m \in C^\infty(\bar{\Omega})$ such as $\mathscr{A}^0 u^m = f^m$, and from (38), $\|u^m\|_{2+\mu,d_\phi}$ remains bounded and $\{u^m\}_{m\in N}$ is a Cauchy sequence in $V^\nu(\bar{\Omega}, d_\phi)$ for any $\nu \in [0, \mu)$. From (b), the limit u of $\{u^m\}_{m\in N}$ in $V^\nu(\bar{\Omega}; d_\phi)$ is an element of $V^\mu(\bar{\Omega}; d_\phi)$ and by taking the limits of the both sides of $\mathscr{A}^0 u^m = f^m$ in $C^\nu(\bar{\Omega}; d_\phi)$, we have $\mathscr{A}^0 u = f$. Thus, the proposition 4 is proved.

Q. E. D.

End of the proof of the theorem 2. Now we complete the proof of the theorem 2 by virtue of the proposition 3 and the proposition 4. We consider the family $\{\mathscr{A}_t\}_{0\leqq t\leqq 1}$ of operators defined by

(40) $$\mathscr{A}_t = (1-t)\mathscr{A}^0 + t\mathscr{A}.$$

Then for each $t \in [0, 1]$, $\mathscr{A}_t$ is linear and continuous from $V^\mu(\bar{\Omega}, d_\phi)$ into $C^\mu(\bar{\Omega}; d_\phi)$ and satisfies (38) with $\mathscr{A} = \mathscr{A}_t$ with some C independent of t. Moreover, $\mathscr{A}_0 = \mathscr{A}^0$ is an isomorphism from $V^\mu(\bar{\Omega}; d_\phi)$ onto $C^\mu(\bar{\Omega}; d_\phi)$. Therefore, by the method of continuity (cf. [1]), each of $\mathscr{A}_t$ is an isomorphism from $V^\mu(\bar{\Omega}; d_\phi)$ onto $C^\mu(\bar{\Omega}; d_\phi)$, in particular, so is $\mathscr{A} = \mathscr{A}_1$. The theorem 2 is completely proved. Q. E. D.

Proof of the theorem 3. If we can prove that the operator $\mathscr{A}$ is injective from $V^\mu(\bar{\Omega}; d_\phi)$ into $C^\mu(\bar{\Omega}; d_\phi)$, then it follows from the method of continuity (see (40) above) that the operator $\mathscr{A}$ is an isomorphism from $V^\mu(\bar{\Omega}; d_\phi)$ onto $C^\mu(\bar{\Omega}; d_\phi)$. Since $a_j(x)$'s are complex valued, we can not use the maximum principle to prove the uniqueness. But, we can replace this argument by L^2-theory, that is to say,

Lemma 4. *Under the same hypotheses as in the theorem* 3, *the operator $\mathscr{A}$ is injective from $V^\mu(\bar{\Omega}; d_\phi)$ into $C^\mu(\bar{\Omega}; d_\phi)$.*

Proof of the lemma 4. Let us set $f = \mathscr{A}u$ for $u \in V^\mu(\bar{\Omega}; d_\phi)$. Multiplying $\bar{u}$ to the both sides of $f = \mathscr{A}u$ and integrating by parts for the second order terms, we have

(a) $$(f, u)-(a_0u, u)-\sum_{j,k=1}^{n}(a_{jk}\partial_j u, \partial_k u)=\sum_{j=1}^{n}(b_j\partial_j u, u),$$

where (,) stands for the scalar product in $L^2(\Omega)$ and

$$b_j(x)=a_j(x)+\sum_{k=1}^{n}\partial_k a_{jk}(x), \qquad 1\leqq j\leqq n.$$

Let us set $\nabla\phi\equiv\nabla\phi(x)=(\phi_1,\dots,\phi_n)$ for $x\in\partial\Omega$, then $\nabla\phi/|\nabla\phi|$ is interior normal to $\partial\Omega$. Again, integrating by parts for $\mathrm{Re}(b_j\partial_j u, u)$, we have

$$2\sum_{j=1}^{n}\mathrm{Re}(b_j\partial_j u, u)=2\sum_{j=1}^{n}\mathrm{Re}\{i((\mathrm{Im}\, b_j)\partial_j u, u)\}-\sum_{j=1}^{n}\mathrm{Re}((\partial_j b_j)u, u)$$
$$-\int_{\partial\Omega}\beta|u|^2 dS,$$

where $\beta=\sum_{j=1}^{n}b_j\phi_j/|\nabla\phi|$.
And we have

$$|\nabla\phi|\beta=\sum_{j=1}^{n}a_j\phi_j-\sum_{j,k=1}^{n}a_{jk}\phi_{jk}+\sum_{j,k=1}^{n}\partial_k(a_{jk}\phi_j)$$
$$=(\alpha(x)+1)g(x) \qquad \left(\text{since } \sum_{j,k=1}^{n}a_{jk}\phi_j\ (1\leqq j\leqq n) \text{ are of } O(\phi)\right),$$

where $\alpha(x)g(x)$ is defined by (14).
Since $\alpha\leqq -1$ by hypothesis, we have

(b) $$-\int_{\partial\Omega}\beta|u|^2 dS\geqq 0.$$

Moreover, for any $\varepsilon\in(0, 1)$, there is a constant $K=K(\varepsilon)>0$ such that

(c) $$\left|\sum_{j=1}^{n}((\mathrm{Im}\, b_j)\partial_j u, u)\right|\leqq\varepsilon\sum_{j,k=1}^{n}(a_{jk}\partial_j u, \partial_k u)+K|||u|||^2$$

holds for any $u\in V^\mu(\bar\Omega; d_\phi)$, where $|||\ \ |||$ denotes the norm in $L^2(\Omega)$. Admitting this for a moment, we have $\mathrm{Re}(f-a_0u, u)\geqq -K'(u, u)$, that is to say, $|||\mathscr{A}u|||\geqq K''|||u|||$, where $K''=\mathrm{Inf}\, a_0(x)-K'$ and $K'=K+\mathrm{Sup}(\sum_{j=1}^{n}|\partial_j b_j(x)|)/2$.
Therefore, if $\mathrm{Re}\, a_0(x)$ is large enough so that $K''\geqq 0$, $u\in V^\mu(\bar\Omega; d_\phi)$ and $\mathscr{A}u=0$ imply $u=0$.

Proof of the inequality (c). (c) is a consequence of the following inequality. There is a positive constant C such that we have

$$\left|\sum_{j=1}^{n}\zeta_j\,\mathrm{Im}\, b_j(x)\right|\leqq C\left(\sum_{j,k=1}^{n}a_{jk}(x)\zeta_j\bar\zeta_k\right)^{1/2},$$

for any $\zeta\in C^n$ and $x\in\bar\Omega$.
And, since $\mathrm{Im}\, b_j(x)=O(\phi)$ $(1\leqq j\leqq n)$, this inequality is obvious. Q. E. D.

And the theorem 3 is now established.

Proof of the theorem 4. Evidently, by almost the same procedure as in the proof of the theorem 1, we can derive the inequality (20) from the lemma 3 in §5 (at first for $\Omega=\boldsymbol{R}^n_+$ and $\mathscr{A}=L_\alpha$, and secondly for general Ω and $\mathscr{A}$). Here, $\sqrt{\lambda}$ is regarded as the dual variable of x_0 (see (32) and (33)). This inequality (20) assures the injectivity of the operator $\mathscr{A}+\lambda$ for $\lambda\in\Lambda_\psi\cap\{|\lambda|>\lambda_0\}$. And the surjectivity is also easy to see by the method of continuity and the compactness of the operator $(\mathscr{A}+\lambda)^{-1}$ (when it exists). At last, we prove the inequality (19). Since all the coefficients are real, we can apply the maximum principle. Take any $f\in C^\mu(\bar{\Omega};\, d_\phi)$, and suppose that u is the unique solution in $V^\mu(\bar{\Omega};\, d_\phi)$ of the equation "$(\mathscr{A}+\lambda)u=f$". Let us set $u_+=\max(u, 0)$ and $u_-=\max(-u, 0)$. Then, by the maximum principle, we have

$$\lambda u_+\leqq\|f\|_\infty \quad\text{and}\quad \lambda u_-\leqq\|f\|_\infty, \quad (\text{since } a_0(x)\geqq 0).$$

Since $\|u\|_\infty=\sup_\Omega \max(u_+, u_-)$, the inequality (19) is proved. Q. E. D.

§7. Appendix 1 (The proof of [P–1] in §5)

It suffices to consider the case where $\lambda=0$ (see the Remark 2 of §4). We introduce a new representation formula of $E_\alpha(x, y)$.

$$E_\alpha(x, y)=\gamma(n, \alpha)y_n^{-\alpha-1}\int_0^1 B(x, y, \theta)^{2\alpha-n+3}\Psi_\alpha(\theta)d\theta. \tag{A.1}$$

Where, $B(x, y, \theta)=\{(x_n+y_n+|x'-y'|^2/4)^2-4x_ny_n(1-\theta)\}^{1/4}$,

$$\psi_\alpha(\theta)=\theta^{-(n+3+2\alpha)/4}(1-\theta)^{(n-5-2\alpha)/4},$$

$$\gamma(n, \alpha)=(2\pi)^{-n/2}2^{-\alpha-3/2}\Gamma\left(\frac{n-3-2\alpha}{4}\right)\Big/\Gamma\left(\frac{1-n-2\alpha}{4}\right)$$

and $\quad \operatorname{Re}\alpha<-\dfrac{n-1}{2} \quad (**)$.

Proof. We can expand the right-hand side of (27) into power series of $Q=2\sqrt{x_ny_n}(x_n+y_n+|x'-y'|^2/4)^{-1}$ and rewrite it to obtain (A.1).

Remark (**). By the analytic continuation with respect to α, this formula remains valid for $\operatorname{Re}\alpha\in(-\infty, 0)$, if $n\geqq 3$. The analytic continuation method used in this paper is essentially as follows: Let $f(\theta)$ be a function of class $C^1([0, 1])$ and τ be a complex parameter with $\operatorname{Re}\tau>0$. Then we have the equality:

$$\Gamma(\tau)^{-1}\int_0^1 f(\theta)\theta^{\tau-1}d\theta=\Gamma(\tau)^{-1}\int_0^1\{f(\theta)-f(0)\}\theta^{\tau-1}d\theta+\Gamma(\tau+1)^{-1}f(0).$$

The right-hand side can be continued analytically to $\operatorname{Re}\tau>-1$. Thus we can extend the definition of the left-hand side to $\operatorname{Re}\tau>-1$ by this equality.

We also introduce the following auxiliary function:

$$T(a, b, X, Y)=\int_0^1 (X+Y\theta)^a\theta^{b-1}d\theta, \tag{A.2}$$

where $X>0$, $Y\geqq 0$, $a+b<0$ and $b>0$.
Then we have

Lemma A.1. *Suppose that $b>0$ and $a+b<0$. Then, there is a constant $C=C(a, b)>0$ such that, for any $X>0$ and $Y\geqq 0$, we have*

$$\text{(A.3)} \qquad T(a, b, X, Y)\leqq CX^{a+b}(X+Y)^{-b}.$$

Proof. It suffices to consider the case where $X=1$. Both of the following inequalities are valid:

$$T(a, b, 1, Y)\leqq 1/b \quad \text{and} \quad Y^{b}T(a, b, 1, Y)\leqq \Gamma(b)\Gamma(-a-b)/\Gamma(-a).$$

Q. E. D.

Let us set

$$\text{(A.4)} \qquad J(p, q)\equiv J(p, q, x, y)=T\left(-\frac{n+p+2q}{2}, q, d(x, y)^2, 2\sqrt{x_n y_n}\right),$$

where $n\geqq 3$, $p\geqq -2$, $q>0$ and $(x, y)\in R^n_+\times R^n_+$ with $x\neq y$.

Since $B(x, y, \theta)^4\leqq \check{d}(x, y)^2\{d(x, y)^2+2\sqrt{x_n y_n \theta}\}$ holds, we can estimate $E_\alpha(x, y)$ and its derivatives by $J(p, q)$'s. After a long calculus, we obtain:

Lemma A.2. *Suppose that $n\geqq 3$ and $\mathrm{Re}\,\alpha$ is contained in a compact set K of the interval $\left(-\infty, -\frac{n-1}{2}\right]$, and put $\beta=-(2\alpha+n-1)/4$. Then, there is a constant $C=C(K)>0$ such that, for any $(x, y)\in \boldsymbol{R}^n_+\times \boldsymbol{R}^n_+$ with $x\neq y$ and $|\gamma|+m\leqq 3$, we have*

$$\text{(A.5)} \qquad |\partial_n^m D^\gamma E_\alpha(x, y)|\leqq Cy_n^{-\alpha'-1}\sum_{j=1}^{m}\check{d}(x, y)^{2-n-2\beta'-m+2j}J(|\gamma|+m-2, \beta'+j),$$

where we denote any ∂_j $(1\leqq j\leqq n-1)$ by D and $\mathrm{Re}\,\beta$ by β'.

Evidently, [P–1] follows from (A.3), (A.4) and (A.5), if $\mathrm{Re}\,\alpha\in K$. We can also prove [P–1] in the case where $\mathrm{Re}\,\alpha$ is contained in a compact set $\tilde{K}$ of the interval $(-\infty, 0)$ by the method of analytic continuation (see the Remark (**)), the detailed proof is omitted.

§8. Appendix 2 (Further preliminaries to §5)

In this section, we shall obtain a number of inequalities needed in the verifications of the properties [P–2] and [P–3] of the kernel $K_n(x, y)=\partial_n E_\alpha(x, y)$ in §5.

We use the following notations in this section. Let us set

$$\text{(A.6)} \qquad h(x_n, y_n)=(\sqrt{x_n}-\sqrt{y_n})/(\sqrt{x_n}+\sqrt{y_n}),$$

$$f_n(x_n, y_n, \theta)=-(1-h)^{-2\alpha-2}\Phi_\alpha(h, \theta)/(\sqrt{x_n}+\sqrt{y_n})\sqrt{x_n},$$

$$\Phi_\alpha(h, \theta)=(h\theta+1-\theta)(h^2\theta+1-\theta)^\alpha\Gamma(-\alpha)/\sqrt{\pi}$$

and

$$k_n(x_n, y_n)=\int_{R^{n-1}} K_n(x, y)dy'.$$

Then we have:

Lemma A.3. *Suppose* $n \geqq 3$. *For any compact set* $K \subset (-\infty, 0)$, *there is a function* $M_\alpha(h)$ *with a parameter* α *such that, for any* $x_n > 0$, $y_n > 0$, $x_n \neq y_n$ *and* $\operatorname{Re}\alpha \in K$, *we have*

$$k_n(x_n, y_n) = -\frac{(1-h)^{-2\alpha-2}}{\sqrt{x_n}(\sqrt{x_n}+\sqrt{y_n})} M_\alpha(h), \tag{A.7}$$

with $h = h(x_n, y_n)$ *defined by* (A.6).
Moreover, $M_\alpha(h)$ *satisfies the following properties:*

(i) $M_\alpha(h)$ *is locally summable with respect to* $y_n \in \boldsymbol{R}_+$ *for a fixed* $x_n \in \boldsymbol{R}_+$.

(ii) *If* $\operatorname{Re}\alpha \leqq -1/2$, $M_\alpha(h) = 2(1+h)^{2\alpha+1} Y(h)$,
where $Y(h) = 1$ *for* $h > 0$ *and* $Y(h) = 0$ *for* $h < 0$.

(iii) *If* $-1/2 < \operatorname{Re}\alpha < 0$, $M_\alpha(h) = O\{(1+h)^2\}$, *in a neighborhood of* $h = -1$.

Proof. We obtain at first

$$k_n(x_n, y_n) = \Gamma(-\alpha-1/2)^{-1} \int_0^1 f_n(x_n, y_n, \theta)\{\theta(1-\theta)\}^{-\alpha-3/2} d\theta \tag{A.8}$$

and
$$M_\alpha(h) = \Gamma(-\alpha-1/2)^{-1} \int_0^1 \Phi_\alpha(h, \theta)\{\theta(1-\theta)\}^{-\alpha-3/2} d\theta.$$

Then, the assertions of (i) and (ii) follow by a direct calculation, and to prove (iii), it suffices to remark $M_\alpha(-1) = (d/dh)M_\alpha(-1) = 0$ (see also the Remark (**) of §7).
Q. E. D.

To establish [P–2], we introduce the following auxiliary function $U_n(x, y_n)$ defined for $x \in \boldsymbol{R}^n_+$ and $0 < y_n \neq x_n$:

$$U_n(x, y_n) = \int_{\boldsymbol{R}^{n-1}} K_n(x, y) a(y') dy', \tag{A.9}$$

where $a(x')$ is a fixed element of $C_0^\infty(\boldsymbol{R}^{n-1})$.
Suppose that F is a compact set of $\boldsymbol{R}^2$ contained in

$$\{(\alpha', \sigma);\ \alpha' < 0 \text{ and } 0 < \sigma < \operatorname{Min}(1, -\alpha')\}, \qquad \text{for} \quad \alpha' = \operatorname{Re}\alpha.$$

Lemma A.4. *Suppose* $n \geqq 3$. *Then, we have, for* $(\alpha', \sigma) \in F$,

$$\left| U_n(x, y_n) + a(x') \frac{(1-h)^{-2\alpha-2}}{\sqrt{x_n}(\sqrt{x_n}+\sqrt{y_n})} M_\alpha(h) \right| \leqq C \frac{y_n^{\sigma-1}}{|\sqrt{x_n}-\sqrt{y_n}|^\sigma},$$

where C *is a positive constant depending only on* (a, F, n).

Proof. By virtue of the lemma A.3, it suffices to show the following inequality:

$$\int_{\boldsymbol{R}^{n-1}} |K_n(x, y)|\, |a(x') - a(y')| dy' \leqq C \frac{y_n^{\sigma-1}}{|\sqrt{x_n}-\sqrt{y_n}|^\sigma}. \tag{A.10}$$

From [P–1], we can show

$$|K_n(x, y)|\,|a(x')-a(y')| \leqq C_1 |a|_\sigma y_n^{\sigma-1} d(x, y)^{1-n-\sigma}.$$

Therefore, we obtain (A.10) by integration on R^{n-1} (we denoted an usual Hölder semi-norm by $|\ |_\sigma$ for $0<\sigma<1$). Q. E. D.

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