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On a Frobenius reciprocity theorem for locally compact groups

Dedicated to Professor Hisaaki YOSHIZAWA on his 60th birthday

By

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§0. Introduction

Let G be a locally compact unimodular group, and S a closed subgroup of G. Suppose that there exists a compact subgroup K of G with $G=SK$ ($S \cap K$ is not necessarily trivial). This paper is devoted to give a generalization of the Frobenius reciprocity theorem for these G and S.

In particular, let G be a finite group, and S a subgroup of G. Let $\{\mathfrak{H}, T(x)\}$, $\{H, \Lambda(s)\}$ be representations of G, S on finite-dimensional vector spaces $\mathfrak{H}$, H respectively. We shall denote by $\{\mathfrak{H}^{\Lambda}, T^{\Lambda}(x)\}$ the representation of G induced from $\{H, \Lambda(s)\}$, and by $\{\mathfrak{H}, T_S(s)\}$ the restriction of $\{\mathfrak{H}, T(x)\}$ to S. Then the Frobenius reciprocity theorem can be stated in the following three forms which are mutually equivalent.

(1) If $\{\mathfrak{H}, T(x)\}$ and $\{H, \Lambda(s)\}$ are irreducible, then $\{\mathfrak{H}^{\Lambda}, T^{\Lambda}(x)\}$ contains $\{\mathfrak{H}, T(x)\}$ exactly as many times as $\{\mathfrak{H}, T_S(s)\}$ contains $\{H, \Lambda(s)\}$.

(2) $\mathrm{Hom}_S(H, \mathfrak{H}) \cong \mathrm{Hom}_G(\mathfrak{H}^{\Lambda}, \mathfrak{H})$ (linearly isomorphic).

(3) $\mathrm{Hom}_S(\mathfrak{H}, H) \cong \mathrm{Hom}_G(\mathfrak{H}, \mathfrak{H}^{\Lambda})$.

Various generalizations of this theorem were given by many people. Using the direct integral decomposition theory, both F. I. Mautner and G. W. Mackey generalized the theorem as stated in form (1) above. In Mautner's case, G is assumed to be a separable locally compact unimodular group and S a compact subgroup [7]. In Mackey's case, G is a separable locally compact group and S a closed subgroup of G [6]. But, in his case, the Frobenius reciprocity theorem is formulated only for representations which appear in the direct integral decompositions of regular representations. R. Penney also formulated in [9] a generalization of the Frobenius reciprocity theorem in form (1). He dealt with Lie groups and made use of the C^{∞}-vector method.

In the case of C. C. Moore [8], the group G is a locally compact group and S a closed subgroup of G. He assumed that the homogenous space $S\backslash G$ possesses an invariant measure and that both $\{\mathfrak{H}, T(x)\}$, $\{H, \Lambda(s)\}$ are unitary. Nevertheless the induced representation $\{\mathfrak{H}^{\Lambda}, T^{\Lambda}(x)\}$ is defined so as to be an isometric one on a Banach space $\mathfrak{H}^{\Lambda}$. He formulated a version of the Frobenius reciprocity theorem in form (2). After that A. Kleppner obtained in [5] a generalization of the theorem in form (2) too. In his case, representations are not necessarily unitary, but assumed to be isometric representations on Banach spaces. More general induced Banach representations were defined and studied by R. A. Fontenot and I. Schochetman [2]. They also proved a generalization of the theorem in form (2).

Roughly speaking, all of the above people attempted to give the equality of the "multiplicity" of $\{H, \Lambda(s)\}$ in $\{\mathfrak{H}, T_S(s)\}$ and that of $\{\mathfrak{H}, T(x)\}$ in $\{\mathfrak{H}^{\Lambda}, T^{\Lambda}(x)\}$. In contrast to this, J. M. G. Fell considered in [1] the weak Frobenius reciprocity property: for irreducible unitary representations $\{\mathfrak{H}, T(x)\}$, $\{H, \Lambda(s)\}$ of G, S respectively, $\{\mathfrak{H}, T(x)\}$ is weakly contained in $\{\mathfrak{H}^{\Lambda}, T^{\Lambda}(x)\}$ if and only if $\{H, \Lambda(s)\}$ is weakly contained in $\{\mathfrak{H}, T_S(s)\}$. R. W. Henrichs studied when this property holds [4].

On the other hand, M. A. Rieffel [10] and R. Rigelhof [11] made category theoretical arguments on this subject. Of course there are some differences between assumptions or results in their papers, but we say nothing here on these differences. In any case, they found, under some conditions, G-modules ${}^{G}H$ and H^{G} for which relations $\mathrm{Hom}_S(H, \mathfrak{H}) \cong \mathrm{Hom}_G({}^{G}H, \mathfrak{H})$ and $\mathrm{Hom}_S(\mathfrak{H}, H) \cong \mathrm{Hom}_G(\mathfrak{H}, H^{G})$ hold. They also gave concrete representations of G-modules ${}^{G}H$ and H^{G} as vector-valued function spaces on G, which are similar to representation spaces of induced representations in Mackey's sense. But these two are not equal.

Now we make clear our aim in this paper. Let $L(G)$ be the convolution algebra of continuous functions on G with compact supports, and $L(S)$ similarly, where G and S are the same as those given first in this section. There exist $L(G)$-submodules $\mathfrak{H}_0$, $\mathfrak{H}_0^{\Lambda}$ of $\mathfrak{H}$, $\mathfrak{H}^{\Lambda}$ respectively, which are more essential in this paper than whole spaces (for definitions, see §1). They are, at the same time, $L(S)$-submodules. Then our aim is to prove the relation

$$\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H) \cong \mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}_0^{\Lambda}),$$

which is a generalization of the Frobenius reciprocity theorem in form (3). Our technique is different from those of the above people. The author has made attempt to prove a theorem in form (2) for these submodules $\mathfrak{H}_0$, $\mathfrak{H}_0^{\Lambda}$, that is, the relation $\mathrm{Hom}_{L(S)}(H, \mathfrak{H}_0) \cong \mathrm{Hom}_{L(G)}(\mathfrak{H}_0^{\Lambda}, \mathfrak{H}_0)$. But at present he has not yet succeeded.

§1. Notations and a scheme for the proof of main Theorem

Let G be a locally compact unimodular group, and S a closed subgroup of G. We assume that there exists a compact subgroup K of G with $G = SK$ ($S \cap K$ is not necessarily trivial).

Let $L(G)$ be the algebra of all complex valued continuous functions on G with compact supports. For every compact subset C of G, the vector space $L_C(G)$ of all functions f in $L(G)$ whose supports are contained in C is a Banach space with the norm $\|f\| = \sup_{x \in C} |f(x)|$. We shall regard the algebra $L(G)$ to be endowed with the inductive topology generated by these Banach spaces $L_C(G)$. For the subgroup S, the algebra $L(S)$ will be defined in the same way.

Throughout this paper, a topologically irreducible representation $\{\mathfrak{H}, T(x)\}$ of G will be fixed. The representation space $\mathfrak{H}$ is a locally convex Hausdorff topological vector space, not necessarily complete, but we assume that, for all Radon measures α on G with compact supports, the integrals $\int_G T(x)d\alpha(x)$ define continuous linear operators on $\mathfrak{H}$. Especially, for a function f in $L(G)$ or in $L(S)$, we set

$$T(f)=\int_G T(x)f(x)dx \quad \text{or} \quad T(f)=\int_S T(s)f(s)d\mu(s)$$

respectively, where dx denotes a Haar measure on G and $d\mu(s)$ a left Haar measure on S. Both are continuous linear operators on $\mathfrak{H}$, and $\mathfrak{H}$ can be seen as an $L(G)$-module at the same time as an $L(S)$-module.

Now we must impose the condition that the representation $\{\mathfrak{H}, T(x)\}$ shall contain an equivalence class δ of irreducible representations of K finitely many times. We shall denote by p the multiplicity of δ in $\{\mathfrak{H}, T(x)\}$:

$$[T|K:\delta]=p \qquad (0<p<+\infty).$$

Let $\mathfrak{H}(\delta)$ be the space of all vectors in $\mathfrak{H}$ transformed according to δ under $u\to T(u)$ $(u\in K)$, and $\mathfrak{H}_0$ the $L(G)$-submodule of $\mathfrak{H}$ generated by $\mathfrak{H}(\delta)$, then, for every non zero vector v in $\mathfrak{H}(\delta)$, we have

$$\mathfrak{H}_0=\{T(f)v; f\in L(G)\}.$$

This is clearly a G-invariant dense subspace of $\mathfrak{H}$, and can be seen as an $L(S)$-module. In the following, the subspace $\mathfrak{H}_0$ is more essential than the representation space $\mathfrak{H}$ itself.

Let $\{H, \Lambda(s)\}$ be a fixed representation of S. The representation space H is, like $\mathfrak{H}$, a (not necessarily complete) locally convex Hausdorff topological vector space, and, for all Radon measures on S with compact supports, continuous linear operators on H are defined by integrals.

We shall denote by $\mathfrak{H}^\Lambda$ the space of all continuous H-valued functions φ on K satisfying the equality

$$\varphi(mu)=\Lambda(m)\varphi(u) \qquad \text{for all} \quad m\in M=S\cap K.$$

For a system of semi-norms $\{|\cdot|_\iota; \iota\in I\}$ which defines the topology on H, we introduce a locally convex topology in $\mathfrak{H}^\Lambda$ defined by the system of semi-norms

$$\|\varphi\|_\iota=\sup_{u\in K}|\varphi(u)|_\iota \qquad (\iota\in I).$$

For every element x in G, a continuous linear operator $T^\Lambda(x)$ on $\mathfrak{H}^\Lambda$ is defined as

$$(T^\Lambda(x)\varphi)(u)=\Lambda(s)\varphi(k)$$

where $ux=sk$, $k\in K$, $s\in S$. The right hand side is clearly well-defined. Also for the induced representation $\{\mathfrak{H}^\Lambda, T^\Lambda(x)\}$ of G, as in the case of $\{\mathfrak{H}, T(x)\}$, we shall denote by $\mathfrak{H}^\Lambda(\delta)$ the space of all vectors in $\mathfrak{H}^\Lambda$ transformed according to δ under $u\to T^\Lambda(u)$, and by $\mathfrak{H}_0^\Lambda$ the $L(G)$-submodule of $\mathfrak{H}^\Lambda$ generated by $\mathfrak{H}^\Lambda(\delta)$.

Let $\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)$ be the vector space of all linear operators of $\mathfrak{H}_0$ to H which commute with $L(S)$-actions, and $\mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}_0^d)$ the vector space of all linear operators of $\mathfrak{H}_0$ to $\mathfrak{H}_0^d$ which commute with $L(G)$-actions, then our aim is to prove the following Theorem which is a version of the Frobenius reciprocity theorem.

Theorem. *Two vector spaces* $\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)$ *and* $\mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}_0^d)$ *are mutually isomorphic*:

$$\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H) \cong \mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}_0^d).$$

The proof of this Theorem will be pursued as follows. We will define three other vector spaces $\mathrm{Hom}_A(A_r/\mathfrak{M}(\mathfrak{a}_V), C^d \otimes H)$, $\mathrm{Hom}^{\sharp}_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V), C^d \otimes_M H)$ and $\mathrm{Hom}^{\sharp}_{L^\circ(\delta)}(\mathfrak{H}_1(\delta), \mathfrak{H}_1^d(\delta))$ (for definitions see §3, §4, and §5 respectively). The following diagram shows the scheme for the proof of main Theorem.

$$\begin{array}{ccc}
\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H) & \xrightarrow[\text{Prop. 1}]{\cong} & \mathrm{Hom}_A(A_r/\mathfrak{M}(\mathfrak{a}_V), C^d \otimes H) \\
 & & \downarrow \cong \\
\mathrm{Hom}^{\sharp}_{L^\circ(\delta)}(\mathfrak{H}_1(\delta), \mathfrak{H}_1^d(\delta)) & \xleftarrow[\text{Prop. 2}]{\cong} & \mathrm{Hom}^{\sharp}_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V), C^d \otimes_M H) \\
\cong \downarrow \text{Prop. 3} & & \\
\mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}_0^d). & &
\end{array}$$

The notation "$\cong$" means "linearly isomorphic", and we understand that the first $\cong$ is the statement of Proposition 1 and so on. The second $\cong$ is clear by Definition of the vector space $\mathrm{Hom}^{\sharp}_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V), C^d \otimes_M H)$ in §4. §2 is devoted to preparations, and §§3, 5 and §6 are to prove Propositions 1, 2 and 3 respectively.

§2. Group algebras on G and matrix algebras on S

Let du be the normalized Haar measure on K, and $d\mu$ a left Haar measure on S, then $dx = d\mu(s)du$ $(x = su)$ is a Haar measure on G.

We choose an irreducible unitary matricial representation $u \to D(u)$ of K which belongs to the equivalence class δ of irreducible representations of K given in §1. Denote by $d_{ij}(u)$ the (i, j)-coefficient of $D(u)$, by d the degree of δ, and set $\chi_\delta(u) = d \cdot \mathrm{trace}\, D(u)$.

Let A be the vector space of compactly supported continuous functions $F = F(s)$ on S with values in $\mathfrak{M}(d, C)$, the set of all $d \times d$ complex matrices. Then it is an algebra over the complex number field with product

$$F * G(s) = \int_S F(t) G(t^{-1}s) d\mu(t).$$

For every compact subset C of S, A_C denotes the Banach space of all functions F in A whose supports are contained in C with the norm $\|F\| = \max_{1 \leq i,j \leq d} \sup_{s \in C} |f_{ij}(s)|$ where $F = (f_{ij})$. We shall topologize the algebra A as the inductive limit of these Banach spaces A_C. Put $M = K \cap S$ as in §1. Then the set

$$A_r=\{F\in A;\ F(sm)=F(s)\overline{D(m)}\quad \text{for all}\quad m\in M\}$$

is a closed subalgebra of A, where $\overline{D(m)}$ means the complex conjugate of the matrix $D(m)$. Denote by D_M the restriction of D on M, and by dm the normalized Haar measure on M, and put

$$F*\overline{D_M}(s)=\int_M F(sm^{-1})\overline{D(m)}dm,$$

then $F\to F*\overline{D_M}$ is a continuous projection of A onto A_r.

On the other hand, for every function $f\in L(G)$, we put

$$f*\overline{\chi_\delta}(x)=\int_K f(xu^{-1})\overline{\chi_\delta(u)}du,$$

then $L(G)*\overline{\chi_\delta}=\{f*\overline{\chi_\delta};\ f\in L(G)\}$ is a closed subalgebra of $L(G)$.

Definition. A linear mapping Φ of $L(G)*\overline{\chi_\delta}$ into A_r is defined as

$$\Phi(f)(s)=\int_K \overline{D(u)}f(su^{-1})du.$$

Lemma 1. *The linear mapping Φ is bijective and bicontinuous. The inverse Φ^{-1} is given by*

$$\Phi^{-1}(F)(su)=d\cdot \text{trace}\,[F(s)\overline{D(u)}]\,.$$

Proof. We assume $\Phi(f)=0$. Then we have

$$f(su)=f*\overline{\chi_\delta}(su)=d\cdot\text{trace}\int_K \overline{D(v)}f(suv^{-1})dv$$

$$=d\cdot\text{trace}\,[\Phi(f)(s)\overline{D(u)}]=0.$$

Therefore Φ is injective.

For every function F in A_r, the function

$$g(s,\,u)=d\cdot\text{trace}\,[F(s)\overline{D(u)}]$$

on $S\times K$ induces a function f in $L(G)*\overline{\chi_\delta}$ in such a way that $f(su)=g(s,\,u)$. It is easy to show that $\Phi(f)=F$. From this, we obtain the explicit formula of Φ^{-1}. The continuity of Φ and Φ^{-1} is clear. Q.E.D.

For every function $f\in L(G)$, we put

$$f^\circ(x)=\int_K f(uxu^{-1})du,$$

then the set $L^\circ(\delta)=\{f^\circ*\overline{\chi_\delta};\ f\in L(G)\}$ is a closed subalgebra of $L(G)$. This plays an important role in this paper.

Lemma 2. *For any functions $f\in L(G)*\overline{\chi_\delta}$ and $g\in L^\circ(\delta)$, we have the equality*

$$\Phi(f*g)=\Phi(f)*\Phi(g)\,.$$

Proof. This is a consequence of the following simple calculation:

$$\Phi(f*g)(s)=\int_K \overline{D(u)}f*g(su^{-1})du=\int_K\int_G \overline{D(u)}f(x)g(x^{-1}su^{-1})dxdu$$

$$=\int_K\int_G \overline{D(u)}f(x)g(u^{-1}x^{-1}s)dxdu=\int_K\int_G \overline{D(u)}f(xu^{-1})g(x^{-1}s)dxdu$$

$$=\int_K\int_K\int_S \overline{D(u)}f(tvu^{-1})g(v^{-1}t^{-1}s)d\mu(t)dvdu$$

$$=\int_K\int_S\int_K \overline{D(u)}\,\overline{D(v)}f(tu^{-1})g(t^{-1}sv^{-1})du\,d\mu(t)dv$$

$$=\Phi(f)*\Phi(g)(s).$$

Q. E. D.

Lemma 1 shows that Φ is an isomorphism of $L(G)*\overline{\chi_\delta}$ onto A_r both regarded as topological vector spaces. Lemma 2 means that Φ gives an isomorphism of topological algebras $L^\circ(\delta)$ and $A^\circ=\Phi(L^\circ(\delta))$. For every function $F=\Phi(f)\in A_r$ we define

$$F^\circ=\Phi(f^\circ),$$

then $F\to F^\circ$ is a continuous projection of A_r onto A°. For any function $F\in A^\circ$, it is clear that

$$F(m_1sm_2)=\overline{D(m_1)}F(s)\overline{D(m_2)} \qquad \text{for all} \quad m_1,\ m_2\in M.$$

Lemma 3. *For any functions $F\in A$ and $G\in A_r$, we have*

$$(F*G^\circ)^\circ=(F*\overline{D_M})^\circ*G^\circ.$$

Proof. First, we perform the following calculation:

$$(F*G^\circ)(s)=\int_S F(t)G^\circ(t^{-1}s)d\mu(t)$$

$$=\int_S\int_M F(t)\overline{D(m)}G^\circ(m^{-1}t^{-1}s)dmd\mu(t)$$

$$=\int_M\int_S F(tm^{-1})\overline{D(m)}G^\circ(t^{-1}s)d\mu(t)dm$$

$$=((F*\overline{D_M})*G^\circ)(s).$$

Let f and g be functions in $L(G)*\overline{\chi_\delta}$ such that $F*\overline{D_M}=\Phi(f)$ and $G=\Phi(g)$, then it follows that

$$(F*G^\circ)^\circ=(\Phi(f)*\Phi(g^\circ))^\circ=(\Phi(f*g^\circ))^\circ=\Phi((f*g^\circ)^\circ)$$

$$=\Phi(f^\circ*g^\circ)=\Phi(f^\circ)*\Phi(g^\circ)=(F*\overline{D_M})^\circ*G^\circ.$$

Q. E. D.

Let $\mathfrak{a}$ be a (non-trivial) closed regular maximal left ideal in $L^\circ(\delta)$, and $\mathfrak{e}$ a right identity modulo $\mathfrak{a}$. Then $\Phi(\mathfrak{a})$ is a closed regular maximal left ideal in A° and $\mathfrak{E}=\Phi(\mathfrak{e})$ is a right identity modulo $\Phi(\mathfrak{a})$. On the set

$$\mathfrak{M}(\mathfrak{a})=\{F\in A_r;\ (G*F)^\circ\in\Phi(\mathfrak{a})\quad \text{for all}\quad G\in A\},$$

we can prove the following

Lemma 4. (i) $\mathfrak{M}(\mathfrak{a})$ *is a closed left A-invariant subspace of A_r.*

(ii) $F*\mathfrak{E}-F\in\mathfrak{M}(\mathfrak{a})$ *for any* $F\in A_r$.

(iii) $\mathfrak{M}(\mathfrak{a})\cap A^\circ=(\mathfrak{M}(\mathfrak{a}))^\circ=\Phi(\mathfrak{a})$.

Proof. The statement (i) is clear.

Let us prove the statement (ii). For any function $G\in A$, the function $G*F$ belongs to A_r. Thus the equality $G*F*\overline{D_M}=G*F$ holds, and hence, by Lemma 3, we obtain

$$(G*(F*\mathfrak{E}-F))^\circ=((G*F)*\mathfrak{E}-G*F)^\circ=(G*F)^\circ*\mathfrak{E}-(G*F)^\circ.$$

Since the right hand side belongs to $\Phi(\mathfrak{a})$, the statement (ii) is proved.

Let us now show the inclusion $(\mathfrak{M}(\mathfrak{a}))^\circ\subset\Phi(\mathfrak{a})$. Let $\{g_\nu\}\subset L(S)$ be an approximate identity of the unit 1 in S. Then the following $\mathfrak{M}(d,\ \boldsymbol{C})$-valued function

$$G_\nu=\begin{pmatrix} g_\nu & & 0\\ & \ddots & \\ 0 & & g_\nu\end{pmatrix}$$

is an element in A. Since, for any function $F\in A_r$, $G_\nu*F$ tends to F in A_r, it follows that $(G_\nu*F)^\circ\to F^\circ$. If F is an element in $\mathfrak{M}(\mathfrak{a})$, then each function $(G_\nu*F)^\circ$ belongs to $\Phi(\mathfrak{a})$, and consequently so does F°. Therefore we have the inclusion $(\mathfrak{M}(\mathfrak{a}))^\circ\subset\Phi(\mathfrak{a})$.

On the other hand, it is clear that $\Phi(\mathfrak{a})\subset\mathfrak{M}(\mathfrak{a})$ by Lemma 3. From this the relations $\Phi(\mathfrak{a})\subset\mathfrak{M}(\mathfrak{a})\cap A^\circ$ and $\Phi(\mathfrak{a})\subset(\mathfrak{M}(\mathfrak{a}))^\circ$ are also clear. Since $\mathfrak{M}(\mathfrak{a})\cap A^\circ$ is a left ideal in A° which contains the maximal left ideal $\Phi(\mathfrak{a})$, we have only to show that $\mathfrak{M}(\mathfrak{a})\cap A^\circ\neq A^\circ$. Suppose that $\mathfrak{M}(\mathfrak{a})\cap A^\circ=A^\circ$, then $\mathfrak{M}(\mathfrak{a})\supset A^\circ$ must hold. But this contradicts the fact $\Phi(\mathfrak{a})=(\mathfrak{M}(\mathfrak{a}))^\circ\subsetneqq A^\circ$. Q. E. D.

§3. The first step: proof of Proposition 1

We imposed the condition $[T|K:\delta]=p$, $0<p<+\infty$, in §1. This means that $\dim\mathfrak{H}(\delta)=pd$. We now consider the continuous linear operators on $\mathfrak{H}$

$$E(\delta)=\int_K T(u)\overline{\chi_\delta(u)}du,\quad \text{and}\quad E_{ij}(\delta)=d\int_K T(u)\overline{d_{ij}(u)}du$$

for $1\le i,\ j\le d$. Then the subspace $\mathfrak{H}(\delta)=E(\delta)\mathfrak{H}$ is decomposed into the direct sum

$$\mathfrak{H}(\delta)=\mathfrak{H}_1(\delta)\oplus\cdots\oplus\mathfrak{H}_d(\delta)$$

with $\mathfrak{H}_i(\delta)=E_{ii}(\delta)\mathfrak{H}$ $(1\le i\le d)$. These subspaces $\mathfrak{H}_i(\delta)$ of $\mathfrak{H}$ are p-dimensional irreducible $L^\circ(\delta)$-submodules, and mutually isomorphic. In fact $E_{ji}(\delta)$ is an intertwining operator of $\mathfrak{H}_i(\delta)$ onto $\mathfrak{H}_j(\delta)$.

We choose a non-trivial K-irreducible subspace V of $\mathfrak{H}(\delta)$, and a basis $e_1,\dots,e_d$ of V such that

$$T(u)e_j=\sum_{i=1}^{d} d_{ij}(u)e_i \qquad (1\leq j\leq d).$$

We keep these notations in the following. In this situation it is clear that $e_i\in\mathfrak{H}_i(\delta)$.

For the above K-irreducible subspace V, the set

$$\mathfrak{a}_V=\{f\in L^\circ(\delta);\ T(f)V=\{0\}\}$$

is a closed regular maximal left ideal in $L^\circ(\delta)$. As a right identity modulo $\mathfrak{a}_V$, we may adopt a function $\mathfrak{e}\in L^\circ(\delta)$ such that the restriction of $T(\mathfrak{e})$ on $\mathfrak{H}(\delta)$ is the identity operator. For an arbitrary non zero vector $v\in V$ we have $\mathfrak{a}_V=\{f\in L^\circ(\delta);\ T(f)v=0\}$ and hence, especially for e_i, it follows that

$$\mathfrak{a}_V=\{f\in L^\circ(\delta);\ T(f)e_i=0\} \qquad (1\leq i\leq d).$$

Thus the $L^\circ(\delta)$-modules $\mathfrak{H}_i(\delta)$ and $L^\circ(\delta)/\mathfrak{a}_V$ are isomorphic.

As in §2, we define a closed A-invariant subspaces $\mathfrak{M}(\mathfrak{a}_V)$ of A_r as

$$\mathfrak{M}(\mathfrak{a}_V)=\{F\in A_r;\ (G*F)^\circ\in\Phi(\mathfrak{a}_V) \quad \text{for all} \quad G\in A\}.$$

Then $A_r/\mathfrak{M}(\mathfrak{a}_V)$ can be seen as an A-module.

On the other hand, we shall denote by $\boldsymbol{C}^d\otimes H$ the vector spaces of all column vectors $\boldsymbol{a}=\begin{pmatrix}a_1\\ \vdots\\ a_d\end{pmatrix}={}^t(a_1,\dots,a_d)$ with $a_i\in H$ $(1\leq i\leq d)$, where H is the representation space of $\{H, \Lambda(s)\}$ given in §1. This vector space $\boldsymbol{C}^d\otimes H$ can be considered as an A-module in the following way:

$$R(F)\boldsymbol{a}=\begin{pmatrix}\Lambda(f_{11})\cdots\Lambda(f_{1d})\\ \vdots \qquad \vdots\\ \Lambda(f_{d1})\cdots\Lambda(f_{dd})\end{pmatrix}\begin{pmatrix}a_1\\ \vdots\\ a_d\end{pmatrix}=\begin{pmatrix}\sum_{i=1}^{d}\Lambda(f_{1i})a_i\\ \vdots\\ \sum_{i=1}^{d}\Lambda(f_{di})a_i\end{pmatrix}$$

where $F=(f_{ij})\in A$, $\boldsymbol{a}={}^t(a_1,\dots,a_d)\in\boldsymbol{C}^d\otimes H$, and $\Lambda(f_{ij})=\int_S\Lambda(s)f_{ij}(s)d\mu(s)$.

Now let $\mathrm{Hom}_A(A_r/\mathfrak{M}(\mathfrak{a}_V), \boldsymbol{C}^d\otimes H)$ be the vector space of all (algebraic) homomorphisms of $A_r/\mathfrak{M}(\mathfrak{a}_V)$ to $\boldsymbol{C}^d\otimes H$, both regarded as A-modules. Our aim in this section is to prove the following

Proposition 1. *The vector space* $\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)$ *is isomorphic to the vector space* $\mathrm{Hom}_A(A_r/\mathfrak{M}(\mathfrak{a}_V), \boldsymbol{C}^d\otimes H)$:

$$\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)\cong\mathrm{Hom}_A(A_r/\mathfrak{M}(\mathfrak{a}_V), \boldsymbol{C}^d\otimes H).$$

At first, we try to give a linear mapping of $\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)$ to $\mathrm{Hom}_A(A_r/\mathfrak{M}(\mathfrak{a}_V), \boldsymbol{C}^d\otimes H)$. Let α be an element in $\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)$. For every function $F=(f_{ij})=\Phi(f)\in\mathfrak{M}(\mathfrak{a}_V)$ we know that

$$\sum_{j=1}^{d} T(f_{ij})e_j=T(f)e_i=0 \qquad (1\leq i\leq d).$$

(See Lemmas 3 and 4 in [12].) Therefore it turns out that

$$\sum_{j=1}^{d} \Lambda(f_{ij})\alpha(e_j)=\alpha(\sum_{j=1}^{d} T(f_{ij})e_j)=0 \qquad (1\leq i\leq d),$$

i.e., $R(F)^t(\alpha(e_1),\ldots, \alpha(e_d))=0$. Denote by $[F]$ the element in $A_r/\mathfrak{M}(\mathfrak{a}_V)$ of which F is a representative, then the above observation makes it possible for us to define a linear mapping ξ_α of $A_r/\mathfrak{M}(\mathfrak{a}_V)$ to $\boldsymbol{C}^d\otimes H$ as

$$\xi_\alpha([F])=R(F)^t(\alpha(e_1),\ldots, \alpha(e_d)).$$

Moreover the following equalities

$$\begin{aligned}\xi_\alpha(G*[F])&=R(G*F)^t(\alpha(e_1),\ldots, \alpha(e_d))\\&=R(G)R(F)^t(\alpha(e_1),\ldots, \alpha(e_d))\\&=R(G)\xi_\alpha([F])\end{aligned}$$

means that ξ_α is an element in $\mathrm{Hom}_A(A_r/\mathfrak{M}(\mathfrak{a}_V), \boldsymbol{C}^d\otimes H)$.

The mapping $\alpha\to\xi_\alpha$ is clearly linear. Suppose $\xi_\alpha=0$, then $\alpha(T(f)e_i)=0$ $(1\leq i\leq d)$ for all functions $f\in L(G)*\chi_\delta$, so $\alpha=0$. Thus the linear mapping $\alpha\to\xi_\alpha$ is injective.

To complete the proof of Proposition 1, we have only to show that the linear mapping $\alpha\to\xi_\alpha$ is surjective. Let ξ be an arbitrary element in $\mathrm{Hom}_A(A_r/\mathfrak{M}(\mathfrak{a}_V), \boldsymbol{C}^d\otimes H)$. Put $\mathfrak{E}=\Phi(\mathfrak{e})$, where $\mathfrak{e}$ is the function in $L^\circ(\delta)$ already given in this section, then, for any function $F\in A_r$, it follows that

$$\xi([F])=\xi(F*[\mathfrak{E}])=R(F)\xi([\mathfrak{E}]),$$

and hence, particularly for $F=\mathfrak{E}$, we obtain $\xi([\mathfrak{E}])=R(\mathfrak{E})\xi([\mathfrak{E}])$. Putting $\mathfrak{E}=(\mathfrak{e}_{ij})$ and $\xi([\mathfrak{E}])={}^t(a_1(\xi),\ldots, a_d(\xi))$, the last equality means

$$a_i(\xi)=\sum_{j=1}^{d} \Lambda(\mathfrak{e}_{ij})a_j(\xi) \qquad (1\leq i\leq d).$$

Lemma 5. *Let f, g be two functions in $L(G)*\bar{\chi}_\delta$ and $\Phi(f)=(f_{ij})$, $\Phi(g)=(g_{ij})$ the corresponding functions in A_r. If there exists a pair (i, j) such that $T(f)e_i=T(g)e_j$, then it follows that*

$$\sum_{k=1}^{d} \Lambda(f_{ik})a_k(\xi)=\sum_{k=1}^{d} \Lambda(g_{jk})a_k(\xi).$$

Proof. Let E_{ij} be the matrix whose (i, j)-coefficient is equal to 1 and the others are equal to 0. Let h_{ik} be the (i, k)-coefficient of the product $E_{ij}\Phi(g)$, then we have $h_{ik}=g_{jk}$ $(1\leq i\leq d)$. So we obtain

$$T(g)e_j=\sum_{k=1}^{d} T(g_{jk})e_k=\sum_{k=1}^{d} T(h_{ik})e_k=T(h)e_i$$

where $h=\Phi^{-1}(E_{ij}\Phi(g))$, and hence $T(f)e_i=T(h)e_i$. Put $f_i=\Phi^{-1}(E_{ii}\Phi(f))$, $h_i=\Phi^{-1}(E_{ii}\Phi(h))$, then it follows from Corollary in [12] that

$$T(f_i)e_i=T(f)e_i=T(h)e_i=T(h_i)e_i,$$

$$T(f_i)e_k = 0 = T(h_i)e_k \qquad (k \neq i).$$

Therefore we know that $T(f_i - h_i)V = \{0\}$, or equivalently, $[E_{ii}\Phi(f)] = [E_{ii}\Phi(h)]$. Hence we have

$$\begin{aligned} E_{ii}R(\Phi(f))^t(a_1(\xi), \dots, a_d(\xi)) &= R(E_{ii}\Phi(f))\xi([\mathfrak{E}]) = \xi([E_{ii}\Phi(f)]) \\ &= \xi([E_{ii}\Phi(h)]) = E_{ii}R(\Phi(h))^t(a_1(\xi), \dots, a_d(\xi)), \end{aligned}$$

and consequently

$$\sum_{k=1}^{d} \Lambda(f_{ik})a_k(\xi) = \sum_{k=1}^{d} \Lambda(h_{ik})a_k(\xi) = \sum_{k=1}^{d} \Lambda(g_{jk})a_k(\xi).$$

Q. E. D.

For a fixed index i, we set

$$\alpha(T(f)e_i) = \sum_{j=1}^{d} \Lambda(f_{ij})a_j(\xi)$$

with $\Phi(f) = (f_{ij})$. Then Lemma 5 shows that α is a well-defined linear mapping of $\mathfrak{H}_0$ to H and that it is independent of the index i. Particularly for $f = \mathfrak{e}$ it follows that

$$\alpha(e_i) = \alpha(T(\mathfrak{e})e_i) = \sum_{j=1}^{d} \Lambda(\mathfrak{e}_{ij})a_j(\xi) = a_i(\xi) \qquad (1 \leq i \leq d).$$

Let us show that this linear mapping α belongs to $\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)$. For arbitrary functions $h \in L(S)$ and $f \in L(G)*\overline{\chi_\delta}$, set

$$h*f(x) = \int_S h(s)f(s^{-1}x)d\mu(s).$$

Then $h*f$ is a function in $L(G)*\overline{\chi_\delta}$. Since the (i, j)-coefficient of $\Phi(h*f)$ is equal to $h*f_{ij}$, it follows that

$$\begin{aligned} \alpha(T(h)T(f)e_i) = \alpha(T(h*f)e_i) &= \sum_{j=1}^{d} \Lambda(h*f_{ij})a_j(\xi) \\ &= \Lambda(h)\alpha(T(f)e_i). \end{aligned}$$

So we know that α belongs to $\mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)$.

For this linear mapping $\alpha \in \mathrm{Hom}_{L(S)}(\mathfrak{H}_0, H)$, the linear mapping $\xi_\alpha \in \mathrm{Hom}_A(A_r/\mathfrak{M}(\mathfrak{a}_V), \boldsymbol{C}^d \otimes H)$ satisfies

$$\begin{aligned} \xi_\alpha([F]) &= R(F)^t(\alpha(e_1), \dots, \alpha(e_d)) \\ &= R(F)^t(a_1(\xi), \dots, a_d(\xi)) \\ &= R(F)\xi([\mathfrak{E}]) = \xi([F]), \end{aligned}$$

i.e., $\xi_\alpha = \xi$. This shows that the linear mapping $\alpha \to \xi_\alpha$ is surjective.

§4. Definition of $\mathrm{Hom}^{\sharp}_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V),\ C^d\otimes_M H)$

Let $C^d\otimes_M H$ be the space of all vectors $a={}^t(a_1,\dots,a_d)\in C^d\otimes H$ such that

$$\Lambda(m)a_j=\sum_{i=1}^{d} d_{ij}(m)a_i \qquad (1\le j\le d)$$

for all elements $m\in M=K\cap S$, or symbolically,

$$\begin{pmatrix} \Lambda(m) & & 0 \\ & \ddots & \\ 0 & & \Lambda(m)\end{pmatrix} a={}^tD(m)a$$

for all element $m\in M$, where ${}^tD(m)$ is the transposed matrix of $D(m)$. For a function $F\in A^\circ$, the equality $\varepsilon_m * F={}^tD(m)F$ (where $\varepsilon_m * F(s)=F(m^{-1}s)$) holds for all $m\in M$. Therefore for every vector $a\in C^d\otimes_M H$, it follows that

$$\begin{pmatrix} \Lambda(m) & & 0 \\ & \ddots & \\ 0 & & \Lambda(m)\end{pmatrix} R(F)\,a=R(\varepsilon_m * F)a=R({}^tD(m)F)a={}^tD(m)R(F)a.$$

Namely, the subspace $C^d\otimes_M H$ is an A°-submodule of $C^d\otimes H$.

On the other hand, the A°-module $A^\circ/\Phi(\mathfrak{a}_V)$ naturally can be seen as an A°-submodule of $A_r/\mathfrak{M}(\mathfrak{a}_V)$ by Lemma 4.

Lemma 6. *For any element $\xi\in\mathrm{Hom}_A(A_r/\mathfrak{M}(\mathfrak{a}_V),\ C^d\otimes H)$, we have*

$$\xi(A^\circ/\Phi(\mathfrak{a}_V))\subset C^d\otimes_M H.$$

Proof. Keep the notations in §3 such as $\xi([\mathfrak{E}])={}^t(a_1(\xi),\dots,a_d(\xi))$ and $\mathfrak{E}=\Phi(\mathfrak{e})=(\mathfrak{e}_{ij})$. Then for any element $m\in M$,

$$\begin{aligned}\Lambda(m)a_j(\xi)&=\sum_{k=1}^{d}\Lambda(m)\Lambda(\mathfrak{e}_{jk})a_k(\xi)=\sum_{k=1}^{d}\Lambda(\varepsilon_m * \mathfrak{e}_{jk})a_k(\xi)\\ &=\sum_{i,k=1}^{d} d_{ij}(m)\Lambda(\mathfrak{e}_{ik})a_k(\xi)=\sum_{i=1}^{d} d_{ij}(m)a_i(\xi).\end{aligned}$$

Thus we know that $\xi([\mathfrak{E}])\in C^d\otimes_M H$, and from this fact, it follows that $\xi([F])=R(F)\xi([\mathfrak{E}])\in C^d\otimes_M H$ for all functions $F\in A^\circ$. Q.E.D.

Now we shall denote by $\underline{\xi}$ the restriction of ξ onto $A^\circ/\Phi(\mathfrak{a}_V)$. Then Lemma 6 shows that $\underline{\xi}$ is an element in $\mathrm{Hom}_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V),\ C^d\otimes_M H)$. The linear correspondence $\xi\to\underline{\xi}$ is injective. In fact, $\underline{\xi}=0$ means that $\xi([F])=R(F)\xi([\mathfrak{E}])=R(F)\underline{\xi}([\mathfrak{E}])=0$ for all elements $[F]\in A_r/\mathfrak{M}(\mathfrak{a}_V)$.

Definition. We shall denote by $\mathrm{Hom}^{\sharp}_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V),\ C^d\otimes_M H)$ the image of $\mathrm{Hom}_A(A_r/\mathfrak{M}(\mathfrak{a}_V),\ C^d\otimes H)$ by the linear injection $\xi\to\underline{\xi}$.

Lemma 7. *Let ξ' be an element in $\mathrm{Hom}_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V),\ C^d\otimes_M H)$. Then ξ' belongs to $\mathrm{Hom}^{\sharp}_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V),\ C^d\otimes_M H)$ if and only if*

$$\mathfrak{M}(\mathfrak{a}_V)\subset\{F\in A_r;\ R(F)\xi'([\mathfrak{E}])=0\}.$$

Proof. Suppose that $\xi'\in \mathrm{Hom}^{\sharp}_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V),\ C^d\otimes_M H)$, then there exists an element $\xi\in \mathrm{Hom}_A(A_r/\mathfrak{M}(\mathfrak{a}_V),\ C^d\otimes H)$ such that $\xi'=\bar{\xi}$. Therefore, for any function $F\in\mathfrak{M}(\mathfrak{a}_V)$, we have

$$R(F)\xi'([\mathfrak{E}])=R(F)\xi([\mathfrak{E}])=\xi([F])=0.$$

Conversely we assume that $\mathfrak{M}(\mathfrak{a}_V)\subset\{F\in A_r;\ R(F)\xi'([\mathfrak{E}])=0\}$. Then we may define a linear mapping ξ of $A_r/\mathfrak{M}(\mathfrak{a}_V)$ to $C^d\otimes H$ by $\xi([F])=R(F)\xi'([\mathfrak{E}])$. It is clear that ξ belongs to $\mathrm{Hom}_A(A_r/\mathfrak{M}(\mathfrak{a}_V),\ C^d\otimes H)$ and that $\xi'=\bar{\xi}$. Q. E. D.

§5. Identification of two vector spaces $\mathrm{Hom}^{\sharp}_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V),\ C^d\otimes_M H)$ and $\mathrm{Hom}^{\sharp}_{L^\circ(\delta)}(\mathfrak{H}_1(\delta),\ \mathfrak{H}^A_1(\delta))$

In §3, we defined irreducible p-dimensional $L^\circ(\delta)$-modules $\mathfrak{H}_1(\delta),\dots,\mathfrak{H}_d(\delta)$. Since these are mutually isomorphic, we pick up the module $\mathfrak{H}_1(\delta)$.

For the induced representation $\{\mathfrak{H}^A,\ T^A(x)\}$ of G, as in the case of $\{\mathfrak{H},\ T(x)\}$, we consider the continuous linear operators

$$E^A(\delta)=\int_K T^A(u)\overline{\chi_\delta(u)}du,\quad E^A_{ij}(\delta)=d\int_K T^A(u)\overline{d_{ij}(u)}du$$

for $1\le i,\ j\le d$, and put $\mathfrak{H}^A(\delta)=E^A(\delta)\mathfrak{H}^A$, $\mathfrak{H}^A_i(\delta)=E^A_{ii}(\delta)\mathfrak{H}^A$. The $L^\circ(\delta)$-module $\mathfrak{H}^A(\delta)$ is decomposed into the direct sum

$$\mathfrak{H}^A(\delta)=\mathfrak{H}^A_1(\delta)\oplus\cdots\oplus\mathfrak{H}^A_d(\delta),$$

and these $L^\circ(\delta)$-submodules $\mathfrak{H}^A_i(\delta)$ are mutually isomorphic. So we pick up the module $\mathfrak{H}^A_1(\delta)$ as above.

Definition. We shall denote by $\mathrm{Hom}^{\sharp}_{L^\circ(\delta)}(\mathfrak{H}_1(\delta),\ \mathfrak{H}^A_1(\delta))$ the set of elements $\sigma\in\mathrm{Hom}_{L^\circ(\delta)}(\mathfrak{H}_1(\delta),\ \mathfrak{H}^A_1(\delta))$ such that $\sigma=0$ or the $L(G)$-submodules of $\mathfrak{H}^A_0$ generated by $\sigma(\mathfrak{H}_1(\delta))$ are (algebraically) isomorphic to the $L(G)$-module $\mathfrak{H}_0$.

For a non zero element $\sigma\in\mathrm{Hom}_{L^\circ(\delta)}(\mathfrak{H}_1(\delta),\ \mathfrak{H}^A_1(\delta))$, set $\varphi_\sigma=\sigma(e_1)$ and denote by V^A_σ the K-irreducible subspace of $\mathfrak{H}^A(\delta)$ which contains φ_σ. The vectors φ_σ, $E^A_{21}(\delta)\varphi_\sigma,\dots,E^A_{d1}(\delta)\varphi_\sigma$ forms a basis of V^A_σ.

Lemma 8. *Let σ be a non zero element in* $\mathrm{Hom}_{L^\circ(\delta)}(\mathfrak{H}_1(\delta),\ \mathfrak{H}^A_1(\delta))$. *Then σ belongs to* $\mathrm{Hom}^{\sharp}_{L^\circ(\delta)}(\mathfrak{H}_1(\delta),\ \mathfrak{H}^A_1(\delta))$ *if and only if*

$$\mathfrak{M}(\mathfrak{a}_V)\subset\{\Phi(f)\in A_r;\ T^A(f)V^A_\sigma=\{0\}\}.$$

Proof. First of all denote by $\mathscr{H}_\sigma$ the $L(G)$-submodule of $\mathfrak{H}^A_0$ generated by $\sigma(\mathfrak{H}_1(\delta))$, then it is clear that

$$\mathscr{H}_\sigma=\{T^A(f)\varphi_\sigma;\ f\in L(G)\}.$$

Suppose that σ is an element in $\mathrm{Hom}^{\sharp}_{L^\circ(\delta)}(\mathfrak{H}_1(\delta),\ \mathfrak{H}^A_1(\delta))$. Then, by the above

Definition, there exists an isomorphism $\tilde{\sigma}$ of the $L(G)$-module $\mathfrak{H}_0$ onto $\mathscr{H}_\sigma$. As was stated in §1, a general element of $\mathfrak{H}_0$ is of the form $T(f)e_1$ $(f \in L(G))$. It follows from this fact that $\tilde{\sigma}\circ E_{11}(\delta)=E^A_{11}(\delta)\circ\tilde{\sigma}$ on $\mathfrak{H}_0$, and hence $\tilde{\sigma}(\mathfrak{H}_1(\delta))=E^A_{11}(\delta)\mathscr{H}_\sigma$. Since we have $\varphi_\sigma=\sigma(e_1)=\sigma(T(\mathfrak{e})e_1)=T^A(\mathfrak{e})\varphi_\sigma$ (the function $\mathfrak{e}\in L^\circ(\delta)$ was given in §3), the vector $\varphi_\sigma=E^A_{11}(\delta)\varphi_\sigma=E^A_{11}(\delta)T^A(\mathfrak{e})\varphi_\sigma$ is in $E^A_{11}(\delta)\mathscr{H}_\sigma=\tilde{\sigma}(\mathfrak{H}_1(\delta))$. Thus two irreducible $L^\circ(\delta)$-modules $\sigma(\mathfrak{H}_1(\delta))$ and $\tilde{\sigma}(\mathfrak{H}_1(\delta))$ contain at least one common element $\varphi_\sigma\neq 0$. Therefore it follows that not only $\sigma(\mathfrak{H}_1(\delta))=\tilde{\sigma}(\mathfrak{H}_1(\delta))$ but also $\tilde{\sigma}$ is a non zero scalar multiple of σ. Thus we have $\tilde{\sigma}(V)=V^A_\sigma$. Now we know that, for every function $F=\Phi(f)\in\mathfrak{M}(\mathfrak{a}_V)$, $T^A(f)V^A_\sigma=\tilde{\sigma}(T(f)V)=\tilde{\sigma}(\{0\})=\{0\}$.

Let us prove the "only if" part. As was stated before, the subspace $\mathfrak{H}(\delta)$ is the direct sum of mutually isomorphic $L^\circ(\delta)$-modules $\mathfrak{H}_1(\delta),\dots,\mathfrak{H}_d(\delta)$:

$$\begin{aligned}\mathfrak{H}(\delta)&=\mathfrak{H}_1(\delta)\oplus\cdots\oplus\mathfrak{H}_d(\delta)\\&=\mathfrak{H}_1(\delta)\oplus E_{21}(\delta)\mathfrak{H}_1(\delta)\oplus\cdots\oplus E_{d1}(\delta)\mathfrak{H}_1(\delta)\,.\end{aligned}$$

Accordingly σ can be extended in a obvious way to a linear mapping σ' of $\mathfrak{H}(\delta)$ into $\mathfrak{H}^A(\delta)$, namely,

$$\sigma'(v)=\sum_{i=1}^{d}E^A_{i1}(\delta)\sigma(v_i)\qquad\text{for}\quad v=\sum_{i=1}^{d}E_{i1}(\delta)v_i,\ v_i\in\mathfrak{H}_1(\delta)\,.$$

It is not so difficult to show that

$$\begin{aligned}\sigma'\circ T(f)&=T^A(f)\circ\sigma'\qquad&\text{for}\quad f\in L^\circ(\delta)\,,\\ \sigma'\circ T(u)&=T^A(u)\circ\sigma'\qquad&\text{for}\quad u\in K.\end{aligned}$$

Now we must remark the fact that the set $\{\varepsilon_u * f;\ u\in K, f\in L^\circ(\delta)\}$, where $\varepsilon_u * f(x)=f(u^{-1}x)$, is total in $L(\delta)=\overline{\chi_\delta} * L(G) * \overline{\chi_\delta}$ (see Lemma 11 in [3]). Then we know that the pd-dimensional subspace $\sigma'(\mathfrak{H}(\delta))$ is $L(\delta)$-invariant, and that

$$\sigma'\circ T(f)=T^A(f)\circ\sigma'\qquad\text{on}\quad\mathfrak{H}(\delta)$$

for all functions $f\in L(\delta)$. Now set $\mathscr{H}_\sigma(\delta)=E^A(\delta)\mathscr{H}_\sigma$, then it follows that

$$\begin{aligned}\mathscr{H}_\sigma(\delta)&=\{T^A(f)\varphi_\sigma;\ f\in L(\delta)\}=\{\sigma'(T(f)e_1);\ f\in L(\delta)\}\\&=\sigma'(\mathfrak{H}(\delta))\,,\end{aligned}$$

i.e., σ' is an isomorphism of the irreducible $L(\delta)$-module $\mathfrak{H}(\delta)$ onto the $L(\delta)$-module $\mathscr{H}_\sigma(\delta)$.

Now let $\mathscr{K}$ be an $L(G)$-submodule of $\mathscr{H}_\sigma$ which is not equal to $\mathscr{H}_\sigma$. Put $\mathscr{K}(\delta)=E^A(\delta)\mathscr{K}$. Since the operator $T(\mathfrak{e})$ is the identity operator on $\mathfrak{H}(\delta)$, the operator $T^A(\mathfrak{e})$ is also the one on $\mathscr{H}_\sigma(\delta)$. Thus $\mathscr{K}(\delta)=T^A(\mathfrak{e})E^A(\delta)\mathscr{K}=T^A(\mathfrak{e})\mathscr{K}\subset\mathscr{K}$. Suppose $\mathscr{K}(\delta)\neq\{0\}$, then the equality $\mathscr{K}(\delta)=\mathscr{H}_\sigma(\delta)$ holds because of the irreducibility of the $L(\delta)$-module $\mathscr{H}_\sigma(\delta)$. But it is impossible since $\mathscr{K}\subsetneqq\mathscr{H}_\sigma$. Therefore we obtain $\mathscr{K}(\delta)=\{0\}$.

This fact shows that there exists the largest proper $L(G)$-submodule $\mathscr{K}_\infty$ of $\mathscr{H}_\sigma$, and that $\mathscr{K}_\infty(\delta)=E^A(\delta)\mathscr{K}_\infty=\{0\}$. Let f, g be arbitrary two functions in $L(G)$, then

$$\sigma'(E(\delta)T(g)T(f)e_1)=\sigma'(T(\overline{\chi_\delta}*g*f*\overline{\chi_\delta})e_1)$$
$$=T^A(\overline{\chi_\delta}*g*f*\overline{\chi_\delta})\varphi_\sigma=E^A(\delta)T^A(g)T^A(f)\varphi_\sigma.$$

Therefore, if $T(f)e_1=0$, then $\{T^A(g)T^A(f)\varphi_\sigma;\ g\in L(G)\}\subset\mathscr{K}_\infty$. Since the subspace $\mathscr{K}_\infty$ must be closed in $\mathscr{H}_\sigma$ and $T^A(f)\varphi_\sigma\in\mathscr{H}_\sigma$, it follows that $T^A(f)\varphi_\sigma\in\mathscr{K}_\infty$. Now we obtain a linear mapping $\tilde{\sigma}$ of $\mathfrak{H}_0$ to $\mathscr{H}_\sigma/\mathscr{K}_\infty$ such that

$$\tilde{\sigma}(T(f)e_1)=T^A(f)\varphi_\sigma.$$

This is obviously an isomorphism of the $L(G)$-module $\mathfrak{H}_0$ onto the $L(G)$-module $\mathscr{H}_\sigma/\mathscr{K}_\infty$.

The last step of the proof of Lemma 8 is to show $\mathscr{K}_\infty=\{0\}$. Since $\mathscr{H}_\sigma(\delta)$ is isomorphic to the irreducible pd-dimensional $L(\delta)$-module $\mathfrak{H}(\delta)$, there exists a function $h\in L(\delta)$ such that

$$T^A(h)\varphi_\sigma=\varphi_\sigma,\ T^A(h)E^A_{j1}(\delta)\varphi_\sigma=0 \qquad (2\leq j\leq d).$$

Let ψ be an arbitrary element in $\mathscr{K}_\infty$. Find a function $g\in L(G)$ such that $T^A(g)\varphi_\sigma=\psi$, and put $f=g*h$, then the relations

$$T^A(f)\varphi_\sigma=\psi,\ T^A(f)E^A_{j1}(\delta)\varphi_\sigma=0 \qquad (2\leq j\leq d)$$

shows that $T^A(f)V^A_\sigma\subset\mathscr{K}_\infty$. This means that $\tilde{\sigma}(T(f)V)=\{0\}$ and hence that $T(f)V=\{0\}$. Thus the function $F=\Phi(f)$ belongs to $\mathfrak{M}(\mathfrak{a}_V)$, and accordingly $T^A(f)V^A_\sigma=\{0\}$ by the condition $\mathfrak{M}(\mathfrak{a}_V)\subset\{\Phi(f)\in A_r;\ T^A(f)V^A_\sigma=\{0\}\}$. Therefore we have $\psi=T^A(f)\varphi_\sigma=0$. We have now proved that $\mathscr{K}_\infty=\{0\}$. Q.E.D.

Corollary. *The set* $\mathrm{Hom}^\sharp_{L^\circ(\delta)}(\mathfrak{H}_1(\delta),\ \mathfrak{H}^A_1(\delta))$ *is a vector space.*

We shall denote by $d_{i1}a$, for every vector $a\in H$, the H-valued continuous function $u\to d_{i1}(u)a$ on K. Now we identify the function $\varphi=\sum_{i=1}^d d_{i1}a_i$ with the vector $\boldsymbol{a}={}^t(a_1,\dots,a_d)\in \boldsymbol{C}^d\otimes H$. Then it is not difficult to show that $\varphi\in\mathfrak{H}^A_1(\delta)$ if and only if $\boldsymbol{a}\in\boldsymbol{C}^d\otimes_M H$. Moreover, for a function $f\in L^\circ(\delta)$, the function $T^A(f)\varphi$ is given by

$$\begin{aligned}(T^A(f)\varphi)(u)&=(T^A(u)T^A(f)\varphi)\ (1)\\&=(T^A(f)T^A(u)\varphi)\ (1)\\&=\int_K\int_S(T^A(s)T^A(v)T^A(u)\varphi)\ (1)f(sv)d\mu(s)dv\\&=\int_K\int_S A(s)\varphi(vu)f(sv)d\mu(s)dv\\&=\sum_{i=1}^d\int_K\int_S A(s)d_{i1}(vu)a_if(sv)d\mu(s)dv\\&=\sum_{i,j=1}^d d_{j1}(u)\int_K\int_S A(s)d_{ij}(v)a_if(sv)d\mu(s)dv\\&=\sum_{j=1}^d d_{j1}(u)(\sum_{i=1}^d\int_S A(s)a_if_{ji}(s)d\mu(s))\end{aligned}$$

$$= \sum_{j=1}^{d} d_{j1}(u)\Big(\sum_{i=1}^{d} \Lambda(f_{ji})a_i\Big)$$

where $\Phi(f)=(f_{ij})\in A^\circ$. Namely the function $T^\Lambda(f)\varphi$ is identified with the vector $R(F)\boldsymbol{a}$. This shows that, through identification of $L^\circ(\delta)$ and A° via Φ, the $L^\circ(\delta)$-module $\mathfrak{H}_1^\Lambda(\delta)$ is identified with the A°-module $\boldsymbol{C}^d\otimes_M H$. In addition the $L^\circ(\delta)$-module $\mathfrak{H}_1(\delta)$ is also identified with the A°-module $A^\circ/\Phi(\mathfrak{a}_V)$. The following diagrams show these identifications:

$$\begin{array}{ccc}
\mathfrak{H}_1^\Lambda(\delta)\ni\varphi=\sum_{i=1}^{d} d_{i1}a_i & \longleftrightarrow & \boldsymbol{a}=\begin{pmatrix}a_1\\ \vdots\\ a_d\end{pmatrix}\in \boldsymbol{C}^d\otimes_M H\\
\Big\downarrow T^\Lambda(f) & \circlearrowleft & \Big\downarrow R(F)\\
\mathfrak{H}_1^\Lambda(\delta)\ni T^\Lambda(f)\varphi & \longleftrightarrow & R(F)\boldsymbol{a}\in \boldsymbol{C}^d\otimes_M H,
\end{array}$$

$$\begin{array}{ccc}
\mathfrak{H}_1(\delta)\ni T(g)e_1 & \longleftrightarrow & [G]\in A^\circ/\Phi(\mathfrak{a}_V)\\
\Big\downarrow T(f) & \circlearrowleft & \Big\downarrow F*\\
\mathfrak{H}_1(\delta)\ni T(f)T(g)e_1 & \longleftrightarrow & F*[G]\in A^\circ/\Phi(\mathfrak{a}_V)
\end{array}$$

where $f,\ g\in L^\circ(\delta)$ and $F=\Phi(f)\in A^\circ$, $G=\Phi(g)\in A^\circ$. Therefore we may identify the vector space $\mathrm{Hom}_{L^\circ(\delta)}(\mathfrak{H}_1(\delta),\ \mathfrak{H}_1^\Lambda(\delta))$ with the vector space $\mathrm{Hom}_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V),\ \boldsymbol{C}^d\otimes_M H)$.

Proposition 2. *Under the above identifications, the vector space* $\mathrm{Hom}^\sharp_{L^\circ(\delta)}(\mathfrak{H}_1(\delta),\ \mathfrak{H}_1^\Lambda(\delta))$ *is identified with the vector space* $\mathrm{Hom}^\sharp_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V),\ \boldsymbol{C}^d\otimes_M H)$.

Proof Let an element $\sigma\in\mathrm{Hom}_{L^\circ(\delta)}(\mathfrak{H}_1(\delta),\ \mathfrak{H}_1^\Lambda(\delta))$ be identified with $\xi\in\mathrm{Hom}_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V),\ \boldsymbol{C}^d\otimes_M H)$. Then the vector $e_1=T(\mathfrak{e})e_1$ is identified with the element $[\Phi(\mathfrak{e})]=[\mathfrak{E}]\in A^\circ/\Phi(\mathfrak{a}_V)$. Hence the function $\varphi_\sigma=\sigma(e_1)$ is identified with $\xi([\mathfrak{E}])$. So, if $\varphi_\sigma=\sum_{i=1}^{d} d_{i1}a_i$, then $\xi([\mathfrak{E}])={}^t(a_1,\dots,a_d)$. For any elements $s\in S$ and $u\in K$, it follows that

$$(T^\Lambda(su)E_{j1}^\Lambda(\delta)\varphi_\sigma)\,(1)=d\int_K \Lambda(s)\varphi_\sigma(uv)\overline{d_{j1}(v)}dv$$

$$=d\int_K \Lambda(s)\varphi_\sigma(v)\overline{d_{j1}(u^{-1}v)}dv$$

$$=d\sum_{i,k=1}^{d}\overline{d_{jk}(u^{-1})}\Lambda(s)a_i\int_K d_{i1}(v)\overline{d_{k1}(v)}dv$$

$$=\sum_{i=1}^{d}\overline{d_{ji}(u^{-1})}\Lambda(s)a_i.$$

Thus, by Lemmas 7 and 8, we know that

$$\xi\in\mathrm{Hom}^\sharp_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V),\ \boldsymbol{C}^d\otimes_M H)$$

$$\Longleftrightarrow R(F)\xi([\mathfrak{E}])=0\quad\text{for all}\quad F\in\mathfrak{M}(\mathfrak{a}_V)$$

$$\Longleftrightarrow \sum_{i=1}^{d}\Lambda(f_{ji})a_i=0\ (1\le j\le d)\quad\text{for all}\quad F=(f_{ij})\in\mathfrak{M}(\mathfrak{a}_V)$$

$\Longleftrightarrow \sum_{i=1}^{d} \int_S \int_K \overline{d_{ji}(u^{-1})} A(s) a_i f(su) du d\mu(s) = 0\ (1 \le j \le d)$ for all $f \in L(G)*\overline{\chi_\delta}$ satisfying $T(f)V = \{0\}$

$\Longleftrightarrow (T^A(f)E^A_{j1}(\delta)\varphi_\sigma)\ (1) = 0\ (1 \le j \le d)$ for all $f \in L(G)*\overline{\chi_\delta}$ satisfying $T(f)V = \{0\}$

$\Longleftrightarrow (T^A(f)E^A_{j1}(\delta)\varphi_\sigma)\ (1) = 0\ (1 \le j \le d)$ for all $f \in L(G)*\overline{\chi_\delta}$ satisfying $T(u^{-1})T(f)V = \{0\}$ for any $u \in K$

$\Longleftrightarrow (T^A(u)T^A(f)E^A_{j1}(\delta)\varphi_\sigma)\ (1) = 0\ (1 \le j \le d)$ for all $u \in K$ and $f \in L(G)*\overline{\chi_\delta}$ satisfying $T(f)V = \{0\}$

$\Longleftrightarrow T^A(f)E^A_{j1}(\delta)\varphi_\sigma = 0\ (1 \le j \le d)$ for all $f \in L(G)*\overline{\chi_\delta}$ satisfying $T(f)V = \{0\}$

$\Longleftrightarrow T^A(f)V^A_\sigma = \{0\}\ (1 \le j \le d)$ for all $f \in L(G)*\overline{\chi_\delta}$ satisfying $T(f)V = \{0\}$

$\Longleftrightarrow \sigma \in \mathrm{Hom}^\sharp_{L^\circ(\delta)}(\mathfrak{H}_1(\delta), \mathfrak{H}^A_1(\delta))$. Q. E. D.

§6. The last step: proof of Proposition 3

Let σ be an arbitrary non zero element in $\mathrm{Hom}^\sharp_{L^\circ(\delta)}(\mathfrak{H}_1(\delta), \mathfrak{H}^A_1(\delta))$. By the arguments in the proof of Lemma 8, it turns out that there exists an isomorphism $\tilde{\sigma}$ of the $L(G)$-module $\mathfrak{H}_0$ onto $\mathscr{H}_\sigma = \{T^A(f)\varphi_\sigma; f \in L(G)\}$ such that $\tilde{\sigma}(e_1) = \sigma(e_1) = \varphi_\sigma$. Then $\tilde{\sigma}$ is of course an element in $\mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}^A_0)$.

This correspondence $\sigma \to \tilde{\sigma}$ is obviously linear. Suppose $\tilde{\sigma} = 0$, then $\sigma(\mathfrak{H}_1(\delta)) = \{\sigma(T(f)e_1); f \in L^\circ(\delta)\} = \{T^A(f)\sigma(e_1); f \in L^\circ(\delta)\} = \{0\}$. Thus the correspondence is injective.

Now let us prove that the linear mapping $\sigma \to \tilde{\sigma}$ of $\mathrm{Hom}^\sharp_{L^\circ(\delta)}(\mathfrak{H}_1(\delta), \mathfrak{H}^A_1(\delta))$ into $\mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}^A_0)$ is surjective. Let σ' be an arbitrary element in $\mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}^A_0)$. Since the equality $\sigma'(E_{11}(\delta)T(f)e_1) = E^A_{11}(\delta)\sigma'(T(f)e_1)$ holds for all function $f \in L(G)$, we have $\sigma'(\mathfrak{H}_1(\delta)) \subset \mathfrak{H}^A_1(\delta)$. So the restriction σ of σ' onto $\mathfrak{H}_1(\delta)$ is an element in $\mathrm{Hom}^\sharp_{L^\circ(\delta)}(\mathfrak{H}_1(\delta), \mathfrak{H}^A_1(\delta))$ and it is clear that $\tilde{\sigma} = \sigma'$.

Therefore we obtain the following

Proposition 3. *The vector space* $\mathrm{Hom}^\sharp_{L^\circ(\delta)}(\mathfrak{H}_1(\delta), \mathfrak{H}^A_1(\delta))$ *is isomorphic to the vector space* $\mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}^A_0)$:

$$\mathrm{Hom}^\sharp_{L^\circ(\delta)}(\mathfrak{H}_1(\delta), \mathfrak{H}^A_1(\delta)) \cong \mathrm{Hom}_{L(G)}(\mathfrak{H}_0, \mathfrak{H}^A_0).$$

Now by the definition of $\mathrm{Hom}^\sharp_{A^\circ}(A^\circ/\Phi(\mathfrak{a}_V), \boldsymbol{C}^d \otimes_M H)$ and by Propositions 1, 2, and 3, the Theorem in §1 follows.

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