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On analytic and geometric properties of Teichmüller spaces

By

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Introduction

As is well known L. Bers [9] initiated the investigation of boundary groups of Teichmüller spaces. Afterwards many authors have been studying Teichmüller spaces and their boundaries ([1], [2], [16], [19], [20] etc.). Recently some geometric methods and results related to 3-manifolds are used in order to investigate Kleinian groups and their Teichmüller spaces (cf. Thurston [23]).

In contrast with the methods in these studies, we shall investigate, in this paper, the boundaries of Teichmüller spaces by using the methods familiar in the complex function theory. Namely, our main tools are the Grunsky's inequality and some theorems on bounded analytic functions in the unit disk, e.g. the Fatou's theorem and the Riesz' one. The method using the Grunsky's inequality was motivated by the recent work of Žuravlev [24].

In the first part of this paper, we shall show a geometric property of Teichmüller spaces and the holomorphic convexity with respect to a family of holomorphic functions (Corollary 1, Theorem 4).

In the second part of this paper, we shall investigate the boundary behaviour of holomorphic mappings of the unit disk to a Teichmüller space (Theorem 5), and consider the boundary approach in Teichmüller disks as the special case (Theorem 6). Further, we shall study the boundary behaviour of periods of holomorphic differentials of the first kind as functions of the Teichmüller space (Theorem 8).

§ 1. The Bers' embedding of Teichmüller spaces

Let G be a non-elementary Fuchsian group acting on the unit disk Δ. We denote by $Q_n(G)$ the set of all quasiconformal self-mappings of Δ that are compatible with G and leave 1, $\pm i$ fixed. *The Teichmüller space $T(G)$ of G* is the set of all $w|_{\partial\Delta}$ with $w \in Q_n(G)$. The Teichmüller space $T(G)$ is a metric space with the Teichmüller metric $t_{T(G)}$. In particular, we call $T=T(\{1\})$ the universal Teichmüller space and denote by t_T the Teichmüller metric on T. If G is of the first kind, then $T(G)$ is identified with the set of all Fuchsian groups which are quasiconformal deformations of G modulo conformal automorphisms of Δ.

Let $L_\infty(\Delta)_1$ denote the set of all measurable functions μ on Δ such that $\|\mu\|_\infty < 1$, and let $L_\infty(\Delta, G)_1$ denote the set of $\mu \in L_\infty(\Delta)_1$ such that

$$\mu(g(z))\overline{g'(z)}g'(z)^{-1} = \mu(z), \quad g \in G,\ z \in \Delta.$$

For each $\mu \in L_\infty(\Delta, G)_1$ we denote by w^μ the quasiconformal mapping of $\boldsymbol{C}$ which leaves 1, $\pm i$ fixed and satisfies $(w^\mu)_{\bar z}(z) = \mu(z)(w^\mu)_z(z)$ on Δ and $(w^\mu)_{\bar z} \equiv 0$ on $\Sigma = \boldsymbol{C} - \bar\Delta$, and denote by w_μ the quasiconformal self-mapping of Δ which leaves 1, $\pm i$ fixed and $(w_\mu)_{\bar z}(z) = \mu(z)(w_\mu)_z(z)$ on Δ.

Then the Schwarzian derivative $\{w^\mu, z\}$ of w^μ on Σ belongs to $B(G)$, where $B(G)$ is the complex Banach space of holomorphic functions $\phi(z)$ on Σ such that $\phi(g(z))g'(z)^2 = \phi(z)$ $(g \in G,\ z \in \Sigma)$ and $\|\phi\| = \sup_\Sigma \lambda(z)^{-2}|\phi(z)| < \infty$, where λ is the Poincaré metric on Σ. Furthermore, the mapping i: $w_\mu|_{\partial\Delta} \mapsto \{w^\mu, z\}$ is well defined on $T(G)$ and injective. Thus the Bers' embedding of $T(G)$ to $B(G)$ is obtained. In the sequel, we identify $T(G)$ with $i(T(G))$ in $B(G)$. It is known that $T(G)$ is a bounded domain in $B(G)$ and $\overline{T(G)} \subset S(G) \subset B(G)$, where $S(G)$ is the set of Schwarzian derivatives of meromorphic functions schlicht on Σ contained in B(G).

Bers [9] showed that if G is of the first kind, then each ϕ in $\overline{T(G)}$ corresponds to a Kleinian group G^ϕ which is isomorphic to G and for each ϕ on $\partial T(G)$ G^ϕ is a *b-group*, i.e. it has only one simply connected invariant component.

§ 2. Teichmüller spaces and holomorphic mappings

Let $f(z) = z + \sum_{n=1}^\infty a_n z^{-n}$ be a univalent meromorphic function on a neighbourhood V of ∞. We can define *the Grunsky's coefficients* b_{mn} $(m, n = 1, 2, \ldots)$ of f as follows;

$$\log \frac{f(z) - f(w)}{z - w} = -\sum_{m,n=1}^\infty b_{mn} z^{-m} w^{-n}, \quad (z, w) \in V \times V.$$

Then it is known that for every sequence $\{\lambda_n\}_1^\infty$ of complex numbers *the Grunsky's inequality*:

$$|\sum_{m,n=1}^\infty b_{mn}\lambda_m\lambda_n| \leq \sum_{n=1}^\infty |\lambda_n|^2/n \tag{2.1}$$

holds whenever f is univalent on Σ (cf. Pommerenke [22]).

By using the inequality (2.1) Žuravlev [24] showed the following remarkable result.

Proposition 1. *Let F: $\Delta \to B(G)$ be a holomorphic mapping on Δ and continuous on $\bar\Delta$. Suppose that $F(\partial\Delta) \subset S(G)$. Then it holds that*

1) $F(\Delta) \subset S(G)$,
2) *if* $F(\Delta) \cap T \neq \emptyset$, *then* $F(\Delta) \subset T$,
3) *if* $F(\Delta) \cap T(G) \neq \emptyset$, *then* $F(\Delta) \subset T(G)$.

As a corollary, Žuravlev showed that $T(G)$ is the component of Int $S(G)$ in $B(G)$ containing the origin.

Now, we shall extend Proposition 1.

Theorem 1. (a) *Let $F\colon \Delta \to B(G)$ be a bounded holomorphic mapping of Δ. Suppose that for almost all $e^{i\theta} \in \partial\Delta$ $(0 \leqq \theta \leqq 2\pi)$ the cluster set of F at $e^{i\theta}$ is contained in $S(G)$. Then we have*

1) $F(\Delta) \subset S(G)$,

2) *if* $F(\Delta) \cap T \neq \emptyset$, *then* $F(\Delta) \subset T$,

3) *if* $F(\Delta) \cap T(G) \neq \emptyset$, *then* $F(\Delta) \subset T(G)$.

(b) *Let D be a bounded domain in $\boldsymbol{C}^n$ and let $F\colon D \to B(G)$ be a holomorphic mapping of D. Suppose that for every $z \in \partial D$ the cluster set of F at z is contained in $S(G)$. Then the same results as* 1), 2) *and* 3) *of* (a) *are valid for D.*

Proof. Since the proof is essentially the same as that of Proposition 1, we shall prove (a)–1) only.

For each ϕ in $B(G)$ we denote by $X(\phi)(z) = z + \sum_{n=1}^{\infty} c_n z^{-n}$ the locally univalent meromorphic function on Σ such that $\{X(\phi), z\} = \phi(z)$. There exist $r \geqq 1$ and $M = M(r) > 0$ such that $X(F(\zeta))$ is univalent in $\Sigma_r = \{z \in \boldsymbol{C};\ |z| > r\}$ and $\sup_{\Sigma_r} |X(F(\zeta))(z)| \leqq M$ for every ζ in Δ because F is bounded and $\|\phi\| < 2$ implies the univalence of $X(\phi)$. Therefore, we can consider the expansion

$$\log \frac{X(F(\zeta))(z) - X(F(\zeta))(w)}{z - w} = -\sum_{m,n} b_{mn}(F(\zeta)) z^{-m} w^{-n}$$

for $(z, w) \in \Sigma_r \times \Sigma_r$ and $\zeta \in \Delta$.

Since $X(F(\zeta))(z)$ is a holomorphic function of $\zeta \in \Delta$, $b_{mn}(F(\zeta))$ $(m, n = 1, 2, \ldots)$ are holomorphic, too. Put $g^{\zeta}(z) = r^{-1} X(F(\zeta))(rz) = z + \sum_{n=1}^{\infty} a_n z^{-n} r^{-(n+1)}$ $(z \in \Sigma)$, then $g^{\zeta}(z)$ is univalent on Σ. For the Grunsky's coefficients $\tilde{b}_{mn}(\zeta)$ $(m, n = 1, 2, \ldots)$ of g^{ζ},

$$\begin{aligned} -\sum_{m,n} \tilde{b}_{mn}(\zeta) z^{-m} w^{-n} &= \log \frac{g^{\zeta}(z) - g^{\zeta}(w)}{z - w} \\ &= \log \frac{X(F(\zeta))(rz) - X(F(\zeta))(rw)}{rz - rw} \\ &= \sum_{m,n} b_{mn}(F(\zeta))(rz)^{-m}(rw)^{-n} = -\sum_{m,n} b_{mn}(F(\zeta)) r^{-(m+n)} z^{-m} w^{-n} \end{aligned}$$

for $(z, w) \in \Sigma \times \Sigma$.

$$b_{mn}(F(\zeta)) = r^{m+n} \tilde{b}_{mn}(\zeta) \quad (m, n = 1, 2, \ldots). \tag{2.2}$$

On the other hand, from the Grunsky's inequality (2.1)

$$\Big|\sum_{m,n} \tilde{b}_{mn}(\zeta) \lambda_m \lambda_n\Big| \leqq \sum_{n=1}^{\infty} |\lambda_n|^2 / n. \tag{2.3}$$

From (2.2) and (2.3), we have

$$|\sum_{m,n}^{k} b_{mn}(F(\zeta))\lambda_m\lambda_n| = |\sum_{m,n}^{k} \tilde{b}_{mn}(\zeta)(\lambda_m r^m)(\lambda_n r^n)|$$

$$\leqq \sum_{n=1}^{k} |\lambda_n r^n|^2/n \leqq r^{2k} \sum_{n=1}^{k} |\lambda_n|^2/n$$

for arbitrarily fixed $\lambda_1, \ldots, \lambda_k$. Hence $\sum_{m,n}^{k} b_{mn}(F(\zeta))\lambda_m\lambda_n$ is a bounded holomorphic function of ζ in Δ.

Suppose that $F(\zeta_j)$ converges to $\phi(\theta) \in S(G)$ as $\zeta_j \to e^{i\theta}$ in a Stolz domain with vertex $e^{i\theta}$. Then $X(F(\zeta_j))$ converges to $X(\phi(\theta))$ normally on every compact subset in Σ. Therefore, $b_{mn}(F(\zeta_j))$ converges to the corresponding Grunsky's coefficient $b_{mn}(\phi(\theta))$ of $X(\phi(\theta))$ for each m, n.

From the Grunsky's inequality again, we have

$$|\sum_{m,n}^{k} b_{mn}(\phi(\theta))\lambda_m\lambda_n| \leqq \sum_{n=1}^{k} |\lambda_n|^2/n.$$

Hence, as $\zeta \to e^{i\theta}$ non tangentially

$$\lim_{\zeta \to e^{i\theta}} |\sum_{m,n}^{k} b_{mn}(F(\zeta))\lambda_m\lambda_n| \leqq \sum_{n=1}^{k} |\lambda_n|^2/n \tag{2.4}$$

holds for almost all $e^{i\theta} \in \partial\Delta$.

Since $\sum_{m,n}^{k} b_{mn}(F(\zeta))\lambda_m\lambda_n$ is a bounded holomorphic function, it can be represented by the Poisson integral of its non tangential limits. Hence from (2.4) we conclude that

$$|\sum_{m,n}^{k} b_{mn}(F(\zeta))\lambda_m\lambda_n| \leqq \sum_{n}^{k} |\lambda_n|^2/n \tag{2.5}$$

holds for every ζ in Δ. Furthermore, the inequality (2.5) holds for every k and for arbitrarily fixed $\lambda_1, \lambda_2, \ldots, \lambda_k$ in $\boldsymbol{C}$. So, $X(F(\zeta))$ is univalent on Σ and we conclude that $F(\zeta)$ is in $S(G)$ for every ζ in Δ. q.e.d.

Abikoff [2] showed that if ϕ is in $\partial T(G)$ (dim $T(G) < +\infty$) and if ϕ is a cusp or the area of the limit set of G^ϕ is zero, then ϕ is on $\partial(\text{Ext } T(G))$. (But this statement is proved for every ϕ in $\partial T(G)$ as a corollary of Proposition 1.) Here, we shall show more detailed results from the above theorem.

Theorem 2. *Let H_k be a k-dimiensional complex hyperplane in $B(G)$ such that $H_k \cap T(G) \neq \emptyset$, and let V_∞ denote the (unique) component of $H_k - H_k \cap \overline{T(G)}$ which is not relatively compact in H_k. Then every ϕ in $\partial(H_k \cap T(G))$ is contained in ∂V_∞, where the boundary operator ∂ is considered in H_k.*

Proof. Let ϕ be in $\partial(H_k \cap T(G))$. Suppose that ϕ is not in ∂V_∞. There exists a sufficiently small neighbourhood $U(\phi)$ of ϕ in H_k such that $U(\phi) \cap V_\infty = \emptyset$. From Theorem 1, ϕ is in $\partial(H_k - H_k \cap \overline{T(G)})$. In fact, if ϕ is not in $\partial(H_k - H_k \cap \overline{T(G)})$, then $U(\phi) \subset \overline{T(G)} \subset S(G)$, and id.$|_{U(\phi)}$, restriction of the identity mapping to $U(\phi)$, satisfies the condition of Theorem 1 (b)–3). Hence $U(\phi)$ is contained in $T(G)$. This is absurd because ϕ is in $\partial T(G)$.

Let V_i $(i=1, 2,\dots)$ be components of $H_k - H_k \cap \overline{T(G)}$ such that $U(\phi) \cap V_i \neq \emptyset$. Since each V_i is bounded and $\partial V_i \subset \overline{T(G)} \subset S(G)$, we conclude that $V_i \subset S(G)$ and $\overline{\cup V_i} \subset S(G)$ from Theorem 1 (b)–1). Therefore, $U(\phi)$ is contained in $S(G)$. On the other hand, $U(\phi) \cap T(G) \neq \emptyset$. Thus we have a contradiction as above. q.e.d.

Corollary 1 (cf. Abikoff [2]). *If* dim $T(G)$ *is finite, then every* ϕ *in* $\partial T(G)$ *is contained in the boundary of the unbounded component of* Ext $T(G)$.

§3. The Carathéodory metric and the holomorphic convexity of Teichmüller spaces

Let M be a complex manifold. Then *the Carathéodory metric* c_M on M is defined by

$$c_M(x, y) = \sup \{\rho(f(x), f(y));\ f\colon M \to \Delta \text{ holomorphic}\},$$

where ρ is the Poincaré distance on Δ.

Earle [14] showed that the Carathéodory metric on $T(G)$ is complete. At first, we shall give another proof of the following result due to Krushkal [18] and Kra [17].

Theorem 3. *If* $T(G)$ *is finite dimensional, then every closed* $c_{T(G)}$*-bounded set is compact.*

Proof. Let K be a closed set in $T(G)$ such that $\sup\{c_{T(G)}(0, \phi);\ \phi \in K\} = M < \infty$. For arbitrary $\lambda_1, \lambda_2,\dots, \lambda_N$ in C, set $f(\lambda_1, \lambda_2,\dots, \lambda_N\colon \phi) = (\sum_{m,n}^{N} b_{mn}(\phi)\lambda_m\lambda_n) \cdot (\sum_{n=1}^{N} |\lambda_n|^2/n)^{-1}$, where ϕ is in $T(G)$ and $b_{mn}(\phi)(m, n=1, 2,\dots)$ are the Grunsky's coefficients of $X(\phi)$. Then $f(\lambda_1, \lambda_2,\dots, \lambda_N\colon \cdot)$ is holomorphic in $T(G)$ and from the Grunsky's inequality (2.1) we have

$$|f(\lambda_1, \lambda_2,\dots, \lambda_N\colon \phi)| \leqq 1 \quad (\phi \in T(G)).$$

Since $f(\lambda_1, \lambda_2,\dots, \lambda_N\colon 0) = 0$, we have

$$\rho(f(\lambda_1, \lambda_2,\dots, \lambda_N\colon \phi), 0) \leqq M \tag{3.1}$$

for all $\phi \in$ K. Since $\lambda_1, \lambda_2,\dots, \lambda_N$ are arbitrary complex numbers, we conclude that for every $\phi \in$ K

$$|\sum_{m,n}^{N} b_{mn}(\phi)\lambda_m\lambda_n| \leqq (e^M - 1)(e^M + 1)^{-1}(\sum_{n=1}^{N} |\lambda_n|^2/n). \tag{3.2}$$

Therefore, by using a result of the univalent function theory (cf. Pommerenke [22] Sec. 9.4) we verify that $X(\phi)$ has a $K(M)$-quasiconformal extension to Δ for every $\phi \in$ K, where $K(M)$ is a constant depending only on M. That is, K is bounded with respect to the (universal) Teichmüller metric t_T, so K is compact because $T(G)$ is finite dimensional (cf. Kra [17] p. 239). q.e.d.

By using the above theorem, Krushkal [18] and Kra [17] showed that if $\dim T(G)<\infty$, then $T(G)$ is convex with respect to H^∞ (which is the family of bounded holomorphic functions on $T(G)$). This obviously implies the holomorphic convexity of $T(G)$ shown by Bers-Ehrenpreis [13].

Here, we shall give a stronger version of their result.

Theorem 4. *Suppose that* $\dim T(G)=N<\infty$. *Let* $\mathcal{O}(\boldsymbol{C}^N)$ *denote the set of all holomorphic functions in* $\boldsymbol{C}^N$. *Then* $T(G)$, *as a bounded domain in* $\boldsymbol{C}^N$, *is convex with respect to* $\mathcal{O}(\boldsymbol{C}^N)$.

Proof. Let K be a compact subset of $T(G)$. We define the $\mathcal{O}(\boldsymbol{C}^N)$-hull $\hat{\mathrm{K}}$ of K by

$$\hat{\mathrm{K}}=\{x;\ |f(x)|\leqq\sup_{y\in K}|f(y)| \text{ for every } f\in\mathcal{O}(\boldsymbol{C}^N)\}.$$

It suffices to show that $\hat{\mathrm{K}}$ is compact. Since $c_{T(G)}\leqq t_{T(G)}$ (cf. [18]) and K is $t_{T(G)}$-bounded, $M=\sup\{c_{T(G)}(0,\phi);\ \phi\in K\}<\infty$. We can easily verify that for arbitrarily fixed $\lambda_1, \lambda_2,\dots, \lambda_n$ in $\boldsymbol{C}$, $f(\lambda_1, \lambda_2,\dots, \lambda_n: \cdot)$ defined in the proof of Theorem 3 is in $\mathcal{O}(\boldsymbol{C}^N)$. Hence we have

$$\sup\{|f(\lambda_1, \lambda_2,\dots, \lambda_n: \phi)|;\ \phi\in \mathrm{K}\}$$
$$\leqq(e^M-1)(e^M+1)^{-1}.$$

And for every $\phi\in\hat{\mathrm{K}}$

$$|f(\lambda_1, \lambda_2,\dots, \lambda_n: \phi)|\leqq(e^M-1)(e^M+1)^{-1}. \tag{3.3}$$

Since the inequality (3.3) holds for arbitrary $\lambda_1, \lambda_2,\dots, \lambda_n$, we conclude that $\hat{\mathrm{K}}$ is bounded with respect to the Teichmüller metric by the same argument as in the proof of Theorem 3. So, $\hat{\mathrm{K}}$ is compact. q.e.d.

§4. The boundary approach in Teichmüller spaces

In this section, we shall consider a Teichmüller space $T(G)$ of finite dimensions and holomorphic mappings of Δ to $T(G)$.

Theorem 5. *If* $\dim T(G)=N<\infty$ *and* F *is a holomorphic mapping of* Δ *to* $T(G)$. *Then we have the followings.*

(a) *There exist measurable sets* E_1 *and* E_2 *on* $\partial\Delta$ *with* $\mathrm{mes}\,E_1=\mathrm{mes}\,E_2=0$ *such that*

i) *F has a non-tangential limit at every* $e^{i\theta}\in\partial\Delta-E_1$, *and*

ii) *the non-tangential limit at every* $e^{i\theta}\in\partial\Delta-E_1\cup E_2$ *corresponds to a quasi-Fuchsian group or a totally degenerat group.*

(b) *Let* $\{a_n\}_1^\infty$ *be a sequence in* Δ *such that* $\lim_{n\to\infty}F(a_n)=\phi_0\in\partial T(G)$ *exists. Let* $\{b_n\}_1^\infty$ *be another sequence in* Δ *satisfying* $\overline{\lim}_{m\to\infty}\rho(\{a_n\}_1^\infty, b_m)\leqq d$ *and* $\overline{\lim}_{m\to\infty}\rho(a_m, \{b_n\}_1^\infty)\leqq d$ *for a constant* $d<\infty$. *Then*

i) $\mathscr{A}(F(\{b_n\}))\subset\{x\in \boldsymbol{C}^N;\ |x-\phi_0|\leqq c\sqrt{N}(e^d-1)(e^d+1)^{-1}\}$,
where $\mathscr{A}(F(\{b_n\}))$ *is the set of accumulation points of* $\{F(b_n)\}_1^\infty$, $|\cdot|$ *the Euclidean distance in* $\boldsymbol{C}^N$ *and* c *a constant not depending on* $\{a_n\}_1^\infty$ *and* $\{b_n\}_1^\infty$. *Especially, if* $d=0$, $\lim_{n\to\infty} F(b_n)=\phi_0$.

ii) *If* ϕ_0 *corresponds to a cusp, then every* ϕ *in* $\mathscr{A}(F(\{b_n\}))$ *corresponds to a cusp.*

Proof. (a) Since $T(G)$ is identified with a bounded domain in $\boldsymbol{C}^N$, each $F^{(j)}$ is a bounded analytic function on Δ, where $F^{(j)}$ is the j-th coordinate function F as a mapping of Δ to $\boldsymbol{C}^N$. By the Fatou's theorem about bounded analytic functions, each $F^{(j)}$ has non-tangential limits almost everywhere on $\partial\Delta$. Hence we can find the exceptional set E_1 on $\partial\Delta$.

Next, we denote by h_z $(z\in\Delta)$ the group isomorphism of G to the Kleinian group corresponding to $F(z)$ in $T(G)$, which is defined in Bers [9], and put $\hat{g}(z)=(\mathrm{trace}\ h_z(g))^2$ for each $g\in G$. Then we can easily verify that $\hat{g}(z)$ is a bounded analytic function on Δ. By the Riesz' theorem about non-tangential limits of bounded analytic functions, non-tangential limits of $\hat{g}(z)$ exist almost everywhere on $\partial\Delta$ and are not equal to 4 almost everywhere on $\partial\Delta$ whenever g is a hyperbolic transformation. Since a point in $\overline{T(G)}$ corresponding to the non-tangential limit 4 of $\hat{g}$, for a hyperbolic transformation $g\in G$, is a cusp and G consists of a countable number of transformations, there is the exceptional set E_2 on $\partial\Delta$ such that the non-tangential limit of F at $e^{i\theta}\notin E_1\cup E_2$ is a non-cusp. Hence we verify from Maskit [20] Theorem 4 that E_2 is the desired exceptional set.

(b) Let $x=(x_1, x_2,\dots, x_N)$ be any point of $\mathscr{A}(F(\{b_n\}))$. We may assume that $\lim_{n\to\infty} F(b_n)=x$. Since $T(G)$ is a bounded domain, $|F^{(j)}(z)|\leqq c'$ $(j=1, 2,\dots, N)$ for a constant c'. By the assumption, for any $\varepsilon>0$ there exist sufficiently large n and $m=m(n)$ such that

$$\rho(a_m, b_n)\leqq d+\varepsilon. \tag{4.1}$$

Put $H_n^{(j)}(z)=(F^{(j)}(z)-F^{(j)}(a_m))(c'-F^{(j)}(z)\overline{F^{(j)}(a_m)}/c')^{-1}$, then $H_n^{(j)}$ is holomorphic on Δ, $|H_n^{(j)}(z)|\leqq 1$ and $H_n^{(j)}(a_m)=0$. Hence by the Schwarz's lemma we have from (4.1)

$$|H_n^{(j)}(b_n)|\leqq(e^{d+\varepsilon}-1)(e^{d+\varepsilon}+1)^{-1},$$

and this implies

$$|F^{(j)}(b_n)-F^{(j)}(a_m)|\leqq 2c'(e^{d+\varepsilon}-1)(e^{d+\varepsilon}+1)^{-1}, \tag{4.2}$$

$$|F(b_n)-F(a_m)|\leqq 2c'\sqrt{N}(e^{d+\varepsilon}-1)(e^{d+\varepsilon}+1)^{-1}.$$

When n, $m\to\infty$ and $\varepsilon\downarrow 0$, we have the desired inequality

$$|x-\phi_0|\leqq c\sqrt{N}(e^d-1)(e^d+1)^{-1},$$

where $c=2c'$.

If ϕ_0 is a cusp, then there exists a hyperbolic transformation g in G such that

$\lim_{n\to\infty} \hat{g}(a_n)=4$. Since $h_z(g)$ is a loxodromic transformation for every z in Δ, $\hat{g}(z) \notin [0, 4]$. Let W be a conformal mapping of $\hat{C}-[0, 4]$ onto $\{z \in \hat{C};\ \mathrm{Re}\, z>1\}$ such that $W(4)=1$. Then $\log|W\circ\hat{g}(z)|$ is a positive harmonic function on Δ and $\lim_{n\to\infty} W\circ\hat{g}(a_n)=1$. On the other hand, from Harnack's theorem there exists a constant $q(d)(\geqq 1)$ depending only on d such that $q(d)^{-1}\log|W\circ\hat{g}(a_m)| \leqq \log|W\circ\hat{g}(b_n)| \leqq q(d)\log|W\circ\hat{g}(a_m)|$, where m is a natural number satisfying (4.1) for a fixed ε. When $n, m\to\infty$, $\lim_{n\to\infty}|W\circ g(b_n)|=1$ and $\lim_{n\to\infty} g(b_n)=4$. Hence, our statement is proved.

q. e. d.

A special case (Teichmüller disks).

Let S_0 be a Riemann surface of type (g, n) with $3g+n-3>0$. We consider the set of all pairs (S, f) where S is a Riemann surface of type (g, n) and f is a quasiconformal mapping of S_0 onto S, and define the equivalence relation as follows. (S_1, f_1) and (S_2, f_2) are equivalent if $f_2\circ f_1^{-1}$: $S_1\to S_2$ is homotopic to a conformal mapping. We cenote by $T(S_0)$, *the Teichmüller space of* S_0, the set of all eqivalence classes $[(S, f)]$. The origin of $T(S_0)$ is taken as $[(S_0, \mathrm{id.})]$.

The Teichmüller space $T(S_0)$ of S_0 is naturally identified with the Teichmüller space $T(G_0)$ of G_0, where G_0 is a Fuchsian group of the first kind acting on Δ such that $S_0=\Delta/G_0$ (cf. Ahlfors [4]).

Let $Q(S_0)$ be the space of integrable holomorphic quadratic differentials on S_0 with the norm $\|\phi\|=\iint_{S_0}|\phi|dxdy$ for $\phi=\phi(z)dz^2$ in $Q(S_0)$. By the Teichmüller theorem, $T(S_0)$ can be idetified with $Q_1(S_0)$, the open unit ball of $Q(S_0)$. Further, for each $[(S, f)]$ in $T(S_0)$ a quasiconformal mapping f can be taken as so-called Teichmüller mapping, that is, the quasiconformal mapping for the Beltrami coefficient $k\bar{\phi}/|\phi|$ where $0\leqq k<1$ and ϕ is in $Q(S_0)$ (cf. Bers [7]).

We call ϕ in $Q(S_0)$ a *Jenkins-Strebel differential* if all horizontal trajectories of ϕ are simple closed curves except finite number of critical trajectories.

For a fixed ϕ in $Q(S_0)$ we consider the mapping of Δ defined by $z\to -z\bar{\phi}/|\phi|$ $(z\in\Delta)$. And we define the Teichmüller mapping f_z: $S_0\to S_z$ with the Beltrami coefficient $-z\bar{\phi}/|\phi|$. Then, by the Teichmüller theorem the mapping Ψ: $z\to[(S_z, f_z)]$ is a injection of Δ into $T(S_0)$. Furthermore, by the canonical identification of $T(S_0)$ with $T(G_0)$ Ψ is a holomorphic mapping of into $T(G_0)$, i.e. Ψ satisfies the condition of Theorem 5. We define *the Teichmüller disk* $D(\phi)$ by $\Psi(\Delta)$.

Theorem 6. *Let* S_0, G_0, ϕ *and* Ψ *be the same as above.*

(a) *If* ϕ *is a Jenkins-Strebel differential whose closed horizontal trajectories are homotopic to a closed curve on* S_0, *then for every horocycle* H *in* Δ *that is tangent to* $\partial\Delta$ *at* $z=1$, *every point in the cluster set of* Ψ *at* $z=1$ *from the inside of* H *is a cusp. In particular, if* $\dim T(G_0)=1$, *then* Ψ *has a limit from the inside of* H *and its limit is a cusp.*

(b) *Suppose that* $\dim T(G_0)=1$. *Let* E_1 *be the exceptional set on* $\partial\Delta$ *obtained in Theorem* 5 (a) *for* Ψ. *Then for distinct points* $e^{i\theta_1}$ *and* $e^{i\theta_2}$ *on* $\partial\Delta-E_1$, *non-tangential limits of* Ψ *at* $e^{i\theta_1}$ *and* $e^{i\theta_2}$ *are distinct.*

Proof. (a) Let H_0 be the horocycle in Δ passing through $z=0$. It is known (cf. Marden-Masur [19]) that the Dehn twist about the homotopy class of closed horizontal trajectories of ϕ determines a Teichmüller modular transformation and the transformation which can be an automorphism of $D(\phi)$ (hence, an automorphism of Δ by Ψ) corresponds to a Möbius transformation γ on Δ. Further, γ is a parabolic transformation with the fixed point $z=1$, and $\{\Psi(\gamma^n(0))\}_1^{\pm\infty}$ converges to a cusp as $n\to\pm\infty$. Since $\gamma^n(0)\in H_0$ ($n=\pm 1, \pm 2,\ldots$), the first statement of (a) for H_0 follows from Theorem 5 (b). Let H be another horocycle and let $\{b_n\}_1^\infty$ be an arbitrary sequence on H converging to $z=1$. Then we can easily verify that $\{\gamma^n(0)\}_1^{\pm\infty}$ and $\{b_n\}_1^\infty$ satisfy the condition of Theorem 5 (b), that is, $\overline{\lim}_{m\to\infty}\rho\{\gamma^n(0)\}_1^{\pm\infty}, b_m)<+\infty$ and $\overline{\lim}_{m\to\pm\infty}\rho(\gamma^m(0), \{b_n\}_1^\infty)<+\infty$. Hence every $x\in\mathscr{A}(\{\Psi(b_n)\})$ is a cusp. Since $\{b_n\}_1^\infty$ is an arbitrary sequence on H, by the same method as in the proof of Theorem 5 (b) the first statement of (a) is proved.

Since cusps are discrete when $\dim T(G_0)=1$, the second statement immediately follows from the connectivity of the cluster set of Ψ from the inside of H.

(b) Suppose that there are distinct points $e^{i\theta_1}$ and $e^{i\theta_2}$ on $\partial\Delta$ such that the non-tangential limits of Ψ at $e^{i\theta_1}$ and $e^{i\theta_2}$ are the same. Let ℓ_1 and ℓ_2 are the line segments from $z=0$ to $z=e^{i\theta_1}$ and $z=e^{i\theta_2}$ respectively. Then $\Psi(\ell_1)\cup\Psi(\ell_2)$ bounds a Jordan region R and $\Psi|_{\mathrm{D}}\colon \mathrm{D}\to\mathrm{R}$ is a conformal mapping of D to R, where D is a sector bounded by ℓ_1, ℓ_2 and an arc between $e^{i\theta_1}$ and $e^{i\theta_2}$. From the theory of conformal mappings, $\Psi(\mathrm{D})\subsetneqq\mathrm{R}$, that is, there exists $x\in\partial T(G_0)\cap\mathrm{R}$. But this contradicts with Corollary 1 in §2. q.e.d.

Of course, for $\Psi\colon \Delta\to T(G_0)$ defined by any $\phi\in Q(S_0)$ Theorem 5 (a) is valid, and the cluster set of Ψ at each point on $\partial\Delta$ consists of b-groups. Applying the Fubini's theorem for the spherical measure on $\partial Q_1(S_0)$ and the linear measure on $\partial D(\phi)$, we have the following theorem from Theorem 5.

Theorem 7. *There exists a measurable set E_3 on $\partial Q_1(S_0)$ with the spherical measure zero such that every geodesic ray $r(\phi)$ (with respect to the Teichmüller metric) in $T(S_0)$ corresponding to a line segment from zero to $\phi\in\partial Q_1(S_0)-E_3$ in $Q_1(S_0)$ converges to a point on $\partial T(G_0)$, which corresponds to a totally degenerate group by the canonical identification from $T(S_0)$ to $T(G_0)$.*

Denote by $\mathrm{Mod}(G_0)$ the modular group of G_0. Bers [11] showed that every $m\in\mathrm{Mod}(G_0)$ has a limit at every point on $\partial T(G_0)$ corresponding to a totally degenerate group. Therefore, from the above theorem we can show immediately the following.

Corollary 2. *Let E_3 be the same as in Theorem 7. Then for every $m\in\mathrm{Mod}(G_0)$ and for every geodesic ray $r(\phi)$ ($\phi\in\partial Q_1(S_0)-E_3$) $m(r(\phi))$ terminates to a point on $\partial T(G_0)$.*

§5. The boundary behaviour of holomorphic differentials

In this section, we assume that S_0 $(=\Delta/G_0)$ is a compact Riemann surface of genus $g>1$. Denote by $\{A_j, B_j\}_1^g$ a homology basis on S_0, and denote by θ_j ($j=1, 2,\dots, g$) the first order holomorphic differentials on S_0 satisfying

$$\int_{A_k} \theta_j=\delta_{jk} \qquad (j,\ k=1,\ 2,\dots,\ g). \tag{5.1}$$

For a holomorphic mapping F: $\Delta \to T(G_0)$ we denote by $[(S^z, f^z)]$ ($\in T(S_0)$) a point identified with $F(z)$. Then $\{f^z(A_j), f^z(B_j)\}_1^g$ is a homology basis on S^z. Denote by $\theta_j(z)$ ($j=1, 2,\dots, g$) the holomorphic differentials on S^z satisfying

$$\int_{f^z(A_k)} \theta_j(z)=\delta_{jk} \qquad (j,\ k=1,\ 2,\dots,\ g). \tag{5.2}$$

Then it is known that $\tau_{jk}(z)=\int_{f^z(B_k)} \theta_j(z)$ ($j, k=1, 2,\dots, g$) are well defined and holomorphic as functions of $z \in \Delta$ (cf. Bers [8]).

Theorem 8. *Let F be a holomorphic mapping of Δ to $T(G_0)$. Then each $\tau_{jk}(z)$ has finite non-tangential limits almost everywhere on $\partial\Delta$.*

Proof. It is well known that the $g \times g$ matrix $(\operatorname{Im} \tau_{jk})_{j,k=1}^g$ is positive definite for each z. Hence our assertion can be led from the following lemma.

Lemma. *Suppose that $g \times g$ matrix $(f_{jk}(z))_{j,k=1}^g$ is symmetric and holomorphic on Δ and $(\operatorname{Im} f_{jk})_{j,k=1}^g$ is positive definite for each $z \in \Delta$. Then each f_{jk} has finite non-tangential limits almost everywhere on $\partial\Delta$.*

Proof of the lemma. Since $(\operatorname{Im} f_{jk}(z))_{j,k=1}^g$ is positive definite, $\operatorname{Im} f_{jj}(z)>0$ ($j=1, 2,\dots, g$). Consequently, $\exp(-\sqrt{-1} f_{jj}(z))$ is a bounded analytic function on Δ. By the Fatou's theorem $\exp(\sqrt{-1} f_{jj}(z))$ has non-tangential limits almost everywhere on $\partial\Delta$, and by the Riesz' theorem the set on which $\exp(-\sqrt{-1} f_{jj}(z))$ has non-tangential limit zero is measure zero. This implies that the statement is true for $f_{jj}(z)$ ($j=1, 2,\dots, g$).

Since $(\operatorname{Im} f_{jk}(z))_{j,k=1}^g$ is positive definite, we have

$$\det\begin{pmatrix} \operatorname{Im} f_{jj}(z) & \operatorname{Im} f_{jk}(z) \\ \operatorname{Im} f_{jk}(z) & \operatorname{Im} f_{kk}(z) \end{pmatrix}>0 \qquad (\operatorname{Im} f_{jk}=\operatorname{Im} f_{kj}),$$

$$\begin{aligned} |\operatorname{Im} f_{jk}(z)| &< (\operatorname{Im} f_{jj}(z))^{1/2}(\operatorname{Im} f_{kk}(z))^{1/2} \\ &\leqq (\operatorname{Im} f_{jj}(z)+\operatorname{Im} f_{kk}(z))/2. \end{aligned}$$

Thus $|\operatorname{Im} f_{jk}(z)|$ has a harmonic majorant, and this implies that $\operatorname{Im} f_{jk}(z)$ is represented by $u_1(z)-u_2(z)$ where u_i ($i=1, 2$) are positive harmonic functions on Δ. By the above argument, the statement is also proved for f_{jk} ($j \neq k$).

Corollary 3. *Let F and $[(S^z, f^z)]$ be the same as above. We have a fixed holomorphic differential Θ on S_0 and denote by $\Theta(z)$ ($z \in \Delta$) the unique holomorphic differential on S^z such that*

$$\int_{f^z(A_j)} \Theta(z) = \int_{A_j} \Theta \qquad (j=1, 2,\dots, g),$$

where $\{A_j, B_j\}_1^g$ *is a fixed homology basis on* S_0. *Then* $\int_{f^z(B_j)} \Theta(z)$ $(j=1, 2,\dots, g)$ *and* $\|\Theta(z)\|_{S^z}^2 = \iint_{S^z} \Theta(z)\overline{\Theta(z)}$ *have finite non-tangential limits almost everywhere on* $\partial\Delta$.

Proof. Set $a_j = \int_{A_j} \Theta$ $(j=1, 2,\dots, g)$, then $\Theta(z) = \sum_{j=1}^{g} a_j \theta_j(z)$ where $\theta_j(z)$ are holomorphic differentials satisfying (5.2). Hence from Theorem 8, $\int_{f^z(B_k)} \Theta(z) = \sum_{j=1}^{g} a_j \int_{f^z(B_k)} \theta_j(z) = \sum_{j=1}^{g} a_j \tau_{jk}(z)$ has finite non-tangential limits almost everywhere on $\partial\Delta$. Furthermore, from the Riemann's bilinear relation (cf. Ahlfors-Sario [6]), we have

$$\begin{aligned}\|\Theta(z)\|_{S^z}^2 &= \sqrt{-1} \sum_{j=1}^{g} \left(\int_{f^z(A_j)} \Theta(z) \overline{\int_{f^z(B_j)} \Theta(z)} \right. \\ &\qquad \left. - \int_{f^z(B^j)} \Theta(z) \overline{\int_{f^z(A_j)} \Theta(z)} \right) \\ &= -2\,\mathrm{Im}\left(\sum_{j=1}^{g} a_j \int_{f^z(B_j)} \Theta(z) \right).\end{aligned}$$

Hence our assertion is also true for $\|\Theta(z)\|_{S^z}^2$. q.e.d.

Remark. From the above result, it is easy to show that *B-periods* and *norms* of holomorphic differentials with prescribed *A-periods*, as functions on $T(G_0)$, have finite limits along geodesic rays almost everywhere in the sense of Theorem 7.

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