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Mixed problems for evolution equations

By

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Researches for heat equations or wave equations have a long history. In 1938, Petrowski studied Cauchy problem for evolution equations as a generalization of above equations. Moreover, he specialized two essential types of evolution equations, i.e. p-parabolic equations and strictly hyperbolic equations ([1]). After him, many authors studied the two types of equations in separate ways. On the other hand, recently, Volevich-Gindikin gave a concept of dominantly correct evolution equations, which are H^∞-well posed under any change of lower order terms in the sense of Newton polygon ([2]).

In this paper, we shall show the H^∞-well posedness of mixed problems for dominantly correct evolution equations, assuming the uniform Lopatinski conditions. The process of the analysis of our problem is just pararel to that in [3].

Our problem is to seek a solution u satisfying

$$(\mathrm{P})\quad \begin{cases} A(t, x; D_t, D_x)u = f & \text{in } (0, T)\times\Omega, \\ B_j(t, x; D_t, D_x)u = g_j & \text{on } (0, T)\times\partial\Omega \quad (j=1,\dots, m_+), \\ D_t^j u = u_j & \text{on } \{t=0\}\times\Omega \quad (j=0,\dots,\mu-1), \end{cases}$$

where $\{f, g_j, u_j\}$ are given datas and Ω is a domain in $\boldsymbol{R}^n$ (we only deal with the case when $\Omega=\boldsymbol{R}^n_+$). Our main result in this paper is

Theorem *Under the assumptions* (A), (B) *and* (B^*), *the problem* (P) *is* H^∞-*well posed.*

§1. Newton polygon

1.1. Newton polygon. For a polynomial

$$A(\tau, \xi) = \sum_{\sigma, \nu} a_{\sigma\nu}\tau^\sigma\xi^\nu,$$

we define the Newton polygon of A by

$$N_A = \text{convex hull of } \Delta_A \cup \Delta'_A,$$

where

$$\Delta_A=\{(\sigma, k);\ \sum_{|\nu|=k} |a_{\sigma\nu}|\neq 0\},$$

$$\Delta'_A=\{(\sigma, 0);\ (\sigma, k)\in\Delta_A\}\cup\{(0, k);\ (\sigma, k)\in\Delta_A\}\cup\{(0, 0)\}.$$

Let

$$O=(0, 0),\quad P_0=(\mu_1+\cdots+\mu_l, 0),\quad P_1(\mu_2+\cdots+\mu_l, m_1),$$

$$\ldots,\ P_{l-1}=(\mu_l, m_1+\cdots+m_{l-1}),\quad P_l=(0, m_1+\cdots+m_l)$$

be vertices of N_A, where $\mu_1,\ldots, \mu_l, m_1,\ldots, m_l$ are positive integers ($\sum \mu_i=\mu$, $\sum m_i=m$), then we have $+\infty\geqslant p_1>p_2>\cdots>p_l\geqslant 0$, $0\leqslant q_1<q_2<\cdots<q_l\leqslant+\infty$, where $\frac{m_i}{\mu_i}=p_i=q_i^{-1}$. If $+\infty>p_1>\cdots>p_l>0$, we say that N_A is normal. Hereafter we consider the case when N_A is normal. Defining non-decreasing positive numbers $(\gamma_1,\ldots, \gamma_m)$ such that

$$\begin{aligned}0&<\gamma_1=\gamma_2=\cdots=\gamma_{m_1}=q_1\\ &<\gamma_{m_1+1}=\gamma_{m_1+2}=\cdots=\gamma_{m_1+m_2}=q_2\\ &\cdots\cdots\\ &<\gamma_{m_1+\cdots+m_{l-1}+1}=\cdots=\gamma_{m_1+\cdots+m_l}=q_l<+\infty,\end{aligned}$$

we have

$$\begin{aligned}A(\tau, \xi)&=\sum_{(\sigma,|\nu|)\in N_A} a_{\sigma\nu}\tau^\sigma\xi^\nu\\ &=\sum_{k=0}^{m}\sum_{\sigma\leqslant\gamma_{k+1}+\cdots+\gamma_m}\Big(\sum_{|\nu|=k} a_{\sigma\nu}\xi^\nu\Big)\tau^\sigma,\end{aligned}$$

therefore we say that

$$\begin{aligned}A_0(\tau, \xi)&=\sum_{(\sigma,|\nu|)\in\bigcup_{i=1}^{l}\overline{P_{i-1}P_i}} a_{\sigma\nu}\tau^\sigma\xi^\nu\\ &=\sum_{k=0}^{m}\Big(\sum_{|\nu|=k} a_{\gamma_{k+1}+\cdots+\gamma_m\nu}\xi^\nu\Big)\tau^{\gamma_{k+1}+\cdots+\gamma_m}\\ &=\sum_{k=0}^{m} a_k^0(\xi)\tau^{\gamma_{k+1}+\cdots+\gamma_m}\end{aligned}$$

is the principal part of $A(\tau, \xi)$. Moreover we denote

$$A^{(0)}=a_0^0,$$

$$\begin{aligned}A^{(i)}(\tau, \xi)&=\sum_{(\sigma,|\nu|)\in\overline{P_{i-1}P_i}} a_{\sigma\nu}\xi^\nu\tau^{\sigma-(\mu_{i+1}+\cdots+\mu_l)}\\ &=\sum_{k=0}^{m_i} a^0_{m_1+\cdots+m_{i-1}+k}(\xi)\tau^{q_i(m_i-k)}\qquad(i=1,\ldots, l).\end{aligned}$$

Now let us denote

$$\tau^{q_i}=\omega_i\quad(i=1,\ldots, l),$$

then we have

$$A_0(\tau, \xi)=a_0^0\tau^\mu+\sum_{i=1}^{l}\sum_{k=1}^{m_i} a^0_{m_1+\cdots+m_{i-1}+k}(\xi)\tau^{q_i(m_i-k)+q_{i+1}m_{i+1}+\cdots+q_lm_l}$$

$$=a_0^0\omega_1^{m_1}\cdots\omega_l^{m_l}$$

$$+\sum_{i=1}^{l}\sum_{k=1}^{m_i} a^0_{m_1+\cdots+m_{i-1}+k}(\xi)\omega_i^{m_i-k}\omega_{i+1}^{m_{i+1}}\cdots\omega_l^{m_l}$$

$$=\hat{A}_0(\omega_1,\dots,\omega_l,\xi)$$

$$A^{(i)}(\tau,\xi)=\sum_{k=0}^{m_i} a^0_{m_1+\cdots+m_{i-1}+k}(\xi)\tau^{q_i(m_i-k)}$$

$$=\sum_{k=0}^{m_i} a^0_{m_1+\cdots+m_{i-1}+k}(\xi)\omega_i^{m_i-k}=\hat{A}^{(i)}(\omega_i,\xi),$$

where $\hat{A}_0(\omega,\xi)$ is homogencous of order m and $\hat{A}^{(i)}(\omega_i,\xi)$ is homogeneous of order $m_1+\cdots+m_i$ with respect ot (ω,ξ). Remarking that

$$\left|\frac{\omega_{i-1}}{\omega_i}\right|=|\tau|^{-(q_i-q_{i-1})}\leqslant K^{-\delta},$$

where $\operatorname{Im}\tau\leqslant -K$ $(K\geqslant 1)$, $\delta=\min_i(q_i-q_{i-1})>0$, we define

$$\Delta_i=\{z\in C^1;\ -\pi q_i\leqslant \arg z\leqslant 0\},$$

$$\Omega_K=\{\omega\in\Delta_1\times\cdots\times\Delta_l;\ |\omega_{i-1}|\leqslant K^{-\delta}|\omega_i|\ (i=2,\dots,l)\},$$

$$D_{i(K,\alpha,\beta)}=\{(\omega,\xi)\in\Omega_K\times C^n;\ \frac{|\omega_{i-1}|}{|\xi|}\leqslant\alpha,\ \frac{|\omega_{i+1}|}{|\xi|}\geqslant\beta\}\quad(i=0,1,\dots,l,\ 0<\alpha<1,\ \beta>1).$$

Then we have

Lemma 1.1. *There exists a positive constant C (independent of K,α,β) such that*

$$|\hat{A}_0(\omega,\xi)\omega_{i+1}^{-m_{i+1}}\cdots\omega_l^{-m_l}-\hat{A}^{(i)}(\omega_i,\xi)|\leqslant C\left\{\alpha\left(\frac{|\omega_i|}{|\xi|}\right)^{m_i}+\beta^{-1}\right\}|\xi|^{m_1+\cdots+m_i}$$

in $D_{i(K,\alpha,\beta)}$ $(i=0,1,\dots,l)$.

Proof. Since

$$\hat{A}_0(\omega,\xi)\omega_{i+1}^{-m_{i+1}}\cdots\omega_l^{-m_l}-\hat{A}^{(i)}(\omega_i,\xi)$$

$$=\sum_{k=1}^{i-1}\{\hat{A}^{(k)}(\omega_k,\xi)-a^0_{m_1+\cdots+m_k}(\xi)\}\omega_{k+1}^{m_{k+1}}\cdots\omega_i^{m_i}$$

$$+\sum_{k=i+1}^{l}\{\hat{A}^{(k)}(\omega_k,\xi)-a^0_{m_1+\cdots+m_{k-1}}(\xi)\omega_k^{m_k}\}\omega_{i+1}^{-m_{i+k}}\cdots\omega_k^{-m_k},$$

we have

$$\{\hat{A}_0(\omega, \xi)\omega_{i+1}^{-m_{i+1}}\cdots\omega_l^{-m_l} - \hat{A}^{(i)}(\omega_i, \xi)\}|\xi|^{-(m_1+\cdots+m_i)}$$

$$= \sum_{k=1}^{i-1}\left\{\hat{A}^{(k)}\left(\frac{\omega_k}{|\xi|}, \frac{\xi}{|\xi|}\right) - a^0_{m_1+\cdots+m_k}\left(\frac{\xi}{|\xi|}\right)\right\}\left(\frac{\omega_{k+1}}{|\xi|}\right)^{m_{k+1}}\cdots\left(\frac{\omega_i}{|\xi|}\right)^{m_i}$$

$$+ \sum_{k=i+1}^{l}\left\{\hat{A}^{(k)}\left(\frac{\omega_k}{|\xi|}, \frac{\xi}{|\xi|}\right) - a^0_{m_1+\cdots+m_{k-1}}\left(\frac{\xi}{|\xi|}\right)\left(\frac{\omega_k}{|\xi|}\right)^{m_k}\right\}\left(\frac{\omega_{i+1}}{|\xi|}\right)^{-m_{i+1}}\cdots\left(\frac{\omega_k}{|\xi|}\right)^{-m_k}$$

$$= I_1 + I_2,$$

where

$$|I_1| \leqslant C\frac{|\omega_{i-1}|}{|\xi|}\left(\frac{|\omega_i|}{|\xi|}\right)^{m_i} \leqslant C\alpha\left(\frac{|\omega_i|}{|\xi|}\right)^{m_i},$$

$$|I_2| \leqslant C\left(\frac{|\omega_{i+1}|}{|\xi|}\right)^{-1} \leqslant C\beta^{-1}. \qquad \blacksquare$$

Here we remark

Lemma 1.2.

$$\{\hat{A}_0(\omega, \xi) - \hat{A}^{(l)}(\omega_l, \xi)\}\omega_l^{-m}$$

$$= \sum_{i=1}^{l-1}\sum_{k=0}^{\mu_i-1} a^0_{m_1+\cdots+m_{i-1}+p_ik}\left(\frac{\xi}{\omega_i}\right)\left(\frac{\omega_i}{\omega_l}\right)^{p_i(\mu_i-k)}\left(\frac{\omega_{i+1}}{\omega_l}\right)^{p_{i+1}\mu_{i+1}}\cdots\left(\frac{\omega_{l-1}}{\omega_l}\right)^{p_{l-1}\mu_{l-1}}$$

$$= \sum_{i=1}^{l-1}\left(\frac{\omega_i}{\omega_l}\right)^{p_i}\varphi_i\left(\frac{\omega}{\omega_l}, \frac{\xi}{\omega_l}\right),$$

where $\varphi_i(\omega, \xi)$ is a homogeneous polynomial of degree $m - p_i$.

Now, in our case, we consider a polynomial with respect to (τ, ξ) with coefficients depending parameters (t, x), that is,

$$A(t, x; \tau, \xi) = \sum_{(\sigma, |\nu|)\in N_A} a_{\sigma\nu}(t, x)\tau^\sigma\xi^\nu,$$

where $a_{\mu 0}(t, x) = 1$, $a_{\sigma\nu}(t, x) \in \mathscr{B}^\infty(\boldsymbol{R}^{n+1})$, and $a_{\sigma\nu}$ are constant outside a ball, whose Newton polygon is normal and independent of (t, x). Moreover, we assume

Assumption (A).

i) $a^0_{m_1+\cdots+m_i}(t, x; \xi) \neq 0$ for $(t, x, \xi) \in \boldsymbol{R}^{n+1}\times S^{n-1}$ $(i = 1,\dots, l-1)$, and $a^0_m(t, x; 1, 0,\dots, 0) \neq 0$.

ii) P_i is even and $A^{(i)}(t, x; \tau, \xi) \neq 0$ for $(t, x, \xi) \in \boldsymbol{R}^{n+1}\times S^{n-1}$ and $\operatorname{Im}\tau \leqslant 0$ $(i = 1,\dots, l-1)$.

iii) $P_l = 1$ and zeros of $A^{(l)}(t, x; \tau, \xi)$ with respect to τ are real and distinct for $(t, x, \xi) \in \boldsymbol{R}^{n+1}\times S^{n-1}$.

1.2. Newton polygon with respect to A. Let us consider a boundary polynomial $B(\tau, \xi)$ with respect to a polynomial $A(\tau, \xi)$. Let N_B be the Newton poylgon of

$$B(\tau, \xi)=\sum_{\sigma, \nu} b_{\sigma\nu}\tau^{\sigma}\xi^{\nu}$$

and $N_B \subset N'_A$, where

$$N'_A=(N_A-(0, 1)) \cap N_A,$$

whose vertices are

$$(0, 0), \quad (\mu'_1+\cdots+\mu'_l, 0), \quad (\mu'_2+\cdots+\mu'_l, m'_1), \dots, (0, m'_1+\cdots+m'_l).$$

Now let us define the Newton polygon of B with respect to A by

$$N_B^{(A)}=\bigcap_{(\alpha,\beta)\in I_{AB}} ((N'_A-(\alpha, \beta)) \cap N'_A), \quad I_{AB}=\{(\alpha, \beta);\ N'_A-(\alpha, \beta) \supset N_B\},$$

then $N_B^{(A)}$ is a polygon whose sides are composed of parts of those of N'_A. Especially when $N_B^{(A)}=N'_A$, we say that B is a standard boundary polynomial with respect to A. Let

$$O=(0, 0),\ P_0^0=(\mu_1^0+\cdots+\mu_l^0, 0),\ P_1^0=(\mu_2^0+\cdots+\mu_l^0, m_1^0), \dots,\ P_l^0=(0, m_1^0+\cdots+m_l^0)$$

be vertices of $N_B^{(A)}$, some of which may coinside, such that

$$\frac{m_i^0}{\mu_i^0}=p_i, \ \mu_i^0 \leqslant \mu'_i, \quad m_i^0 \leqslant m'_i,$$

where we denote

$$\mu_i^1=\mu'_i-\mu_i^0, \quad m_i^1=m'_i-m_i^0.$$

We say that

$$B_0(\tau, \xi)=\sum_{(\sigma,|\nu|)\in \bigcup_{i=1}^{l} \overline{P_{i-1}^0P_i^0}} b_{\sigma\nu}\tau^{\sigma}\xi^{\nu}=\sum_{(\sigma,k)\in \bigcup_{i=1}^{l} \overline{P_{i-1}^0P_i^0}} b_k^0(\xi)\tau^{\sigma}$$

is the principal part of B with respect to A. Moreover, we denote

$$B^{(i)}(\tau, \xi=) \sum_{(\sigma,|\nu|)\in \overline{P_{i-1}^0P_i^0}} b_{\sigma\nu}\xi^{\nu}\tau^{\sigma-(\mu_{i+1}^0+\cdots+\mu_l^0)}.$$

Denoting $\omega_i=\tau^{q_i}$, we define $\hat{B}_0$, $\hat{B}^{(i)}$ as follows:

$$\begin{aligned}B_0(\tau, \xi) &= b_0^0\omega_1^{m_1^0}\cdots\omega_l^{m_l^0} \\ &\quad + \sum_{i=1}^{l}\ \sum_{(\sigma,k)\in \overline{P_{i-1}^0P_i^0}/\{P_{i-1}^0\}} b_k^0(\xi)\omega_i^{p_i\sigma-(m_{i+1}^0+\cdots+m_l^0)}\omega_{i+1}^{m_{i+1}^0}\cdots\omega_l^{m_l^0} \\ &= \hat{B}_0(\omega, \xi),\end{aligned}$$

$$B^{(i)}(\tau, \xi)=B^{(i)}(\omega_i^{p_i}, \xi)=\hat{B}^{(i)}(\omega_i, \xi).$$

Moreover, denoting

$$C(\tau, \xi')=\hat{C}(\omega, \xi')=\prod_{j=1}^{l}(\omega_j-i|\xi'|)^{m_j^1}, \quad \xi'=(\xi_2,\dots,\xi_n),$$

we introduce a standardized boundary polynomial $B^\#$ of B with respect to A:

$$B^\#(\tau, \xi)=\hat{B}^\#(\omega, \xi)=\hat{C}(\omega, \xi')\hat{B}_0(\omega, \xi),$$

$$B^{\#(i)}(\tau, \xi)=\hat{B}^{\#(i)}(\omega_i, \xi)=(-i|\xi'|)^{m_1^1+\cdots+m_{i-1}^1}(\omega_i-i|\xi'|)^{m_i^1}\hat{B}^{(i)}(\omega_i, \xi).$$

Lemma 1.3. *There exists a positive constant C (independent of K, α, β) such that*

$$|\hat{B}^\#(\omega, \xi)\omega_{i+1}^{-m'_{i+1}}\cdots\omega_l^{-m'_l}-\hat{B}^{\#(i)}(\omega_i, \xi)|\leqslant C(\alpha+\beta^{-1})|\xi|^{m'_1+\cdots+m'_i}\left(1+\frac{|\omega_i|}{|\xi|}\right)^{m'_i}$$

in $D_{i(K,\alpha,\beta)}$.

Proof. Set

$$\begin{aligned}
&\hat{B}^\#(\omega, \xi)\omega_{i+1}^{-m'_{i+1}}\cdots\omega_l^{-m'_l}-\hat{B}^{\#(i)}(\omega_i, \xi)\\
&=\left(\prod_{j=1}^{l}(\omega_j-i|\xi'|)^{m_j^1}\right)\omega_{i+1}^{-m_{i+1}^1}\cdots\omega_l^{-m_l^1}\\
&\quad\times\{\hat{B}_0(\omega, \xi)\omega_{i+1}^{-m_{i+1}^0}\cdots\omega_l^{-m_l^0}-\hat{B}^{(i)}(\omega_i, \xi)\}\\
&\quad+\left\{\prod_{j=1}^{l}(\omega_j-i|\xi'|)^{m_j^1}-\prod_{j=1}^{i-1}(-i|\xi'|)^{m_j^1}(\omega_i-i|\xi'|)^{m_i^1}\prod_{j=i+1}^{l}\omega_j^{m_j^1}\right\}\\
&\quad\times\hat{B}^{(i)}(\omega_i, \xi)\omega_{i+1}^{-m_{i+1}^1}\cdots\omega_l^{-m_l^1}\\
&=I_1+I_2.
\end{aligned}$$

Since

$$|(\omega_j-i|\xi'|)-(-i|\xi'|)|=|\omega_j|\leqslant\alpha|\xi| \qquad \text{if} \quad j<i,$$

$$|(\omega_j-i|\xi'|)-\omega_j|=|\xi'|\leqslant\beta^{-1}|\omega_j| \qquad \text{if} \quad j>i,$$

we have

$$\begin{aligned}
|I_2|&\leqslant C(\alpha+\beta^{-1})|\xi|^{m_1^1+\cdots+m_{i-1}^1}(|\omega_i|+|\xi|)^{m_i^1}|\hat{B}^{(i)}(\omega_i, \xi)|\\
&\leqslant C'(\alpha+\beta^{-1})|\xi|^{m'_1+\cdots+m'_{i-1}}(|\omega_i|+|\xi|)^{m'_i}.
\end{aligned}$$

On the other hand, since

$$|\hat{B}_0(\omega, \xi)\omega_{i+1}^{-m_{i+1}^0}\cdots\omega_l^{-m_l^0}-\hat{B}^{(i)}(\omega_i, \xi)|\leqslant C(\alpha+\beta^{-1})|\xi|^{m_1^0+\cdots+m_{i-1}^0}(|\omega_i|+|\xi|)^{m_i^0},$$

we have

$$|I_1|\leqslant C'(\alpha+\beta^{-1})|\xi|^{m'_1+\cdots+m'_{i-1}}(|\omega_i|+|\xi|)^{m'_i}. \qquad \blacksquare$$

In our case, we consider a boundary polynomial B with respect to A with coefficients depending on (t, x), whose Newton polygon with respect to A is independent of (t, x), that is,

$$B(t, x; \tau, \xi) = \sum_{(\sigma, |\nu|) \in N_B^{(A)}} b_{\sigma\nu}(t, x)\tau^\sigma \xi^\nu, \quad b_{\sigma\nu}(t, x) \in \mathscr{B}^\infty(\boldsymbol{R}^{n+1}),$$

where $b_{\sigma\nu}$ are constant outside a ball.

§2. Uniform Lopatinski conditions

Hereafter we change the notations of variables:

$$(x_1, \dots, x_n) \longrightarrow (x, y_1, \dots, y_{n-1}) = (x, y),$$

$$(\xi_1, \dots, \xi_n) \longrightarrow (\xi, \eta_1, \dots, \eta_{n-1}) = (\xi, \eta).$$

For short, sometimes we shall write $A(\tau, \xi, \eta)$ instead of $A(t, x, y; \tau, \xi, \eta)$, omitting the variables (t, x, y).

From the assumption (A), we have

$$A^{(l)}(\tau, \xi, 0) = a^0_{m-m_l}(\xi, 0)\tau^{m_l} + \cdots + a^0_m(\xi, 0)$$

$$= c\xi^{m-m_l} \prod_{j=1}^{m_l} (\xi - \kappa_{lj}\tau),$$

where $\{\kappa_{lj}\}_{j=1,\dots,m}$ are real, distinct and non-zero. Let m_l^+ of them be negative and m_l^- be positive, that is,

$$\kappa_{l1} < \kappa_{l2} < \cdots < \kappa_{lm_l^+} < 0 < \kappa_{lm_l^+ + 1} < \cdots < \kappa_{lm_l}.$$

Let

$$m_i^\pm = \frac{1}{2} m_i \qquad (i = 1, \dots, l-1) \quad \text{and} \quad m_\pm = \sum_{i=1}^{l} m_i^\pm,$$

then we have

Lemma 2.1. *m_+ (resp. m_-) zeros of $A_0(\tau, \xi, \eta)$ with respect to ξ have positive (resp. negative) imaginary parts, if* $\operatorname{Im} \tau < -K$ *(K: large enough) and $\eta \in \boldsymbol{R}^{n-1}$.*

The proof of Lemma 2.1 will be shown later in §3. We denote

$$\hat{A}_0(\omega, \xi, \eta) = c \prod_{j=1}^{m_+} (\xi - \xi_j^+(\omega, \eta)) \prod_{j=1}^{m_-} (\xi - \xi_j^-(\omega, \eta)) = C\hat{A}_0^+(\omega, \xi, \eta)\hat{A}_0^-(\omega, \xi, \eta),$$

where $\operatorname{Im} \xi_j^\pm(\omega, \eta) \gtrless 0$ if $\omega_i = \omega_i(\tau) = \tau^{q_i}$, $\operatorname{Im} \tau < -K$ and $\eta \in \boldsymbol{R}^{n-1}$.

2.1. Lopatinski determinant. Let us denote

$$\boldsymbol{B}(\tau, \xi, \eta) = \begin{pmatrix} B_1(\tau, \xi, \eta) \\ \vdots \\ B_{m_+}(\tau, \xi, \eta) \end{pmatrix}, \quad \boldsymbol{B}^\#(\tau, \xi, \eta) = \begin{pmatrix} B_1^\#(\tau, \xi, \eta) \\ \vdots \\ B_{m_+}^\#(\tau, \xi, \eta) \end{pmatrix},$$

where $\{B_j(\tau, \xi, \eta)\}_{j=1,\dots,m_+}$ are boundary polynomials with respect to $A(\tau, \xi, \eta)$, and $\{B_j^\#(\tau, \xi, \eta)\}_{j=1,\dots,m_+}$ are their standardizations. We define the basic Lopatinski determinant for $\{A, \boldsymbol{B}\}$ by

$$\hat{R}(\omega, \eta)=\det\left(\frac{1}{2\pi i}\oint\frac{\hat{\boldsymbol{B}}_0^{\#}(\omega, \xi, \eta)\xi^{m_+-k}}{\hat{A}_0^+(\omega, \xi, \eta)}d\xi\right)_{k=1,\dots,m_+}.$$

Let us denote a $m_+\times(m_1^++\cdots+m_i^+)$-matrix

$$\hat{B}_+^{(i)}(\omega_i, \eta)=\left(\frac{1}{2\pi i}\oint\frac{\hat{\boldsymbol{B}}^{(i)}(\omega_i, \xi, \eta)\xi^{m_1^++\cdots+m_i^+-k}}{\hat{A}_+^{(i)}(\omega_i, \xi, \eta)}d\xi\right)_{k=1,\dots,m_1^++\cdots+m_i^+},$$

where $\hat{A}_+^{(i)}(\omega_i, \xi, \eta)=\prod_{j=1}^{m_1^++\cdots+m_i^+}(\xi-\xi_j^{(i)}(\omega_i, \eta))$ and $\{\xi_j^{(i)}\}_{j=1,\dots,m_1^++\cdots+m_i^+}$ are zeros of $\hat{A}^{(i)}(\omega_i, \xi, \eta)$ with respect to ξ satisfying $\operatorname{Im}\xi>0$ when $\omega_i\in\Delta_i$. Moreover we denote a $m_+\times m_i^+$-matrix

$$b_+^{(i)}=\left(\left(\frac{1}{2\pi i}\oint\frac{\hat{\boldsymbol{B}}^{(i)}(1, \xi, 0)\xi^{\mu_i-k}}{\prod_{\alpha=1}^{\mu_i}(\xi-e^{i2\pi q_i(j-1)}\kappa_{i\alpha})}d\xi\right)_{k=1,\dots,\mu_i}\right)_{j=1,\dots,\frac{1}{2}p_i}$$

$$(i=1,\dots, l-1),$$

and a $m_+\times m_l^+$-matrix

$$b_+^{(l)}=\left(\frac{1}{2\pi i}\oint\frac{\hat{\boldsymbol{B}}^{(l)}(1, \xi, 0)}{\xi-\kappa_{lj}}d\xi\right)_{j=1,\dots,m_l^+},$$

where

$$\hat{A}^{(i)}(\omega_i, \xi, 0)=a^0_{m_1+\cdots+m_{i-1}}(\xi, 0)\prod_{j=1}^{\mu_i}(\omega_i^{p_i}-\kappa_{ij}^{-p_i}\xi^{p_i}).$$

Here we define

$$\hat{R}^{(i)}(\omega_i, \eta)=\det(\hat{B}_+^{(i)}(\omega_i, \eta)b_+^{(i+1)}b_+^{(i+2)}\cdots b_+^{(l)})\qquad(i=0, 1,\dots, l),$$

then $\hat{R}^{(i)}(\omega_i, \eta)$ is homogeneous of order

$$s_i=\sum_{k=1}^{m_1^++\cdots+m_i^+}(m_1'+\cdots+m_i'+1-k)$$

$$=(m_1^++\cdots+m_i^+)\{m_1'+\cdots+m_i'+\frac{1}{2}-\frac{1}{2}(m_1^++\cdots+m_i^+)\}$$

with respect to (ω_i, η) and we have

Lemma 2.2. *We have*

$$\hat{R}(\omega, \eta)\omega_{i+1}^{-(s_{i+1}-s_i)}\cdots\omega_l^{-(s_l-s_{l-1})}$$

$$=c_i\hat{R}^{(i)}(\omega_i, \eta)+\varepsilon_i(\omega, \eta)|\eta|^{s_i}\qquad(i=0, 1,\dots, l),$$

where $|c_i|>const.\ (>0)$ *and* $\varepsilon_i(\omega, \eta)\to 0$ *as* $\beta K^{-\delta}+\beta^{-1}\to 0$ *in* $\mathscr{D}_{i(K,\beta)}$ $(i=0,\dots, l-1)$ *and* $\varepsilon_l(\omega, \eta)\to 0$ *as* $\beta K^{-\delta}\to 0$ *in* $\mathscr{D}_{l(K,\beta)}$(see the notations in §3).

Proof. Let $(\omega, \eta)\in\mathscr{D}_{i_0(K,\beta)}$ $(0\leqslant i_0\leqslant l-1)$, then we denote

$$\hat{A}_0^+(\xi)=\mathscr{E}_+(\xi)\prod_{i=i_0+1}^{l} E_i^+(\xi)(E_l^+=H_l^+),\qquad E_i^+(\xi)=\prod_{k=1}^{m_i^+}(\xi-\xi_{ik}),$$

in case when $\{\xi_{ik}\}$ are distnct, because the other case is considered as the limit case. Then we have

$$\hat{R}=\det\left(\frac{1}{2\pi i}\oint\frac{\hat{\boldsymbol{B}}_0^\sharp(\xi)\xi^{m_+-k}}{\mathscr{E}_+(\xi)\prod_{i=i_0+1}^{l}E_i^+(\xi)}\,d\xi\right)_{k=1,\ldots,m_+}$$

$$=\det(MM_{i_0+1}\ldots M_l),$$

where

$$M=\left(\frac{1}{2\pi i}\oint_{\partial G^{(i_0)}}\frac{\hat{\boldsymbol{B}}_0^\sharp(\xi)\xi^{m_1^++\cdots+m_{i_0}^+-k}}{\mathscr{E}_+(\xi)}\,d\xi\right)_{k=1,\ldots,m_1^+\cdots+m_{i_0}^+},$$

$$M_i=\left(\frac{1}{2\pi i}\oint_{|\xi-\xi_{ik}|=\delta}\frac{\hat{\boldsymbol{B}}_0^\sharp(\xi)}{\mathscr{E}_+(\xi)\prod_{j=i_0+1}^{i-1}E_j^+(\xi)\prod_{j=1}^{k}(\xi-\xi_{ij})}\,d\xi\right)_{k=1,\ldots,m_i^+}$$

$$(i_0+1\leqslant i\leqslant l).$$

Denoting

$$(M'_{0k})=\left(\frac{1}{2\pi i}\oint_{\partial G^{(i_0)}}\frac{\hat{\boldsymbol{B}}^{\sharp(i_0)}\left(\frac{\omega_i}{|\eta|},\frac{\xi}{|\eta|},\frac{\eta}{|\eta|}\right)\left(\frac{\xi}{|\eta|}\right)^{m_1^++\cdots+m_{i_0}^+-k}}{\hat{A}_+^{(i_0)}\left(\frac{\omega_{i_0}}{|\eta|},\frac{\xi}{|\eta|},\frac{\eta}{|\eta|}\right)}\frac{d\xi}{|\eta|}\right),$$

$$(M'_{ik})=\left(\frac{1}{2\pi i}\oint_{|\xi-\xi_{ik}|=\delta}\frac{\hat{\boldsymbol{B}}^{\sharp(i)}\left(\frac{\omega_i}{|\omega_i|},\frac{\xi}{|\omega_i|},\frac{\eta}{|\omega_i|}\right)}{\prod_{j=1}^{k}\left(\frac{\xi}{|\omega_i|}-\frac{\xi_{ij}}{|\omega_i|}\right)}\frac{d\xi}{|\omega_i|}\right),$$

we have

$$M=((M'_{0k}+o(1))|\eta|^{m_1'+\cdots+m_{i_0}'+1-k}\prod_{i=i_0+1}^{l}\omega_i{}^{m_i'})_k,$$

$$M_i=\left(\frac{1}{\mathscr{E}_+(\xi_{ik})\prod_{j=i_0+1}^{i-1}E_j^+(\xi_{ik})}(M'_{ik}+o(1))|\omega_i|^{m_1'+\cdots+m_i'+1-k}\right.$$

$$\left.\times\prod_{j=i+1}^{l}\omega_j^{m_j'}\right)_k,$$

therefore we have

$$\hat{R}=\{|\eta|^{\sum_{k=1}^{m_1^++\cdots+m_{i_0}^+}(m_1'+\cdots+m_{i_0}'+1-k)}\left(\prod_{j=i_0+1}^{l}\omega_j^{m_j'}\right)^{m_1^++\cdots+m_{i_0}^+}\}$$

$$\times\prod_{i=i_0+1}^{l}\{|\omega_i|^{\sum_{k=1}^{m_i^+}(m_1'+\cdots+m_i'+1-k)}\left(\prod_{j=i+1}^{l}\omega_j^{m_j'}\right)^{m_i^+}\}$$

$$\times\prod_{i=i_0+1}^{l}\prod_{k=1}^{m_i^+}\{\mathscr{E}_+(\xi_{ik})\prod_{j=i_0+1}^{i-1}E_j^+(\xi_{ik})\}^{-1}$$

$$\times\{\det(M'M'_{i_0+1}\cdots M'_l)+o(1)\}$$
$$=\varphi\{\det(M'M'_{i_0+1}\cdots M'_l)+o(1)\},$$

where

$$\varphi=|\eta|^{s_{i_0}}\prod_{i=i_0+1}^{l}|\omega_i|^{s_i-s_{i-1}}\varphi_0 \qquad (0<c_1\leqslant|\varphi_0|\leqslant c_2)$$

and

$$\det(M'M'_{i_0+1}\cdots M'_l)=\hat{R}^{(i_0)}|\eta|^{-s_{i_0}}+o(1). \qquad \blacksquare$$

Now we assume

Assumption (B). [uniform Lopatinski condition for $\{A, \boldsymbol{B}\}$]

$$\inf_{(t,y)\in \boldsymbol{R}^n}|\hat{R}^{(i)}(t, 0, y; \omega_i, \eta)|\neq 0$$

for $\omega_i\in\Delta_i$ and $\eta\in S^{n-2}$ $(i=0, 1,\dots, l)$.

2.2. Examples.

Let us consider

$$A=(\tau-i(\xi^2+\eta^2))(\tau+a\xi+b\eta) \qquad (a, b\text{: real, } a\neq 0),$$

then

$$A_0=\tau^2-i(\xi^2+\eta^2)\tau-i(\xi^2+\eta^2)(a\xi+b\eta)$$

and we have, taking $\omega_1=\tau^{\frac{1}{2}}$ and $\omega_2=\tau$,

$$\hat{A}^{(1)}=\omega_1^2-i(\xi^2+\eta^2)=-i(\xi-\xi^+(\omega_1, \eta))(\xi-\xi^-(\omega_1, \eta)),$$
$$\hat{A}^{(2)}=-i(\xi^2+\eta^2)(\omega_2+a\xi+b\eta).$$

i) [in case of $a<0$ and $B=1$]

$$\hat{B}^\#=(\omega_1-i|\eta|)(\omega_2-i|\eta|), \quad \hat{B}^{\#(1)}=\omega_1-i|\eta|,$$
$$\hat{R}^{(1)}=\frac{1}{2\pi i}\int\frac{\omega_1-i|\eta|}{\xi-\xi^+(\omega_1, \eta)}d\xi=\omega_1-i|\eta|.$$

ii) [in case of $a<0$ and $B=\xi$]

$$\hat{B}^\#=(\omega_2-i|\eta|)\xi,\ \hat{B}^{\#(1)}=\xi,\ \hat{R}^{(1)}=\frac{1}{2\pi i}\int\frac{\xi}{\xi-\xi^+(\omega_1, \eta)}d\xi=\xi^+(\omega_1, \eta).$$

iii) [in case of $a>0$ and $B_1=1$, $B_2=\xi$]

$$\hat{\boldsymbol{B}}^\#=\begin{pmatrix}\hat{B}_1^\#\\ \hat{B}_2^\#\end{pmatrix}=\begin{pmatrix}(\omega_1-i|\eta|)(\omega_2-i|\eta|)\\ (\omega_2-i|\eta|)\xi\end{pmatrix},$$

$$\hat{\boldsymbol{B}}^{\#(1)}=\begin{pmatrix}\omega_1-i|\eta|\\ \xi\end{pmatrix},\quad \hat{\boldsymbol{B}}^{\#(2)}=\begin{pmatrix}-i|\eta|(\omega_2-|\eta|)\\ (\omega_2-i|\eta|\xi)\end{pmatrix},$$

$$\hat{R}^{(1)}=\begin{vmatrix}\frac{1}{2\pi i}\int\frac{\omega_1-i|\eta|}{\xi-\xi^+(\omega_1,\eta)}d\xi & 0\\ \frac{1}{2\pi i}\int\frac{\xi}{\xi-\xi^+(\omega_1,\eta)}d\xi & \frac{1}{2\pi i}\int\frac{\xi}{\xi(\xi+a^{-1})}d\xi\end{vmatrix}$$

$$=\omega_1-i|\eta|,$$

$$\hat{R}^{(2)}=\begin{vmatrix}\frac{1}{2\pi i}\int\frac{-i|\eta|(\omega_2-i|\eta|)\xi}{(\xi-i|\eta|)(\xi+a^{-1}(\omega_2+b\eta))}d\xi & \frac{1}{2\pi i}\int\frac{-i|\eta|(\omega_2-i|\eta|)}{(\xi-i|\eta|)(\xi+a^{-1}(\omega_2+b\eta))}d\xi\\ \frac{1}{2\pi i}\int\frac{(\omega_2-i|\eta|)\xi^2}{(\xi-i|\eta|)(\xi+a^{-1}(\omega_2+b\eta))}d\xi & \frac{1}{2\pi i}\int\frac{(\omega_2-i|\eta|)\xi}{(\xi-i|\eta|)(\xi+a^{-1}(\omega_2+b\eta))}d\xi\end{vmatrix}$$

$$=-i|\eta|(\omega_2-i|\eta|)^2.$$

iv) [in case of $a>0$ and $B_1=1$, $B_2=\xi^2$]

$$\hat{\boldsymbol{B}}^{\#}=\begin{pmatrix}\hat{B}_1^{\#}\\ \hat{B}_2^{\#}\end{pmatrix}=\begin{pmatrix}(\omega_1-i|\eta|)(\omega_2-i|\eta|)\\ \xi^2\end{pmatrix},$$

$$\hat{\boldsymbol{B}}^{\#(1)}=\begin{pmatrix}\omega_1-i|\eta|\\ 0\end{pmatrix},\quad \hat{\boldsymbol{B}}^{\#(2)}=\begin{pmatrix}-i|\eta|(\omega_2-i|\eta|)\\ \xi^2\end{pmatrix},$$

$$\hat{R}^{(1)}=\begin{vmatrix}\frac{1}{2\pi i}\int\frac{\omega_1-i|\eta|}{\xi-\xi^+(\omega_1,\eta)}d\eta & 0\\ 0 & \frac{1}{2\pi i}\int_{|\xi+a^{-1}|=\varepsilon}\frac{\xi^2}{\xi(\xi+a^{-1})}d\xi\end{vmatrix}$$

$$=(\omega_1-i|\eta|)(-a^{-1}),$$

$$\hat{R}^{(2)}=\begin{vmatrix}\frac{1}{2\pi i}\int\frac{-i|\eta|(\omega_2-i|\eta|)\xi}{(\xi-i|\eta|)(\xi+a^{-1}(\omega_2+b\eta))}d\xi & \frac{1}{2\pi i}\int\frac{-i|\eta|(\omega_2-|\eta|)}{(\xi-i|\eta|)(\xi+a^{-1}(\omega_2+b\eta))}d\xi\\ \frac{1}{2\pi i}\int\frac{\xi^3}{(\xi-i|\eta|)(\xi+a^{-1}(\omega_2+b\eta))}d\xi & \frac{1}{2\pi i}\int\frac{\xi^2}{(\xi-i|\eta|)(\xi+a^{-1}(\omega_2+b\eta))}d\xi\end{vmatrix}$$

$$=-i|\eta|(\omega_2-i|\eta|)\begin{vmatrix}\frac{1}{2\pi i}\int\frac{1}{\xi-i|\eta|}d\xi & 0\\ \frac{1}{2\pi i}\int\frac{\xi^2}{\xi-i|\eta|}d\xi & \frac{1}{2\pi i}\int\frac{(i|\eta|-a^{-1}(\omega_2+b\eta))\xi}{(\xi-i|\eta|)(\xi+a^{-1}(\omega_2+b\eta))}d\xi\end{vmatrix}$$

$$=-i|\eta|(\omega_2-i|\eta|)(i|\eta|-a^{-1}(\omega_2+b\eta)).$$

§3. Basic properties

3.1. Zeros of $\hat{A}^{(i)}(\omega_i, \xi, \eta)$.

First, from the assumption (A), there exists $\varepsilon_0>0$ such that

$$\begin{cases} \inf_{(\xi,\eta)\in S^{n-1}} |a^0_{m_1+\cdots+m_i}(\xi,\eta)|>\varepsilon_0 \quad (i=1,\dots,l-1), \quad |a^0_m(1,0)|>\varepsilon_0, \\ \inf_{\substack{(\xi,\eta)\in S^{n-1}\\ \omega_i\in\Delta_i}} |\hat{A}^{(i)}(\omega_i,\xi,\eta)|>\varepsilon_0(|\omega_i|+1)^{m_i} \quad (i=1,\dots,l-1). \end{cases}$$

Hence there exist $c_0>0$ and $C_0>0$ such that

i) $$|a^0_{m_1+\cdots+m_i}(\xi,\eta)|\geqslant \frac{1}{2}\varepsilon_0(\sqrt{|\xi|^2+|\eta|^2})^{m_1+\cdots+m_i} \quad (i=1,\dots,l-1)$$

in $\{(\xi,\eta)\in C^1\times R^{n-1};\ \xi\notin G_+\cup G_-\}$, and

$$|a^0_m(\xi,\eta)|\geqslant \frac{1}{2}\varepsilon_0|\xi|^m$$

in $\{(\xi,\eta)\in C^1\times R^{n-1};\ \xi\notin G\}$, where

$$G=\{\xi\in C^1;\ |\xi|<C_0|\eta|\},\quad G_\pm=G\cap\{\operatorname{Im}\xi\gtrless\pm c_0|\eta|\},$$

ii) $$|\hat{A}^{(i)}(\omega_i,\xi,\eta)|\geqslant \frac{1}{2}\varepsilon_0(\sqrt{|\xi|^2+|\eta|^2})^{m_1+\cdots+m_{i-1}}(|\omega_i|+\sqrt{|\xi|^2+|\eta|^2})^{m_i}$$

in $\{(\omega_i,\xi,\eta)\in\Delta_i\times C^1\times R^{n-1};\ \xi\notin G^{(i)}_+\cup G^{(i)}_-\}$ $(i=1,\dots,l-1)$, and

$$|\hat{A}^{(l)}(\omega_l,\xi,\eta)|\geqslant \frac{1}{2}\varepsilon_0|\xi|^m$$

in $\{(\omega_l,\xi,\eta)\in\Delta_l\times C^1\times R^{n-1};\ \xi\notin G^{(l)}\}$, where

$$G^{(i)}=\{\xi\in C^1;\ |\xi|<C_0(|\omega_i|+|\eta|)\},\ G^{(i)}_\pm=G^{(i)}\cap\{\operatorname{Im}\xi\gtrless\pm c_0|\eta|\}.$$

Next we shall consider zeros of $\hat{A}^{(i)}(\omega_i,\xi,0)$. For $1\leqslant i\leqslant l-1$, we have

$$\begin{aligned} A^{(i)}(\tau,\xi,0)&=a^0_{m_1+\cdots+m_{i-1}}(\xi,0)\prod_{j=1}^{\mu_i}(\tau-\kappa_{ij}^{-p_i}\xi^{p_i}) \\ &=a^0_{m_1+\cdots+m_{i-1}}(\xi,0)\prod_{j=1}^{\mu_i}(-\kappa_{ij}^{-p_i})\prod_{j=1}^{\mu_i}\prod_{k=1}^{p_i}(\xi-e^{i2\pi q_i(k-1)}\kappa_{ij}\omega_i), \end{aligned}$$

where $\operatorname{Im}\kappa_{ij}^{-p_i}>0$ and $\pi q_i<\arg\kappa_{ij}<2\pi q_i$. Therefore we can choose $d>0$ such that μ_i zeros of $\hat{A}^{(i)}(\omega_i,\xi,0)$ with respect to ξ belong to

$$\begin{aligned} G_{ik}=\{\xi\in C^1;\ \pi q_i+2\pi(k-1)q_i+d<\arg\left(\frac{\xi}{\omega_i}\right)<2\pi q_i \\ +2\pi(k-1)q_i-d,\ d<\frac{|\xi|}{|\omega_i|}<d^{-1}\} \end{aligned}$$

for eahc $k=1,\dots, p_i$ $(i=1,\dots, l-1)$. Let $\omega_i \in \Delta_i$, then

$$G_{ik} \subset \{\xi \in C^1;\ 2\pi(k-1)q_i < \arg \xi < 2\pi k q_i\},$$

therefore

$$G_{i1},\dots, G_{i\frac{1}{2}p_i} \subset C_+^1 = \{\xi;\ \operatorname{Im} \xi > 0\},$$

$$G_{i\frac{1}{2}p_i+1},\dots, G_{ip_i} \subset C_-^1 = \{\xi;\ \operatorname{Im} \xi < 0\}.$$

For $i=l$, we have

$$A^{(l)}(\tau, \xi, 0) = a^0_{m_1+\cdots+m_{l-1}}(\xi, 0) \prod_{j=1}^{m_l} (\tau - \kappa_{lj}^{-1}\xi)$$

$$= a^0_{m_1+\cdots+m_{l-1}}(\xi, 0) \prod_{j=1}^{m_l} (-\kappa_{lj}^{-1}) \prod_{j=1}^{m_l} (\xi - \kappa_{lj}\tau),$$

where $\{\kappa_{lj}\}$ are real and distinct:

$$\kappa_{l1} < \kappa_{l2} < \cdots < \kappa_{lm_l^+} < 0 < \kappa_{lm_l^+ +1} < \cdots < \kappa_{lm_l}.$$

Therefore we cna choose $d>0$ such that

$$G_{lk} = \{\xi \in C^1;\ \left|\frac{\xi}{\omega_l} - \kappa_{lk}\right| < d\} \qquad (k=1,\dots, m_l)$$

are disjoint.

Here we have

$$|\hat{A}^{(i)}(\omega_i, \xi, 0)| \geqslant \varepsilon_1 |\xi|^{m_1+\cdots+m_{i-1}} (|\omega_i| + |\xi|)^{m_i} \qquad \text{for} \quad \xi \notin \bigcup_k G_{ik} \qquad (i=1,\dots, l),$$

and

Lemma 3.1. *There exists $C_1 > 0$ $(C_1 > C_0)$ such that*

$$|\hat{A}^{(i)}(\omega_i, \xi, \eta)| \geqslant \frac{1}{2} \varepsilon_1 |\xi|^{m_1+\cdots+m_{i-1}} (|\omega_i| + |\xi|)^{m_i}$$

where $(\omega_i, \xi, \eta) \in \Delta_i \times C^1 \times R^{n-1}$, $|\xi| > C_1|\eta|$, and $\xi \notin \bigcup_k G_{ik}$ $(1 \leqslant i \leqslant l)$.

Finally let us see the zeros of $\hat{A}^{(l)}(\omega_l, \xi, \eta)$ in $|\xi| \leqslant C_1|\eta|$. We shall see the zeros of $\hat{A}^{(l)}$ for fixed $(t^0, x^0, y^0) \in \boldsymbol{R}^{n+1}$ and $(\omega_l^0, \eta^0) \in \Delta_l \times S^{n-2}$. Let $\{\xi_1^0,\dots, \xi_r^0\}$ be the real zeros of $\hat{A}^{(l)}(t^0, x^0, y^0; \omega_l^0, \xi, \eta^0)$, where ξ_j^0 is a σ_j-ple zero $(\sigma = m - \sum_{j=1}^{r} \sigma_j \geqslant 0)$. Then we can choose disjoint domains $\mathfrak{G}^+$, $\mathfrak{G}^-$, $\mathfrak{G}_1,\dots, \mathfrak{G}_r$ such that

$$\mathfrak{G}^{\pm} = \left\{\xi \in C^1;\ \frac{|\xi|}{|\eta|} < C,\ \frac{\operatorname{Im} \xi}{|\eta|} \gtrless \pm\delta\right\} \qquad (\delta > 0)$$

and

$$\mathfrak{G}_j = \left\{\xi \in C^1;\ \left|\frac{\xi}{|\eta|} - \xi_j^0\right| < \delta\right\},$$

where non-real zeros of $\hat{A}^{(l)}(t^0, x^0, y^0; \omega_l^0, \xi, \eta^0)$ belong to $\mathfrak{G}^+ \cup \mathfrak{G}^-$. Therefore we can find $\varepsilon > 0$ and $c > 0$ such that

$$|\hat{A}^{(l)}(t, x, y; \omega_l, \xi, \eta)| \geqslant c|\eta|^m$$

if ξ belongs to the boundary of $\mathfrak{G}^+ \cup \mathfrak{G}^- \cup \mathfrak{G}_1 \cup \cdots \cup \mathfrak{G}_r$ and $(t, x, y, \omega_l, \eta)$ belongs to

$$U(t^0, x^0, y^0, \omega_l^0, \eta^0)$$

$$=\left\{|t-t^0|<\varepsilon,\ |x-x^0|<\varepsilon,\ |y-y^0|<\varepsilon,\ \left|\frac{\omega_l}{|\eta|}-\omega_l^0\right|<\varepsilon,\ \left|\frac{\eta}{|\eta|}-\eta^0\right|<\varepsilon\right\},$$

which we call a neighbourhood of $(t^0, x^0, y^0, \omega_l^0, \eta^0)$ satisfying the contition $(\mathfrak{G})$.

3.2. Zeros of $\hat{A}_0(\omega, \xi, \eta)$. Let us denote for large parameters K, β

$$G_{i(\beta)}=\left\{\xi;\ |\omega_i|\leqslant \frac{\beta}{C_1}|\xi|\leqslant|\omega_{i+1}|\right\} \qquad (i=0,\dots, l),$$

$$\mathscr{D}_{i(K,\beta)}=\{(\omega, \eta)\in\Omega_K\times \boldsymbol{R}^{n-1};\ |\omega_i|\leqslant\beta|\eta|\leqslant|\omega_{i+1}|\} \qquad (i=0,\dots, l),$$

then the complex ξ plane is covered by $\bigcup_{i=0}^{l} G_{i(\beta)}$ and $\Omega_K\times\boldsymbol{R}^{n-1}$ is covered by $\bigcup_{i=0}^{l}\mathscr{D}_{i(K,\beta)}$. Let $\beta^{-1}+\beta K^{-\delta}\leqslant \min(dC_1, dC_1^{-1}, C_0(C_1-C_0)^{-1}, C_0^{-1}(C_1-C_0))$, then we have

$$G_{i(\beta)}\supset\bigcup_j G_{ij} \quad\text{and}\quad G_{i(\beta)}\supset\partial G^{(i)}|_{(\omega,\eta)\in\mathscr{D}_{i(k,\beta)}} \qquad (i=1,\dots, l).$$

Lemma 3.2. *There exists $\delta_0>0$ as follows. Let$K^{-\delta}\beta+\beta^{-1}<\delta_0$, then μ_i zeros of $\hat{A}_0(\omega, \xi, \eta)$ are contained in G_{ik} for each $k=1,\dots, p_i$ $(i_0+1\leqslant i\leqslant l-1)$, one zero of $\hat{A}_0(\omega, \xi, \eta)$ is contained in G_{lk} for each $k=1,\dots, m_l$ for $(\omega, \eta)\in\mathscr{D}_{i_0(K,\beta)}$ $(i_0=0, 1,\dots, l-1)$, the rest ones, $m_1+\cdots+m_{i_0}$ zeros of $\hat{A}_0(\omega, \xi, \eta)$, are contained in $G_{\pm}^{(i_0)}$ for $(\omega, \eta)\in\mathscr{D}_{i_0(K,\beta)}$ $(1\leqslant i_0\leqslant l-1)$.*

Proof. (i) Let $(\omega, \eta)\in\mathscr{D}_{i_0(K,\beta)}$ and $\xi\in G_{i(\beta)}$ $(i\geqslant i_0)$, that is,

$$|\omega_{i_0}|\leqslant\beta|\eta|\leqslant|\omega_{i_0+1}|\leqslant|\omega_{i_0+2}|\leqslant,\dots,$$

$$|\omega_i|\leqslant\frac{\beta}{C_1}|\xi|\leqslant|\omega_{i+1}|$$

then we have

$$K^{\delta}|\omega_{i-1}|\leqslant|\omega_i|\leqslant\frac{\beta}{C_1}(|\xi|^2+|\eta|^2)^{\frac{1}{2}}\leqslant\sqrt{2}|\omega_{i+1}|,$$

there$_\varepsilon$ore we have $(\omega, \xi, \eta)\in D_{i(K,K^{-\delta}\beta C_1^{-1},\beta\sqrt{2}^{-1}C_1^{-1})}$. Let $K^{-\delta}\beta+\beta^{-1}<\delta_1$, then we from lemma 1.1,

$$|\hat{A}_0(\omega, \xi, \eta)\omega_{i+1}^{-m_{i+1}}\dots\omega_l^{-m_l}-\hat{A}^{(i)}(\omega_i, \xi, \eta)|\leqslant\frac{1}{4}\varepsilon_1(|\xi|+|\eta|)^{m_1+\cdots+m_i}$$

for $(\omega, \eta)\in\mathscr{D}_{i_0(K,\beta)}$, $\xi\in G_{i(\beta)}(i\geqslant i_0)$, where ε_1 is given from lemma 2.1:

$$|\hat{A}^{(i)}(\omega_i, \xi, \eta)|\geqslant\frac{1}{2}\varepsilon_1(|\xi|+|\eta|)^{m_1+\cdots+m_i}$$

for $\xi \in (\bigcup_j G_{ij})^c \cap \{|\xi| > C_1|\eta|\}$. Here we have $\hat{A}_0(\omega, \xi, \eta) \neq 0$ for $(\omega, \eta) \in \mathscr{D}_{i_0(K,\beta)}$ and $\xi \in G_{i(\beta)} \cap (\bigcup_j G_{ij})^c \cap \{|\xi| > C_1|\eta|\}$, and the number of zeros of $\hat{A}_0(\omega, \xi, \eta)$ is equal to that of $\hat{A}^{(i)}(\omega_i, \xi, \eta)$ in G_{ij} for $(\omega, \eta) \in \mathscr{D}_{i_0(K,\beta)}$ by using Rouché's theorem.

Let $(\omega, \eta) \in \mathscr{D}_{i_0(K,\beta)}$ and $\xi \in G^{(i_0)}$. Then we have

$$K^\delta|\omega_{i_0-1}| \leqslant |\omega_{i_0}| \leqslant \beta(|\xi|^2 + |\eta|^2)^{\frac{1}{2}}$$
$$\leqslant \beta(C_0|\omega_{i_0}| + (C_0+1)|\eta|) \leqslant (\beta C_0 K^{-\delta} + C_0 + 1)|\omega_{i_0+1}|,$$

that is, $(\omega, \xi, \eta) \in D_{i_0(K, K^{-\delta}\beta, \beta(\beta C_0 K^{-\delta} + C_0 + 1)^{-1})}$. Hence, from lemma 1.1,

$$|\hat{A}_0(\omega, \xi, \eta)\omega_{i_0+1}^{-m_{i_0+1}} \cdots \omega_l^{-m_l} - \hat{A}^{(i_0)}(\omega_{i_0}, \xi, \eta)| \leqslant \frac{1}{4}\varepsilon_1|\eta|^{m_1+\cdots+m_{i_0}},$$

if $K^{-\delta}\beta + \beta^{-1} < \delta_2$. On the other hand, the zeros of $\hat{A}^{(i_0)}(\omega_{i_0}, \xi, \eta)$ are contained in $G_\pm^{(i_0)}$ and

$$|\hat{A}^{(i_0)}(\omega_{i_0}, \xi, \eta)| \geqslant \frac{1}{2}\varepsilon_1|\eta|^{m_1+\cdots+m_{i_0}}$$

on $\partial G_\pm^{(i_0)}$. Hence the number of zeros of $\hat{A}_0(\omega, \xi, \eta)$ is equal to that of $\hat{A}^{(i_0)}(\omega_{i_0}, \xi, \eta)$ in $G_\pm^{(i_0)}$. ∎

3.3. Decomposition of $\hat{A}_0(\omega, \xi, \eta)$. Let $(\omega, \eta) \in \mathscr{D}_{i_0(K,\beta)}(0 \leqslant i_0 \leqslant l-1)$, then, we have, following lemma 2.2, the decomposition of $\hat{A}_0$:

$$\hat{A}_0(\omega, \xi, \eta) = c\mathscr{E}_+(\omega, \xi, \eta)\mathscr{E}_-(\omega, \xi, \eta) \prod_{i=i_0+1}^{l-1} E_i(\omega, \xi, \eta) \cdot H_l(\omega, \xi, \eta),$$

$$E_i(\omega, \xi, \eta) = \prod_{j=1}^{p_i} E_{ij}(\omega, \xi, \eta), \quad H_l(\omega, \xi, \eta) = \prod_{j=1}^{m_l} H_{lj}(\omega, \xi, \eta),$$

where the zeros of $\mathscr{E}_\pm = \mathscr{E}_\pm^{(i_0)}$ ($\mathscr{E}_\pm^{(0)} = 1$) with respect to ξ belong to $G_\pm^{(i_0)}$, those of E_{ij} belong to G_{ij}, and those of H_{lj} belong to G_{lj}. Then we define

$$\mathscr{E}_\pm(\omega, \xi, \eta) = |\eta|^{m_1^\pm+\cdots+m_{i_0}^\pm}\mathscr{E}'_\pm(\omega, \frac{\xi}{|\eta|}, \eta),$$

$$E_{ij}(\omega, \xi, \eta) = |\omega_i|^{\mu_i}E'_{ij}(\omega, \frac{\xi}{|\omega_i|}, \eta),$$

$$H_{lj}(\omega, \xi, \eta) = |\omega_l|H'_{lj}(\omega, \frac{\xi}{|\omega_l|}, \eta).$$

Now let us denote $a(t, x, y; \omega, \eta) \in \tilde{\mathscr{B}}^\infty(X)$ if

$$\sup_{(t,x,y,\omega,\eta)\in X} |D_{t,x,y} \prod_{i=1}^{l} ((|\omega_i| + |\eta|)D_{\mathrm{Re}\,\omega_i})^{\nu_i} \prod_{i=1}^{l} ((|\omega_i| + |\eta|)D_{\mathrm{Im}\,\omega_i})^{\nu_i'}$$
$$\times \{(|\omega_1| + |\eta|)D_\eta\}^\mu a(t, x, y; \omega, \eta)| < +\infty$$

for any (α, ν, ν', μ), then we have

Lemma 3.3. *In the above decomposition of* $\hat{A}_0$ *in* $\mathscr{D}_{i_0(K,\beta)}$ $(0\leqslant i_0\leqslant l-1)$, *the coefficients of* $\mathscr{E}'_{\pm}$, E'_{ij} $(i_0+1\leqslant i\leqslant l-1)$ *and* H'_{lj} *belong to* $\tilde{\mathscr{B}}^{\infty}(\mathscr{D}_{i_0(K,\beta)})$ *and have the following properties*:

i) *absolute values of imaginary parts of the zeros of* $\mathscr{E}'_{\pm}$ *and* E'_{lj} *are greater than* $c(>0)$.

ii) *we have*

$$H'_{lj}(\omega, \xi, \eta)=\xi-\xi'_j(\omega, \eta),$$

$$\xi'_j(\omega, \eta)=a_j(\omega_l, \eta)+i\frac{\operatorname{Im}\omega_l}{|\omega_l|}b_j(\omega_l, \eta)+\sum_{k=1}^{l-1}\left(\frac{\omega_k}{\omega_l}\right)^{d_k}c_{jk}(\omega, \eta),$$

where a_j, b_j, $c_{jk}\in\tilde{\mathscr{B}}^{\infty}(\bigcup_{i=0}^{l-1}\mathscr{D}_{i(K,\beta)})$, a_j *and* b_j *are real valued and satisfy*

$$b_j<-c \quad (i=1,\dots, m_l^+), \qquad b_j>c \quad (i=m_l^++1,\dots, m_l) \qquad (c>0).$$

Proof. The proof of regularity is based on the integeral representations of the coefficients of $\mathscr{E}'_{\pm}$ and etc. (see [3]). Let $\xi_j^0(\tau, \eta)$ be the zero of $A^{(l)}(\tau, \xi, \eta)$ in G_{lj}, then $\xi_j^0(\tau, \eta)$ is real for $\tau\in\boldsymbol{R}^1$, $\eta\in\boldsymbol{R}^{n-1}$, therefore we can write

$$\operatorname{Im}\xi_j^0(\tau, \eta)=\operatorname{Im}\tau\cdot\tilde{b}_j(\tau, \eta).$$

Let $\xi_j(\omega, \eta)$ be the zero of $\hat{A}_0(\omega, \xi, \eta)$ in G_{lj} and let

$$\xi_j(\omega, \eta)=\xi_j^0(\omega_l, \eta)+\xi_j^1(\omega, \eta).$$

Since we have from lemma 1.2

$$\{\hat{A}_0(\omega, \xi, \eta)-\hat{A}^{(l)}(\omega_l, \xi, \eta)\}\omega_l^{-m}$$
$$=\sum_{k=1}^{l-1}\left(\frac{\omega_k}{\omega_l}\right)^{p_k}\varphi_k\left(\frac{\omega}{\omega_l}, \frac{\xi}{\omega_l}, \frac{\eta}{\omega_l}\right),$$

we have

$$\xi_j^1(\omega, \eta)\omega_l^{-1}=\sum_{k=1}^{l-1}\left(\frac{\omega_k}{\omega_l}\right)^{p_k}\tilde{c}_{jk}\left(\frac{\omega}{\omega_l}, \frac{\eta}{\omega_l}\right). \qquad \blacksquare$$

3.4. Let $(\omega, \eta)\in\mathscr{D}_{i_0(K,\beta)}$ $(0\leqslant i_0\leqslant l-1)$, then we have

$$\hat{A}_0=c\mathscr{E}_+\mathscr{E}_-\Big(\prod_{i=i_0+1}^{l-1}\prod_{j=1}^{p_i}E_{ij}\Big)\Big(\prod_{j=1}^{m_l}H_{lj}\Big),$$

and we denote

$$\mathscr{E}_k^{\pm}=\sum_{\alpha=0}^{m_1^{\pm}+\cdots+m_{i_0}^{\pm}-k}c'_{\pm\alpha}(\omega, \eta)\left(\frac{\xi}{|\eta|}\right)^{m_1^{\pm}+\cdots+m_{i_0}^{\pm}-k-\alpha} \quad (0\leqslant k\leqslant m_1^{\pm}+\cdots+m_{i_0}^{\pm}),$$

$$E_{ijk}=\sum_{\alpha=0}^{\mu_i-k}e'_{ij\alpha}(\omega, \eta)\left(\frac{\xi}{|\omega_i|}\right)^{\mu_i-k-\alpha} \quad (0\leqslant k\leqslant\mu_i),$$

where $\mathscr{E}_0^{\pm}=\mathscr{E}'_{\pm}\left(\omega, \frac{\xi}{|\eta|}, \eta\right)$ and $E_{ij0}=E'_{ij}\left(\omega, \frac{\xi}{|\omega_i|}, \eta\right)$.

Denoting

$$\mathscr{V}^{\pm}=\begin{pmatrix}\mathscr{V}_1^{\pm}\\ \vdots\\ \mathscr{V}^{\pm}_{m_1^{\pm}+\cdots+m_{i_0}^{\pm}}\end{pmatrix}=\begin{pmatrix}\mathscr{E}_1^{\pm}\\ \vdots\\ \mathscr{E}^{\pm}_{m_1^{\pm}+\cdots+m_{i_0}^{\pm}}\end{pmatrix}\frac{\hat{A}_0}{\mathscr{E}_0^{\pm}}|\eta|^{-1},$$

$$V_{ij}=\begin{pmatrix}V_{ij1}\\ \vdots\\ V_{ij\mu_i}\end{pmatrix}=\begin{pmatrix}E_{ij1}\\ \vdots\\ E_{ij\mu_i}\end{pmatrix}\frac{\hat{A}_0}{E_{ij0}}|\omega_i|^{-1}\qquad (i_0+1\leqslant i\leqslant l-1,\ 1\leqslant j\leqslant p_i),$$

$$V_{lj}=\frac{\hat{A}_0}{H_{lj}}\qquad (1\leqslant j\leqslant m_l),$$

$$V_i^+=\begin{pmatrix}\mathbf{V}_{i1}\\ \vdots\\ V_{i\frac{1}{2}p_i}\end{pmatrix},\quad V_i^-=\begin{pmatrix}V_{i\frac{1}{2}p_i+1}\\ \vdots\\ V_{ip_i}\end{pmatrix}\qquad (i_0+1\leqslant i\leqslant l-1),$$

$$V_l^+=\begin{pmatrix}V_{l1}\\ \vdots\\ V_{lm_l^+}\end{pmatrix},\quad V_l^-=\begin{pmatrix}V_{lm_l^++1}\\ \vdots\\ V_{lm_l}\end{pmatrix},$$

we define

$$V^{\pm}=\begin{pmatrix}\mathscr{V}^{\pm}\\ V^{\pm}_{i_0+1}\\ \vdots\\ V^{\pm}_l\end{pmatrix},\quad V=\begin{pmatrix}V^+\\ V^-\end{pmatrix}.$$

We denote $V=V^{(i_0)}$, if we need to remark their dependency on i_0.

Lemma 3.4. *Let* $P(\tau,\xi,\eta)=\sum_{(\sigma,j+|\nu|)\in N'_A} c_{\sigma j\nu}\tau^{\sigma}\xi^j\eta^{\nu}$, then we have

$$\hat{P}(\omega,\xi,\eta)=C^{(i_0)}(\omega,\eta)V^{(i_0)}(\omega,\xi,\eta),\quad C^{(i_0)}(\omega,\eta)\in\tilde{\mathscr{B}}^{\infty}(\mathscr{D}_{i_0(K,\beta)}),$$

for $(\omega,\eta)\in\mathscr{D}_{i_0(K,\beta)}$ $(0\leqslant i_0\leqslant l-1)$. More precisely, denoting

$$\hat{P}(\xi)=\sum_{j=1}^{m_1^++\cdots+m_{i_0}^+}c_j^+\mathscr{V}_j^+(\xi)+\sum_{j=1}^{m_1^-+\cdots+m_{i_0}^-}c_j^-\mathscr{V}_j^-(\xi)$$
$$+\sum_{i=i_0+1}^{l-1}\sum_{j=1}^{p_i}\sum_{k=1}^{\mu_i}c_{ijk}V_{ijk}(\xi)+\sum_{j=1}^{m_l}c_{lj}V_{lj}(\xi),$$

we have

$$c_j^{\pm}=\frac{1}{2\pi i}\int_{\partial G_{\pm}^{(i_0)}}\frac{\hat{P}(\xi)}{\hat{A}_0(\xi)}\left(\frac{\xi}{|\eta|}\right)^{j-1}d\xi,$$

$$c_{ijk}=\frac{1}{2\pi i}\int_{\partial G_{ij}}\frac{\hat{P}(\xi)}{\hat{A}_0(\xi)}\left(\frac{\xi}{|\omega_i|}\right)^{k-1}d\xi\qquad (i_0+1\leqslant i\leqslant l-1),$$

$$c_{lj}=\frac{1}{2\pi i}\int_{\partial G_{lj}}\frac{\hat{P}(\xi)}{\hat{A}_0(\xi)}d\xi$$

Now let us denote

$$C^{(i_0)'}=((c_j^{+'})_{j=1,\dots,m_1^++\cdots+m_{i_0}^+},\ (c_j^{-'})_{j=1,\dots,m_1^-+\cdots+m_{i_0}^-},$$
$$(c'_{ijk})_{i=i_0+1,\dots,l-1;j=1,\dots,p_i;k=1,\dots,\mu_i},\ (c'_{lj})_{j=1,\dots,m_l})$$

where

$$c_j^{\pm'}=\frac{1}{2\pi i}\int_{\partial G_{\pm}^{(i_0)}}\frac{\hat{P}^{(i_0)}\left(\frac{\omega_{i_0}}{|\eta|},\frac{\xi}{|\eta|},\frac{\eta}{|\eta|}\right)}{\hat{A}^{(i_0)}\left(\frac{\omega_{i_0}}{|\eta|},\frac{\xi}{|\eta|},\frac{\eta}{|\eta|}\right)}\left(\frac{\xi}{|\eta|}\right)^{j-1}\frac{d\xi}{|\eta|},$$

$$c'_{ijk}=\frac{1}{2\pi i}\int_{\partial G_{ij}}\frac{\hat{P}^{(i)}\left(\frac{\omega_i}{|\omega_i|},\frac{\xi}{|\omega_i|},0\right)}{\hat{A}^{(i)}\left(\frac{\omega_i}{|\omega_i|},\frac{\xi}{|\omega_i|},0\right)}\left(\frac{\xi}{|\omega_i|}\right)^{k-1}\frac{d\xi}{|\omega_i|},$$

$$c'_{lj}=\frac{1}{2\pi i}\int_{\partial G_{lj}}\frac{\hat{P}^{(l)}\left(\frac{\omega_l}{|\omega_l|},\frac{\xi}{|\omega_l|},0\right)}{\hat{A}^{(l)}\left(\frac{\omega_l}{|\omega_l|},\frac{\xi}{|\omega_l|},0\right)}\frac{d\xi}{|\omega_l|}.$$

Then we have, due to Lemma 1.1 and Lemma 1.2,

$$C^{(i_0)}(\omega,\eta)=C^{(i_0)}(\omega,\eta)+o(1)$$

in $\mathscr{D}_{i_0(K,\beta)}$, where $o(1)$ can be arbitrarily small if $K^{-\delta}\beta+\beta^{-1}$ is small enough.

Lemma 3.5.
i) $\boldsymbol{B}^{\#}{}_0(\omega,\xi,\eta)=C_+^{(i)}(\omega,\eta)V_+^{(i)}(\omega,\xi,\eta)+C_-^{(i)}(\omega,\eta)V_-^{(i)}(\omega,\xi,\eta)$,
$C_\pm^{(i)}(\omega,\eta)=C_\pm'^{(i)}(\omega,\eta)+o(1)$
for $(\omega,\eta)\in\mathscr{D}_{i(K,\beta)}$ $(0\leqslant i\leqslant l-1)$, *where* $o(1)\longrightarrow 0$ as $K^{-\delta}\beta+\beta^{-1}\longrightarrow 0$.
ii) $\det C_+'^{(i)}(\omega,\eta)=c_i(\omega,\eta)\hat{R}^{(i)}(\omega_i,\eta)$,
$|c_i(\omega,\eta)|\geqslant c(|\omega_i|+|\eta|)^{-s_i}\quad(c>0)$
for $(\omega,\eta)\in\mathscr{D}_{i(K,\beta)}$ $(0\leqslant i\leqslant l-1)$.

As a corollary of Lemma 3.5, there exists $\delta_0'>0$ such that $\det C_+^{(i)}(\omega,\eta)\neq 0$ for $(\omega,\eta)\in\mathscr{D}_{i(k,\beta)}$ $(0\leqslant i\leqslant l-1)$ if $K^{-\delta}\beta+\beta^{-1}<\delta_0'$.
Here we fix $\beta_0^{-1}=\frac{1}{2}\min(\delta_0,\delta_0')$, $\beta_1>\beta_0$, $K_0^{-\delta}=\beta_1^{-1}\beta_0^{-1}$, where δ_0 is given in Lemma 3.2, and we consider $\mathscr{D}_{i(K,\beta)}$ for $\beta_0<\beta<\beta_1$, $K\geqslant K_0$.

3.5. Let

$$\mathscr{M}^0=\{(t,x,y,\omega_l,\eta)\in R^{n+1}_{t,x,y}\times\Delta_l\times S^{n-2};\ t^2+x^2+|y|^2\leqslant M,\ |\omega_l|\leqslant\beta_1\},$$

$$\mathscr{M}=\left\{(t,x,y,\omega_l,\eta)\in R^{n+1}_{t,x,y}\times\Delta_l\times\boldsymbol{R}^{n-1};\ \left(t,x,y,\frac{\omega_l}{|\eta|},\frac{\eta}{|\eta|}\right)\in\mathscr{M}^0\right\}.$$

For any $X^0=(t^0,x^0,y^0,\omega_l^0,\eta^0)\in\mathscr{M}^0$, there exists a neighbourhood $U^0(X^0)$ such that

$$U(X^0)=\left\{(t,x,y,\omega_l,\eta)\in R^{n+1}_{t,x,y}\times\Delta_l\times\boldsymbol{R}^{n-1};\ \left(t,x,y,\frac{\omega_l}{|\eta|},\frac{\eta}{|\eta|}\right)\in U^0(X^0)\right\}$$

satisfies the condition ($\mathfrak{G}$) (see §3.1). Since $\mathscr{M}^0$ is compact, there exist $\{X^j\}_{j=1,\dots,N}$, a finite number of points in $\mathscr{M}^0$, such that $\mathscr{M}\subset\bigcup_{j=1}^{N} U(X^j)$, where we denote $U_j=U(X^j)$. Hence we have

$$\begin{aligned}
&\mathscr{D}_{l(K,\beta_l)}\cap\{t^2+x^2+|y|^2\leqslant M\}\\
&=\{(t,x,y,\omega,\eta)\in \boldsymbol{R}^{n+1}\times\Omega_K\times\boldsymbol{R}^{n-1};\,(t,x,y,\omega_l,\eta)\in\mathscr{M}\}\\
&=\bigcup_{j=1}^{N}\{(t,x,y,\omega,\eta)\in \boldsymbol{R}^{n+1}\times\Omega_K\times\boldsymbol{R}^{n-1};\,(t,x,y,\omega_l,\eta)\in U_j\}\\
&=\bigcup_{j=1}^{N}\mathscr{U}_{j(K)}
\end{aligned}$$

In the following, we sometimes write $\mathscr{U}_{(K)}$,omitting the index j.

Now for $(t,x,y,\omega,\eta)\in\mathscr{U}_{(K)}$, we consider the zeros of $\hat{A}_0$, the smooth decomposition of $\hat{A}_0$ and etc. just in the same wasy as for $(t,x,y,\omega,\eta)\in\mathscr{D}_{i(K,\beta)}$ $(i=0,\dots,l-1)$.

Lemma 3.2′. *There exists $K_0'(\geqslant K_0)$ such that zeros $\hat{A}_0(t,x,y;\omega,\xi,\eta)$ are contained in $\mathfrak{G}^+\cup\mathfrak{G}^-\cup\mathfrak{G}_1\cup\cdots\cup\mathfrak{G}_r$ for $(t,x,y,\omega,\eta)\in\mathscr{U}_{(K)}$ and $K\geqslant K_0'$.*

Hence, for $(t,x,y,\omega,\eta)\in\mathscr{U}_{(K)}$ $(K>K_0')$, we have

$$\hat{A}_0(t,x,y;\omega,\xi,\eta)=c\mathscr{E}_+(t,x,y;\omega,\xi,\eta)\mathscr{E}_-(t,x,y;\omega,\xi,\eta)\times\prod_{j=1}^{r}\mathscr{H}_j(t,x,y;\omega,\xi,\eta),$$

where the zeros of $\mathscr{E}_\pm$ with respect to ξ belong to $\mathfrak{G}^\pm$, and those of $\mathscr{H}_j$ belong to $\mathfrak{G}_j$. Let us denote

$$\mathscr{E}_\pm(\omega,\xi,\eta)=|\eta|^{\sigma_\pm}\mathscr{E}'_\pm\left(\omega,\frac{\xi}{|\eta|},\eta\right),$$

$$\mathscr{H}_j(\omega,\xi,\eta)=|\eta|^{\sigma_j}\mathscr{H}'_j\left(\omega,\frac{\xi}{|\eta|},\eta\right).$$

Lemma 3.3′. *The coefficients of $\mathscr{E}'_\pm(\omega,\xi,\eta)$ and $\mathscr{H}'_j(\omega,\xi,\eta)$ as polynomials with respect to ξ belong to $\mathscr{B}^\infty(\mathscr{U}_{(K)})$ $(K\geqslant K_0')$, and moreover*

i) absolute values of imaginary parts of the zeros of $\mathscr{E}'_\pm(\omega,\xi,\eta)$ are greater than $c(>0)$,

ii) denoting

$$\mathscr{H}'_j(\omega,\xi,\eta)=\sum_{k=0}^{\sigma_j}h'_{jk}(\omega,\eta)\xi^{\sigma_j-k},$$

we have

$$h'_{jk}(\omega,\eta)=a_{jk}(\omega,\eta)+i\,\frac{\operatorname{Im}\omega_l}{|\eta|}\,b_{jk}(\omega,\eta)+\sum_{s=1}^{l-1}\left(\frac{\omega_s}{|\eta|}\right)^{p_s}c_{jks}(\omega,\eta),$$

where a_{jk}, b_{jk}, $c_{jks}\in\mathscr{B}^\infty(\mathscr{U}_{(K)})$, a_{jk} and b_{jk} are real valued and $|b_{j\sigma_j}|>$ const. (>0).

For the decomposition

$$\hat{A}_0 = c\mathscr{E}_+\mathscr{E}_- \prod_{j=1}^{r} \mathscr{H}_j,$$

we define

$$\mathscr{E}_k^{\pm} = \sum_{\alpha=0}^{\sigma_\pm - k} e'_{\pm\alpha}(t, x, y; \omega, \eta)\left(\frac{\xi}{|\eta|}\right)^{\sigma_\pm - k - \alpha} \qquad (0 \leqslant k \leqslant \sigma_\pm),$$

$$\mathscr{H}_{jk} = \sum_{\alpha=0}^{\sigma_j - k} h'_{j\alpha}(t, x, y; \omega, \eta)\left(\frac{\xi}{|\eta|} - \xi_j^0\right)^{\sigma_j - k - \alpha} \qquad (0 \leqslant k \leqslant \sigma_j),$$

where

$$\mathscr{E}_0^{\pm} = \mathscr{E}'_{\pm}\left(\omega, \frac{\xi}{|\eta|}, \eta\right) \text{and } \mathscr{H}_{j0} = \mathscr{H}'_j\left(\omega, \frac{\xi}{|\eta|}, \eta\right).$$

Denoting

$$\mathscr{V}^{\pm} = \begin{pmatrix} \mathscr{V}_1^{\pm} \\ \vdots \\ \mathscr{V}_{\sigma\pm}^{\pm} \end{pmatrix} = \begin{pmatrix} \mathscr{E}_1^{\pm} \\ \vdots \\ \mathscr{E}_{\sigma\pm}^{\pm} \end{pmatrix} \frac{\hat{A}_0}{\mathscr{E}_0^{\pm}} \cdot |\eta|^{-1},$$

$$\mathscr{V}_j = \begin{pmatrix} \mathscr{V}_{j1} \\ \vdots \\ \mathscr{V}_{j\sigma_j} \end{pmatrix} = \begin{pmatrix} \mathscr{H}_{j1} \\ \vdots \\ \mathscr{H}_{j\sigma_j} \end{pmatrix} \frac{\hat{A}_0}{\mathscr{H}_{j0}} \cdot |\eta|^{-1},$$

$$\mathscr{V}_j^- = \begin{pmatrix} \mathscr{V}_{j1} \\ \vdots \\ \mathscr{V}_{j\sigma_j^-} \end{pmatrix}, \quad \mathscr{V}_j^+ = \begin{pmatrix} \mathscr{V}_{j\sigma_j^-+1} \\ \vdots \\ \mathscr{V}_{j\sigma_j} \end{pmatrix} \qquad (j = 1, \dots, r),$$

we define

$$V^{\pm} = \begin{pmatrix} \mathscr{V}^{\pm} \\ \mathscr{V}_1^{\pm} \\ \vdots \\ \mathscr{V}_r^{+} \end{pmatrix}, \quad V = \begin{pmatrix} V^+ \\ V^- \end{pmatrix}.$$

We denote $V = V^{(U)}$, $W = W^{(U)}$ if we need to remark their dependency on U.

Lemma 3.4′. *Let P be given in Lemma 3.4. We have*

$$\hat{P}(t, x, y; \omega, \xi, \eta) = C^{(U)}(t, x, y; \omega, \eta) V^{(U)}(t, x, y; \omega, \xi, \eta),$$

$$C^{(U)}(t, x, y; \omega, \eta) \in \mathscr{B}^{\infty}(\mathscr{U})$$

for $(t, x, y, \omega, \eta) \in \mathscr{U}$. More precisely, denoting

$$\hat{P}(\xi) = \sum_{j=1}^{\sigma_+} c_j^+ \mathscr{V}_j^+(\xi) + \sum_{j=1}^{\sigma_-} c_j^- \mathscr{V}_j^-(\xi)$$

$$+ \sum_{i=1}^{r} \sum_{j=1}^{\sigma_i} c_{ij} \mathscr{V}_{ij}(\xi),$$

we have

$$c_j^{\pm}=\frac{1}{2\pi i}\int_{\partial\mathfrak{G}_\pm}\frac{\hat{P}(\xi)}{\hat{A}_0(\xi)}\left(\frac{\xi}{|\eta|}\right)^{j-1}d\xi,$$

$$c_{ij}=\frac{1}{2\pi i}\int_{\partial\mathfrak{G}_i}\frac{\hat{P}(\xi)}{\hat{A}_0(\xi)}\left(\frac{\xi}{|\eta|}-\xi_i^0\right)^{j-1}d\xi.$$

Now let us denote

$$C'^{(U)}=((c_j^{+\prime})_{j=1,\dots,\sigma_+},(c_j^{-\prime})_{j=1,\dots,\sigma_-},(c'_{ij})_{j=1,\dots,\sigma_i}\ (i=1,\dots,r)),$$

where

$$c_j^{\pm\prime}=\frac{1}{2\pi i}\int_{\partial\mathfrak{G}^\pm}\frac{\hat{P}^{(l)}\left(\frac{\omega_l}{|\eta|},\frac{\xi}{|\eta|},\frac{\eta}{|\eta|}\right)\frac{|\eta|}{|\operatorname{Im}\omega_l|}}{\hat{A}^{(l)}\left(\frac{\omega_l}{|\eta|},\frac{\xi}{|\eta|},\frac{\eta}{|\eta|}\right)}\left(\frac{\xi}{|\eta|}\right)^{j-1}\frac{d\xi}{|\eta|},$$

$$c'_{ij}=\frac{1}{2\pi i}\int_{\partial\mathfrak{G}_i}\frac{\hat{P}^{(l)}\left(\frac{\omega_l}{|\eta|},\frac{\xi}{|\eta|},\frac{\eta}{|\eta|}\right)}{\hat{A}^{(l)}\left(\frac{\omega_l}{|\eta|},\frac{\xi}{|\eta|},\frac{\eta}{|\eta|}\right)}\left(\frac{\xi}{|\eta|}-\xi_i^0\right)^{j-1}\frac{d\xi}{|\eta|}.$$

Then we have, due to Lemma 1.1 and Lemma 1.3,

$$C^{(U)}(\omega,\eta)=C'^{(U)}(\omega_l,\eta)+o(1)$$

in $\mathscr{U}_{(K)}$, where $o(1)\to 0$ as $K^{-1}\to 0$.

Lemma 3.5′.

i) $\hat{\boldsymbol{B}}_0^\#(t,x,y;\omega,\xi,\eta)=\boldsymbol{C}_+^{(U)}(t,x,y;\omega,\eta)\boldsymbol{V}_+^{(U)}(t,x,y;\omega,\xi,\eta)$

$$+\boldsymbol{C}_-^{(U)}(t,x,y;\omega,\eta)\boldsymbol{V}_-^{(U)}(t,x,y;\omega,\xi,\eta),$$

$$\boldsymbol{C}_\pm^{(U)}(t,x,y;\omega,\eta)=\boldsymbol{C}_\pm'^{(U)}(t,x,y;\omega_l,\eta)+o(1)$$

for $(t,x,y,\omega,\eta)\in\mathscr{U}_{(K)}$ $(K>K'_0)$, *where* $o(1)\to 0$ *as* $K^{-1}\to 0$.

ii) $\det C_+'^{(U)}(t^0,x^0,y^0;\omega_l^0,\eta^0)=c\hat{R}^{(l)}(t^0,x^0,y^0;\omega_l^0,\eta^0)$ $(c\neq 0)$.

As a corollary of Lemma 3.5′, there exists $K''_0(\geqslant K'_0)$ such that $\det C_+^{(U)}(t,x,y;\omega,\eta)\neq 0$ for $(t,x,y,\omega,\eta)\in\mathscr{U}_{(K)}$ if $K\geqslant K''_0$. Hereafter we fix $K=K''_0$.

§4. Energy inequalities

4.1. Pseudo-differential operators. Let us consider a function $a(t,x,y;\tau,\eta)\in C^\infty(\boldsymbol{R}^{n+1}\times C_K^1\times\boldsymbol{R}^{n-1})$ satisfying

$$(*)\qquad \sup_{\boldsymbol{R}^{n+1}\times C_K^1\times\boldsymbol{R}^{n-1}}\{|D_{t,x,y}^\nu D_{\sigma,\gamma,\eta}^\mu a(t,x,y;\sigma-i\gamma,\eta)|\times(|\sigma-i\gamma|^{q_1}+|\eta|)^{|\mu|}\}<+\infty,$$

where $C_K^1=\{\tau\in C^1;\ \operatorname{Im}\tau<-K\}$. Then it is a symbol of pseudo-differential operator with parameters $\{\gamma>K,\ x\in\boldsymbol{R}^1\}$. We define

$$a(t, x, y; D_t - i\gamma, D_y)u(t, y) = \frac{1}{(2\pi)^n}\int_{R^n\times R^n} e^{-i(t'\sigma+y'\eta)}a(t, x, y; \sigma - i\gamma, \eta)u(t+t', y+y')dt'dy'd\sigma d\eta,$$

$$a'(D_t - i\gamma, D_y; t, x, y)u(t, y) = \frac{1}{(2\pi)^n}\int_{R^n\times R^n} e^{-i(t'\sigma+y'\eta)}a(t+t', x, y+y'; \sigma - i\gamma, \eta) \times u(t+t', y+y')dt'dy'd\sigma d\eta,$$

$$a(t, x, y; D_t - i\gamma, D_y)\circ b(t, x, y; D_t - i\gamma, D_y)u(t, y) = \frac{1}{(2\pi)^n}\int_{R^n\times R^n} e^{-i(t'\sigma+y'\eta)}a(t, x, y; \sigma - i\gamma, \eta)b(t, x, y; \sigma - i\gamma, \eta) \times u(t+t', y+y')dt'dy'd\sigma d\eta.$$

Then we have not only

$$\|a\|_{\mathscr{L}(H^k(R^n), H^k(R^n))}, \quad \|a'\|_{\mathscr{L}(H^k(R^n), H^k(R^n))}$$

are bounded w.r.t. $\{\gamma \geqslant K, x \in R^1\}$, but also

$$\|a\|_{\mathscr{L}(H^k(R^{n+1}), H^k(R^{n+1}))}, \quad \|a'\|_{\mathscr{L}(H^k(R^{n+1}), H^k(R^{n+1}))}$$

are bounded w.r.t. $\{\gamma \geqslant K\}$ (see Kumano-go [4]).

Lemma 4.1. *Let $f(t, x, y; \omega, \eta) \in \tilde{\mathscr{B}}^\infty(R^{n+1}\times \Omega_K \times R^{n-1})$ and $\omega_i(\tau) = \tau^{q_i}$, then $f(t, x, y; \omega(\tau), \eta)$ satisfy (*).*

Proof. We only remark

$$|D_{\sigma,\gamma} f(t, x, y; \omega(\tau), \eta)| = |\sum_i f_{\mathrm{Re}\,\omega_i}(t, x, y; \omega(\tau), \eta) D_{\sigma,\gamma}\mathrm{Re}\,\omega_i(\tau)$$
$$+ i\sum_i f_{\mathrm{Im}\,\omega_i}(t, x, y; \omega(\tau), \eta) D_{\sigma,\gamma}\mathrm{Im}\,\omega_i(\tau)| \leqslant C(|\omega_1(\tau)| + |\eta|)^{-1},$$

$$|D_\eta f(t, x, y; \omega(\tau), \eta)| \leqslant C(|\omega_1(\tau)| + |\eta|)^{-1}. \quad \blacksquare$$

Now let $\varphi(r) \in \mathscr{B}^\infty(R_+^1)$ be non-negative real valued function satisfying

$$\varphi(r) = \begin{cases} c_1 & \text{for } r < r_1, \\ c_2 & \text{for } r > r_2 \quad (r_2 > r_1 > 0), \end{cases}$$

then we have

$$\varphi\left(\frac{|\omega_i|}{|\eta|}\right) \in \tilde{\mathscr{B}}^\infty(R^{n+1}\times \Omega_K \times R^{n-1}).$$

Let us denote $\varphi = \varphi_\lambda$ if $c_1 = 1$, $c_2 = 0$, $r_1 = \lambda - h$, $r_2 = \lambda$, and denote $\varphi = \psi_\lambda$ if $c_1 = 0$, $c_2 = 1$, $r_1 = \lambda$, $r_2 = \lambda + h$ $(0 < 2hl < \beta_1 - \beta_0)$, then

$$\begin{cases} \varphi_\lambda\left(\frac{|\omega_i|}{|\eta|}\right)\varphi_\lambda\left(\frac{|\omega_{i+1}|}{|\eta|}\right)=1 \quad \text{in } \mathscr{D}_{i(K,\lambda-h)}\cap\mathscr{D}_{i(K,\lambda+h)}=\left\{\frac{|\omega_i|}{\lambda-h}\leqslant|\eta|\leqslant\frac{|\omega_{i+1}|}{\lambda+h}\right\}, \\ \operatorname{supp}\left[\varphi_\lambda\left(\frac{|\omega_i|}{|\eta|}\right)\psi_\lambda\left(\frac{|\omega_{i+1}|}{|\eta|}\right)\right]\subset\mathscr{D}_{i(K,\lambda)}, \end{cases}$$

$$\begin{cases} \varphi_{\lambda+h}\left(\frac{|\omega_i|}{|\eta|}\right)\psi_{\lambda-h}\left(\frac{|\omega_{i+1}|}{|\eta|}\right)=1 \text{ in } \mathscr{D}_{i(K,\lambda)}, \\ \operatorname{supp}\left[\varphi_{\lambda+h}\left(\frac{|\omega_i|}{|\eta|}\right)\varphi_{\lambda-h}\left(\frac{|\omega_{i+1}|}{|\eta|}\right)\right]\subset\mathscr{D}_{i(K,\lambda+h)}\cup\mathscr{D}_{i(K,\lambda-h))}. \end{cases}$$

Let $\beta_0<\lambda_1<\cdots<\lambda_l<\beta_1$ satisfy $\lambda_i+h<\lambda_{i+1}-h$, then

$$\bigcup_{i=0}^{l}\left\{\frac{|\omega_i|}{\lambda_i-h}\leqslant|\eta|\leqslant\frac{|\omega_{i+1}|}{\lambda_i+h}\right\}\supset\bigcup_{i=0}^{l}\left\{\frac{|\omega_i|}{\lambda_i-h}\leqslant|\eta|\leqslant\frac{|\omega_{i+1}|}{\lambda_{i+1}-h}\right\}=\boldsymbol{R}^{n+1}\times\Omega_K\times\boldsymbol{R}^{n-1}.$$

Here we define

$$\hat{\Phi}_i(\omega,\eta)=\varphi_{\lambda_i}\left(\frac{|\omega_i|}{|\eta|}\right)\psi_{\lambda_i}\left(\frac{|\omega_{i+1}|}{|\eta|}\right),$$

$$\hat{\Psi}_i(\omega,\eta)=\varphi_{\lambda_i+1}\left(\frac{|\omega_i|}{|\eta|}\right)\psi_{\lambda_i-1}\left(\frac{|\omega_{i+1}|}{|\eta|}\right),$$

and $\Phi_i(\tau,\eta)=\hat{\Phi}_i(\omega(\tau),\eta)$, $\Psi_i(\tau,\eta)=\hat{\Psi}_i(\omega(\tau),\eta)$.

Denoting

$$\tilde{\Lambda}(\tau,\eta)=(|\sigma-i\gamma|^{2q_1}+|\eta|^2)^{\frac{1}{2}}$$

we have

Lemma 4.2. (Kumano-go ([4])) *Let a, b satisfy $(*)$, then*

$$\|\tilde{\Lambda}(D_t-i\gamma,D_y)\{a(t,x,y;D_t-i\gamma,D_y)-a'(D_t-i\gamma,D_y;t,x,y)\}\|_{\mathscr{L}(H^k(\boldsymbol{R}^{n+1}),H^k(\boldsymbol{R}^{n+1}))}$$

$$\|\tilde{\Lambda}(D_t-i\gamma,D_y)\{a(t,x,y;D_t-i\gamma,D_y)b(t,x,y;D_t-i\gamma,D_y)-a(t,x,y;D_t-i\gamma,D_y)\circ b(t,x,y;D_t-i\gamma,D_y)\}\|_{\mathscr{L}(H^k(\boldsymbol{R}^{n+1}),H^k(\boldsymbol{R}^{n+1}))}$$

are bounded in $\{\gamma\geqslant K\}$.

4.2. Norms. Let

$$|||u|||=\sum_{k=0}^{l}\||D_t-i\gamma|^{\mu'_{k+1}+\cdots+\mu'_l}|D_{x,y}|^{m'_1+\cdots+m'_k}u\|,$$

$$\langle\!\langle u\rangle\!\rangle=\sum_{k=0}^{l}\langle|D_t-i\gamma|^{\mu'_{k+1}+\cdots+\mu'_l}|D_{x,y}|^{m'_1+\cdots+m'_k}u\rangle,$$

where

$$\|u\|=\|u\|_{L^2(\boldsymbol{R}^{n+1}_+)},\ \||D_{x,y}|^s u\|=\sum_{0\leqslant j\leqslant s}\|D_x^j|D_y|^{s-j}u\|,$$

$$\langle u\rangle = \|u\|_{L^2(R^n),x=0}.$$

It follows easily from the definitions of norms:

Lemma 4.3.

$$\sum_{(\sigma,k)\in N''_A} \langle (D_t-i\gamma)^\sigma D^k_{x,y}u\rangle \leqslant C\gamma^{-q_1}\langle\!\langle u\rangle\!\rangle,$$

where

$$N''_A=(N_A-(0,2))\cap N_A.$$

From Lemma 4.2, we have

Lemma 4.4. *For $i=0, 1,\dots, l$, we have*

i) $\|\{\Phi_i(D_t-i\gamma, D_y)A(t, x, y; D_t-i\gamma, D_x, D_y)$

$$-A(t, x, y; D_t-i\gamma, D_x, D_y)\Phi_i(D_t-i\gamma, D_y)\}u\| \leqslant C|||u|||,$$

ii) $\langle\{\Phi_i(D_t-i\gamma, D_y)\boldsymbol{B}^\#(t, x, y; D_t-i\gamma, D_x, D_y)$

$$-\boldsymbol{B}^\#(t, x, y; D_t-i\gamma, D_x, D_y)\Phi_i(D_t-i\gamma, D_y)\}u\rangle \leqslant C\gamma^{-q_1}\langle\!\langle u\rangle\!\rangle.$$

4.3. Energy inequalities. Now let

$$\hat{A}^{((i))}(t, x, y; \omega, \xi, \eta)=\hat{\Psi}_i(\omega, \eta)\hat{A}(t, x, y; \omega, \xi, \eta)$$
$$+(1-\hat{\Psi}_i(\omega, \eta))\hat{A}^{(i)}(t, x, y; \omega_i, \xi, \eta)\omega_{i+1}^{m_{i+1}}\cdots\omega_l^{m_l},$$

$$A^{((i))}(t, x, y; \tau, \xi, \eta)=\hat{A}^{((i))}(t, x, y; \omega(\tau), \xi, \eta),$$

then we have

$$A^{((i))}(t, x, y; \tau, \xi, \eta)\Phi_i(\tau, \eta)=A(t, x, y; \tau, \xi, \eta)\Phi_i(\tau, \eta)$$

for $i=0, 1,\dots, l-1$. For $i=l$, more fine locallizations of $\{\mathscr{U}_j\}_{j=1,\dots,N}$ need to be given. Let

$$\hat{\Phi}_l(\omega, \eta)=\sum_{j=1}^N \hat{\Phi}_{U_j}(t, x, y; \omega, \eta)$$

$$\hat{\Psi}_{U_j}=1 \quad \text{on} \quad \text{supp}\,[\hat{\Phi}_{U_j}],$$

$$\text{supp}\,[\hat{\Psi}_{U_j}]\subset\mathscr{U}_j,$$

and let

$$\hat{A}^{((U_j))}(t, x, y; \omega, \xi, \eta)=\hat{\Psi}_{U_j}(t, x, y; \omega, \eta)\hat{A}(t, x, y; \omega, \xi, \eta)$$
$$+(1-\hat{\Psi}_{U_j}(t, x, y; \omega, \eta))\hat{A}^{(l)}(t^j, x^j, y^j; |\eta|\omega^j_l, \xi, |\eta|\eta^j),$$
$$A^{((U_j))}(t, x, y; \tau, \xi, \eta)=\hat{A}^{((U_j))}(t, x, y; \omega(\tau), \xi, \eta),$$

where $(t^j, x^j, y^j, \omega^j_l, \eta^j)\in U_j$ ($|\eta^j|=1$), then we have

$$A^{((U_j))}(t, x, y; \tau, \xi, \eta)\Phi_{U_j}(t, x, y; \tau, \eta)=A(t, x, y; \tau, \xi, \eta)\Phi_{U_j}(t, x, y; \tau, \eta).$$

Analogously, we define $B^{\sharp((i))}$ for $i=0, 1,\dots, l$. Since we have the decompositions

$$\hat{A}_0^{((i))}(t, x, y; \omega, \xi, \eta)=c\mathscr{E}_+\mathscr{E}_-(\prod_{j=i+1}^{l-1} E_j)H_l \qquad (i=0, 1,\dots, l-1),$$

$$\hat{A}_0^{((U))}(t, x, y; \omega, \xi, \eta)=c\mathscr{E}_+\mathscr{E}_-(\prod_{j=1}^{r} \mathscr{H}_j),$$

where $\mathscr{E}_\pm$, $H_j,\dots$ have the same properties as those in §3, even though they were defined in local, $V^{((i))}$ are defined by using $\mathscr{E}_\pm$, $H_j,\dots$ in the same way as in §3.

From Lemma 3.4 and Lemma 3.4′, we have

Lemma 4.5.

$$|||u||| \leqslant C(\sum_{i=0}^{l-1} \| V^{((i))}\Phi_i u\| + \sum^j \| V^{((U_j))}\Phi_{U_j}u\| + \gamma^{-q_1}|||u|||),$$

$$\langle\!\langle u\rangle\!\rangle \leqslant C(\sum_{i=0}^{l-1} \langle V^{((i))}\Phi_i u\rangle + \sum_j \langle V^{((U_j))}\Phi_{U_j}u\rangle + \gamma^{-q_1}\langle\!\langle u\rangle\!\rangle).$$

From Lemmas 3.3, 3.3′, 3.5 and 3.5′, we have

Lemma 4.6.

i) *Under the Assumption* (*A*), *we have*

$$\langle V_-^{((i))}u\rangle + \mu\gamma^{\frac{1}{2}q_1}\| V^{((i))}u\| \leqslant C\mu\langle V_+^{((i))}u\rangle + C_\mu\gamma^{-\frac{1}{2}q_1}\{\| A_0^{((i))}u\| + \langle\!\langle u\rangle\!\rangle + |||u|||\}$$
$$(i=0, 1,\dots, l)$$

for any $0<\mu<\mu_0$.

ii) *Under the Assumption* (*B*), *we have*

$$\langle V_+^{((i))}u\rangle \leqslant C\{\langle V_-^{((i))}u\rangle + \langle B^{\sharp((i))}u\rangle + \gamma^{-q_1}\langle\!\langle u\rangle\!\rangle\} \qquad (i=0, 1,\dots, l).$$

As a corollary of Lemma 4.6, we have

$$\langle V^{((i))}u\rangle + \gamma^{\frac{1}{2}q_1}\| V^{((i))}u\| \leqslant C\{\langle B^{\sharp((i))}u\rangle + \gamma^{-\frac{1}{2}q_1}\| A_0^{(i)}u\| + \langle\!\langle u\rangle\!\rangle + |||u|||\}$$
$$(i=0,\dots, l).$$

Replacing u by $\Phi_i u$ in the above, we sum up them about $i=0,\dots, l$. Then we have, from Lemma 4.4 and Lemma 4.5,

Proposition 4.7. *Unedr the assumptions* (*A*) *and* (*B*), *there exists* C *such that*

$$\langle\!\langle u\rangle\!\rangle + \gamma^{\frac{1}{2}q_1}|||u||| \leqslant C\{\langle B^\sharp(D_t - i\gamma, D_x, D_y)u\rangle + \gamma^{-\frac{1}{2}q_1}\| A(D_t - i\gamma, D_x, D_y)u\|\}$$

for $\gamma \geqslant C$.

4.4. Energy inequalities of higher or lower order. Let us denote

$$|||u|||_h = \sum_{k+j=h} |||(D_t - i\gamma)^k D_{x,y}^j u|||,$$

$$\langle\!\langle u\rangle\!\rangle_h=\sum_{k+j=h}\langle\!\langle (D_t-i\gamma)^k D_y^j u\rangle\!\rangle$$

for $h=0, 1, 2,\dots$, then we have

Proposition 4.8.

$$\gamma^{\frac{1}{2}q_1}|||u|||_h+\langle\!\langle u\rangle\!\rangle_h\leqslant C_h\{\gamma^{-\frac{1}{2}q_1}\|A(D_t-i\gamma, D_x, D_y)u\|_{H^h}+\langle \boldsymbol{B}^\sharp(D_t-i\gamma, D_x, D_y)u\rangle_{H^h}\}$$

for $\gamma\geqslant C_h$ $(h=0, 1, 2,\dots)$.

Proof. We remark that

$$\sum_{k+j=h}\|\{(D_t-i\gamma)^k D_y^j A-A(D_t-i\gamma)^k D_y^j\}u\|\leqslant C_h|||u|||_h$$

$$\leqslant C_h'(\sum_{k+j=h}|||(D_t-i\gamma)^k D_y^j u|||+\|A(D_t-i\gamma, D_x, D_y)u\|_{H^{h-1}}),$$

$$\sum_{k+j=h}\langle\{(D_t-i\gamma)^k D_y^j \boldsymbol{B}^\sharp-\boldsymbol{B}^\sharp(D_t-i\gamma)^k D_y^j\}u\rangle\leqslant C_h\langle\!\langle u\rangle\!\rangle_{h-1}\leqslant C_h\gamma^{-q_1}\langle\!\langle u\rangle\!\rangle_h.$$

Then we have from Proposition 4.7

$$\gamma^{\frac{1}{2}q_1}|||u|||_h+\langle\!\langle u\rangle\!\rangle_h\leqslant C\{\gamma^{-\frac{1}{2}q_1}\sum_{k+j=h}\|A(D_t-i\gamma)^k D_y^j u\|+\sum_{k+j=h}\langle \boldsymbol{B}^\sharp(D_t-i\gamma)^k D_y^j u\rangle+\|Au\|_{H^{h-1}}\}$$

$$\leqslant C(\gamma^{-\frac{1}{2}q_1}\|Au\|_{H^h}+\langle \boldsymbol{B}^\sharp u\rangle_{H^h})+C_h\gamma^{-\frac{1}{2}q_1}(|||u|||_h+\langle\!\langle u\rangle\!\rangle_h),$$

therefore

$$\gamma^{\frac{1}{2}q_1}|||u|||_h+\langle\!\langle u\rangle\!\rangle_h\leqslant C_h'(\gamma^{-\frac{1}{2}q_1}\|Au\|_{H^h}+\langle \boldsymbol{B}^\sharp u\rangle_{H^h})$$

for $\gamma\geqslant C_h'$. ■

In the same way, we have

Proposition 4.9.

$$\gamma^{\frac{1}{2}q_1}|||\Lambda(D_t-i\gamma, D_y)^{-h}u|||+\langle\!\langle \Lambda(D_t-i\gamma, D_y)^{-h}u\rangle\!\rangle$$

$$\leqslant C_h\{\gamma^{-\frac{1}{2}q_1}\|\Lambda(D_t-i\gamma, D_y)^{-h}A(D_t-i\gamma, D_x, D_y)u\|+\langle \Lambda(D_t-i\gamma, D_y)^{-h}\boldsymbol{B}^\sharp(D_t-i\gamma, D_x, D_y)u\rangle\}$$

for $\gamma\geqslant C_h$ $(h=0, 1, 2,\dots)$, where $\Lambda(\tau, \eta)=(|\tau|^2+|\eta|^2)^{\frac{1}{2}}$.

Proof. We remark

$$\|(\Lambda^{-h}A-A\Lambda^{-h})u\|\leqslant C_h|||\Lambda^{-h}u|||,$$

$$\langle(\Lambda^{-h}\boldsymbol{B}^\sharp-\boldsymbol{B}^\sharp\Lambda^{-h})u\rangle\leqslant C_h\langle\!\langle \Lambda^{-h-1}u\rangle\!\rangle\leqslant C_h\gamma^{-q_1}\langle\!\langle \Lambda^{-h}u\rangle\!\rangle.$$

Then we have from Proposition 4.7

$$\gamma^{\frac{1}{2}q_1}|||\Lambda^{-h}u||| + \langle\!\langle \Lambda^{-h}u \rangle\!\rangle \leqslant C\{\gamma^{-\frac{1}{2}q_1}\|A\Lambda^{-h}u\| + \langle B^{\#}\Lambda^{-h}u\rangle\}$$

$$\leqslant C(\gamma^{-\frac{1}{2}q_1}\|\Lambda^{-h}Au\| + \langle \Lambda^{-h}B^{\#}u\rangle) + C_h\gamma^{-\frac{1}{2}q_1}(|||\Lambda^{-h}u||| + \langle\!\langle \Lambda^{-h}u\rangle\!\rangle). \qquad \blacksquare$$

§5. Adjoint problem

5.1. Dirichlet set. Let us denote

$$N_{A,j} = (N_A - (0, m-j)) \cap N_A$$

and

$$\mathcal{N}_{A,j} = (N_A - (\gamma_{j+1} + \gamma_{j+2} + \cdots + \gamma_m, 0)) \cap N_A,$$

then we have

$$\mathcal{N}_{A,j} \subset N_{A,j} \qquad (j = 1, 2, \ldots, m).$$

Moreover we have

Lemma 5.1. *Let $j + k \leqslant m$.*

i) Let $N_P \subset \mathcal{N}_{A,j}$ and $N_Q \subset \mathcal{N}_{A,k}$, then $N_{PQ} \subset \mathcal{N}_{A,j+k}$.

ii) Let $N_P \subset N_{A,j}$ and $N_Q \subset \mathcal{N}_{A,k}$, then $N_{PQ} \subset N_{A,j+k}$.

Now let us say that $\{B_i(\tau, \xi, \eta)_{i=1,\ldots,m}$ is a Dirichlet set with respect to $A(\tau, \xi, \eta)$ if

$$\begin{pmatrix} B_1(\tau, \xi, \eta) \\ B_2(\eta, \xi, \eta) \\ \vdots \\ \vdots \\ B_m(\tau, \xi, \eta) \end{pmatrix} = \begin{pmatrix} 1 & b_{11}(\tau, \eta) & b_{12}(\tau, \eta) \cdots b_{1m-1}(\tau, \eta) \\ & 1 & b_{21}(\tau, \eta) \cdots b_{2m-2}(\tau, \eta) \\ & & \ddots \quad \ddots \quad \vdots \\ & & \ddots \quad \vdots \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} \xi^{m-1} \\ \xi^{m-2} \\ \vdots \\ \vdots \\ 1 \end{pmatrix},$$

where

$$N_{b_{ij}} \subset N_{A,j}.$$

Moreover, it is said to be a special Dirichlet set if

$$N_{b_{ij}} \subset \mathcal{N}_{A,j}.$$

We have from Lemma 5.1

Lemma 5.2. *We assume that $\{B_i(\tau, \xi, \eta)\}_{i=1,\ldots,m}$ is a (special) Dirichlet set with respect to $A(\tau, \xi, \eta)$. Denoting*

$$\begin{pmatrix} B_1 \\ \vdots \\ \vdots \\ B_m \end{pmatrix} = \begin{pmatrix} 1 & b_{11} & \cdots & b_{1m-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} \xi^{m-1} \\ \vdots \\ \vdots \\ 1 \end{pmatrix},$$

we set

$$\begin{pmatrix} C_1 \\ \vdots \\ \vdots \\ C_m \end{pmatrix} = \begin{pmatrix} 1 & b_{11} & \cdots & b_{1m-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix}^{-1} \begin{pmatrix} \xi^{m-1} \\ \vdots \\ \vdots \\ 1 \end{pmatrix},$$

then $\{C_i(\tau, \xi, \eta)\}_{i=1,\dots,m}$ *is also a (special) Driichlet set wieht respect to* $A(\tau, \xi, \eta)$.

5.2. Adjoint boundary operators. Let us denote

$$A(\tau, \xi, \eta) = a_0\xi^m + a_1(\tau, \eta)\xi^{m-1} + \cdots + a_m(\tau, \eta),$$

where we assume $a_0 = 1$ and Assumption (A), then we have

$$N_{a_j} \subset N_{A,j} \qquad (j = 1, \dots, m).$$

Moreover, we have

$$(A(D)u, v)_{L^2(R_+^{n+1})} - (u, A^*(\bar{D})v)_{L^2(R_+^{n+1})}$$

$$= \sum_{j=0}^{m} \{(a_{m-j}(D)D_x^j u, v) - (u, D_x^j a_{m-j}^*(\bar{D})v)\}$$

$$= i\sum_{j=0}^{m}\sum_{k=0}^{j-1} \langle D_x^{j-1-k}u, D_x^k a_{m-j}^*(\bar{D})v\rangle_{L^2(R^n), x=0}$$

$$= i\left\langle \begin{pmatrix} 1 & a_{11}(D)\cdots\cdots & a_{1m-1}(D) \\ & 1\ a_{21}(D)\cdots & a_{2m-2}(D) \\ & \ddots \quad \ddots & \vdots \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} D_x^{m-1} \\ D_x^{m-2} \\ \vdots \\ 1 \end{pmatrix} u, \begin{pmatrix} 1 \\ D_x \\ \vdots \\ D_x^{m-1} \end{pmatrix} v \right\rangle,$$

where $D = (D_t - i\gamma, D_x, D_y)$, $\bar{D} = (D_t + i\gamma, D_x, D_y)$, and

$$a_{ij}(\tau, \eta) = a_j(\tau, \eta) + \sum_{\substack{|\nu| \geq 1 \\ k < j}} c_{ijk\nu} D_{t,x,y}^{\nu} a_k(\tau, \eta) \quad \text{i.e.} \quad N_{a_{ij}} \subset \bigcup_{k=1}^{j} N_{a_k} \subset N_{A,j}.$$

Let us say that a set of boundary operators $\{B_i(D_t, D_x, D_y)\}_{i=1,\dots,m_+}$ be (specially) normal with respect to $A(D_t, D_x, D_y)$ if we get a (special) Dirichlet set with respect to $A(\tau, \xi, \eta)$

$$\{B_i(\tau, \xi, \eta)\}_{i=1,\dots,m_+} \cup \{\xi^{s_i}\}_{i=1,\dots,m_-} \qquad (m_+ + m_- = m)$$

for some $0 \leq s_i \leq m-1$, which we denote

$$\begin{pmatrix} \tilde{B}_1(\tau,\xi,\eta) \\ \vdots \\ \vdots \\ \tilde{B}_m(\tau,\xi,\eta) \end{pmatrix} = \begin{pmatrix} 1 & \tilde{b}_{11}(\tau,\eta) & \cdots & \tilde{b}_{1m-1}(\tau,\eta) \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} \xi^{m-1} \\ \vdots \\ \vdots \\ 1 \end{pmatrix}.$$

Now we denote

$$\begin{pmatrix} 1 & \tilde{b}'_{11}(D) & \cdots & \tilde{b}'_{1m-1}(D) \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a_{11}(D) & \cdots & a_{1m-1}(D) \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & \tilde{\tilde{b}}_{11}(D) & \cdots & \tilde{\tilde{b}}_{1m-1}(D) \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} 1 & \tilde{\tilde{b}}_{11}(D) & \cdots & \tilde{\tilde{b}}_{1m-1}(D) \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & \tilde{b}_{11}(D) & \cdots & \tilde{b}_{1m-1}(D) \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix} = I,$$

and we denote

$$(\tilde{B}'_1(D) \cdots \tilde{B}'_m(D)) = (1\, D_x \cdots D_x^{m-1}) \begin{pmatrix} 1 & \tilde{b}'_{11}(D) & \cdots & \tilde{b}'_{1m-1}(D) \\ & \ddots & \ddots & \vdots \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix},$$

then we have

$$(A(D)u, v)_{L^2(\boldsymbol{R}_+^{n+1})} - (u, A^*(\bar{D})v)_{L^2(\boldsymbol{R}_+^{n+1})}$$

$$= i \sum_{j=1}^{m} \langle \tilde{B}_j(D)u, \tilde{B}_j'^*(\bar{D})v \rangle_{L^2(\boldsymbol{R}^n), x=0}.$$

Denoting

$$B'_j = \tilde{B}'_{m-s_j} \qquad (j=1, \dots, m_-),$$

we have

$$(A(D)u, v)-(u, A^*(\bar{D})v)=i\{\sum_{j=1}^{m+}\langle B_j(D)u, C_j'^*(\bar{D})v\rangle+\sum_{j=1}^{m-}\langle D_x^{s_j}u, B_j'^*(\bar{D})v\rangle .$$

We say $\{B_j'^*\}_{j=1,\dots,m_-}$ a system of adjoint boundary operators for $\{A, \{B_j\}_{j=1,\dots,m_+}\}$.

Here we remark that $\{B_j'\}$ is not always normal with respect to A even if $\{B_j\}$ is normal with respect to A. In fact, let

$$\begin{cases} A=\tau^2-i(\xi^2+\eta^2)\tau-i(\xi^2+\eta^2)(\xi+\eta), \\ B_1=\xi^2, \quad B_2=\xi+i\tau, \end{cases}$$

then

$$B_1'=-i\tau(\tau+\eta)+\eta^2+(-i\tau+(\tau+\eta))\xi+\xi^2,$$

and therefore

$$N_{B_1'}\not\subset N_{A,2}.$$

We have from Lemmas 5.1 and 5.2

Lemma 5.3. *Let $\{B_i\}_{i=1,\dots,m_+}$ be specially normal with respect to A, then $\{B_i'\}_{i=1,\dots,m_-}$ is normal with resepct to A.*

5.3. Adjoint problem. For the problem:

$$(P)_\gamma \quad \begin{cases} A(D_t-i\gamma, D_x, D_y)u=f & \text{in } \boldsymbol{R}_+^{n+1}, \\ B_j(D_t-i\gamma, D_x, D_y)u|_{x=0}=g_j & \text{on } \boldsymbol{R}^n \quad (j=1,\dots, m_+), \end{cases}$$

we define its adjoint problem:

$$(P^*)_\gamma \quad \begin{cases} A^*(D_t+i\gamma, D_x, D_y)v=\varphi & \text{in } \boldsymbol{R}_+^{n+1}, \\ B_j'^*(D_t+i\gamma, D_x, D_y)v|_{x=0}=\psi_j & \text{on } \boldsymbol{R}^n \quad (j=1,\dots, m_-). \end{cases}$$

Assumption (B^*). $\{B_i\}_{i=1,\dots,m_+}$ is specially normal with respect to A, and $\{A^*(\bar{\tau}, \xi, \eta), \{B_i'^*(\bar{\tau}, \xi, \eta)\}_{i=1,\dots,m_-}\}$ satisfies the condition stated in Assumption (B).

Example. Let

$$\begin{cases} A=\tau^2-i(\xi^2+\eta^2)\tau-i(\xi^2+\eta^2)(\xi+\eta), \\ B_1=1, \quad B_2=\xi, \end{cases}$$

then $B_1'=1$ and $\{A, \{B_i\}_{i=1,2}\}$ satisfies Assumptions (A), (B), and (B^*).

Corresponding to Propositions 4.7–4.9, we have

Proposition 5.4. *Under the assumptions (A) and (B^*), we have*

$$\gamma^{\frac{1}{2}q_1}|||u|||_h+\langle\!\langle u\rangle\!\rangle_h\leqslant C_h\{\gamma^{-\frac{1}{2}q_1}\|A^*(D_t+i\gamma, D_x, D_y)u\|_{H^h}$$
$$+\langle \boldsymbol{B}'^{**}(D_t+i\gamma, D_x, D_y)u\rangle_{H^h}\},$$

$$\gamma^{\frac{1}{2}q_1}|||\Lambda^{-h}u||| + \langle\!\langle \Lambda^{-h}u \rangle\!\rangle \leqslant C_h\{\gamma^{-\frac{1}{2}q_1}\|\Lambda^{-h}A^*(D_t+i\gamma, D_x, D_y)u\| + \langle \Lambda^{-h}\boldsymbol{B}'^{*\sharp}(D_t+i\gamma, D_x, D_y)u\rangle\}$$

for $\gamma \geqslant C_h$ $(h=0, 1, 2,\dots)$.

§6. Existence theorem

In case when the adjoint problem is set well, we can get the existence theorem for the original problem.

Proposition 6.1. *Under the assumptions* (A) *and* (B^*), *we have a solution* $u \in H^{h-m}$ *satisfying*

$$(P)_\gamma \quad \begin{cases} A(D_t-i\gamma, D_x, D_y)u=f & \text{in } \boldsymbol{R}_+^{n+1}, \\ B_j(D_t-i\gamma, D_x, D_y)u|_{x=0}=g_j & \text{on } \boldsymbol{R}^n \quad (j=1,\dots, m_+) \end{cases}$$

for any $\{f \in H^h(\boldsymbol{R}_+^{n+1}), g_j \in H^h(\boldsymbol{R}^n)\ (j=1,\dots, m_+)\}(\gamma \geqslant C_h, h \geqslant 2m)$.

Proof Let us introduce

$$(v, w)_{\mathscr{H}} = (\Lambda^{-h}A^*(\bar{D})v, \Lambda^{-h}A^*(\bar{D})w) + \langle \Lambda^{-h}\boldsymbol{B}'^{*\sharp}(\bar{D})v, \Lambda^{-h}\boldsymbol{B}'^{*\sharp}(\bar{D})w\rangle,$$

$$\|v\|^2_{\mathscr{H}} = (v, v)_{\mathscr{H}}$$

for $\gamma \geqslant C_h$, where $\bar{D}=(D_t+i\gamma, D_x, D_y)$, and we define a Hilbert space $\mathscr{H}$ as a closure of H^∞ by $\|\cdot\|_{\mathscr{H}}$. Here we remark

$$|||\Lambda^{-h}v||| + \langle\!\langle \Lambda^{-h}v \rangle\!\rangle \leqslant C_h\|v\|_{\mathscr{H}}.$$

Now let

$$l(v) = (v, f) - \sum_{j=1}^{m_+} \langle C_j'^*(\bar{D})v, g_j\rangle,$$

then

$$\begin{aligned} |l(v)| &\leqslant C(\|\Lambda^{-h}v\|\,\|\Lambda^h f\| + \langle\!\langle \Lambda^{-h}v \rangle\!\rangle \textstyle\sum \langle \Lambda^h g_j\rangle) \\ &\leqslant C(|||\Lambda^{-h}v||| + \langle\!\langle \Lambda^{-h}v \rangle\!\rangle)(\|\Lambda^h f\| + \textstyle\sum \langle \Lambda^h g_j\rangle) \\ &\leqslant C'\|v\|_{\mathscr{H}}(\|\Lambda^h f\| + \textstyle\sum \langle \Lambda^h g_j\rangle). \end{aligned}$$

By Riesz's theorem, there exists $w \in \mathscr{H}$ such that

$$(w, v)^{\mathscr{H}} = (f, v) - \sum \langle g_j, C_j'^*(\bar{D})v\rangle.$$

We define

$$u = \Lambda^{-2h}A^*(\bar{D})w,$$

then we have $u \in H^{h-m}$ by the usual technique (see [5]) and it satisfies

$$\begin{cases} A(D)u=f & \text{in } \boldsymbol{R}_+^{n+1}, \\ B_j(D)u|_{x=0}=g_j & \text{on } \boldsymbol{R}^n \quad (j=1,\dots, m_+). \end{cases} \qquad \blacksquare$$

Now we consider the problem

$$(P)\quad \begin{cases} A(D_t, D_x, D_y)u=f & \text{in } \boldsymbol{R}_+^{n+1}, \\ B_j(D_t, D_x, D_y)u|_{x=0}=g_j & \text{on } \boldsymbol{R}^n \quad (j=1,\dots, m_+). \end{cases}$$

Proposition 6.2. *We assume the assumptions (A), (B) and (B*). Let*

$$e^{-\gamma t}f\in H^{\infty}(\boldsymbol{R}_+^{n+1}),\quad e^{-\gamma t}g_j\in H^{\infty}(\boldsymbol{R}^n)$$

for any $\gamma\geqslant C_0$, then there exists a unique solution u of (P), satisfying $e^{-\gamma t}u\in H^h(\boldsymbol{R}_+^{n+1})$ for any $\gamma\geqslant C_h$ and $h\geqslant m$.

Proof. Let

$$u_\gamma=e^{-\gamma t}u,\quad f_\gamma=e^{-\gamma t}f,\quad g_{j\gamma}=e^{-\gamma t}g_j,$$

then (P) becomes

$$(P)_\gamma\quad \begin{cases} A(D_t-i\gamma, D_x, D_y)u_\gamma=f_\gamma\in H^{\infty}(\boldsymbol{R}_+^{n+1}), \\ B_j(D_t-i\gamma, D_x, D_y)u_\gamma|_{x=0}=g_{j\gamma}\in H^{\infty}(\boldsymbol{R}^n) \quad (j=1,\dots, m_+). \end{cases}$$

Owing to Proposition 6.1, $(P)_\gamma$ has a unique solution $u_\gamma\in H^\infty$ for $\gamma\geqslant C_h$. Since we can show $e^{\gamma t}u=e^{\gamma' t}y_\gamma$ (see [5]), we have

$$u=e^{\gamma t}u_\gamma,\quad u_\gamma\in H^h$$

for $\gamma\geqslant C_h$. ■

As a corollary of Proposition 6.2, we have

Theorem. *We assume the assumptions (A), (B) and (B*). Let*

$$f\in H^{\infty}((-\infty, T)\times \boldsymbol{R}_+^n),\quad \operatorname{supp}[f]\subset[0,\infty)\times\overline{\boldsymbol{R}_+^n},$$

$$g_j\in H^{\infty}((-\infty, T)\times \boldsymbol{R}^{n-1}),\quad \operatorname{supp}[g_j]\subset[0,\infty)\times \boldsymbol{R}^{n-1}$$

then there exists unique solution u such that

$$u\in H^{\infty}((-\infty, T)\times \boldsymbol{R}_+^n),\quad \operatorname{supp}[u]\subset[0,\infty)\times\overline{\boldsymbol{R}_+^n}$$

and

$$(P)\quad \begin{cases} A(D_t, D_x, D_y)u=f & \text{in } (0, T)\times \boldsymbol{R}_+^n \\ B_j(D_t, D_x, D_y)u|_{x=0}=g_j & \text{on } (0, T)\times \boldsymbol{R}^{n-1} \quad (j=1,\dots, m_+). \end{cases}$$

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References

[1] I. G. Petrowsky, Über das Cauchysche Problem für ein System linearer partieller Differentialgleichungen im Gebiete der nichtanalytischen Funktionen, Bull. Univ. État, Moscow (1938).

[2] L. R. Volevich and S. G. Gindikin, The problem of Cauchy for differential operators with a dominant principal part, Functional Anal. Appl., **2** (1968).

[3] R. Sakamoto, Mixed problems for degenerate hyperbolic equations, J. Math. Kyoto Univ., **23** (1983)

[4] H. Kumano-go, Pseudo-differential operators, MIT Press (1982).

[5] R. Sakamoto, Mixed problems for hyperbolic equations I, II, J. Math. Kyoto Univ., **10** (1970).