J. Math. Kyoto Univ. (JMKYAZ)
24-4 (1984) 725-740

On Ahlfors' weak finiteness theorem

By

Hiromi OHTAKE

(Received September 3, 1983)

§ 0. Introduction.

Let $\boldsymbol{R}^n$ be the n-dimensional euclidean space ($n\geqq 2$), and set $\boldsymbol{H}=\boldsymbol{H}^n=\{x\in\boldsymbol{R}^n;\ x_n>0\}$, then $\boldsymbol{H}$ becomes the n-dimensional hyperbolic space with respect to the hyperbolic metric $\rho(x)|dx|$, where $\rho(x)=x_n^{-1}$. And let Γ be a group of isometries of $\boldsymbol{H}$, which acts discontinuously on $\boldsymbol{H}$. L. V. Ahlfors showed, in his lecture note [5], the weak finiteness theorem: if Γ is finitely generated, then the dimension of a certain class $Q(\Gamma)$ of mixed tensor densities, automorphic under Γ, is finite, which is an extension to higher dimensions of analytic parts of his famous finiteness theorem [1].

Our main aim is to introduce another certain class $\tilde{Q}(\Gamma)$ containing $Q(\Gamma)$, for which the dimension of $\tilde{Q}(\Gamma)$ is still finite (Corollary 3). In order to investigate $\tilde{Q}(\Gamma)$, we shall study a class $P(\Gamma)$ of quasiconformal deformations and derive properties of $\tilde{Q}(\Gamma)$ from those of $P(\Gamma)$ (Theorems 5 and 6, and Corollary 4).

In §1 we shall define some notations and state Ahlfors' weak finiteness theorem. In §2 we shall study quasiconformal deformations, and derive some new facts (Theorems 3 and 4). In §3 we shall state our main results, which will be proven in §5, after providing some lemmas in §4. And in the last §6 we shall state some remarks for the case $n=3$, particularly that our class $\tilde{Q}(\Gamma)$ turns out to be 0-dimensional.

The author wishes to express his deepest gratitude to Professor Y. Kusunoki for his encouragement and valuable comments, and for bringing this problem to author's attention. And the author also thanks to Drs. M. Taniguchi and H. Shiga, and Mr. M. Masumoto for their advices and comments during the preparation of this paper.

§ 1. Notations and Ahlfors' weak finiteness theorem.

By column vectors we denote the points in $\boldsymbol{R}^n$, and by ${}^tX$ the transpose of a matrix X.

Let f be an $\boldsymbol{R}^n$-valued smooth function defined on an open subset U of $\boldsymbol{R}^n$, then we define a linear operator $S=S_n$ by

$$(1.1)\qquad S_nf=\frac{1}{2}\{Df+{}^t(Df)\}-\frac{1}{n}(\mathrm{tr}Df)I_n,$$

where Df is the Jacobian matrix of f, and I_n is the unit matrix. We remark that S_2f corresponds to $f_{\bar{z}}$ (cf. Lemma 6). For any smooth functions ϕ with compact support, each entry $\nu_{ij}=(S_nf)_{ij}$ of S_nf satisfies the following equality

$$(1.2)\qquad \int_U\nu_{ij}\phi dx=-\int_U\frac{1}{2}(f_i\partial_j\phi+f_j\partial_i\phi)-\frac{1}{n}\delta_{ij}\sum_{k=1}^{n}f_k\partial_k\phi dx,$$

where by ∂_k we mean $\dfrac{\partial}{\partial x_k}$, and δ_{ij} is the Kronecker's delta.

For a matrix $X=(x_{ij})$, we define a norm $\|\cdot\|$ by

$$(1.3)\qquad \|X\|=\sqrt{\mathrm{tr}({}^tXX)}=\sqrt{\sum_{i,j}x_{ij}^2}.$$

It satisfies

$$(1.4)\qquad \|XY\|\leqq\|X\|\cdot\|Y\|,\ \text{and}\ \|X+Y\|\leqq\|X\|+\|Y\|.$$

A continuous function $f\colon \boldsymbol{R}^n\to\boldsymbol{R}^n$ is called *a quasiconformal (q. c.) deformation of* $\boldsymbol{R}^n$ if there exists S_nf in the distributional sense and $\|S_nf\|\in L^\infty(\boldsymbol{R}^n)$, that is, if there exist n^2 elements $\nu_{ij}(1\leqq i,\ j\leqq n)$ of $L^\infty(\boldsymbol{R}^n)$, which satisfy (1.2) for any smooth functions ϕ with compact support. We remark that the above definition of q. c. deformations is slightly different from Ahlfors' original one, but from Theorem 1 and Lemma 2 of Chapter VIII in [5], it can be seen that two definitions are equivalent.

By SM^n we denote the set of $n\times n$ matrices X such that ${}^tX=X$ and $\mathrm{tr}X=0$. Let $\varphi=(\varphi_{ij})$ be an SM^n-valued smooth function on an open subset of $\boldsymbol{R}^n$, then we define another linear operator $S^*=S_n^*$ by

$$(1.5)\qquad (S_n^*\varphi)_i=\sum_{j=1}^{n}\partial_j\varphi_{ij}.$$

We remark that $S_2^*\varphi$ corresponds to the complex derivative φ_z of φ.

Let $\mathscr{M}_n$ be the group of all Möbius transformations of $\hat{\boldsymbol{R}}^n=\boldsymbol{R}^n\cup\{\infty\}$ (the one-point compactification of $\boldsymbol{R}^n$). $\mathscr{M}_n$ is generated by reflections with respect to $(n-1)$-dimensional planes and spheres in $\boldsymbol{R}^n$. For $\gamma\in\mathscr{M}_n$, its Jacobian matrix $\gamma'(x)$ (for the elements of $\mathscr{M}_n$ we use the notation $\gamma'(x)$ instead of $D\gamma(x)$) can be written with $k>0$ and $V\in O(n)$ as

$$(1.6)\qquad \gamma'(x)=kV,$$

where k and V may depend on x. We denote this positive number k by $|\gamma'(x)|$. Then we get

$$(1.7)\qquad {}^t\gamma'(x)\gamma'(x)=|\gamma'(x)|^2.$$

By $\mathscr{M}_n(\boldsymbol{H})$ we mean the subgroup of the elements of $\mathscr{M}_n$, which leave $\boldsymbol{H}^n$ fixed. Then it is known that the hyperbolic metric $\rho(x)\,|dx|$, where $\rho(x)=x_n^{-1}$, is $\mathscr{M}_n(\boldsymbol{H})$-invariant, that is,

(1.8) $$\rho(\gamma x)\,|\gamma'(x)|=\rho(x) \qquad \text{for any } \gamma\in\mathscr{M}_n(\boldsymbol{H}).$$

In particular,

(1.9) $$|\gamma'(e_n)|={}^te_n\gamma(e_n) \qquad \text{for any } \gamma\in\mathscr{M}_n(\boldsymbol{H}),$$

where ${}^te_n=(0,\dots,0,1)$. We identify, in this paper, $\boldsymbol{R}^{n-1}$ with $\{x\in\boldsymbol{R}^n;\ x_n=0\}$, then for $n\geqq 3$ it is known that each element of $\mathscr{M}_{n-1}$ is uniquely and canonically extensible to an element of $\mathscr{M}_n(\boldsymbol{H})$, and conversely the restriction to $\hat{\boldsymbol{R}}^{n-1}$ of each element of $\mathscr{M}_n(\boldsymbol{H})$ becomes an element of $\mathscr{M}_{n-1}$. Hence we identify $\mathscr{M}_{n-1}$ with $\mathscr{M}_n(\boldsymbol{H})$, and use the same letters for their elements.

Let Γ be a subgroup of $\mathscr{M}_n(\boldsymbol{H})$, which acts discontinuously on $\boldsymbol{H}^n$. From now on we assume that Γ has this property. By $\Lambda=\Lambda(\Gamma)$ and $\Omega=\Omega(\Gamma)$ we mean, as usual, the limit set and the set of discontinuity of Γ in $\hat{\boldsymbol{R}}^n$, respectively. It is known that Λ is the set of the accumulation points of $\Gamma e_n=\{\gamma(e_n);\ \gamma\in\Gamma\}$ in $\hat{\boldsymbol{R}}^n$, $\Omega=\hat{\boldsymbol{R}}^n-\Lambda$ is open, and if $\Omega\cap\hat{\boldsymbol{R}}^{n-1}\neq\emptyset$ then $\Omega\cap\hat{\boldsymbol{R}}^{n-1}$ is dense in $\hat{\boldsymbol{R}}^{n-1}$.

For $\gamma\in\mathscr{M}_n$ and for an $\boldsymbol{R}^n$-valued function f on an open subset U of $\boldsymbol{R}^n$, we define an $\boldsymbol{R}^n$-valued function f_γ on $\gamma^{-1}(U)\cap\boldsymbol{R}^n$ by

(1.10) $$f_\gamma=(\gamma')^{-1}f\circ\gamma.$$

Definition 1. A continuous function $f\colon \boldsymbol{R}^n\longrightarrow\boldsymbol{R}^n$ is called *a q. c. deformation of* $\hat{\boldsymbol{R}}^n$ if f is a q. c. deformation of $\boldsymbol{R}^n$ and if f_γ is continuously extensible to $\gamma^{-1}(\infty)$ for some $\gamma\in\mathscr{M}_n$ such that $\gamma(\infty)\neq\infty$.

Let $Q(\Gamma)$ be the set of the SM^n-valued smooth functions φ on $\boldsymbol{H}^n$, which satisfy the next conditions (i)~(iv).

(i) $S_n^*\varphi=0$ on $\boldsymbol{H}^n$,

and by means of some q. c. deformation f of $\hat{\boldsymbol{R}}^n$ such that ${}^te_nf=0$ on $\boldsymbol{R}^{n-1}$, φ can be written as

$$\varphi=\rho^nS_nf \qquad \text{on } \boldsymbol{H}^n.$$

(ii) For any $\gamma\in\Gamma$ and all $x\in\boldsymbol{H}^n$

(1.11) $$|\gamma'(x)|^n\gamma'(x)^{-1}\varphi(\gamma x)\gamma'(x)=\varphi(x).$$

(iii) $$\sup_{x\in\boldsymbol{H}^n}\rho^{-n}(x)\|\varphi(x)\|<\infty, \text{ and}$$

(1.12) $$\int_{\boldsymbol{H}^n/\Gamma}\|\varphi(x)\|dx<\infty.$$

(iv) If $\Omega\cap\hat{\boldsymbol{R}}^{n-1}\neq\emptyset$, then φ has a smooth extension to $\Omega\cap\hat{\boldsymbol{R}}^{n-1}$ and this extension satisfies

$$Q(e_n)\varphi=\varphi Q(e_n) \qquad \text{on } \Omega\cap \boldsymbol{R}^{n-1},$$

where $Q(x)=x^t x/|x|^2$.

Remarks. 1) From (1.3), (1.7) and (1.11) we get

$$\|\varphi(x)\|=|\gamma'(x)|^n\|\varphi(\gamma x)\| \quad \text{for any } \gamma\in\Gamma,$$

that is, $\|\varphi(x)\|dx$ is Γ-invariant. Hence the integral of (1.12) can be taken over any fundamental set.

2) φ is said to have a smooth extension to $x=\infty$ if for $\gamma\in\mathcal{M}_n(\boldsymbol{H})$ such that $\gamma(\infty)\neq\infty$, $|\gamma'(x)|^n\gamma'(x)^{-1}\varphi(\gamma x)\gamma'(x)$ has a smooth extension to a neighbourhood of $x=\gamma^{-1}(\infty)$.

3) For $n=2$, $Q(\Gamma)$ is the vector space of anti-analytic functions φ on the upper half-plane such that $\bar{\varphi}$ are bounded and integrable automorphic forms of weight (-4) under Γ, and have the symmetric extension across $\Omega\cap\boldsymbol{R}$.

In [5] Ahlfors showed the following theorem.

Theorem 1. *(the weak finiteness theorem) If Γ is finitely generated, then the dimension of $Q(\Gamma)$ is finite.*

In this paper, instead of $Q(\Gamma)$, we are interested in a certain class $\tilde{Q}(\Gamma)$, which contain $Q(\Gamma)$ and is finitely dimensional for finitely generated Γ. Theorem 1 will be obtained as a consequence of our consideration of $\tilde{Q}(\Gamma)$.

§ 2. Q. c. deformations and harmonic functions.

The operators S_n and S_n^* defined in the previous section have the following properties.

Lemma 1. *For an $\boldsymbol{R}^n$-valued smooth function f and an SM^n-valued smooth function φ, and for $\gamma\in\mathcal{M}_n$*

(2.1) $S_n(f_\gamma)=(\gamma')^{-1}\{(S_nf)\circ\gamma\}\gamma'$,

(2.2) $S_n^*\{|\gamma'|^n(\gamma')^{-1}(\varphi\circ\gamma)\gamma'\}=|\gamma'|^{n+2}(\gamma')^{-1}(S_n^*\varphi)\circ\gamma$,

and, in particular, for $\gamma\in\mathcal{M}_n(\boldsymbol{H})$

(2.3) $S_n^*\{\rho^nS_n(f_\gamma)\}=|\gamma'|^{n+2}(\gamma')^{-1}\{S_n^*(\rho^nS_nf)\}\circ\gamma$.

(2.1) and (2.2) are shown in [3] (Lemma 1) and [5] ((2.1) is Lemma 4 of Chapter VIII, and (2.2) is shown in pp. 127∼129.). (2.3) is a consequence of (2.1), (2.2) and (1.8). We remark that the above relations hold in the distributional sense.

Proposition 1. *Every q. c. deformation f on $\hat{\boldsymbol{R}}^n$ has the following properties.*

(i) *For any* $\gamma\in\mathcal{M}_n$ *such that* $\gamma(\infty)\neq\infty$, f_γ *is continuously extensible to* $\gamma^{-1}(\infty)$.

(ii) $|f(x)|/(1+|x|^2)$ *is bounded on* $\boldsymbol{R}^n$.

(iii) *For any* $\gamma\in\mathcal{M}_n$, f_γ *is a q. c. deformation of* $\hat{\boldsymbol{R}}^n$.

Proof. From Definition 1 there exists some $\sigma\in\mathcal{M}_n$, $\sigma(\infty)\neq\infty$, such that f_σ is continuously extensible to $\sigma^{-1}(\infty)$. Set $\sigma^{-1}\gamma=\eta$, then we get $f_\gamma=(\eta')^{-1}f_\sigma\circ\eta$. The right hand side of this equlity is continuously extensible to $\gamma^{-1}(\infty)=\eta^{-1}\sigma^{-1}(\infty)$. Hence (i) is concluded.

Let J be the inversion with respect to the unit sphere, then we find that $J(x)=x/|x|^2$ and $|J'(x)|=|x|^{-2}$. Since f_J is bounded on $\{|x|<1\}$ from (i), and $|f_J(Jx)|=|f(x)|/|x|^2$, we see that $|f(x)|/|x|^2$ is bounded on $\{|x|>1\}$. This implies (ii).

In order to prove (iii), it is enough to show the following equalities: for any smooth functions ϕ with compact support

$$\tag{2.4} \begin{aligned}&\int_{R^n}[(\gamma')^{-1}\{(Sf)\circ\gamma\}\gamma']_{ij}\phi dx\\ &=-\int_{R^n}\frac{1}{2}\{(f_\gamma)_i\partial_j\phi+(f_\gamma)_j\partial_i\phi\}-\frac{1}{n}\delta_{ij}\sum_{k=1}^n(f_\gamma)_k\partial_k\phi dx.\end{aligned}$$

Set $\varphi=\left\{\frac{1}{2}(E_{ij}+E_{ji})-\frac{1}{n}\delta_{ij}I_n\right\}\phi$, where E_{ij} is the $n\times n$ matrix with entries $E_{ij,kl}=\delta_{ki}\delta_{lj}$. Then (2.4) can be rewritten as

$$\tag{2.5} \int_{R^n}\mathrm{tr}[(\gamma')^{-1}\{(Sf)\circ\gamma\}\gamma'\varphi]dx=-\int_{R^n}{}^t(f_\gamma)S^*\varphi dx.$$

We assume first that $\mathrm{supp}\varphi\not\ni\gamma^{-1}(\infty)$, This means that $|\eta'|^n(\eta')^{-1}(\varphi\circ\eta)\eta'$ is an SM^n-valued smooth function with compact support, where $\eta=\gamma^{-1}$. Then

$$\begin{aligned}&\int_{R^n}\mathrm{tr}[(\gamma')^{-1}\{(Sf)\circ\gamma\}\gamma'\varphi]dx\\ &=\int_{R^n}\mathrm{tr}[(\gamma'\circ\eta)^{-1}(Sf)(\gamma'\circ\eta)(\varphi\circ\eta)]|\eta'|^n dx\\ &=\int_{R^n}\mathrm{tr}[(Sf)|\eta'|^n(\eta')^{-1}(\varphi\circ\eta)\eta']dx\\ &=-\int_{R^n}{}^tfS^*\{|\eta'|^n(\eta')^{-1}(\varphi\circ\eta)\eta'\}dx\\ &=-\int_{R^n}{}^tf|\eta'|^{n+2}(\eta')^{-1}(S^*\varphi)\circ\eta dx\\ &=-\int_{R^n}{}^t(f\circ\gamma)|\gamma'|^{-2}\gamma'(S^*\varphi)dx=-\int_{R^n}{}^tf_\gamma S^*\varphi dx.\end{aligned}$$

Next we consider the general case. We may assume without loss of generality that $\gamma^{-1}(\infty)=0$. Let λ be a smooth function such that $0\leqq\lambda\leqq1$ on $\boldsymbol{R}^n$, $\lambda=1$ on $\{|x|\leqq1\}$, and $\mathrm{supp}\lambda=\{|x|\leqq2\}$. For $\delta>0$, set $\lambda_\delta(x)=\lambda(x/\delta)$, then we have $|\mathrm{grad}\ \lambda_\delta|\leqq\mathrm{const.}\delta^{-1}$. Since (2.5) holds for

$(1-\lambda_\delta)\varphi,\ \delta>0,$

$$\left|\int_{R^n}\mathrm{tr}[(\gamma')^{-1}\{(Sf)\circ\gamma\}\gamma'\varphi]dx+\int_{R^n}{}^tf_\gamma S^*\varphi dx\right|$$
$$=\left|\int_{R^n}\mathrm{tr}[(\gamma')^{-1}\{(Sf)\circ\gamma\}\gamma'(\lambda_\delta\varphi)]dx+\int_{R^n}{}^tf_\gamma S^*(\lambda_\delta\varphi)dx\right|$$
$$\leqq\int_{R^n}\|(Sf)\circ\gamma\|\ \|\varphi\|\lambda_\delta dx$$
$$+\int_{R^n}|f_\gamma|(|S^*\varphi|\lambda_\delta+\|\varphi\|\,|\mathrm{grad}\ \lambda_\delta|)dx$$
$$\leqq \text{const. }\delta^n+\text{const. }\delta^{n-1}.$$

By letting $\delta\to 0$ we see that (2.5) holds for any smooth φ with compact support, q. e. d.

An $\boldsymbol{R}^n$-valued smooth function f on $\boldsymbol{H}^n$ is said to be *harmonic* if $S_n^*(\rho^nS_nf)=0$ on $\boldsymbol{H}^n$.

Theorem 2. *Let f be an $\boldsymbol{R}^n$-valued continuous function on $\boldsymbol{H}^n\cup\boldsymbol{R}^{n-1}$. Suppose that f is harmonic, $|f|/(1+|x|^2)$ is bounded on $\boldsymbol{H}^n$ and ${}^te_nf=0$ on $\boldsymbol{R}^{n-1}$. Then we have*

$$c_nf(x)=2^{n-1}\int_{R^{n-1}}\frac{\{I_n-2Q(y-x)\}f(y)}{\rho^{n+1}(x)\,|y-x|^{2n}}dy, \tag{2.6}$$

where $c_n=\dfrac{2(n-1)}{n}\omega_n$ ($\omega_n=2\pi^{n/2}\Gamma(n/2)$ is the $(n-1)$-dimensional measure of the $(n-1)$-dimensional unit sphere.)

The above theorem is shown in Theorem 3 of [3], implicitly. It is assumed there that f is bounded, but if f_T is bounded for $T(x)=e_n+2(x^*-e_n)^*$, where $x^*=J(x)=x/|x|^2$, then the proof in [3] is applicable. It is easily seen that the boundedness of $|f|/(1+|x|^2)$ implies that of f_T. Hence (2.6) holds.

Let h be an $\boldsymbol{R}^{n-1}$-valued continuous function on $\boldsymbol{R}^{n-1}$ such that $|h|/(1+|x|^2)$ is bounded on $\boldsymbol{R}^{n-1}$. From Theorem 2 it can be seen that there exists a harmonic function f on $\boldsymbol{H}^n$ such that $f|_{\boldsymbol{R}^{n-1}}=h$, and $|f|/(1+|x|^2)$ is bounded on $\boldsymbol{H}^n$, and that such a harmonic function f is unique. We call such f *the canonical harmonic extension of* h, and denote it by Hh.

Corollary 1. *Under the same assumption as in Theorem* 2,

$$\lim_{t\to 0}f(x+te_n)=f(x)\qquad\text{for } x\in\boldsymbol{R}^{n-1},$$

and this convergence is uniform on any compact subsets of $\boldsymbol{R}^{n-1}$.

This corollary is shown by the same argument as in the case of the Poisson integral for the upper half-plane.

Lemma 2. *Let U be an open subset of $\boldsymbol{R}^n$. For a smooth function f: U*

$\longrightarrow \boldsymbol{R}^n$ *we define the smooth function* $h: U\cap \boldsymbol{R}^{n-1}\longrightarrow \boldsymbol{R}^{n-1}$ *by* $h=(I_{n-1}\ 0)f|_{U\cap \boldsymbol{R}^{n-1}}$, *where* $(I_{n-1}\ 0)$ *is the* $(n-1)\times n$ *matrix of the form* $\begin{pmatrix} 1 & 0 & 0 \\ & \ddots & \vdots \\ 0 & 1 & 0 \end{pmatrix}$. *Then the relation between* S_nf *and* $S_{n-1}h$ *is*

$$S_nf=\begin{pmatrix} S_{n-1}h-\alpha I_{n-1} & a \\ {}^ta & (n-1)\alpha \end{pmatrix}, \tag{2.7}$$

where $\alpha=\frac{1}{n}\left\{\partial_nf_n-\frac{1}{n-1}(\mathrm{tr}\,Dh)\right\}$, *and* a *is the* $(n-1)$*-dimensional vector with components* $a_j=\frac{1}{2}(\partial_jf_n+\partial_nf_j)$. *In particular,*

$$||S_{n-1}h||\leqq\sqrt{2}\,||S_nf||. \tag{2.8}$$

Proof. (2.7) is seen from the definition (1.1) of S_nf and $S_{n-1}h$, and by simple calculation. And (2.8) is shown from the next estimate.

$$\begin{aligned}||Sh||^2&\leqq(||Sh-\alpha I_{n-1}||+||\alpha I_{n-1}||)^2\\ &\leqq 2(||Sh-\alpha I_{n-1}||^2+||\alpha I_{n-1}||^2)\\ &=2\{||Sh-\alpha I_{n-1}||^2+(n-1)\alpha^2\}\\ &\leqq 2\{||Sh-\alpha I_{n-1}||^2+2|a|^2+(n-1)^2\alpha^2\}\\ &=2||Sf||^2.\end{aligned}$$

Lemma 3. *Let* U *be an open subset of* $\boldsymbol{R}^n$ *and* $f: U\longrightarrow \boldsymbol{R}^n$. *If there exists a sequence of smooth functions* $f^{(k)}: U\longrightarrow \boldsymbol{R}^n$ *such that* $f^{(k)}$ *converges to* f *uniformly on every compact subset of* U, *and for any* k

$$||S_nf^{(k)}||\leqq M<\infty \qquad \text{on } U, \tag{2.9}$$

then there exists S_nf *on* U *and we have*

$$||S_nf||\leqq nM \qquad \text{on } U. \tag{2.10}$$

Proof. Let $C_0^\infty(U)$ be the set of smooth functions with compact support in U. (We regard $C_0^\infty(U)$ as a linear subspace of $L^1(U)$.) For any $\phi\in C_0^\infty(U)$ we have

$$\begin{aligned}&\int_U (Sf^{(k)})_{ij}\phi dx\\ &\quad=-\int_U \frac{1}{2}\{f_i^{(k)}\partial_j\phi+f_j^{(k)}\partial_i\phi\}-\frac{1}{n}\delta_{ij}\sum_m f_m^{(k)}\partial_m\phi dx.\end{aligned} \tag{2.11}$$

Define linear functionals $\ell_{k,ij}$ on $C_0^\infty(U)$ by

$$\ell_{k,ij}(\phi)=\int_U (Sf^{(k)})_{ij}\phi dx \qquad \text{for } \phi\in C_0^\infty(U). \tag{2.12}$$

Then from (2.11) there exist linear functionals ℓ_{ij} on $C_0^\infty(U)$ such that $\ell_{ij}(\phi)=\lim_{k\to\infty}\ell_{k,ij}(\phi)$. From (2.9) and (2.12) we see that ℓ_{ij} are bounded, in fact, $\|\ell_{ij}\|_{C_0^\infty(U)} \leqq M$. By the Hahn-Banach theorem, there exist linear functionals $\tilde{\ell}_{ij}$ on $L^1(U)$ such that $\|\tilde{\ell}_{ij}\|_{L^1(U)} = \|\ell_{ij}\|_{C_0^\infty(U)}$ and $\tilde{\ell}_{ij}(\phi) = \ell_{ij}(\phi)$ for $\phi \in C_0^\infty(U)$. Corresponding to $\tilde{\ell}_{ij}$, there exist elements ν_{ij} of $L^\infty(U) = L^1(U)^*$ such that $\|\nu_{ij}\|_\infty = \|\tilde{\ell}_{ij}\|_{L^1(U)}$ and $\tilde{\ell}_{ij}(g) = \int_U \nu_{ij}\, g dx$ for $g \in L^1(U)$. By letting $k \to \infty$, for any $\phi \in C_0^\infty(U)$ we get

$$\int_U \nu_{ij} \phi dx = -\int_U \frac{1}{2} \{f_i \partial_j \phi + f_j \partial_i \phi\} - \frac{1}{n} \delta_{ij} \sum_m f_m \partial_m \phi dx.$$

Hence, from the definition of Sf, we see $Sf = (\nu_{ij})$. And from $\|\nu_{ij}\|_\infty \leqq M$, it is seen that $\|Sf\| \leqq nM$ on U.

Lemma 4. *Let f be a q. c. deformation of $\boldsymbol{R}^n$. Then there exists a sequence of smooth q. c. deformations $f^{(k)}$ of $\boldsymbol{R}^n$ such that $f^{(k)} \to f$ uniformly on any compact subsets of $\boldsymbol{R}^n$, and*

$$\|S_n f^{(k)}\| \leqq \operatorname*{ess\,sup}_{\boldsymbol{R}^n} \|S_n f\| \qquad \textit{for every } k.$$

Proof. Let δ_ε be a smooth positive function with support in $\{x \in \boldsymbol{R}^n; |x| < \varepsilon\}$ such that $\int_{\boldsymbol{R}^n} \delta_\varepsilon(x) dx = 1$. And set

$$f_\varepsilon(x) = \int_{\boldsymbol{R}^n} \delta_\varepsilon(y-x) f(y) dy.$$

Then we have

$$\partial_j (f_\varepsilon)_i(x) = -\int_{\boldsymbol{R}^n} (\partial_j \delta_\varepsilon)(y-x) f_i(y) dy.$$

So we get from the definition of Sf that

$$\begin{aligned}
&S(f_\varepsilon)_{ij}(x) \\
&\quad = -\int_{\boldsymbol{R}^n} \frac{1}{2} \{(\partial_j \delta_\varepsilon)(y-x) f_i(y) + (\partial_i \delta_\varepsilon)(y-x) f_j(y)\} \\
&\qquad - \frac{1}{n} \delta_{ij} \sum_m (\partial_m \delta_\varepsilon)(y-x) f_m(y) dy \\
&\quad = \int_{\boldsymbol{R}^n} \delta_\varepsilon(y-x) (Sf)_{ij}(y) dy.
\end{aligned}$$

Thus

$$\begin{aligned}
\|S(f_\varepsilon)(x)\|^2 &= \sum_{i,j} \left(\int_{\boldsymbol{R}^n} \delta_\varepsilon(y-x) (Sf)_{ij}(y) dy \right)^2 \\
&\leqq \sum_{i,j} \int_{\boldsymbol{R}^n} \delta_\varepsilon(y-x) (Sf)_{ij}^2(y) dy \\
&= \int_{\boldsymbol{R}^n} \delta_\varepsilon(y-x) \|Sf(y)\|^2 dy \\
&\leqq \operatorname{ess\,sup} \|Sf\|^2.
\end{aligned}$$

Hence by setting $f^{(k)} = f_{1/k}$ we get a desired sequence.

The following Theorems 3 and 4 are consequences of Lemmas 2, 3 and 4.

Theorem 3. *Let $f^{(k)}$ be q. c. deformations of $\boldsymbol{R}^n$ such that for any k $\|S_n f^{(k)}\| \leqq M < \infty$ on $\boldsymbol{R}^n$. If $f^{(k)}$ converges to f uniformly on compact subsets of $\boldsymbol{R}^n$, then f is a q. c. deformation.*

Theorem 4. *Let f be a q. c. deformation of $\boldsymbol{R}^n$ $(n \geqq 3)$, and set $h = (I_{n-1}0) f|_{\boldsymbol{R}^{n-1}}$, then h is a q. c. deformation of $\boldsymbol{R}^{n-1}$.*

Corollary 2. *In the above theorem if f is a q. c. deformation of $\hat{\boldsymbol{R}}^n$ $(n \geqq 3)$, then h is a q. c. deformation of $\hat{\boldsymbol{R}}^{n-1}$.*

§ 3. Main results.

In this section we define two classes $\tilde{Q}(\Gamma)$ and $P(\Gamma)$, and state our main results.

Definition 2. For $n \geqq 3$, by $\tilde{Q}(\Gamma)$ we denote the vector space of all SM^n-valued functions φ on $\boldsymbol{H}^n$, which satisfy the following conditions (i) $\sim$(iii).

(i) There is some q. c. deformation h of $\hat{\boldsymbol{R}}^{n-1}$, and by means of the canonical harmonic extension $f = Hh$ of h, φ can be written as

$$\varphi = \rho^n S_n f \quad \text{on} \quad \boldsymbol{H}^n. \tag{3.1}$$

(ii) For any $\gamma \in \Gamma$ and all $x \in \boldsymbol{H}^n$

$$|\gamma'(x)|^n \gamma'(x)^{-1} \varphi(\gamma x) \gamma'(x) = \varphi(x). \tag{3.2}$$

(iii) If $\Omega(\Gamma) \cap \hat{\boldsymbol{R}}^{n-1} \neq \emptyset$, then $\rho^{-n}\varphi$ has a continuous extension to $\Omega(\Gamma) \cap \hat{\boldsymbol{R}}^{n-1}$, and this extension satisfies

$$\rho^{-n}\varphi = 0 \quad \text{on } \Omega(\Gamma) \cap \hat{\boldsymbol{R}}^{n-1}. \tag{3.3}$$

Remarks. 1) In case of $\Omega(\Gamma) \cap \hat{\boldsymbol{R}}^{n-1} \ni \infty$, $\rho^{-n}\varphi$ is said to have a continuous extension to $x = \infty$ if $(\rho \circ \gamma)^{-n} (\gamma')^{-1} (\varphi \circ \gamma) \gamma'$ has a continuous extension to a neighbourhood of $x = \gamma^{-1}(\infty)$ for some $\gamma \in \mathscr{M}_n(\boldsymbol{H})$.

2) We do not define $\tilde{Q}(\Gamma)$ for $n = 2$. For $n \geqq 3$, from Corollary 2, it turns out that $Q(\Gamma) \subset \tilde{Q}(\Gamma)$. So Theorem 1, for $n \geqq 3$, comes from Corollary 3 below. In case of $n = 2$, Theorem 1 follows from Ahlfors' finiteness theorem [1].

Definition 3. For $n \geqq 3$, by $P(\Gamma)$ we mean the vector space of all q. c. deformations h of $\hat{\boldsymbol{R}}^{n-1}$, which satisfy the next conditions.

$$S_{n-1} h = (\gamma')^{-1} [(S_{n-1} h) \circ \gamma] \gamma' \quad \text{for any } \gamma \in \Gamma. \tag{3.4}$$

$$S_{n-1} h = 0 \quad \text{on } \Omega(\Gamma) \cap \boldsymbol{R}^{n-1}. \tag{3.5}$$

Theorem 5. *If Γ is finitely generated, then the dimension of $P(\Gamma)$ is finite.*

Theorem 6. *The linear mapping $h \longrightarrow \varphi = \rho^n S_n(Hh)$ is a mapping of $P(\Gamma)$ onto $\tilde{Q}(\Gamma)$ and the kernel of this mapping is $\{h \in P(\Gamma) ; S_{n-1} h = 0$ on $\boldsymbol{R}^{n-1}\}$.*

Corollary 3. *If Γ is finitely generated, then the dimension of $\tilde{Q}(\Gamma)$ is finite.*

Corollary 4. *The following conditions* (a) *and* (b) *are equivalent.*

(a) *If $h \in P(\Gamma)$, then $S_{n-1}h = 0$ on $\boldsymbol{R}^{n-1}$.*

(b) $\tilde{Q}(\Gamma) = \{0\}$.

The proofs of the above two theorems will be given in §5.

§4. Lemmas.

In this section we state some lemmas to show our main results.

Lemma 5. *Let h be a q. c. deformation of $\hat{\boldsymbol{R}}^{n-1} (n \geqq 3)$, then for $\gamma \in \mathcal{M}_n(\boldsymbol{H}) = \mathcal{M}_{n-1}$*

$$H(h_\gamma) = (Hh)_\gamma. \tag{4.1}$$

We remark that γ in the left hand side of (4.1) is regarded as $\gamma \in \mathcal{M}_{n-1}$, on the other hand, γ in the right hand side is regarded as $\gamma \in \mathcal{M}_n(\boldsymbol{H})$.

Proof. It follows from (2.3) that $(Hh)_\gamma$ is harmonic. And it is easily seen that $|(Hh)_\gamma|/(1+|x|^2)$ is bounded on $\boldsymbol{H}^n$ and the boundary value of $(Hh)_\gamma$ is h_γ. Hence $(Hh)_\gamma$ is the canonical harmonic extension of h_γ, so (4.1) holds.

Lemma 6. 1) *For $n \geqq 3$, $S_n f = 0$ on a domain U in $\boldsymbol{R}^n$ if and only if f is of the from*

$$f(x) = a + \lambda x + Bx + c|x|^2 - 2x^t xc \quad \text{on } U, \tag{4.2}$$

where a and c are constant vectors, λ is a constant scalar, and B is a constant matrix such that $B = -{}^tB$.

2) *$S_2 f = 0$ and $|f|/(1+|x|^2)$ is bounded on $\boldsymbol{R}^2$ if and only if f has the form* (4.2) *on $\boldsymbol{R}^2$.*

Proof. In [5] 1) is shown (see Lemma 1 of Chapter VIII.), so we show here only 2). By the same argument as used in the proof of 1) in [5], we may assume that f is smooth. From the definition (1.1), it is seen that $S_2 f = 0$ if and only if $\partial_1 f_1 - \partial_2 f_2 = 0$ and $\partial_2 f_1 + \partial_1 f_2 = 0$, that is, $F = f_1 + \sqrt{-1} f_2$ is an analytic function of $z = x_1 + \sqrt{-1} x_2$. Since $|F|/(1+|z|^2)$ is bounded on the complex plane $\boldsymbol{C}$, we get that $F(z) = \alpha + \beta z + \delta z^2$ on $\boldsymbol{C}$, where α, β, $\delta\in\boldsymbol{C}$. So we have the equality

$$f(x)=\begin{pmatrix}\mathrm{Re}\ \alpha\\ \mathrm{Im}\ \alpha\end{pmatrix}+(\mathrm{Re}\ \beta)x+(\mathrm{Im}\ \beta)\begin{pmatrix}0&-1\\1&0\end{pmatrix}x$$
$$+\begin{pmatrix}-\mathrm{Re}\ \delta\\ \mathrm{Im}\ \delta\end{pmatrix}|x|^2-2x^tx\begin{pmatrix}-\mathrm{Re}\ \delta\\ \mathrm{Im}\ \delta\end{pmatrix}.$$

This concludes the proof of 2).

Lemma 7. 1) *If h is of the form (4.2) on $\boldsymbol{R}^{n-1}$ $(n-1\geqq 2)$, then $f=Hh$ has the following form on $\boldsymbol{H}^n\cup\boldsymbol{R}^{n-1}$;*

$$f(x)=\begin{pmatrix}a\\0\end{pmatrix}+\lambda x+\begin{pmatrix}B&0\\0&0\end{pmatrix}x+\begin{pmatrix}c\\0\end{pmatrix}|x|^2-2x^tx\begin{pmatrix}c\\0\end{pmatrix}, \tag{4.3}$$

where for $(n-1)\times 1$ matrices a and c, and an $(n-1)\times(n-1)$ matrix B by $\begin{pmatrix}a\\0\end{pmatrix}$, $\begin{pmatrix}c\\0\end{pmatrix}$ and $\begin{pmatrix}B&0\\0&0\end{pmatrix}$ we denote the $n\times 1$ matrices and the $n\times n$ matrix obtained by adding zeros, respectively.

2) *If f is of the form (4.2) on $\boldsymbol{H}^n\cup\boldsymbol{R}^{n-1}$ $(n-1\geqq 2)$ and ${}^te_nf=0$ on $\boldsymbol{R}^{n-1}$, then f has the form (4.3).*

Proof. 2) is seen by simple calculation. Let g be the right hand side of (4.3), then $g|_{\boldsymbol{R}^{n-1}}=h$, $|g|/(1+|x|^2)$ is bounded on $\boldsymbol{H}^n$, and g is harmonic from Lemma 6. Since the canonical harmonic extension is unique, we get $g=f$. Hence 1) is proved.

Lemma 8. *Let h be a q. c. deformation of $\hat{\boldsymbol{R}}^{n-1}$ $(n-1\geqq 2)$, and set $f=Hh$, and $\varphi=\rho^nS_nf$. Then the next three conditions are equivalent.*

(a) $S_{n-1}(h_\gamma)=S_{n-1}h$ *for any* $\gamma\in\Gamma$.

(b) $S_n(f_\gamma)=S_nf$ *for any* $\gamma\in\Gamma$.

(c) φ *satisfies* (3.2).

Proof. The equivalence of (b) and (c) comes from (1.8) and (2.1). Suppose that (a) holds. Then $S_{n-1}(h_\gamma-h)=0$, hence it is seen from Lemma 6 that $h_\gamma-h$ has the form (4.2) on $\boldsymbol{R}^{n-1}$. Since, from Lemma 5, $f_\gamma-f$ is the canonical harmonic extension of $h_\gamma-h$, it follows that $S_n(f_\gamma-f)=0$. Thus (b) holds.

Conversely, if (b) holds, then Lemmas 6 and 7 implies $S_{n-1}(h_\gamma-h)=0$, for ${}^te_n(f_\gamma-f)=0$ on $\boldsymbol{R}^{n-1}$. Hence (a) holds.

Lemma 9. *Let h be a q. c. deformation of $\hat{\boldsymbol{R}}^{n-1}$ $(n-1\geqq 2)$, which defines some $\varphi\in\tilde{Q}(\Gamma)$, then $h\in P(\Gamma)$.*

Proof. From Lemma 8 and (3.2) we get $S(h_\gamma)=Sh$. Hence (3.4) holds.

Let U be an arbitrary open subset of $\boldsymbol{R}^{n-1}$, which is relatively compact in $\Omega\cap\boldsymbol{R}^{n-1}$. Define a sequence of smooth functions $h^{(k)}$ on U by

$$h^{(k)}(x)=(I_{n-1}\ 0)\,Hh\Big(x+\frac{e_n}{k}\Big) \qquad \text{for } x\in U.$$

From Lemma 2 we have

$$\|S_{n-1}h^{(k)}(x)\|\leqq\sqrt{2}\,\|S_n(Hh)\Big(x+\frac{e_n}{k}\Big)\|.$$

On the other hand, from (i) and (iii) in Definition 2 we obtain

$$\lim_{k\to\infty}\ \operatorname*{ess\,sup}_{x\in U}\ \|S_n(Hh)\Big(x+\frac{e_n}{k}\Big)\|=0.$$

Hence, from Corollary 1 and Lemma 3 we see

$$S_{n-1}h=0 \qquad \text{on } U.$$

Thus we have (3.5) from the arbitrariness of U, q. e. d.

By σ_{hk} we denote the SM^n matrix with entries

$$\sigma_{hk,ij}=\delta_{ik}\delta_{jh}+\delta_{jk}\delta_{ih}-\frac{2}{n}\delta_{ij}\delta_{hk}, \tag{4.4}$$

and we set

$$\Gamma_{hk}(x)=|x|^{-2n}\{I_n-2Q(x)\}\,\sigma_{hk}\{I_n-2Q(x)\}. \tag{4.5}$$

The following lemma is shown in [3] (pp. 91∼92), implicitly.

Lemma 10. *Let h be a q. c. deformation of $\boldsymbol{R}^{n-1}(n-1\geqq2)$. Suppose that $|h|/(1+|x|^2)$ is bounded on $\boldsymbol{R}^{n-1}$. Then for $x\in\boldsymbol{H}^n$*

$$c_n\rho(x)^nS_n(Hh)(x)=2^{n-1}(n+1)\sum_{k=1}^{n-1}\int_{R^{n-1}}\Gamma_{nk}(x-y)\,h_k(y)\,dy. \tag{4.6}$$

§5. Proofs of main results.

1. Proof of Theorem 5. Suppose that the limit set Λ is of measure zero, then from Lemma 6 we get the conclusion, for such q. c. deformations have the form (4.2).

So we assume that Λ has positive measure. For each $\gamma\in\Gamma$, and $h\in P(\Gamma)$, as in the case of $n-1=2$, we define the periods $p_\gamma h$ of h under the mapping γ by $p_\gamma h=h_\gamma-h$. This periods satisfy the conditions

$$p_{\gamma\eta}h=(\eta')^{-1}(p_\gamma h)\circ\eta+p_\eta h, \tag{5.1}$$

and

$$S(p_\gamma h)=0. \tag{5.2}$$

From (5.2) it follows that $p_\gamma h$ is of the form (4.2) on $\boldsymbol{R}^{n-1}$. Suppose that Γ is generated by $\gamma_1,\dots,\gamma_N$. And consider a linear map p of $P(\Gamma)$ by

$$p: h \longmapsto (p_{\gamma_1}h, \dots, p_{\gamma_N}h).$$

Then in order to prove Theorem 5, it is enough to show that the dimension of the kernel of p is finite, for the dimension of image of p is finite.

Suppose $h \in \ker p$. Then, from (5.1), for all $\gamma \in \Gamma$, $p_\gamma h = 0$, that is,

$$h_\gamma = h. \tag{5.3}$$

It follows from (5.3) and Lemma 5 that the canonical harmonic extension $f = Hh$ of h satisfies

$$f_\gamma = f. \tag{5.4}$$

Let x be an arbitrary point in $\Lambda \cap \boldsymbol{R}^{n-1}$, then there is a sequence of elements γ_k of Γ such that $\lim_{k\to\infty} \gamma_k(e_n) = x$. From (5.4) and (1.9) we see

$$|f \circ \gamma_k(e_n)| = |\gamma_k'(e_n)|\, |f(e_n)| = {}^t e_n \gamma_k(e_n)\, |f(e_n)|.$$

So we have

$$|h(x)| = \lim_{k\to\infty} |f \circ \gamma_k(e_n)| = |f(e_n)| \lim_{k\to\infty} {}^t e_n \gamma_k(e_n) = 0.$$

Hence we get

$$h = 0 \quad \text{on } \Lambda \cap \boldsymbol{R}^{n-1}. \tag{5.5}$$

So, in particular, in case of $\Lambda = \hat{\boldsymbol{R}}^{n-1}$, the conclusion is true.

The remaining cases are divided into the following three: (1) $n-1=2$, (2) $n-1 \geqq 3$ and the number of components of $\Omega \cap \hat{\boldsymbol{R}}^{n-1}$ is not less than two, and (3) $n-1 \geqq 3$ and $\Omega \cap \hat{\boldsymbol{R}}^{n-1}$ consists of only one component. We show that the dimension of the kernel of p is finite in each case of the above three.

The case (1). As seen in the proof of Lemma 6, if we identify h with $h_1 + \sqrt{-1}h_2$, then we see, from $Sh|_{\Omega \cap \boldsymbol{R}^2} = 0$, that h is an analytic function of $z = x_1 + \sqrt{-1}x_2$ on $\Omega \cap \boldsymbol{C}$. By this identification and the relation (5.3), it turns out that hdz^{-1} is Γ-invariant on $\Omega \cap \hat{\boldsymbol{C}}$, that is, h becomes a holomorphic inverse differential on the Riemann surfaces $(\Omega \cap \hat{\boldsymbol{C}})/\Gamma$. Thus $h = 0$ on $\Omega \cap \boldsymbol{R}^2$. Hence $\ker p = \{0\}$. (Under these situations, Theorem 6 is proved in [2], the above proof is the same as Ahlfors'.)

The case (2). Let Ω' be an arbitrary component of $\Omega \cap \hat{\boldsymbol{R}}^{n-1}$. Via conjugation of some element of $\mathcal{M}_{n-1} = \mathcal{M}_n(\boldsymbol{H})$, if necessary, we may assume without loss of generality that Ω' is relatively compact in $\boldsymbol{R}^{n-1}$, and $0 \in \Omega'$. From Lemma 6,

$$h(x) = a + \lambda x + Bx + c|x|^2 - 2x{}^t xc \quad \text{on } \Omega'.$$

For an arbitrary but fixed i, j ($i \neq j$ and $1 \leqq i, j \leqq n-1$), we identify $\boldsymbol{R}^2$ with $\{x \in \boldsymbol{R}^{n-1};\ x_k = 0\ (k \neq i, j)\}$. Since for $x \in \Omega' \cap \boldsymbol{R}^2$

$$h_i(x)=a_i+\lambda x_i+B_{ij}x_j+c_i|x|^2-2x_i{}^t xc,$$
$$h_j(x)=a_j+\lambda x_j-B_{ij}x_i+c_j|x|^2-2x_j{}^t xc$$

(we remark that $B_{ii}=B_{jj}=0$ and $B_{ij}=-B_{ji}$), if we set

$$z=x_i+\sqrt{-1}x_j,\ \alpha=a_i+\sqrt{-1}a_j,\ \beta=\lambda+\sqrt{-1}B_{ij}$$
$$\text{and }\ \delta=c_i+\sqrt{-1}c_j,$$

then $g(z)=h_i(x)+\sqrt{-1}h_j(x)$ can be written as

$$g(z)=\alpha+\bar{\beta}z-\bar{\delta}z^2.$$

From (5.5) we get $g|_{\partial(\Omega'\cap \boldsymbol{R}^2)}=0$, where the boundary is considered in $\boldsymbol{R}^2$. Since Ω' is relatively compact, $\partial(\Omega'\cap\boldsymbol{R}^2)$ consists of more than two points, thus $g=0$ on $\Omega'\cap\boldsymbol{R}^2$, that is, $\alpha=\beta=\delta=0$. From the arbitrariness of i and j, we see that $a=c=0$, $\lambda=0$ and $B=0$. Hence $h|_{\Omega'}=0$. This means that $h=0$ on $\boldsymbol{R}^{n-1}$, that is, $\ker p=\{0\}$.

The case (3). From Lemma 6, h has the form (4.2) on $\Omega\cap\boldsymbol{R}^{n-1}$. Since $\Omega\cap\boldsymbol{R}^{n-1}$ is dense in $\hat{\boldsymbol{R}}^{n-1}$, and h is continuous on $\boldsymbol{R}^{n-1}$, h has also the form (4.2) on $\boldsymbol{R}_{n-1}$. Hence the dimension of the kernel of p is finite. This completes the proof.

2. Proof of Theorem 6. It is obvious that the image of the mapping: $P(\Gamma)\ni h\longmapsto\varphi=\rho^n S_n(Hh)$ contains $\tilde{Q}(\Gamma)$. And the last statement is easily obtained from Lemmas 6 and 7.

Let h be an arbitrary element of $P(\Gamma)$, and set $f=Hh$ and $\varphi=\rho^n S_n f$. Since $S_{n-1}(h_\gamma)=S_{n-1}h$ for any $\gamma\in\Gamma$, it follows from Lemma 8 that φ satisfies (3.2). Thus if $\Lambda=\hat{\boldsymbol{R}}^{n-1}$, then $\varphi\in\tilde{Q}(\Gamma)$. In contrast with this, if $\Lambda=\emptyset$ then $S_{n-1}h=0$ on $\boldsymbol{R}^{n-1}$. Hence, from Lemmas 6 and 7, we see that $\varphi=0\in\tilde{Q}(\Gamma)$.

It remains to consider the case where $\Omega\cap\hat{\boldsymbol{R}}^{n-1}\neq\emptyset$ and $\Lambda\neq\emptyset$. We may assume that $\infty\in\Lambda$. Let Ω' be an arbitrary component of $\Omega\cap\hat{\boldsymbol{R}}^{n-1}$.

Suppose first that $n-1\geqq 3$. Since $S_{n-1}h=0$ on Ω', h has the form (4.2) on Ω'. Let q be the function of the form (4.2) such that $h-q$ vanishes on Ω'. Set $g=h-q$, then it is easily seen that g is a q. c. deformation of $\boldsymbol{R}^{n-1}$ and $|g|/(1+|x|^2)$ is bounded on $\boldsymbol{R}^{n-1}$. Since $S_{n-1}g=S_{n-1}h$, it follows from Lemmas 6 and 7 that $\rho^n S_n(Hg)=\rho^n S_n(Hh)=\varphi$. Hence from Lemma 10 we get

$$c_n\varphi(x)=2^{n-1}(n+1)\sum_{k=1}^{n-1}\int_{\boldsymbol{R}^{n-1}-\Omega'}\Gamma_{nk}(x-y)g_k(y)dy. \tag{5.6}$$

From (1.4) and (4.5) we find

$$\|\Gamma_{nk}(x)\|\leqq \text{const.}\,|x|^{-2n}\qquad\text{for } x\in\boldsymbol{R}^n.$$

So we get

$$\|\varphi(x)\|\leqq\text{const.}\int_{\boldsymbol{R}^{n-1}-\Omega'}\frac{1+|y|^2}{|x-y|^{2n}}dy.$$

Thus for each compact subset K of Ω' we have

$$\rho(x)^{-n}\|\varphi(x)\| \leqq \text{const.}x_n^n \quad \text{uniformly on } K\times(0,\ 1].$$

This implies that φ satisfies (iii) in Definition 2. Therefore $\varphi\in\tilde{Q}(\Gamma)$.

Finally suppose that $n-1=2$. Since $S_2h=0$ on Ω', $\tilde{h}(\zeta)=h_1(y)+\sqrt{-1}h_2(y)$ is an analytic function of $\zeta=y_1+\sqrt{-1}y_2$. Let y' be an arbitrary point in Ω'. There is a neighbourhood U of y' where $\tilde{h}$ can be represented by a power series converging normally. We may assume without loss of generality that $y'=0$ and U is the unit disk. Set $V_j=\{|\zeta|<1/(j+2)\}$ $(j=1,\ 2)$, and define a function $g(\,.\,,.\,)$ on $\boldsymbol{C}\times V_2$, analytic on $\Omega'\times V_2$ by

$$g(\zeta,\ z)=\tilde{h}(\zeta)-\left\{\tilde{h}(z)+\frac{d\tilde{h}(z)}{dz}(\zeta-z)+\frac{1}{2}\,\frac{d^2\tilde{h}(z)}{dz^2}(\zeta-z)^2\right\}$$

for $\zeta\in\boldsymbol{C}$, and $z\in V_2$. Then we find

$$\sup\frac{|g(\zeta,\ z)|}{1+|\zeta|^2}<\infty,$$

where the supremum is taken over all $\zeta\in\boldsymbol{C}$ and all $z\in V_2$. Since we have

$$g(\zeta,\ z)=\sum_{j=3}^{\infty}\frac{1}{j!}\,\frac{d^j\tilde{h}(z)}{dz^j}(\zeta-z)^j \qquad \text{for } \zeta\in V_1 \text{ and } z\in V_2,$$

we get

$$\sup\frac{|g(\zeta,\ z)|}{|\zeta-z|^3}<\infty,$$

where the supremum is taken over all $\zeta\in V_1$ and all $z\in V_2$. It follows from Lemmas 6 and 7 that for any $z\in V_2$ $g(\cdot,z)$ defines the same φ as h. For $x\in\boldsymbol{H}^3$ we set $z=x_1+\sqrt{-1}x_2$ and $t=x_3$. Then, from Lemma 10, we get for $x\in V_2\times(0,\ 1]$

$$\begin{aligned}\|\varphi(x)\| &\leqq \text{const.}\int_C \frac{|g(\zeta,\ z)|}{(|\zeta-z|^2+t^2)^3}d\xi d\eta\\ &\leqq \text{const.}\int_{C-V_1}\frac{1+|\zeta|^2}{(|\zeta-z|^2+t^2)^3}d\xi d\eta\\ &\quad+\text{const.}\int_{V_1}\frac{|\zeta-z|^3}{(|\zeta-z|^2+t^2)^3}d\xi d\eta,\end{aligned}$$

where $\zeta=\xi+\sqrt{-1}\eta$. The first integral is uniformly bounded for $x\in V_2\times(0,\ 1]$ and the second integral does not exceed $(1/t)\int_C|\zeta|^3/(|\zeta|^2+1)^3d\xi d\eta$. Thus $\rho^{-3}\varphi$ has a continuous extension to V_2 and this extension vanishes on V_2. This implies that φ satisfies (iii) in Definition 2. Hence $\varphi\in\tilde{Q}(\Gamma)$,

q. e. d.

§ 6. Remarks on the case $n=3$.

In this section we consider the case $n=3$ and state some remarks.

We identify $\boldsymbol{R}^2$ with $\boldsymbol{C}$. In this case Γ becomes a so-called kleinian group. It may, however, be of the first kind. (A kleinian group is said to be of the first kind if $\Omega(\Gamma)\cap\hat{C}=\emptyset$.) Our notations can be rewritten in familiar forms;

$$S_2h=h_{\bar z},\ \|S_2h\|=\sqrt{2}\,|h_{\bar z}|,\ S_2^*\varphi=\varphi_z \text{ and}$$
$$(\gamma')^{-1}\{(S_2h)\circ\gamma\}\gamma'=\{(h_{\bar z})\circ\gamma\}\frac{\overline{\gamma'}}{\gamma'}.$$

Hence for $h\in P(\Gamma)$ it turns out that $\mu=S_2h$ is a Beltrami differential compatible with Γ and supp $\mu\subset\Lambda(\Gamma)$. And therefore h becomes a potential for μ, that is, h is of the form

$$h(z)=-\frac{1}{\pi}\int_{\Lambda(\Gamma)}\frac{z(z-1)\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)}d\xi d\eta$$
$$+(\text{a quadratic polynomial of } z).$$

It is known that if Γ is finitely generated then the action of Γ on the (topological) limit set $\Lambda(\Gamma)$ is conservative, that is, for any measurable subset X of $\Lambda(\Gamma)$ with meas$X>0$, $\#\{\gamma\in\Gamma;\ \text{meas}(X\cap\gamma X)>0\}=+\infty$. (Ahlfors [2], [4] and Sullivan [6]) It is an open question whether or not the above fact is also true for higher dimensions. Sullivan [6] showed that there is no Beltrami differential compatible with Γ and supported on the conservative part of the action of Γ. Hence we see, from Corollary 4, $Q(\Gamma)\subset\tilde{Q}(\Gamma)=\{0\}$ for $n=3$. This implies that the situation of 3-dimensional case is much different from that of 2-dimensional case.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] L. V. Ahlfors, Finitely generated Kleinian groups, Amer. J. Math., **86** (1964), 413-429 and **87** (1965), 759.

[2] L. V. Ahlfors, Some remarks on Kleinian groups, Proceedings of the conference on quasiconformal mappings, moduli, and discontinuous groups, Tulane University (1967), 7-13.

[3] L. V. Ahlfors, Quasiconformal deformations and mappings in $\boldsymbol{R}^n$, J. d'Analyse Math., **30** (1976), 74-97.

[4] L. V. Ahlfors, Ergodic properties of groups of Möbius transformations, Lecture Notes in Math. 798, Springer, 1981, 1-9.

[5] L. V. Ahlfors, Möbius transformations in several dimensions, Ordway professorship lectures in mathematics, University of Minnesota, 1981.

[6] D. Sullivan, On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook conference, Ann. of Math. Studies, **97** (1981), 465-496.