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Phase transition in one-dimensional Ising models with spatially inhomogeneous potentials

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Gibbs distributions in one-dimension have attracted interest of several authors [1-9]. In usual cases there can be no phase transition. But we have examples of phase transition in the following cases:

1. The set of values of random fields is $\{-1\ +1\}$ and the interaction is of long range. Dyson [1].
2. The random fields take values in a countably infinite set and the interaction is of nearest neighbour. Spitzer [7].
3. The random fields take values in $\{-1, +1\}$ and the interaction is of nearest neighbour but it is spatially inhomogeneous. Sullivan [8].

We consider Ising models in one-dimenson with spatially inhomogeneous potentials. Let $\Omega=\{-1, +1\}^Z$. For a configuration $\boldsymbol{\sigma}\in\Omega$, let σ_k be the spin of $\boldsymbol{\sigma}$ at $k\in Z$. The formal Hamiltonian $H(\boldsymbol{\sigma})$ is equal to

$$H(\boldsymbol{\sigma})=\sum_{k\in Z}J_k\sigma_k\sigma_{k+1}-\sum_{k\in Z}h_k\sigma_k \qquad (\boldsymbol{\sigma}\in\Omega).$$

We call $J=\{J_k\}$ and $h=\{h_k\}$ *interaction* and *external field*, respectively. In ordinary Ising models, J_k and h_k do not depend on k, but in our case they depend on k. For $n\leqq m\in Z$ and $\boldsymbol{\sigma}=(\sigma_n, \sigma_{n+1}, \dots, \sigma_m)\in\{-1, +1\}^{[n,m]}$, let

$$H^{[n,m]}(\boldsymbol{\sigma}\,|\sigma_{n-1}, \sigma_{m+1})=\sum_{k=n-1}^{m}J_k\sigma_k\sigma_{k+1}-\sum_{k=n}^{m}h_k\sigma_k,$$

$$q^{[n,m]}_{\sigma_{n-1},\sigma_{m+1}}(\boldsymbol{\sigma})=\Xi^{[n,m]}(\sigma_{n-1}, \sigma_{m+1})^{-1}\exp\{-H^{[n,m]}(\boldsymbol{\sigma}\,|\sigma_{n-1}, \sigma_{m+1})\},$$

where $\Xi^{[n,m]}(\sigma_{n-1}, \sigma_{m+1})=\sum_{\sigma}\exp\{-H^{[n,m]}(\boldsymbol{\sigma}\,|\sigma_{n-1}, \sigma_{m+1})\}$. A probability measure μ on Ω is called *Gibbs distribution with potential* (J, h), if for each $\boldsymbol{\sigma}\in\{-1,+1\}^{[n,m]}$

$$\mu(\boldsymbol{\sigma}\,|\mathscr{B}_{[n,m]^c})=q^{[n,m]}_{\sigma_{n-1},\sigma_{m+1}}(\boldsymbol{\sigma}) \qquad a.\ e.\ (\mu).$$

The set of all Gibbs distributions with (J, h) is denoted by $\mathscr{G}(J, h)$. It is well known that if J_k and h_k do not depend on k, then $\mathscr{G}(J, h)$ consists of a unique measure μ. The measure μ is a distribution of a time-homogeneous Markov chain.

Our aim is to clarify under what conditions the uniqueness of $\mathcal{G}(J, h)$ is broken. Put $\hat{J}_n = -|J_n|$ and $\hat{h}_n = s_n h_n$, where

$$s_n = \begin{cases} \prod_{k=0}^{n-1} \mathrm{sgn}(-J_k), & \text{if } n \geqq 1, \\ 1 \quad , & \text{if } n=0, \\ \prod_{k=n}^{-1} \mathrm{sgn}(-J_k), & \text{if } n \leqq -1. \end{cases}$$

Define a tranformation $\boldsymbol{\sigma} \rightarrow \hat{\boldsymbol{\sigma}}$ of Ω by

$$\hat{\sigma}_n = s_n \sigma_n.$$

Remark that the formal Hamiltonians induced by (J, h) and $(\hat{J}, \hat{h})$ coincide, i. e.,

$$\sum_{n \in Z} J_n \sigma_n \sigma_{n+1} - \sum_{n \in Z} h_n \sigma_n = \sum_{n \in Z} \hat{J}_n \hat{\sigma}_n \hat{\sigma}_{n+1} - \sum_{n \in Z} \hat{h}_n \hat{\sigma}_n.$$

Therefore, $\mathcal{G}(J, h)$ and $\mathcal{G}(\hat{J}, \hat{h})$ are isomorphic to each other. Let $\hat{q}$ be the conditional Gibbs distribution induced by $(\hat{J}, \hat{h})$. Since the potential $(\hat{J}, \hat{h})$ is ferromagnetic, we can see by the FKG inequality [10] that the limit

$$\hat{q}_{\tau',\tau} \equiv \lim_{\substack{n \to -\infty \\ m \to +\infty}} \hat{q}^{[n,m]}_{\tau',\tau}$$

exists for constant boundary conditions $\sigma_{n-1} = \tau'$ and $\sigma_{m+1} = \tau$ with $\tau', \tau = \pm 1$. Let $\mathcal{G}_{ex}(\hat{J}, \hat{h})$ be the set of all extremal measures of the convex set $\mathcal{G}(\hat{J}, \hat{h})$. It is easy to see that

$$\mathcal{G}_{ex}(\hat{J}, \hat{h}) = \{\hat{q}_{\tau',\tau};\ \tau', \tau = \pm 1\}.$$

Returning to (J, h), we have

$$\mathcal{G}_{ex}(J, h) = \{q_{\tau',\tau};\ \tau', \tau = \pm 1,\},$$

where $q_{\tau',\tau} = \lim q^{[n,m]}_{s_{n-1}\tau', s_{m+1}\tau}$.

For a 2×2-matrix A, we denote the (σ', σ)-component of A by $A(\sigma', \sigma)$, i. e.,

$$A = \begin{pmatrix} A(-1, -1) & A(-1, +1) \\ A(+1, -1) & A(+1, +1) \end{pmatrix}.$$

Let $\boldsymbol{a}(\sigma)$ and $\boldsymbol{b}(\sigma)$ be the σ-th column and σ-th row of A, respectively. Put

$$\tilde{A} = (\boldsymbol{a}(-1)/\|\boldsymbol{a}(-1)\|,\ \boldsymbol{a}(+1)/\|\boldsymbol{a}(+1)\|),$$

$$\tilde{A}^* = \begin{pmatrix} \boldsymbol{b}(-1)/\|\boldsymbol{b}(-1)\| \\ \boldsymbol{b}(+1)/\|\boldsymbol{b}(+1)\| \end{pmatrix},$$

where $\|\ \ \|$ is the Euclidean norm. Put

$$Q_k=\begin{pmatrix} e^{-J_k-h_k} & e^{J_k-h_k} \\ e^{J_k+h_k} & e^{-J_k+h_k} \end{pmatrix},$$

i. e., $Q_k(\sigma', \sigma)=e^{-J_k\sigma'\sigma+h_k\sigma'}$ and put

$$\Pi_c^d=\begin{cases} Q_cQ_{c+1}\cdots Q_d, & \text{if } d\geqq c, \\ E \quad\quad , & \text{if } d=c-1. \end{cases}$$

Let $n<l\leqq r<m$ and let $\boldsymbol{\sigma}=(\sigma_l, \sigma_{l+1}, \dots, \sigma_r)\in\{-1, +1\}^{[l,r]}$. We have

$$q^{[n,m]}_{\sigma_{n-1},\sigma_{m+1}}(\boldsymbol{\sigma})=\frac{\Pi_{n-1}^{l-1}(\sigma_{n-1}, \sigma_l)\prod_{k=l}^{r-1}Q_k(\sigma_k, \sigma_{k+1})\,\Pi_r^m(\sigma_r, \sigma_{m+1})}{\Pi_{n-1}^m(\sigma_{n-1}, \sigma_{m+1})}$$

$$=\frac{\tilde{\Pi}^{*\,l-1}_{n-1}(\sigma_{n-1}, \sigma_l)\prod_{k=l}^{r-1}Q_k(\sigma_k, \sigma_{k+1})\,\tilde{\Pi}_r^m(\sigma_r, \sigma_{m+1})}{\tilde{\Pi}^{*\,l-1}_{n-1}\,\Pi_l^{r-1}\,\tilde{\Pi}_r^m(\sigma_{n-1}, \sigma_{m+1})}.$$

Since $\tilde{\Pi}^{*\,l-1}_{n-1}$ and $\tilde{\Pi}_r^m$ are bounded, we can extract subsequences $n_i\to-\infty$ and $m_j\to+\infty$ such that

$$\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau, \sigma)\equiv\lim_i \tilde{\Pi}^{*\,l-1}_{n_i-1}(s_{n_i-1}\tau, \sigma),$$

$$\tilde{\Pi}_r^{+\infty}(\sigma, \tau)\equiv\lim_j \tilde{\Pi}_r^{m_j}(\sigma, s_{m_j+1}\tau)$$

exist for all l, r, τ and σ. Hence, we have

$$q_{\tau',\tau}(\boldsymbol{\sigma})=\frac{\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau', \sigma_l)\prod_{k=l}^{r-1}Q_k(\sigma_k, \sigma_{k+1})\,\tilde{\Pi}_r^{+\infty}(\sigma_r, \tau)}{\tilde{\Pi}^{*\,l-1}_{-\infty}\,\Pi_l^{r-1}\,\tilde{\Pi}_r^{+\infty}(\tau', \tau)}.$$

We can see by the following Lemma 1 that the matrices $\tilde{\Pi}_r^{+\infty}$ and $\tilde{\Pi}^{*\,l-1}_{-\infty}$ are positive.

Lemma 1. *There are sequences* $\{\rho_r(\sigma)\}_r$ and $\{\rho_l^*(\sigma)\}_l$ *of positive functions such that*

$$\tilde{\Pi}_r^{+\infty}(\sigma', \sigma)=Q_r\,\tilde{\Pi}_{r+1}^{+\infty}(\sigma', \sigma)\rho_r(\sigma),$$
$$\tilde{\Pi}^{*\,l}_{-\infty}(\sigma', \sigma)=\rho_l^*(\sigma')\,\tilde{\Pi}^{*\,l-1}_{-\infty}Q_l(\sigma', \sigma).$$

Proof is easy.

We say that $\{\tilde{\Pi}_r^{+\infty}\}$ is *asymptotically non-singular* if $\tilde{\Pi}_r^{+\infty}$ is non-singular for sufficiently large r. If not, Lemma 1 shows that $\tilde{\Pi}_r^{+\infty}$ is singular for all r. Asymptotic non-singularity of $\tilde{\Pi}^{*\,l-1}_{-\infty}$ is defined in the same way.

We prove the following Theorems.

Theorem 1. *Put*

$$\mathcal{M}_{+\infty}(J, h)=\begin{cases} \{-1, +1\}, & \textit{if } \{\tilde{\Pi}_r^{+\infty}\} \textit{ is asymptotically non-singular}, \\ \{+1\} \quad\quad , & \textit{if otherwise}. \end{cases}$$

A set $\mathcal{M}_{-\infty}(J, h)$ *is defined analogously. Put*

$$\mathcal{M}(J, h)=\mathcal{M}_{-\infty}(J, h)\times\mathcal{M}_{+\infty}(J, h).$$

The set $\mathcal{G}_{ex}(J, h)$ *is isomorphic to* $\mathcal{M}(J, h)$. *The mapping*

$$q=q_{\tau',\tau}: \mathcal{M}(J, h)\to\mathcal{G}_{ex}(J, h)$$

is an isomorphism. If $\mathcal{M}_{+\infty}(J, h)=\{-1, +1\}$, *then for* $(\tau', \tau)\in\mathcal{M}(J, h)$,

$$q_{\tau',\tau}(\sigma_n=s_n\tau \text{ for all but finitely many } n\geqq 0)=1.$$

We call $\mathcal{M}(J, h)$ the *Martin-Dynkin boundary of* $\mathcal{G}(J, h)$.

Theorem 2. 1) *Assume that* $\mathcal{M}_{-\infty}(J, h)$ *or* $\mathcal{M}_{+\infty}(J, h)$ *is equal to* $\{+1\}$. *Then, any* $\mu\in\mathcal{G}(J, h)$ *is a Markov chain.*
2) *Assume that* $\mathcal{M}_{-\infty}(J, h)=\mathcal{M}_{+\infty}(J, h)=\{-1, +1\}$. *Then, a measure*

$$\mu=\sum\lambda_{\tau',\tau}q_{\tau',\tau}\in\mathcal{G}(J, h)\quad(\sum\lambda_{\tau',\tau}=1,\ \lambda_{\tau',\tau}\geqq 0)$$

is a Markov chain, if and only if

$$\det(\lambda_{\tau',\tau}/\tilde{\Pi}^{*}{}_{-\infty}^{l-1}\,\Pi_{l}^{r-1}\,\tilde{\Pi}_{r}^{+\infty}(\tau', \tau))=0.$$

This condition is consistent in the sense that it is equivalent to

$$\det(\lambda_{\tau',\tau}/\tilde{\Pi}^{*}{}_{-\infty}^{l'-1}\,\Pi_{l'}^{r'-1}\,\overset{*}{\Pi}{}_{r'}^{+\infty}(\tau', \tau))=0$$

for any $l'\leqq r'$.

Theorem 3. *If* $\sum^{+\infty}e^{-2|J_n|}=+\infty$, *then* $\mathcal{M}_{+\infty}(J, h)=\{+1\}$ *for any* $h=\{h_n\}$.

Theorem 4. *If*

$$\sum^{+\infty}\exp 2(-|J_n|+|\sum_{k=0}^{n}s_kh_k|)<+\infty,$$

then $\mathcal{M}_{+\infty}(J, h)=\{-1, +1\}$. *Conversely, if*

$$\sum^{+\infty}\exp 2(-|J_n|+|\sum_{k=k}^{n}s_kh_k|)=+\infty$$

and if in addition $s_kh_k\geqq 0$ *(or* $\leqq 0$*) for all* $k\geqq 0$, *then* $\mathcal{M}_{+\infty}(J, h)=\{+1\}$.

As for homogeneous external fields, we have

Theorem 5. *Assume that the external field* h_n *is equal to a constant* h *and* $J_n\leqq 0$ *for all* $n\geqq 0$. *Then,* $\mathcal{M}_{+\infty}(J, h)=\{+1\}$ *if and only if*

$$\sum^{+\infty}e^{2(J_n+n|h|)}=+\infty.$$

Theorem 6. *Assume that the external field* h_n *is equal to a constant and* $J_n\geqq 0$ *for all* $n\geqq 0$. *Then,* $\mathcal{M}_{+\infty}(J, h)=\{+1\}$ *if and only if*

$$\sum^{+\infty}e^{-2J_n}=+\infty.$$

Theorems 5 and 6 are corollaries to Theorems 3 and 4.

We will discuss the one-dimensional Widom-Rowlison models in a forthcoming paper [5].

Proof of Theorem 1. The first statement in Theorem 1 follows immediately from Lemma 2 below. To prove the second statement, put

$$B_{\tau',\tau}=\{\boldsymbol{\sigma}=(\sigma_m)_m;\ \lim_{n,m} q^{[n,m]}_{s_{n-1}\tau',\sigma_{m+1}}=q_{\tau',\tau}\}.$$

By Theorem 1 in [11], $q_{\tau',\tau}(B_{\tau',\tau})=1$. Assume $\tilde{\Pi}_r^{+\infty}(\sigma,\ -1)\neq\tilde{\Pi}_r^{+\infty}(\sigma,\ +1)$ for sufficiently large r. Then, $\boldsymbol{\sigma}\in B_{\tau',\tau}$ if and only if $\sigma_m=s_m\tau$ for all but finitely many $m\geqq 0$. Q. E. D.

Lemma 2. *It holds* $q_{\tau'_1,\tau_1}=q_{\tau'_2,\tau_2}$, *if and only if* $(\tilde{\Pi}_r^{+\infty}(\sigma,\ \tau_i))_{\sigma,i}$ *and* $(\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau'_i,\ \sigma))_{i,\sigma}$ *are singular for all l and r.*

Proof. Assume $q_{\tau'_1,\tau_1}=q_{\tau'_2,\tau_2}$. Take any $\boldsymbol{\sigma}=(\sigma_l,\ \sigma_{l+1},\dots,\ \sigma_r)\in\{-1,\ +1\}^{[l,r]}$. Put $P=\tilde{\Pi}^{*\,l-1}_{-\infty}\,\Pi_l^{r-1}\,\tilde{\Pi}_r^{+\infty}$. From

$$\frac{\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau'_1,\ \sigma_r)\prod_{k=l}^{r-1}Q_k(\sigma_k,\ \sigma_{k+1})\,\tilde{\Pi}_r^{+\infty}(\sigma_r,\ \tau_1)}{P(\tau'_1,\ \tau_1)}=\frac{\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau'_2,\ \sigma_l)\prod_{k=l}^{r-1}Q_k(\sigma_k,\ \sigma_{k+1})\,\tilde{\Pi}_r^{+\infty}(\sigma_r,\ \tau_2)}{P(\tau'_2,\ \tau_2)},$$

it follows that

$$\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau'_1,\ \sigma_l)\,\tilde{\Pi}_r^{+\infty}(\sigma_r,\ \tau_1)/P(\tau'_1,\ \tau_1)=\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau'_2,\ \sigma_l)\,\tilde{\Pi}_r^{+\infty}(\sigma_r,\ \tau_2)/P(\tau'_2,\ \tau_2).$$

Since $\sum_{\sigma_l}\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau'_i,\ \sigma_l)^2=\sum_{\sigma_r}\tilde{\Pi}_r^{+\infty}(\sigma_r,\ \tau_i)^2=1$, we have $P(\tau'_1,\ \tau_1)=P(\tau'_2,\ \tau_2)$. Therefore,

$$\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau'_1,\ \sigma_l)\,\tilde{\Pi}_r^{+\infty}(\sigma_r,\ \tau_1)=\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau'_2,\ \sigma_l)\,\tilde{\Pi}_r^{+\infty}(\sigma_r,\ \tau_2).$$

By the same argument as above, we have

$$\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau'_1,\ \sigma_l)=\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau'_2,\ \sigma_l),$$
$$\tilde{\Pi}_r^{+\infty}(\sigma_r,\ \tau_1)=\tilde{\Pi}_r^{+\infty}(\sigma_r,\ \tau_2).$$

Conversely, assume that $(\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau'_i,\ \sigma))_{i,\sigma}$ and $(\tilde{\Pi}_r^{+\infty}(\sigma,\ \tau_i))_{\sigma,i}$ are singular. Then, $\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau'_1,\ \sigma)=\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau'_2,\ \sigma)$ and $\tilde{\Pi}_r^{+\infty}(\sigma,\ \tau_1)=\tilde{\Pi}_r^{+\infty}(\sigma,\ \tau_2)$, because the rows of $\tilde{\Pi}^{*\,l-1}_{-\infty}$ and the columns of $\tilde{\Pi}_r^{+\infty}$ are non-negative and normalized, which implies $q_{\tau'_1,\tau_1}=q_{\tau'_2,\tau_2}$. Q. E. D.

Proof of Theorem 2. Take any $\mu=\sum_{\tau',\tau}\lambda_{\tau',\tau}q_{\tau',\tau}\in\mathcal{G}(J,\ h)$ and $\boldsymbol{\sigma}=(\sigma_l,\ \sigma_{l+1},\dots,\ \sigma_r)\in\{-1,\ +1\}^{[l,r]}$.

Put $\hat{\lambda}_{\tau',\tau}=\lambda_{\tau',\tau}/\tilde{\Pi}^{*\,l-1}_{-\infty}\,\Pi_l^{r-1}\,\tilde{\Pi}_r^{+\infty}(\tau',\tau)$. We have

$$\begin{aligned}\mu(\boldsymbol{\sigma})&=\sum_{\tau',\tau}\hat{\lambda}_{\tau',\tau}\,\tilde{\Pi}^{*\,l-1}_{-\infty}(\tau',\sigma_l)\prod_{k=l}^{r-1}Q_k(\sigma_k,\sigma_{k+1})\,\tilde{\Pi}_r^{+\infty}(\sigma_r,\tau)\\&=\prod_{k=l}^{r-1}Q_k(\sigma_k,\sigma_{k+1})L(\sigma_r,\sigma_l),\end{aligned}$$

where $L=\tilde{\Pi}_r^{+\infty\,t}(\hat{\lambda}_{\tau',\tau})\tilde{\Pi}^{*\,l-1}_{-\infty}$. Since

$$\mu(\sigma_l,\sigma_{l+1},\dots,\sigma_{r-1})=\prod_{k=l}^{r-2}Q_k(\sigma_k,\sigma_{k+1})\sum_{\sigma}Q_{r-1}(\sigma_{r-1},\sigma)L(\sigma,\sigma_l),$$

we have

$$\begin{aligned}&\mu(\sigma_r\,|\,\sigma_l,\sigma_{l+1},\dots,\sigma_{r-1})\\&\quad=Q_{r-1}(\sigma_{r-1},\sigma_r)L(\sigma_r,\sigma_l)/\sum_{\sigma}Q_{r-1}(\sigma_{r-1},\sigma)L(\sigma,\sigma_l)\\&\quad=\mu(\sigma_r\,|\,\sigma_l,\sigma_{r-1}).\end{aligned}$$

Consequently,

$$\begin{aligned}&\mu(\sigma_r\,|\,\sigma_l=+1,\sigma_{r-1})-\mu(\sigma_r\,|\,\sigma_l=-1,\sigma_{r-1})=\\&\quad=\prod_{\sigma}\{Q_{r-1}(\sigma_{r-1},\sigma)/Q_{r-1}L(\sigma_{r-1},\sigma)\}\det L\\&\quad=\prod_{\sigma}\{Q_{r-1}(\sigma_{r-1},\sigma)/Q_{r-1}L(\sigma_{r-1},\sigma)\}\det\tilde{\Pi}^{*\,l-1}_{-\infty}\det\tilde{\Pi}_r^{+\infty}\det(\hat{\lambda}_{\tau',\tau}).\end{aligned}$$

If one of above three determinants vanishes, then

$$\mu(\sigma_r\,|\,\sigma_l=+1,\sigma_{r-1})=\mu(\sigma_r\,|\,\sigma_l=-1,\sigma_{r-1})=\mu(\sigma_r\,|\,\sigma_{r-1}),$$

from which follows $\mu(\sigma_r\,|\,\sigma_l,\sigma_{l+1},\dots,\sigma_{r-1})=\mu(\sigma_r\,|\,\sigma_{r-1})$.

Consistency in 2) is clear from Lemma 1. Q. E. D.

Let A be a 2×2-matrix whose columns are $\boldsymbol{a}$ and $\boldsymbol{a}'$. Put

$$\Delta(A)=|\det\tilde{A}|=|\det A|/(\|\boldsymbol{a}\|\ \|\boldsymbol{a}'\|).$$

Denote the inner product of $\boldsymbol{a}$ and $\boldsymbol{a}'$ by $\langle\boldsymbol{a},\boldsymbol{a}'\rangle$. It is easy to see

Lemma 3.

1) $\Delta(A)\leqq|\det A/\langle\boldsymbol{a},\boldsymbol{a}'\rangle|$.

2) *Let* $D=\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}$ *with* $\alpha,\beta\neq 0$. *Then,* $\Delta(AD)=\Delta(A)$.

Let us write $A\cong B$. if $A=\alpha B$ with a scalar $\alpha\neq 0$. Remark that $\Delta(A)=\Delta(B)$ if $A\cong B$. For $H(h)=\begin{pmatrix}e^{-h}&0\\0&e^{h}\end{pmatrix}$ and $K(J)=\begin{pmatrix}e^{-J}&e^{J}\\e^{J}&e^{-J}\end{pmatrix}$, we have

$$H(h)K(J)H(h)^{-1}\cong E+e^{2J}H(h)^2F,$$

where $F=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Put $H_k=H(\hat{h}_k)$, $K_k=K(\hat{J}_k)$, $Q_k=H_kK_k$ and

$$P_k=E+e^{2\hat{J}_k}\Big(\prod_{j=r}^{k}H_j\Big)^2F.$$

Remark $\det P_k = 1 - e^{4f_k}$. It holds that

$$\begin{aligned}\Pi_r^m &= \prod_{k=r}^m Q_k = \prod_{k=r}^m H_k K_k \\ &= \prod_{k=r}^m \{(\prod_{j=r}^k H_j) K_k (\prod_{j=r}^k H_j)^{-1}\} \prod_{j=r}^m H_j \\ &\cong \prod_{k=r}^m \{E + e^{2f_k} (\prod_{j=r}^k H_j)^2 F\} \prod_{j=r}^m H_j \\ &= \prod_{k=r}^m P_k \prod_{j=r}^m H_j.\end{aligned}$$

Therefore, by 2) in Lemma 3, we have

$$\Delta(\tilde{\Pi}_r^m) = \Delta(\Pi_r^m) = \Delta(\prod_{k=r}^m P_k).$$

Let $\boldsymbol{a}_m(\sigma)$ be the σ-th column of $\prod_{k=r}^m P_k$. It is easy to see by direct calculations

Lemma 4. *Put* $\hat{I}_m = \sum_{k=r}^m \hat{h}_k$ *and* $\boldsymbol{b}_m = e^{\hat{I}_m} \boldsymbol{a}_{m-1}(+1) - e^{-\hat{I}_m} \boldsymbol{a}_{m-1}(-1)$.

1) $\boldsymbol{a}_m(-1) = \boldsymbol{a}_{m-1}(-1) + e^{2(f_m + \hat{I}_m)} \boldsymbol{a}_{m-1}(+1)$,
$\boldsymbol{a}_m(+1) = e^{2(f_m - \hat{I}_m)} \boldsymbol{a}_{m-1}(-1) + \boldsymbol{a}_{m-1}(+1)$.

2) $\langle \boldsymbol{a}_m(-1), \boldsymbol{a}_m(+1)\rangle = e^{2f_m} \|\boldsymbol{b}_m\|^2 + (1 + e^{2f_m})^2 \langle \boldsymbol{a}_{m-1}(-1), \boldsymbol{a}_{m-1}(+1)\rangle$.

Proof of Theorem 3. We prove that $\mathscr{M}_{+\infty}(\hat{J}, \hat{h}) = \{+1\}$ for any $\hat{h} = \{\hat{h}_n\}$ if $\sum^{+\infty} e^{2f_n} = +\infty$. By 2) in Lemma 4, we have

$$\langle \boldsymbol{a}_m(-1), \boldsymbol{a}_m(+1)\rangle \geqq \prod_{k=r+1}^m (1 + e^{2f_k})^2 \langle \boldsymbol{a}_r(-1), \boldsymbol{a}_r(+1)\rangle.$$

Therefore,

$$\begin{aligned}\Delta(\tilde{\Pi}_r^m) &= \Delta(\prod_{k=r}^m P_k) \leqq \prod_{k=r}^m \det P_k / \langle \boldsymbol{a}_m(-1), \boldsymbol{a}_m(+1)\rangle \\ &\leqq \{\prod_{k=r+1}^m (1 + e^{2f_k})^2 \langle \boldsymbol{a}_r(-1), \boldsymbol{a}_r(+1)\rangle\}^{-1}.\end{aligned}$$

The right-hand side diverges to 0 as $m \to +\infty$. Hence $\Delta(\tilde{\Pi}_r^{+\infty}) = 0$ for all r, which implies $\mathscr{M}_{+\infty}(\hat{J}, \hat{h}) = \{+1\}$ by Theorem 1. Q. E. D.

Next, we prove the second statement in Theorem 4.

Lemma 5. *If* $\hat{h}_n \geqq 0$ *for all* $n \geqq 0$, *then*

$$\boldsymbol{b}_k \geqq \prod_{j=r+2}^k e^{\hat{h}_j} \{1 - e^{2(f_{j-1} - \hat{h}_j)}\} \boldsymbol{b}_{r+1}.$$

Proof. We see by 1) in Lemma 4,

$$\begin{aligned}\boldsymbol{b}_k &= e^{\hat{I}_k} \boldsymbol{a}_{k-1}(+1) - e^{-\hat{I}_k} \boldsymbol{a}_{k-1}(-1) \\ &= (e^{\hat{h}_k} - e^{2f_{k-1} - \hat{h}_k}) e^{\hat{I}_{k-1}} \boldsymbol{a}_{k-2}(+1) - (e^{-\hat{h}_k} - e^{2f_{k-1} + \hat{h}_k}) e^{-\hat{I}_{k-1}} \boldsymbol{a}_{k-2}(-1).\end{aligned}$$

Since $e^{h}-e^{2f-h}\geqq e^{-h}-e^{2f+h}$ if $h\geqq 0$, we have

$$\boldsymbol{b}_k\geqq (e^{h_k}-e^{2f_{k-1}-h_k})\boldsymbol{b}_{k-1}$$
$$=e^{h_k}\{1-e^{2(f_{k-1}-h_k)}\}\boldsymbol{b}_{k-1}. \qquad \text{Q. E. D.}$$

Proof of the second statement in Theorem 4. It is enough to prove that $\mathscr{M}_{+\infty}(f,h)=\{+1\}$ if $\sum^{+\infty}\exp 2(f_n+\sum_{k=0}^{n}h_k)=+\infty$ and if $h_n\geqq 0$ for all $n\geqq 0$. In case $\sum^{+\infty}e^{2f_n}=+\infty$, $\mathscr{M}_{+\infty}(f,h)=\{+1\}$ by Theorem 3. So, we prove in case $\sum^{+\infty}e^{2f_n}<+\infty$.

We replace f_r by $f'_r=-h_{r+1}$. Then,

$$\boldsymbol{b}_{r+1}=e^{h_r+h_{r+1}}\boldsymbol{a}_r(+1)-e^{-h_r-h_{r+1}}\boldsymbol{a}_r(-1)$$
$$=\begin{pmatrix} e^{-h_r}(e^{2f'_r+h_{r+1}}-e^{-h_{r+1}}) \\ e^{h_r}(e^{h_{r+1}}-e^{2f'_r-h_{r+1}}) \end{pmatrix}$$
$$=\begin{pmatrix} 0 \\ e^{h_r}(e^{h_{r+1}}-e^{-3h_{r+1}}) \end{pmatrix}\geqq \boldsymbol{0}.$$

Therefore, we can see by Lemma 5,

$$\|\boldsymbol{b}_k\|\geqq \prod_{j=r+2}^{k} e^{h_j}\{1-e^{2(f_{j-1}-h_j)}\}\|\boldsymbol{b}_{r+1}\|$$
$$\geqq b\ \exp\sum_{j=r+2}^{k}h_j,$$

where $b=\|\boldsymbol{b}_{r+1}\|\prod_{j=r+2}^{+\infty}(1-e^{2f_{j-1}})>0$ by our assumption $\sum^{+\infty}e^{2f_n}<+\infty$. On the other hand, by 2) in Lemma 4, we have

$$\langle \boldsymbol{a}_m(-1),\ \boldsymbol{a}_m(+1)\rangle=\sum_{k=r+1}^{m}\prod_{j=k+1}^{m}(1+e^{2f_j})^2e^{2f_k}\|\boldsymbol{b}_k\|^2$$
$$+\prod_{j=r+1}^{m}(1+e^{2f_j})^2\langle \boldsymbol{a}_r(-1),\ \boldsymbol{a}_r(+1)\rangle$$
$$\geqq \sum_{k=r+1}^{m}e^{2f_k}\|\boldsymbol{b}_k\|^2$$
$$\geqq b^2\sum_{k=r+1}^{m}\exp 2(f_k+\sum_{j=r+2}^{k}h_j).$$

The right-hand side diverges to $+\infty$ as $m\to+\infty$.

Put $P'_r=E+e^{2f'_r}H_r^2F$. Since

$$\Delta(P'_r\prod_{k=r+1}^{m}P_k)\leqq \det P'_r\prod_{k=r+1}^{m}\det P_k/\langle \boldsymbol{a}_m(-1),\ \boldsymbol{a}_m(+1)\rangle$$
$$\leqq \langle \boldsymbol{a}_m(-1),\ \boldsymbol{a}_m(+1)\rangle^{-1},$$

we have $\lim_{m\to+\infty}\Delta(P'_r\prod_{k=r+1}^{m}P_k)=0$. Therefore,

$$\Delta(\tilde{\Pi}_r^{+\infty})=\lim_{m\to+\infty}\Delta(\prod_{k=r}^{m}P_k)$$
$$=\lim_{m\to+\infty}\Delta(P_rP_r'^{-1}\{P'_r\prod_{k=r+1}^{m}P_k\})=0.$$

Thus, $\mathcal{M}_{+\infty}(\mathfrak{J}, \mathfrak{h})=\{+1\}$ by Theorem 1. Q. E. D.

We proceed to the proof of the first statement in Theorem 4. Let $||A||$ be the operator norm of a matrix A, i. e.,

$$||A||=\sup_x ||A\boldsymbol{x}||/||\boldsymbol{x}||.$$

Lemma 6. *Let $\{A_k\}_k$ be a sequence of matrices. If*

$$\sum^{+\infty} ||A_k||<+\infty,$$

then $\prod_{k=1}^{m} (E+A_k)$ converges as $m\to+\infty$.

Proof. Let $n>m$. We have

$$\begin{aligned}||\prod_{k=1}^{n} (E+A_k)-\prod_{k=1}^{m} (E+A_k)||\\ &\leqq||\prod_{k=1}^{m} (E+A_k)||\,||\prod_{k=m+1}^{n} (E+A_k)-E||\\ &\leqq\prod_{k=1}^{m} (1+||A_k||)\{\prod_{k=m+1}^{n} (1+||A_k||)-1\}.\end{aligned}$$

The right-hand side converges to 0 as $n>m\to+\infty$. Q. E. D.

Proof of the first statement in Theorem 4. We show that $\mathcal{M}_{+\infty}(\mathfrak{J}, \mathfrak{h})=\{-1, +1\}$ if $\sum^{+\infty} \exp 2(\mathfrak{J}_n+|\sum_{k=0}^{n} \mathfrak{h}_k|)<+\infty$. Since

$$\begin{aligned}\sum_{k=r}^{+\infty} ||e^{2\mathfrak{J}_k}(\prod_{j=r}^{k} H_j)^2F||&\leqq\sum_{k=r}^{+\infty} e^{2\mathfrak{J}_k}||(\prod_{j=r}^{k} H_j)^2||\,||F||\\ &=\sum_{k=r}^{+\infty} \exp 2(\mathfrak{J}_k+|\sum_{j=r}^{k} \mathfrak{h}_j|)<+\infty,\end{aligned}$$

$\prod_{k=r}^{m} P_k=\prod_{k=r}^{m} \{E+e^{2\mathfrak{J}_k}(\prod_{j=r}^{k} H_j)^2F\}$ converges as $m\to+\infty$ by Lemma 6.

On the other hand, for sufficiently large r,

$$\det \prod_{k=r}^{+\infty} P_k=\prod_{k=r}^{+\infty} (1-e^{4\mathfrak{J}_k})>0.$$

Therefore, we have

$$\Delta(\tilde{\Pi}_r^{+\infty})=\Delta(\prod_{k=r}^{+\infty} P_k)>0,$$

which implies $\mathcal{M}_{+\infty}(\mathfrak{J}, \mathfrak{h})=\{-1, +1\}$ by Theorem 1. Q. E. D.

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