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Stochastic differential equations of jump type and Lévy processes in diffeomorphisms group

By

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Introduction.

Consider a stochastic differential equation with respect to Brownian motions. It is a known fact that the solution defines a stochastic flow of homeomorphisms, provided that coefficients of the equation are Lipschitz continuous. See e.g. Bismut [2], Kunita [6]. Hence the solution defines a Brownian motion in homeomorphisms group G. Recently, Brownian motion in homeomorphisms group itself is studied a lot. See Harris [4]. Le Jan [9], Baxendale [1], Kunita [7] and Le Jan-Watanabe [10]. They characterize it by the infinitesimal mean and covariance, called local characteristics.

The purpose of this paper is to study the related problem for jump processes. The stochastic differential (integral) equations considered in this paper is based on the Lévy process $X_t=X_t(x)$, $x\in R^d$ (process with independent increments, continuous in probability), which takes values in the vector space of continuous maps (or continuous vector fields). We call it a C-valued Lévy process. In Section 1, we study systematically the C-valued Lévy process. We introduce the system (a, b, ν, U) which characterizes the law of C-valued process and discuss the existence problem of C-valued Lévy process associated with a given characteristics. The Lévy process with values in the vector space of C^m-maps is also considered.

In Section 2, we consider the stochastic differential equation based on C-valued Lévy process $X_t(x)$. The equation is written in short by $d\xi_t=dX_t(\xi_{t-})$ or $d\xi_t/dt=X_t(\xi_{t-})$. Hence it is a natural extension of a stochastic differential equation defined by a finite dimensional Brownian motion and a Poisson point process. Generally, the solution does not define a stochastic flow of homeomorphisms, owing to the jump part of the C-valued Lévy process. In fact, we prove that the solution defines a *Lévy process with values in the semigroup of continuous maps*, under Lipschitz conditions to the characteristics of the C-valued Lévy process. Further, if the characteristics are smooth, then the solution defines a Lévy process with values in the semigroup of smooth maps. In order that the solution defines a Lévy process with values in homeomorphisms group (or dif-

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feomorphisms group), we had to introduce a specific condition to the characteristics. The details will be discussed at Section 2.5.

Conversely suppose we are given a Lévy process ξ_t with values in the semigroup G_+ of continuous maps (or in the group of homeomorphisms). We prove in Section 3 that under a few condition to ξ_t there is a unique C-valued Lévy process such that ξ_t is represented as the solution of the stochastic differential equation based on the C-valued Lévy process. Hence the associated C-valued Lévy process is the *(pathwise) infinitesimal generator* of the G_+-valued Lévy process. In particular, a Brownian motion in homeomorphisms group G is represented as a solution of the stochastic differential equation based on C-valued Brownian motion (=continuous Gaussian process in C with independent increments). The latter result is closely related to those of [9] and [1]. The case where the Lévy process ξ_t takes values in the semigroup of smooth maps is also discussed in Section 3.

Our objective is a complete characterization to Lévy process with values in homeomorphisms group or diffeomorphisms group via stochastic differential equation or infinitesimal generator. Results presented in this paper give a rather satisfactory characterization both to *Lévy processes with values in the semigroup of continuous (or smooth) maps* and to *Brownian motions with values in homeomorphisms (or diffeomorphisms) group.* However, there are still gaps between the construction theorem and the representation theorem of the Lévy process with values in homeomorphisms (or diffeomorphisms) group. This problem will be discussed elsewhere.

§1. Lévy process with values in the vector space of continuous maps.

1.1. Preliminaries. Let $C=C(R^d;R^d)$ be the totality of continuous maps from R^d into itself equipped with the compact uniform topology

$$\rho(f, g)=\sum_{N=1}^{\infty}\frac{1}{2^N}\frac{\sup_{|x|\leq N}|f(x)-g(x)|}{1+\sup_{|x|\leq N}|f(x)-g(x)|}.$$

Then C is a Fréchet space. Let $k=(k_1, \cdots, k_d)$ be a multi-index of nonnegative integers. We denote by D^k the differential operator $(\partial/\partial x_1)^{k_1}\cdots(\partial/\partial x_d)^{k_d}$. For a positive integer m, we denote by $C^m=C^m(R^d;R^d)$, the subspace of C consisting of C^m-maps. It is again a Fréchet space by the metric ρ_m:

$$\rho_m(f, g)=\sum_{|k|\leq m}\rho(D^k f, D^k g)$$

Let $X_t=X_t(\omega)$, $t\in[0, T]$ be a stochastic process with values in C (or C^m) defined on the probability space $(\Omega, \mathcal{F}, P)$. It is called a *Lévy process* if it is continuous in probability, right continuous with the left hand limit in ρ-topology (or ρ_m, resp.), and has the *independent increments*: i.e., $X_{t_{i+1}}-X_{t_i}$, $i=0, \cdots, n-1$ are independent for any $0\leqq t_0<t_1<\cdots<t_n\leqq T$. In particular, if X_t is continuous in t, it is called a *Brownian motion.* In this paper, we always assume that the C-valued (or C^m-valued) Lévy process is *stationary,* i.e., the law of X_t-X_s

depends only on $t-s$ and that $X_0=0$.

Given a C-valued Lévy process X_t, we define a point process associated with it. Set $D_p=\{s\in(0, T]\,;\Delta X_s\neq 0\}$, where $\Delta X_s=X_s-X_{s-}$ and $X_{s-}=\lim_{r\uparrow s} X_r$. Let p_t be a *C-valued point process* defined by $p_t=\Delta X_t$ if $t\in D_p$. Let $N_p((0, t], A)$ be the *counting measure* of p_t;

$$N_p((0, t], A)=\#\{s\in D_p\cap(0, t]\,;\, p_s\in A\}\,,$$

where A is a Borel subset of C. Then it is a *stationary Poisson random measure.* The *intesity measure*

$$\nu((0, t]\times A)=E[N_p((0, t], A)]$$

is then written as $\nu(A)t$. The measure ν satisfies the following property:

(1.1) It holds that

$$\nu(\{f : f=0\})=0\,.$$

There is an open neighborhood U of 0 in C such that $\nu(U^c)<\infty$ and $\int_U |f(x)|^2\nu(df)<\infty$ holds for any x.

Let $X_t(x)$ denote the restriction of the C-valued Lévy process X_t at the point $x\in R^d$. Then for any $x_1, \cdots, x_n\in R^d$, the n-point process $(X_t(x_1), \cdots, X_t(x_n))$ is an nd-dimensional Lévy process. Hence the characteristic function admits the *Lévy-Khinchin's formula*:

(1.2)
$$\begin{aligned} &E[e^{i\sum_{k=1}^{n}(\alpha_k, X_t(x_k))}] \\ &=\exp t\Big\{i\sum_{k=1}^{n}(\alpha_k, b(x_k))-\frac{1}{2}\sum_{k,l}\alpha_k a(x_k, x_l)\alpha_l \\ &\quad+\int_U (e^{i\sum_k(\alpha_k, f(x_k))}-1-i\sum_k(\alpha_k, f(x_k)))\nu(df) \\ &\quad+\int_{U^c}(e^{i\sum_k(\alpha_k, f(x_k))}-1)\nu(df)\Big\} \end{aligned}$$

where,

(1.3) $b(x)$ is an R^d-valued function,

(1.4) $a(x, y)$ is a $d\times d$-matrix valued function such that $a^{kl}(x, y)=a^{lk}(y, x)$ for any $k, l=1, \cdots, n$ and $x, y\in R^d$, and $\sum_{i,j}\alpha_i a(x_i, x_j)\alpha_j\geqq 0$ for any $x_i, a_i\in R^d$, $i=1, \cdots, n$.

Note that the law of $X_t(\omega)$ is uniquely determined by the system (a, b, ν, U). It is called the *characteristics* of X_t.

Here is an example of a C-valued Brownian motion.

Example. Let $B_t^1, B_t^2, \cdots$ be finite or infinite sequence of independent standard one dimensional Brownian motions. Let $X_k(x)$, $k=1, 2, \cdots$ be elements in C. Suppose that each $X_k(x)$ is Lipschitz continuous such that the Lipschitz constants

$L_k : |X_k(x)-X_k(y)| \leqq L_k |x-y|$ satisfies $\sum L_k^2 < \infty$. Then

$$X_t(x) \equiv \sum_k X_k(x) B_t^k \tag{1.5}$$

is a C-valued Brownian motion with characteristics

$$a(x, y) = \sum_k X_k(x) X_k(y)'^{\dagger)}, \qquad b(x) = 0, \tag{1.6}$$

if (1.5) is a finite sum. We will see at Section 1.2 that the fact is valid even if (1.5) is an infinite sum.

Conversely, let $X_t(x)$ be a C-valued Brownian motion with mean 0. Then there is a finite or infinite sequence of one dimensional standard Brownian motions B_t^k, $k=1, 2, \cdots$ and elements of C; $X_k(x)$, $k=1, 2, \cdots$ with the above property such that X_t is represented as (1.5). For the proof of this we proceed as follows. Let t be in $[0, T]$ and let H_t be the closed linear span of $X_t^i(x)$, $x \in R^d$, $i=1, \cdots, d$. Then it is a Gaussian space. Choose a sequence $\{x_n\}$ from R^d such that $X_T^i(x_n)$, $n \geqq 1$, $i=1, \cdots, d$ are linearly independent and dense in H_T. Then $\{X_t^i(x_n)\}$ is dense in H_t for any t. We shall reorder the latter sequence and write it as $\{X_t^n\}$. We define one dimensional process B_t^1 by

$$B_t^1 = \frac{1}{\|X_1^1\|} X_t^1$$

where $\| \ \|$ is the L^2-norm. It is a standard Brownian motion. We next define B_t^2 by

$$B_t^2 = (X_t^2 - (X_1^2, B_1^1) B_t^1) \| X_1^2 - (X_1^2, B_1^1) B_1^1 \|^{-1}$$

where $(,)$ denotes the L^2-inner product. Then B_t^2 is a standard Brownian motion independent of B_t^1. We will define B_t^k. $k \geqq 3$ by induction:

$$B_t^k = \Big(X_t^k - \sum_{i=1}^{k-1} (X_1^k, B_1^i) B_t^i \Big) \Big\| X_1^k - \sum_{i=1}^{k-1} (X_1^k, B_1^i) B_1^i \Big\|^{-1} \tag{1.7}$$

Then B_t^k, $k=1, 2, \cdots$ are independent standard Brownian motions. Obviously $\{B_t^k, k=1, 2, \cdots\}$ is an orthogonal basis of H_t for each t. Therefore $X_t(x)$ has the Fourier expansion (1.5), where $X_k(x) = (X_1(x), B_1^k)$. The coefficients $X_k(x)$ are continuous and satisfies (1.6).

1.2. Existence of C-valued Lévy process associated to the characteristics. Suppose we are given a system (a, b, ν, U) satisfying (1.1), (1.3) and (1.4). We are concerned with the existence of a C-valued Lévy process with the characteristics (a, b, ν, U). For this purpose, we introduce the following conditions.

(C, I) $a(x, y)$ is *bi*-Lipschitz continuous in the following sense: There is a positive constant L such that it holds

$$\|a(x, x) - 2a(x, y) + a(y, y)\| \leqq L |x-y|^2, \qquad \forall x, y \in R^d \tag{1.8}$$

where $\|a\| = \sum_i a_{ii}$.

$\dagger)$ $X_k(x)$ is a row vector function and $X_k(x)'$ is the transpose.

(C, II) $b(x)$ is Lipschitz continuous, i.e., there is a positive constant L such that

(1.9) $$|b(x)-b(y)|\leqq L|x-y|, \qquad \forall x, y\in R^d.$$

(C, III$_p$) There is a positive constant L such that

(1.10) $$\int_U |f(x)-f(y)|^{p'}\nu(df)\leqq L|x-y|^{p'}, \qquad \forall x, y\in R^d,$$

(1.11) $$\int_U |f(x)|^{p'}\nu(df)\leqq L(1+|x|)^{p'}, \qquad \forall x\in R^d$$

holds for any $p'\in[2, p]$.

Here p is a positive number not less than 2.

Theorem 1.1. *Let* (a, b, ν, U) *be a system satisfying* (1.1), (1.3), (1.4) *and* (C, I), (C, II) *and* (C, III$_p$) *for some* $p>d$. *Then there is a C-valued Lévy process with the characteristics* (a, b, ν, U). *In particular, if* ν *is identically* 0, *it is a C-valued Brownian motion, continuous in* t *with respect to the metric* ρ.

Before the proof of the theorem, we prepare a lemma, which is a modification of Kolmogorov's criterion for the continuity of random fields.

Lemma 1.1. *Let* $X_t(x)$, $t\in[0, T]$, $x\in R^d$ *be stochastic processes with parameter* x. *Suppose that for each* x $X_t(x)$ *is continuous (right continuous with the left hand limits) in* t *a.s., and satisfies*

$$E[\sup_t |X_t(x)|^\alpha]<\infty, \qquad \forall x\in R^d,$$

$$E[\sup_t |X_t(x)-X_t(y)|^\alpha]\leqq K|x-y|^{d+\beta}, \qquad \forall x, y\in R^d$$

for some positive α, β. *Then* $X_t(x)$ *has a modification such that it is a continuous (resp. right continuous with the left hand limits) C-valued process.*

Proof. Let $C([0, T]; R^d)$ (resp. $D([0, T]; R^d)$) be the set of continuous (right continuous with the left hand limits) maps from $[0, T]$ into R^d equipped with the norm $\|\phi\|=\sup_t |\phi(t)|$. Then $X_t(x)$ may be regarded as a $C([0, T]; R^d)$ (or $D([0, T]; R^d)$)-valued random field satisfying

$$E[\|X_\cdot(x)\|^\alpha]<\infty, \qquad E[\|X_\cdot(x)-X_\cdot(y)\|^\alpha]\leqq K|x-y|^{d+\beta}.$$

Then Kolmogorov's theorem states that there is a modification of the random field denoted by the same letter $X_\cdot(x)$ such that it is continuous in x, i.e., $\lim_{y\to x}\|X_\cdot(x)-X_\cdot(y)\|=0$ holds for any all x a.s. See e.g. Chap. I, Appendix in Kunita [8].

The above modification is what we want. We prove this in case where X_t is right continuous. Let $t\in[0, T]$ and $\varepsilon>0$ be any given numbers. For each x, there is a positive number δ_x such that $|X_t(x)-X_{t+h}(x)|<\varepsilon$ if $|h|<\delta_x$. Let $U(x)$ be an open neighborhood of x in R^d such that $\|X_\cdot(x)-X_\cdot(y)\|<\varepsilon$ for any

$y\in U(x)$. Then it holds $|X_t(y)-X_{t+h}(y)|<3\varepsilon$ for $y\in U(x)$. Choose $x_1, \cdots, x_n$ from $B_N=\{x||x|\leqq N\}$ such that $\bigcup_{i=1}^{n} U(x_i)\supset B_N$. Set $\delta=\min(\delta_{x_1}, \cdots, \delta_{x_n})$. Then we have $|X_t(x)-X_{t+h}(x)|<3\varepsilon$ for all $x\in B_N$ if $|h|<\delta$. This proves that X_t is a right continuous C-valued process. The existence of the left hand limit can be proved similarly.

Proof of Theorem 1.1. Associated with the pair (a, b), we can construct a Gaussian random field $X_t^c(x)$, $t\in[0, T]$, $x\in R^d$ with values in R^d on a suitable probability space $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P})$ with the following properties

$$\hat{E}[X_t^c(x)]=tb(x),$$

$$\hat{E}[(X_t^c(x)-tb(x))(X_s^c(y)-sb(y))']=(t\wedge s)a(x, y).$$

Then, for each fixed $\boldsymbol{x}=(x_1, \cdots, x_n)$, $X_t^c(\boldsymbol{x})=(X_t^c(x_1), \cdots, X_t^c(x_n))$ is an R^{nd}-valued Brownian motion.

We shall prove that $Y_t^c(x)\equiv X_t^c(x)-tb(x)$ has a modification which is continuous in (t, x). Note that for each x and y, $Y_t^c(x)-Y_t^c(y)$ is a Brownian motion with zero mean and covariance $(a(x, x)-a(x, y)-a(y, x)+a(y, y))t$. Hence it is a continuous martingale. By Doob's inequality and the moment property of the Gaussian random variable, we have for any $p'>2$

$$\text{(1.12)}\qquad E[\sup_{0\leq s\leq t}|Y_s^c(x)-Y_s^c(y)|^{p'}]\leqq C_1E[|Y_t^c(x)-Y_t^c(y)|^{p'}]$$
$$\leqq C_1'\|a(x, x)-2a(x, y)+a(y, y)\|^{p'/2}t^{p'/2}.$$

The last member is dominated by $C_1'L^{(p'/2)}|x-y|^{p'}t^{(p'/2)}$ in view of hypothesis (C, I). Then by Lemma 1.1 $Y_t^c(x)$ is a continuous C-valued process.

Now let $N_p(dt, df)$ be a stationary Poisson random measure on $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P})$ with the intensity measure ν. For arbitrary fixed x, define a Lévy process by

$$X_t^d(x)=\int_U f(x)\tilde{N}_p((0, t], df)+\int_{C-U} f(x)N_p((0. t], df),$$

where

$$\tilde{N}_p((0, t], df)=N_p((0, t], df)-\nu(df)t.$$

Obviously, the last member of the above is a C-valued Lévy process, since it is written as a finite sum of $f(x)$'s. Now, denote the first member of the right hand side by $Y_t^d(x)$. It is an L^2-martingale with zero-mean and covariance $t\int_U f^i(x)f^j(x)\nu(df)$. In order to prove that $Y_t^d(x)$ is a C-valued process, we shall get the L^p-estimate of $N_t\equiv Y_t^d(x)-Y_t^d(y)$, where x, y are fixed for a moment. By Itô's formula for discontinuous semimartingales (Ikeda-Watanabe [5]), we have for any $p'\in[2, p]$,

$$|N_t|^{p'}=\text{a martingale with zero-mean}$$
$$+\int_0^t\int_U \{|N_r+f(x)-f(y)|^{p'}-|N_r|^{p'}$$

$$-p'\sum_i (f^i(x)-f^i(y))|N_r|^{p'-2}N_r^i\}\nu(df)dr\,,$$

where $f^i(x)$ is the i-th component of $f(x)$. The absolute value of the integrant $\{\cdots\}$ is dominated by

$$\frac{1}{2}p'(p'-1)|f(x)-f(y)|^2|N_r+\theta_r(f(x)-f(y))|^{p'-2}$$
$$\leqq C_2\{|f(x)-f(y)|^2|N_r|^{p'-2}+|f(x)-f(y)|^{p'}\}$$

where $|\theta_r|\leqq 1$. Therefore we get

$$(1.13)\qquad E[\sup_{u\leq t}|Y_u^d(x)-Y_u^d(y)|^{p'}]$$
$$\leqq C_3E[|Y_t^d(x)-Y_t^d(y)|^{p'}]$$
$$\leqq C_3'\Big\{\int_0^t E[\sup_{0\leq r\leq u}|Y_r^d(x)-Y_r^d(y)|^{p'-2}]du\cdot\int_U|f(x)-f(y)|^2\nu(df)$$
$$+t\int_U|f(x)-f(y)|^{p'}\nu(df)\Big\}$$
$$\leqq C_3''\Big\{|x-y|^2\int_0^t E[\sup_{0\leq r\leq u}|Y_r^d(x)-Y_r^d(y)|^{p'-2}]du+t|x-y|^{p'}\Big\}.$$

In case $p'=2$, the above inequality shows that

$$E[\sup_{r\leq t}|Y_r^d(x)-Y_r^d(y)|^2]\leqq C_4t|x-y|^2.$$

Then by Hölder's inequality, it holds

$$E[\sup_{r\leq t}|Y_r^d(x)-Y_r^d(y)|^{p'}]\leqq (C_4t)^{p'/2}|x-y|^{p'}$$

for any $0<p'\leqq 2$. Substitute this to (1.13), and we get

$$(1.14)\qquad E[\sup_{0\leq r\leq t}|Y_r^d(x)-Y_r^d(y)|^{p'}]\leqq C_5t|x-y|^{p'}$$

for any $2\leqq p'\leqq 4\wedge p$. Repeating this argument inductively, we obtain (1.14) for any $p'\in[2, p]$.

We can prove similarly as the proof of (1.14) that

$$(1.15)\qquad E[\sup_{0\leq u\leq t}|Y_u^d(x)|^{p'}]\leqq C_6t(1+|x|)^{p'}\,,\qquad \forall x\in R^d$$

holds for any $p'\in[2, p]$. Then using Kolmogorov's criterion again, we see that $Y_t^d(x)$ is a right continuous C-valued process. The sum $X_t(x)=X_t^c(x)+X_t^d(x)$ is then a C-valued Lévy process with characteristics (a, b, ν, U).

Remark. L_p-estimates (1.14) and (1.15) are rather local in the following sense. Suppose that (C, I), (C, II) and (C, III$_p$) are valid for x, y with $|x|, |y|\leqq N$, where N is a positive constant. Then inequalities (1.14) and (1.15) are valid for x, y with $|x|, |y|\leqq N$, too. Hence local Lipschitz continuities for a, b, ν are sufficient for Theorem 1.1.

Remark. Applying Theorem 1.1, we see immediately that the infinite sum (1.5) is a C-valued Brownian motion. Indeed, $a(x, y)$ of (1.6) is *bi*-Lipschitz continuous since

$$\|a(x, x)-2a(x, y)+a(y, y)\|=\sum_k \|X_k(x)-X_k(y)\|^2$$
$$\leqq(\sum_k L_k^2)|x-y|^2.$$

Conversely, the Fourier coefficients $X_k(x)$ of a given C-valued Brownian motion $X_t(x)$ are Lipschitz continuous, if it satisfies (C, I). In fact, we have

$$\begin{aligned}(1.16)\qquad \sum_k |X_k(x)-X_k(y)|^2 &=\sum_k E((X_1(x)-X_1(y))B_1^k)^2\\ &=(E\|X_1(x)-X_1(y)\|^2)\\ &=\|a(x, x)-2a(x, y)+a(y, y)\|.\end{aligned}$$

We next discuss the problem converse to Theorem 1.1. We restrict our attention to Lévy processes with p-th moment.

Theorem 1.2. *Let $X_t(x)$, $t\in[0, T]$, $x\in R^d$ be a random field such that the n-point process $(X_t(x_1), \cdots, X_t(x_n))$ is an nd-dimensional stationary Lévy process for any n and $x_1, \cdots, x_n$. Suppose that there is $p>d\vee z$ and constant $K>0$ such that*

$$(1.17)\qquad E[|X_t(x)-X_t(y)|^{p'}]\leqq Kt|x-y|^{p'}, \qquad \forall x, y\in R^d$$

$$(1.18)\qquad E[|X_t(x)|^{p'}]\leqq Kt(1+|x|)^{p'}, \qquad \forall x\in R^d.$$

for any $p'\in[2, p]$. Then $X_t(x)$ has a modification of C-valued Lévy process. Furthermore, its characteristics satisfy (C, I), (C, II) *and* (C, III$_p$) *with* U=C.

Proof. The existence of a modification of C-valued process is immediate from Kolmogorov's criterion. We shall prove that the characteristics satisfy (C, I), (C, II) and (C, III$_p$). For each x, the process $X_t(x)$ admits the Lévy-Itô decomposition $X_t(x)=Y_t^c(x)+b(x)t+Y_t^d(x)$, where $Y_t^c(x)$ is a Brownian motion with zero-mean and covariance $a(x, x)$, and $Y_t^d(x)$ is a discontinuous Lévy process with zero-mean. Since both are independent, we have

$$\begin{aligned}&E[|X_t(x)-X_t(y)|^2]\\ &=E[|Y_t^c(x)-Y_t^c(y)|^2]+t^2|b(x)-b(y)|^2+E[|Y_t^d(x)-Y_t^d(y)|^2]\\ &=t\Big\{\|a(x, x)-2a(x, y)+a(y, y)\|+t|b(x)-b(y)|^2\\ &\quad+\int_C |f(x)-f(y)|^2\nu(df)\Big\}.\end{aligned}$$

Therefore, (1.17) with $p'=2$ implies (C, I) and (C, II).

For the proof of (C, III$_p$) we proceed as follows. Observe that the variance of $Y_t^c(x)-Y_t^c(y)$ is $t(a(x, x)-a(x, y)-a(y, x)+a(y, y))$. Then we haves imilarly as (1.12) that

(1.19) $$E[\sup_{0\leq s\leq t}|Y_s^c(x)-Y_s^c(y)|^{p'}]\leqq C_1''t^{p'/2}|x-y|^{p'}.$$

Hence $Y_t^d(x)$ itself satisfies (1.17) (with different constant K). Set $Z_t=Y_t^{d,i}(x)-Y_t^{d,i}(y)$, where x, y and i are fixed. Then it is a discontinuous L^2-martingale. Let $[Z]_t$ be the quadratic variation of Z_t. Then it holds

(1.20) $$[Z]_t=\int_0^t\int_C|f^i(x)-f^i(y)|^2N_p(ds, df).$$

Therefore, by Itô's formula, we have for any $p'\leqq p$

$$[Z]_t^{p'/2}=\int_0^t\int_C\{([Z]_{s-}+|f^i(x)-f^i(y)|^2)^{p'/2}-[Z]_{s-}^{p'/2}\}N_p(ds, df)$$

$$\geqq\int_0^t\int_C|f^i(x)-f^i(y)|^{p'}N_p(ds, df).$$

Then by Burkholder's inequality,

(1.21) $$E[|Z_t|^{p'}]\geqq C_6E[[Z]_t^{p'/2}]\geqq tC_6\int_C|f^i(x)-f^i(y)|^{p'}\nu(df).$$

The left hand side is dominated by $Ct|x-y|^{p'}$. Hence (1.10) follows. Inequality (1.11) can be proved similarly. The proof is complete.

In case of Brownian motion, we can replace condition (1.17) to a weaker one. Indeed, the proof of Theorem 1.2 shows the followings.

Corollary 1. *Let $X_t(x)$, $t\in[0, T]$, $x\in R^d$ be a random fiela sucn that the n-point process $(X_t(x_1), \cdots, X_t(x_n))$ is an nd-dimensional Brownian motion for any n and $x_1, \cdots, x_n$. Suppose that there is a constant K such that*

(1.17′) $$E[|X_t(x)-X_t(y)|^2]\leqq Kt|x-y|^2,\qquad \forall x, y\in R^d.$$

(1.18′) $$E[|X_t(x)|^2]\leqq Kt(1+|x|)^2,\qquad \forall x\in R^d.$$

Then $X_t(x)$ has a modification of a C-valued Brownian motion.

Corollary 2. *Suppose that C-valued Lévy process X_t satisfies* (1.17) *and* (1.18). *Then it admits the Lévy-Itô decomposition $X_t^c+X_t^d$, where X_t^c is a C-valued Brownian motion and X_t^d is a C-valued discontinuous Lévy process with zero mean represented as $X_t^d(x)=\int_C f(x)\tilde{N}_p((0, t], df)$.*

1.3. Smoothness of C-valued Lévy process. We shall next discuss the smoothness of the C-valued Lévy process. Let m be *a* positive integer. For the system (a, b, ν, U), we introduce the following hypotheses.

(C^m, I) $a(x, y)=(a^{ij}(x, y))$ are m-times continuously differentiable in both x and y. Further, $D_x^kD_y^ka(x, y)$ is *bi*-Lipschitz continuous for any k with $|k|\leqq m$.

(C^m, II) $b(x)=(b^i(x))$ is a C^m-function and $D^kb(x)$ is Lipschitz continuous for any

k with $|k|\leqq m$.

(C^m, III_p) The measure ν is supported by C^m. There is a positive constant L such that

(1.22) $$\int_U |D^k f(x)-D^k f(y)|^{p'}\nu(df)\leqq L|x-y|^{p'}, \qquad \forall x, y\in R^d$$

(1.23) $$\int_U |D^k f(x)|^{p'}\nu(df)\leqq L, \qquad \forall x\in R^d$$

hold for any k with $1\leqq|k|\leqq m$ and (1.11) for $p'\in[2, p]$.

Theorem 1.3. (i) *Suppose that the characteristics of the C-valued Lévy process X_t satisfy* (C^m, I), (C^m, II) *and* (C^m, III_p) *for some $p>d+1$. Then it is a C^m-valued Lévy process. Furthermsre, in case $U=C$, there is a positive constant K such that*

(1.24) $$E[|D^k X_t(x)-D^k X_t(y)|^{p'}]\leqq Kt|x-y|^{p'}, \qquad \forall x, y\in R^d$$

(1.25) $$E[|D^k X_t(x)|^{p'}]\leqq Kt, \qquad \forall x\in R^d$$

for any k with $1\leqq|k|\leqq m$ and (1.18) *for $p'\in[2, p]$.*

(ii) *Conversely if X_t is a C^m-valued Lévy process satisfying* (1.24) *and* (1.25) *for any k with $1\leqq|k|\leqq m$ and $p'\in[2, p]$. Then the characteristics satisfy* (C^m, I), (C^m, II) *and* (C^m, III_p) *with $U=C$.*

Proof. We shall prove (i) in the case $m=1$ and $U=C$ only. Let $e_i=(0, \cdots, 0, 1, 0, \cdots, 0)$ (1 is at the i-th component) and let $y\in R^1-\{0\}$. Set

$$N_t(x, y)=\frac{1}{y}\{X_t(x+ye_i)-X_t(x)\}.$$

Then using hypotheses (C^1, I), (C^1, II) and (C^1, III_p), $p>d$, we can prove similarly as in the proof of Theorem 1.1 that there is a constant C such that

(1.26) $$E[\sup_{s\leq t}|N_s(x, y)-N_s(x', y')|^{p'}]\leqq Ct\{|x-x'|^{p'}+|y-y'|^{p'}\},$$

(1.27) $$E[\sup_{s\leq t}|N_s(x, y)|^{p'}]\leqq Ct,$$

hold for any x, x', y, y'. (Details are omitted). Then, by Kolmogorov's criterion, $N_t(x, y)$ has a continuous extension at $y=0$. This proves that $X_t(x)$ is continuously differentiable and it holds $N_t(x, 0)=\partial_i X_t(x)$ for any x a.s. Let y tend to 0 in (1.26) and (1.27). Then we get

$$E[\sup_{s\leq t}|\partial_i X_s(x)-\partial_i X_s(x')|^{p'}]\leqq Ct|x-x'|^{p'}, \qquad \forall x, x'\in R^d,$$

$$E[\sup_{s\leq t}|\partial_i X_s(x)|^{p'}]\leqq Ct, \qquad \forall x\in R^d.$$

Suppose conversely that X_t is a C^m-valued Lévy process satisfying (1.24), (1.25) for any k with $|k|\leqq m$. It is clear that $b(x)=E[X_1(x)]$ satisfies (C^m, II). Let $Y_t(x)=X_t(x)-b(x)t=Y_t^c(x)+Y_t^d(x)$ be the Lévy-Itô decomposition. We shall prove that both $Y_t^c(x)$ and $Y_t^d(x)$ are differentiable with respect to x up to k, with $|k|\leqq m$ and they satisfy (1.24) and (1.25). The first derivative $\partial_i Y_t(x)$ is a C-valued process. It satisfies

$$E[|\partial_i Y_t(x)-\partial_i Y_t(y)|^p]\leqq Kt|x-y|^p, \qquad \forall x,\ y\in R^d$$

with a positive constant K because of (1.24). By Corollary 2 to Theorem 1.2, $\partial_i Y_t$ admits Lévy-Itô decomposition $\partial_i Y_t(x)=\tilde{Y}_t^c(x)+\tilde{Y}_t^d(x)$, where $\tilde{Y}_t^c(x)$ is a C-valued Brownian motion and $\tilde{Y}_t^d(x)$ is a discontinuous C-valued Lévy process. By the mean value theorem, we have

$$\begin{aligned} Y_t(x+he_i)-Y_t(x) &= h\int_0^1 \partial_i Y_t(x+vhe_i)dv \\ &= h\int_0^1 \tilde{Y}_t^c(x+vhe_i)dv+h\int_0^1 \tilde{Y}_t^d(x+vhe_i)dv \end{aligned}$$

By the uniqueness of the Lévy-Itô decomposition, it holds

$$Y_t^c(x+he_i)-Y_t^c(x)=h\int_0^1 \tilde{Y}_t^c(x+vhe_i)dv\,.$$

This shows that the partial derivative $\partial_i Y_t^c(x)$ exists for all x a.s. and coincides with the C-valued process $\tilde{Y}_t^c(x)$. Therefore $Y_t^c(x)$ is a C^1-valued Brownian motion. Repeating the argument inductively, we see that $Y_t^c(x)$ is a C^m-valued Brownian motion. As a consequence, $Y_t^d(x)$ is also a C^m-valued Lévy process.

We shall next prove that both Y_t^c and Y_t^d satisfy (1.24). Note that the quadratic variation of $D^kY_t(x)-D^kY_t(y)$ is $[D^kY^c(x)-D^kY^c(y)]_t+[D^kY^d(x)-D^kY^d(y)]_t$. Then (1.24) together with Burkholder's inequality proves that $E[[D^kY^c(x)-D^kY^c(y)]_t^{p'/2}]$ and $E[[D^kY^d(x)-D^kY^d(y)]_t^{p'/2}]$ are dominated by $Ct|x-y|^{p'}$. Then again by Burkholder's inequality, $E[|D^kY_t^c(x)-D^kY_t^c(y)|^{p'}]$ and $E[|D^kY_t^d(x)-D^kY_t^d(y)|^{p'}]$ are dominated by $C't|x-y|^{p'}$, showing the inequality (1.24) for Y_t^c and Y_t^d. By the same argument, we can show that both Y_t^c and Y_t^d satisfy (1.25).

Consider the characteristics $a(x, y)$. It is differentiable up to k with $|k|\leqq m$ and satisfies

$$D_x^k D_y^k a(x,\ y)=E[D^kY_1^c(x)D^kY_1^c(y)']\,,$$

which is *bi*-Lipschitz continuous because of (1.24) for Y_t^c. Consider next Y_t^d. It holds that

$$D^kY_t^d(x)=\int_C D^kf(x)\tilde{N}_p((0,\ t],\ df) \quad \text{a.s.}$$

for any x. Then we can show similarly as in the proof of Theorem 1.2 that there is a constant C independent of x such that

$$E[|D^kY_t^d(x)-D^kY_t^d(y)|^{p'}]\geqq Ct\int_C |D^kf(x)-D^kf(y)|^{p'}\nu(df)\,.$$

Therefore inequality (1.22) is verified. Inequality (1.23) can be verified similarly. The proof is complete.

In case of Brownian motion, we can replace conditions (1.24) and (1.25) to weaker ones.

Corollary. *Let X_t be a C-valued Brownian motion with characteristics* (a, b). *It is a C^m-valued Brownian motion satisfying*

$$E[|D^k X_t(x)-D^k X_t(y)|^2]\leqq Kt|x-y|^2, \qquad \forall x, y\in R^d \tag{1.24'}$$

$$E[|D^k X_t(x)|^2]\leqq Kt, \qquad \forall x\in R^d, \qquad 1\leqq|k|\leqq m \tag{1.25'}$$

and (1.18′) *if and only if* (a, b) *satisfies* (C^m, I), (C^m, II) *and* (C^m, III_2).

§ 2. Stochastic differential equation governed by C-valued Lévy process.

2.1. Stochastic integrals by C-valued Lévy process. Let $X_t(x)$, $x\in R^d$, $t\in[0, T]$ be a C-valued Lévy process with characteristics (a, b, ν, U) satisfying (C, I), (C, II) and (C, III_p), $p>d$. Let $s<t$ and $\mathcal{F}_{s,t}$ be the least sub σ-field of $\mathcal{F}$ for which X_u-X_v; $s\leqq u\leqq v\leqq t$ are measurable. Then for each s and x, $X_t(x)-X_s(x)$, $t\in[s, T]$ is an $\mathcal{F}_{s,t}$-adapted semimartingale. $X_t(x)$ is decomposed to the sum of the process of bounded variation $B_t(x)\equiv b(x)t+\int_{U^c} f(x)N_p((0, t], df)$ and an L^2-martingale $Y_t(x)\equiv X_t(x)-B_t(x)$. Let $\langle Y^i(x), Y^i(y)\rangle_t$ be the continuous process of bounded variation such that

$$(Y_t^i(x)-Y_s^i(x))(Y_t^j(y)-Y_s^j(y))-\{\langle Y^i(x), Y^j(y)\rangle_t-\langle Y^i(x), Y^j(y)\rangle_s\} \tag{2.1}$$

is an $\mathcal{F}_{s,t}$-martingale. Then it holds $\langle Y^i(x), Y^j(y)\rangle_t=tA^{ij}(x, y)$, where

$$A^{ij}(x, y)=a^{ij}(x, y)+\int_U f^i(x)f^j(y)\nu(df). \tag{2.2}$$

Let $s>0$ be a fixed number and let $\phi_t(\omega)$ be an $\mathcal{F}_{s,t}$-adapted R^d-valued process, right continuous with the left hand limits. Itô integral of ϕ_t by dY_t is defined by

$$\int_s^t dY_r^i(\phi_{r-})=\lim_{|\delta|\to 0}\sum_{k=0}^{n-1}\{Y^i_{t_{k+1}\wedge t}(\phi_{t_k\wedge t})-Y^i_{t_k\wedge t}(\phi_{t_k\wedge t})\}, \tag{2.3}$$

where δ are partitions $\{s=t_0<t_1<\cdots<t_n=T\}$. The limit exists in probability and is a local martingale. Let $\psi_t(\omega)$ be an $\mathcal{F}_{s,t}$-adapted process having the same property as ϕ_t. Then it holds

$$\left\langle\int_s^t dY_r^i(\phi_{r-}), \int_s^t dY_r^j(\psi_{s-})\right\rangle=\int_s^t A^{ij}(\phi_{r-}, \psi_{r-})dr. \tag{2.4}$$

See Le Jan [9] and Le Jan-Watanabe [10]. Now the stochastic integral by C-valued Lévy process X_t is defined by

$$\int_s^t dX_r(\phi_{r-})\equiv\int_s^t dY_r(\phi_{r-})+\int_s^t b(\phi_{r-})dr+\int_s^t\int_{U^c} f(\phi_{r-})N_p(dr, df). \tag{2.5}$$

2.2. Stochastic differential equation. We shall consider the stochastic differential equation defined by the C-valued Lévy process X_t:

$$d\xi_t = dX_t(\xi_{t-}). \tag{2.6}$$

The definition of the solution is as follows. Given a time s and state x, an R^d-valued $\mathcal{F}_{s,t}$-adapted process ξ_t, right continuous with the left hand limits is called a *solution of equation* (2.6) if it satisfies

$$\xi_t = x + \int_s^t dX_r(\xi_{r-}). \tag{2.7}$$

Note that this is an extension of a standard stochastic differential equation with respect to Poisson point process. Indeed, let $(B_t^1, \cdots, B_t^n)$ be a standard Brownian motion and let $X_0(x), \cdots, X_n(x)$ be Lipschitz continuous R^d-valued functions and let $N_p((0, t], A)$ be a Poisson random measure on C such that the intensity measure ν satisfies (C, III$_p$). Then

$$X_t(x) = \sum_{k=1}^n X_k(x)B_t^k + X_0(x)t + \int_U f(x)\tilde{N}_p((0, t], df) + \int_{U^c} f(x)N_p((0, t], df)$$

is a C-valued Lévy process. The corresponding stochastic differential equation is written by

$$d\xi_t = \sum_{k=1}^n X_k(\xi_{t-})dB_t^k + X_0(\xi_{t-})dt$$
$$+\int_U f(\xi_{t-})\tilde{N}_p(dt, df) + \int_{U^c} f(\xi_{t-})N_p(dt, df).$$

Theorem 2.1. *For each s, x, the equation* (2.7) *has a univue solution.*

Proof. We first consider the case $C=U$. We construct a solution by the successive approximation. Set $\xi_t^0 = x$ and

$$\xi_t^{n+1} = x + \int_s^t dX_r(\xi_{r-}^n), \qquad n=0, 1, 2, \cdots \tag{2.8}$$

Then it holds

$$E[\sup_{s\le u\le t} |\xi_u^{n+1} - \xi_u^n|^2] \le C_1\Big\{E\Big[\Big|\int_s^t dY_r(\xi_{r-}^n) - \int_s^t dY_r(\xi_{r-}^{n-1})\Big|^2\Big]$$
$$+E\Big[\Big|\int_s^t b(\xi_{r-}^n) - b(\xi_{r-}^{n-1})dr\Big|^2\Big]\Big\}$$
$$\le C_1\Big\{E\Big[\int_s^t \|A(\xi_{r-}^n, \xi_{r-}^n) - 2A(\xi_{r-}^n, \xi_{r-}^{n-1}) + A(\xi_{r-}^{n-1}, \xi_{r-}^{n-1})\|dr\Big]$$
$$+(t-s)E\Big[\int_s^t |b(\xi_{r-}^n) - b(\xi_{r-}^{n-1})|^2 dr\Big]\Big\}.$$

Using (C, I), (C, II) and (C, III$_p$), $p=2$, we get

$$E[\sup_{s\le u\le t} |\xi_u^{n+1} - \xi_u^n|^2] \le C_1(1+T)L\int_s^t E[\sup_{s\le u\le r} |\xi_u^n - \xi_u^{n-1}|^2]dr$$
$$\le \{C_1(1+T)L\}^n \frac{(t-s)^n}{n!} E[\sup_{s\le u\le t} |\xi_u^1 - \xi_u^0|^2].$$

Then ξ_u^{n+1} converges to a right continuous $\mathcal{F}_{s,u}$-adapted process ξ_u in L^2-sense. It is a solution of the equation (2.7). The uniqueness of the solution can be proved similarly.

We next consider the general case. Let $s=\sigma_0<\sigma_1<\cdots$ be a sequence of jumping times of $N_p(t, U^c)\equiv N_p((0, t], U^c)$:

$$\begin{aligned} \sigma_n &= \inf\{t>\sigma_{n-1}; N_p(t, U^c)-N_p(\sigma_{n-1}, U^c)\geqq 1\}, \quad n\geqq 1 \\ &= \infty \qquad \text{if} \quad \{\ \}=\phi. \end{aligned} \tag{2.9}$$

Then it holds $\lim_{n\to\infty}\sigma_n=\infty$. Set

$$\tilde{X}_t(x)=X_t(x)-\int_{U^c} f(x)N_p((0, t], df)$$

and consider an SDE

$$d\tilde{\xi}_t=d\tilde{X}_t(\tilde{\xi}_{t-}). \tag{2.10}$$

It has a unique solution $\tilde{\xi}_t$ starting at (s, x). We will define a right continuous process ξ_t, $t\in[s, T]$ as follows. For $t\in[s, \sigma_1)$, set $\xi_t=\tilde{\xi}_t$ and for $t=\sigma_1$, set $\xi_{\sigma_1}=\phi_{p_{\sigma_1}}(\tilde{\xi}_{\sigma_1-})$, where $\phi_f=f+id$ [†] and p_t is the Poisson point process of X_t. Let now $\tilde{\xi}_t$, $t\in[\sigma_1, \sigma_2)$ be the solution of (2.10) starting at $(\sigma_1, \xi_{\sigma_1})$. For $t\in[\sigma_1, \sigma_2)$ we define $\xi_t=\tilde{\xi}_t$ and for $t=\sigma_2$, we define $\xi_{\sigma_2}=\phi_{p_{\sigma_2}}(\tilde{\xi}_{\sigma_2-})$. Repeating this argument inductively, we can define ξ_t for all $t\in[s, T]$. It is easy to see that this ξ_t is a unique solution of equation (2.7).

2.3. The continuity of the solution with respect to the initial data. Let $\xi_{s,t}(x)$, $t\in[s, T]$ be the solution of equation (2.7) starting at (s, x). In this section, we will show that there is a modification of the solution such that the map $\xi_{s,t}; R^d\to R^d$ is continuous a.s. For this purpose, we get some L^p-estimates for the solution and then apply Kolmogorov's criterion for the continuity of random fields.

Lemma 2.1. *Suppose that the characteristics of the C-valued Lévy process X_t satisfy* (C, I), (C, II) *and* (C, III$_p$) *for some $p>d$ and $U=C$. Then there is a positive constant M depending only on constants L and p appearing in* (C, I), (C, II), (C, III$_p$) *and T such that*

$$E[\sup_{s\leq r\leq t}|\xi_{s,r}(x)-x-\xi_{s,r}(y)+y|^{p'}]\leqq M(t-s)|x-y|^{p'}, \tag{2.11}$$

$$E[\sup_{s\leq r\leq t}|\xi_{s,r}(x)-x|^{p'}]\leqq M(t-s)(1+|x|)^{p'} \tag{2.12}$$

holds for any $s, t\in[0, T]$, $x, y\in R^d$ and $p'\in[2, p]$.

Proof. We shall give the proof of (2.11) only, since the proof of (2.12) is similar. In the following, we write $\xi_{s,t}(x)$ as $\xi_t(x)$ and write $\eta_t=\xi_t(x)-\xi_t(y)$ since s, x, y are fixed. Let $X_t(x)=Y_t^c(x)+Y_t^d(x)+b(x)t$ be the decomposition

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[†] *id* is the identity map.

such that $Y_t^c(x)$ is a C-valued Brownian motion with zero-mean and $Y_t^d(x)$ is a discontinuous Lévy process with zero-mean. Then we have

$$(2.13)\qquad \eta_t-(x-y)=\int_s^t dY_r^c(\xi_{r-}(x))-\int_s^t dY_r^c(\xi_{r-}(y))$$
$$+\int_s^t dY_r^d(\xi_{r-}(x))-\int_s^t dY_r^d(\xi_{r-}(y))$$
$$+\int_s^t \{b(\xi_{r-}(x))-b(\xi_{r-}(y))\}\,dr$$
$$=M_t^c+M_t^d+A_t\,.$$

We shall compute the p-th moments of the supremum of M_t^c, M_t^d and A_t one by one. For the convenience, we write p' as p. By (C, II),

$$(2.14)\qquad E[\sup_{s\le r\le t}|A_r|^p]\le|t-s|^{p-1}E\Big[\int_s^t|b(\xi_{r-}(x))-b(\xi_{r-}(y))|^p dr\Big]$$
$$\le|t-s|^{p-1}L^p\int_s^t E[|\eta_{r-}|^p]dr\,.$$

Next, we have by Doob's inequality and Burkholder's inequality,

$$(2.15)\qquad E[\sup_{s\le r\le t}|M_r^c|^p]\le C_1E[|M_t^c|^p]$$
$$\le C_1'E\Big[\Big\{\int_s^t\|a(\xi_{r-}(x),\,\xi_{r-}(x))-2a(\xi_{r-}(x),\,\xi_{r-}(y))$$
$$+a(\xi_{r-}(y),\,\xi_{r-}(y))\|dr\Big\}^{p/2}\Big]$$
$$\le C_1'|t-s|^{(p/2)-1}L^{p/2}\int_s^t E[|\eta_{r-}|^p]dr\,.$$

The last inequality follows from (C, I).

For the computation of $E[\sup|M_r^d|^p]$, we proceed as follows. Apply Itô's formula to discontinuous martingale M_t^d. Then we have

$$(2.16)\qquad |M_t^d|^p=\text{a martingale with zero-mean}$$
$$+\int_s^t\int_C\{|M_{r-}^d+g_f(r)|^p-|M_{r-}^d|^p-p\sum_i g_f^i(r)|M_{r-}^d|^{p-2}M_{r-}^{d,i}\}\,\nu(df)dr\,,$$

where $g_f(r)=f(\xi_{r-}(x))-f(\xi_{r-}(y))$, and $M_r^{d,i}$ is the i-th component of d-vector process M_t^d. By the mean value theorem, the absolute value of the integrant $\{\cdots\}$ is dominated by

$$\frac{1}{2}p(p-1)|g_f(r)|^2|M_{r-}^d+\theta g_f(r)|^{p-2}\le C_2\{|g_f(r)|^2|M_{r-}^d|^{p-2}+|g_f(r)|^p\}\,,$$

where $|\theta|\le1$. By condition (C, III$_p$), it holds

$$\int|g_f(r)|^{\tilde p}\nu(df)\le C_3|\eta_{r-}|^{\tilde p}$$

for any $2\leqq\tilde{p}\leqq p$. Consequently,

$$E[\sup_{s\leq r\leq t}|M_r^d|^p]\leqq C_4E[|M_t^d|^p] \tag{2.17}$$

$$\leqq C_4'\left\{\int_s^t E[|\eta_{r-}|^p]dr+\int_s^t E[|\eta_{r-}|^2|M_{r-}^d|^{p-2}]dr\right\}.$$

Substitute $M_r^d=\eta_r-(x-y)-M_r^c-A_r$ to the last member of the above and then apply inequalities (2.14) and (2.15). Then we get

$$E[|\eta_{r-}|^2|M_{r-}^d|^{p-2}]\leqq C_5\{|x-y|^{p-2}E[|\eta_{r-}|^2]+E[\sup_{s\leq v\leq r}|\eta_v|^p]\}. \tag{2.18}$$

Substitute the above to (2.17) and then sum up (2.14), (2.15) and (2.17). Then we get

$$E[\sup_{s\leq r\leq t}|\eta_r-(x-y)|^p] \tag{2.19}$$

$$\leqq C_6\left\{|x-y|^{p-2}\int_s^t E[|\eta_{r-}|^2]dr+\int_s^t E[\sup_{s\leq v\leq r}|\eta_v|^p]dr\right\}$$

$$\leqq C_7\left\{(t-s)|x-y|^p+|x-y|^{p-2}\int_s^t E[|\eta_{r-}-(x,\ y)|^2]dr\right.$$

$$\left.+\int_s^t E[\sup_{s\leq v\leq r}|\eta_v-(x-y)|^p]dr\right\}.$$

In case $p=2$, the above inequality is written as

$$E[\sup_{s\leq r\leq t}|\eta_r-(x-y)|^2]\leqq C_7(t-s)|x-y|^2 \tag{2.20}$$

$$+2C_7\int_s^t E[\sup_{s\leq v\leq r}|\eta_v-(x-y)|^2]dr.$$

Gronwall's lemma implies

$$E[\sup_{s\leq r\leq t}|\eta_r-(x-y)|^2]\leqq C_8(t-s)|x-y|^2.$$

Substitute the above to (2.19). Then we get (2.20) replacing the 2-nd power by the p-th power. Then, by Gronwall's inequality, we get (2.11). The proof is complete.

We can now state the main result of this section.

Theorem 2.2. *Let $X_t(x)$ be a C-valued Lévy process with characteristics (a, b, ν, U) satisfying* (C, I), (C, II) *and* (C, III$_p$) *for some $p>d$. Then the solution of equation* (2.7) *has a modification $\xi_{s,t}$ with the following properties.*

(i) *For each s, $\xi_{s,t}$, $t\in[s, T]$ is a right continuous C-valued process with the left hand limits.*

(ii) *For any $0\leqq t_0<t_1<\cdots<t_n$, $\xi_{t_i,t_{i+1}}$, $i=0, \cdots, n-1$ are mutually independent.*

(iii) *For each s, it holds that $\xi_{s,u}=\xi_{t,u}\circ\xi_{s,t}$ a.s. for any $s<t<u$*

Proof. Suppose first that the characteristics (a, b, ν, U) satisfies (C, I), (C, II), (C, III_p), $p>d$ with $U=C$. Then the solution $\xi_{s,t}(x)$ satisfies (2.11) and (2.12). Then by Lemma 1.1, we see that $\xi_{s,t}$ satisfies (i). In case $U\neq C$, the solution of equation (2.7) is written by

$$(2.21)\qquad \xi_{s,t}(x)=\tilde{\xi}_{\sigma_n,t}\circ\phi_{p_{\sigma_n}}\circ\tilde{\xi}_{\sigma_{n-1},\sigma_n}\circ\cdots\circ\phi_{p_{\sigma_1}}\circ\tilde{\xi}_{s,\sigma_1}(x),\qquad \text{if}\quad t\in[\sigma_n,\sigma_{n+1})$$

where $\tilde{\xi}_{s,t}(x)$ is a C-valued process satisfying (2.10) (See the proof of Theorem 2.1). The above $\xi_{s,t}$ is obviously a right continuous C-valued process with the left hand limits.

The second assertion is clear since $\xi_{t_i,t_{i+1}}$ is $\mathcal{F}_{t_i,t_{i+1}}$-measurable and $\mathcal{F}_{t_i,t_{i+1}}$, $i=0,\cdots,n-1$ are mutually independent. Next observe that both of $\xi_{s,u}(x)$ and $\hat{\xi}_{s,u}(x)\equiv\xi_{t,u}\circ\xi_{s,t}(x)$, $u\in[t,T]$ are solutions of equation (2.7) starting at t at $\xi_{s,t}(x)$. Then by the uniqueness theorem, we have $\xi_{s,u}(x)=\hat{\xi}_{s,u}(x)$, for all $u\in[t,T]$. Then the assertion (iii) follows. The proof is complete.

Let us define the product of two elements f, g of $C(R^d;R^d)$ by the composition $f\circ g$ of the maps. Then $C(R^d;R^d)$ becomes a topological semigroup by the topology ρ. We denote the semigroup by G_+. Then the solution $\xi_{s,t}$ defines a *Lévy process* in the semigroup G_+ because of properties (i)~(iii) of Theorem 2.2. The associated C-valued Lévy process X_t is called the *infinitesimal generator* of $\xi_{s,t}$ and $\xi_{s,t}$ is said to be *generated by* the C-valued Lévy process X_t.

2.4. Regularity of the solution with respect to the initial data. In this section, we shall study the smoothness of the solution with respect to the initial value under some regularity assumptions to the characteristics (a, b, ν, U). We denote by G_+^m the sub-semigroup of G_+ consisting of C^m-maps. It is a topological semigroup by the metric ρ_m. Lévy process with values in G_+^m is defined similarly as that with values in G_+.

Theorem 2.3. *Suppose that the characteristics of a C-valued Lévy process satisfy* (C^m, I), (C^m, II) *and* (C^m, III_p) *for some* $p>(m+1)^2d$. *Then the solution* $\xi_{s,t}(x)$ *of equation* (2.7) *has a modification such that it is a* G_+^m*-valued Lévy process. Furthermore, in case* $U=C$, *there is a constant* M *such that*

$$(2.22)\qquad E[\sup_{s\le r\le t}|D^k\xi_{s,r}(x)-D^k\xi_{s,r}(y)|^{p'}]\le M(t-s)|x-y|^{p'},\qquad \forall x,y\in R^d,$$

$$(2.23)\qquad E[\sup_{s\le r\le t}|D^k(\xi_{s,r}(x)-x)|^{p'}]\le M(t-s),\qquad \forall x\in R^d$$

hold for any k *with* $1\le|k|\le m$ *and* $p'\in[2, p/(m+1)^2]$.

We prove the theorem in case $m=1$ mainly. Our argument is based on the following lemma.

Lemma 2.2. *Suppose* $U=C$. *Let* $e_i=(0,\cdots,0,1,0,\cdots,0)$ (1 *is at the i-th component*) *and* $y\in R^1-\{0\}$ *Set*

$$N_{s,t}(x,y)=\frac{1}{y}\{\xi_{s,t}(x+ye_i)-\xi_{s,t}(x)\}.$$

Then there is a positive constant C such that

(2.24) $$E[\sup_{s\le r\le t}|N_{s,r}(x, y)-N_{s,r}(x', y')|^{p'}]\le C\{|x-x'|^{p'}+|y-y'|^{p'}\}(t-s)$$

holds for any $x, x'\in R^d$, $y, y'\in R^1-\{0\}$ *and* $p'\in[2, p/2]$.

Proof. For simplicity, we only prove (2.24) in case $x=x'$. We will write $N_{s,t}(x, y)$ by $N_t(y)$ since s and x are fixed. By inequality (2.11), it holds that for any $p'\in[2, p]$

(2.25) $$E[\sup_{s\le r\le t}|N_r(y)-e_i|^{p'}]\le M(t-s).$$

The process $N_t(y)$ is written by

(2.26) $$N_t(y)=e_i+\int_s^t dZ_r(y)N_r(y)+\int_s^t\int_C g_f(r, y)N_{r-}(y)\tilde{N}_p(dr, df)$$
$$+\int_s^t B_r(y)N_r(y)dr,$$

where $Z_r(y)=\int_0^1 Y_r^{c\prime}(\zeta_r(x, y, v))dv$, $g_f(r, y)=\int_0^1 f'(\zeta_{r-}(x, y, v))dv$, $B_r(y)=\int_0^1 b'(\zeta_r(x, y, v))dv$, and

$$\zeta_r(x, y, v)=\xi_r(x)+v(\xi_r(x+ye_i)-\xi_r(x)).$$

Here $Y_r^{c\prime}(x)$ etc. are $d\times d$-matrix $(\partial Y_r^{c,i}(x)/\partial x_j)$ etc. Let $\eta_t=N_t(y)-N_t(y')$ and $\eta_t=M_t^c+M_t^d+A_t$ (d-vector) be the decomposition such that M_t^c is a continuous martingale, M_t^d is a purely discontinuous martingale and A_t is a process of bounded variation, respectively. Let $M_t^{c,i}$ be the i-th component of M_t^c. Then

$$\langle M^{c,i}\rangle_t\le 2\sum_{k,l}\int_s^t (N_r^k(y)-N_r^k(y'))(N_r^l(y)-N_r^l(y'))a_{kl}^{ii}(y, y)dr$$
$$+2\sum_{k,l}\int_s^t N_r^k(y)N_r^l(y)\{a_{kl}^{ii}(y, y)-2a_{kl}^{ii}(y, y')+a_{kl}^{ii}(y', y')\}dr$$

where

$$a_{kl}^{ii}(y, y')=\int_0^1\int_0^1\left(\frac{\partial^2}{\partial x_k\partial y_l}a^{ii}\right)(\zeta_r(x, y, v), \zeta_r(x, y', v'))dvdv'$$

It holds by (C^m, I) that

$$|a_{kl}^{ii}(y, y'|\le C_1, \qquad \forall y, y'\in R^d,$$
$$|a_{kl}^{ii}(y, y)-2a_{kl}^{ii}(y, y')+a_{kl}^{ii}(y, y')$$
$$\le C_2|\xi_r(x+ye_i)-\xi_r(x+y'e_i)|^2$$

Then by Burkholder's inequality, we have

$$E[\sup_{s\le r\le t}|M_r^c|^{p'}]\le C_3\sum_i E[\langle M^{c,i}\rangle_t^{p'/2}]$$

$$\leqq C_4 E\left[\int_s^t |N_r(y)-N_r(y')|^{p'} dr\right]$$

$$+C_5 E\left[\int_s^t |N_r(y)|^{p'} |\xi_r(x+ye_i)-\xi_r(x+y'e_i)|^{p'} dr\right]$$

The last member is dominated by $C_5'|y-y'|^{p'}(t-s)$ in view of Lemma 2.1 and (2.25). Therefore we have

$$E[\sup_{s\leqq r\leqq t} |M_t^c|^{p'}] \leqq C_4 \int_s^t E[|\eta_r|^{p'}]dr + C_5'|y-y'|^{p'}(t-s).$$

By a similar calculation, we have

$$(2.27)\qquad E[\sup_{s\leqq r\leqq t} |A_r|^{p'}] \leqq C_6 \int_s^t E[|\eta_r|^{p'}]dr + C_7|y-y'|^{p'}(t-s).$$

For the estimate of M_t^d, we proceed as follows. By Itô's formula,

$$(2.28)\qquad |M_t^d|^{p'} = \text{martingale with mean } 0$$

$$+\int_s^t\int_C \{|M_r^d+h_f(r)|^{p'}-|M_r^d|^{p'}-p\sum_i h_f^i(r)|M_r^d|^{p'-2}M_r^{d,i}\}\nu(df)dr,$$

where

$$h_f(r)=g_f(r, y)N_r(y)-g_f(r, y')N_r(y').$$

The integrant of the last member of (2.28) is dominated by

$$C_8\{|h_f(r)|^{p'}+|h_f(r)|^2|M_r^d|^{p'-2}\}$$

as before. Since $\int \|f'(x)\|^{p'}\nu(df)$[†] is bounded in x and $\int_C \|f'(x)-f'(x')\|^{p'}\nu(df)$ $\leqq$const $|x-x'|^{p'}$ by (C^m, III_p), we have

$$(2.29)\qquad \int_C |h_f(r)|^{p'}\nu(df)$$

$$\leqq C_8\left\{|\eta_r|^{p'}\int_C \|g_f(r, y)\|^{p'}\nu(df)+|N_r(y)|^{p'}\int_C \|g_f(r, y)-g_f(r, y')\|^{p'}\nu(df)\right\}$$

$$\leqq C_9\{|\eta_r|^{p'}+|N_r(y)|^{p'}|\xi_r(x+ye_i)-\xi_r(x+y'e_i)|^{p'}\}.$$

The expectation of the above is dominated by $C_{10}\{E[|\eta_r|^{p'}]+|y-y'|^{p'}\}$. On the other hand,

$$(2.30)\qquad E\left[|M_r^d|^{p'-2}\int_C |h_f(r)|^2\nu(df)\right]$$

$$\leqq C_{11}E[(|\eta_r|^{p'-2}+|M_r^c|^{p'-2}+|A_r|^{p'-2})|\eta_r|^2]$$

$$+C_{12}E[(|\eta_r|^{p'-2}+|M_r^c|^{p'-2}+|A_r|^{p'-2})|N_r(y)|^2|\xi_r(x+ye_i)$$

$$-\xi_r(x+y'e_i)|^2].$$

The first member of the right hand side is dominated by $C_{11}'E[\sup_{s\leqq v\leqq r}|\eta_v|^{p'}]$

[†] $\|a\|$ denotes the matrix norm $\Sigma_{i,j}\ |a_{ij}|$.

$+C'_{11}|y-y'|^{p'}$. Obviously $E[|M_t^d|^{p'}]$ is dominated by the sum of the above two, integrating by dr from s to t. Summing up the above calculations, we arrive at

$$E[\sup_{s\leq r\leq t}|\eta_r|^{p'}]\leqq C_{13}|y-y'|^{p'}(t-s)+C_{14}\int_s^t E[\sup_{s\leq v\leq r}|\eta_v|^{p'}]dr \tag{2.31}$$

$$+C_{15}\int_s^t E[|\eta_r|^{p'-2}+|M_c^c|^{p'-2}+|A_r|^{p'-2})$$

$$\times|N_r(y)|^2|\xi_r(x+ye_i)-\xi_r(x+y'e_i)|^2]dr\,.$$

We will prove by induction on p' that

$$E[\sup_{s\leq r\leq t}|\eta_r|^{p'}]\leqq C_{16}|y-y'|^{p'}(t-s)+C_{17}\int_s^t E[\sup_{s\leq v\leq r}|\eta_v|^{p'}]dv \tag{2.32}$$

holds for any $p'\in[2, p/2]$. In case $p'=2$, (2.31) is written as

$$E[\sup_{s\leq r\leq t}|\eta_r|^2]\leqq(C_{13}+C'_{15})|y-y'|^2(t-s)+C_{14}\int_s^t E[\sup_{s\leq v\leq r}|\eta_v|^2]dr$$

which is nothing but (2.32). By Gronwall's inequality, we have $E[\sup|\eta_r|^2]\leqq C_{18}|y-y'|^2(t-s)$. Suppose next $2\leqq p'\leqq 3\wedge p/2$. Let $\tilde{p}=2/(p'-2)$ and $\tilde{q}$ be its conjugate. The last member of (2.31) is dominated by

$$C_{15}\int_s^t (E[|\eta_r|^2]^{1/\tilde{p}}+E[|M_r^c|^2]^{1/\tilde{p}}+E[|A_r|^2]^{1/\tilde{p}})$$

$$\times E(|N_r(y)|^{2\tilde{q}}|\xi_r(x+ye_i)-\xi_r(x+y'e_i)|^{2\tilde{q}})^{1/\tilde{q}}dr$$

$$\leqq C_{19}\left(\int_s^t E[|\eta_r|^2]^{(p'/2)-1}dr\right)|y-y'|^2\leqq C_{20}|y-y'|^{p'}(t-s)\,.$$

Therefore we get (2.32) in case $2\leqq p'\leqq 3\wedge p/2$.

Consider next $3\wedge p/2\leqq p'\leqq 4\wedge p/2$. Let $\tilde{p}=3/(p'-2)$. Then we can show similarly as the above that the last member of (2.31) is dominated by const. $\times|y-y'|^{p'}(t-s)$. Therefore we get (2.32) in case $3\wedge p/2\leqq p'\leqq 4\wedge p/2$. Repeating this argument inductively, we get (2.32) for any p' less than $p/2$. The proof is complete.

Proof of Theorem 2.3. It is enough to prove the theorem in case $U=C$. By Kolmogorov's theorem, $\partial_i\xi_{s,t}(x)=\lim_{y\to 0}N_t(x, y)$ exists for any t, x, a.s. It satisfies (2.22) with $D^k=\partial_i$ by (2.24) and satisfies (2.23) by (2.25). Therefore $\xi_{s,t}$ is a right continuous C^1 valued process.

Now, let y tend to 0 in (2.26). Then we see that the Jacobian matrix $\partial\xi_{s,t}(x)=(\partial_i\xi_{s,t}(x))$, $i=1, \cdots, d$ satisfies

$$\partial\xi_{s,t}(x)=I+\int_s^t dX_r^{c'}(\xi_{s,r-}(x))\partial\xi_{s,r-}(x)$$

$$+\int_s^t\int_C f'(\xi_{s,r-}(x))\partial\xi_{s,r-}(x)\tilde{N}_p(dr, df)\,.$$

Therefore, the pair $(\xi_{s,t}(x), \partial\xi_{s,t}(x))$ satisfies a closed system of SDE. Apply the same argument to the pair process. Then we see that $\partial\xi_{s,t}(x)$ is a C^1-function of x and in fact that $\xi_{s,t}$ is a right continuous process with values in $C^2(R^d; R^d)$. Repeating the argument inductively on m, we get the assertion of the theorem.

2.5. Homeomorphic property of the solution. We have seen in Section 2.3 that the solution of SDE (2.7) defines a Lévy process with values in the semigroup G_+ of continuous maps. In this section, we shall discuss the case that the maps $\xi_{s,t}: R^d \to R^d$ become homeomorphisms a.s. In case of continuous SDE, i.e., $\nu(C)\equiv 0$, it is known that the maps $\xi_{s,t}$ are homeomorphisms. (See Kunita [6], Bismut [2]). However, in case of discontinuous SDE, additional requirements have to be made to the intensity measure ν of the Poisson point process.

We denote by G the totality of homeomorphisms of R^d. It is a subgroup of G_+, and is a topological group by the metric $d(f, g)=\rho(f, g)+\rho(f^{-1}, g^{-1})$ where ρ is the compact uniform metric on C. However, we will not use the metric d in this paper, but use the metric ρ. The definition of the G-valued Lévy process is similar to that of G_+-valued Lévy process.

We first consider the case that the intensity measure ν of the Poisson point process is a finite measure.

Theorem 2.4. *Let $X_t(x)$ be a C-valued Lévy process satisfying* (C, I), (C, II) *and* (C, III$_p$) *for some $p>d$. Suppose the following.*

(C, IV) *The intensity measure ν is finite and is supported by f such that $\phi_f\equiv f+id\in G$.*

Then the solution of SDE (2.7) *defines a G-valued Lévy process.*

Proof. By the condition (C, IV), the equation (2.7) is written by

$$d\xi_t=dX_t^c(\xi_{t-})+\int_C f(\xi_{t-})N_p(dt, df), \tag{2.33}$$

where X_t^c is a Brownian motion with values in C. Now let $\tilde{\xi}_{s,t}(x)$ be the solution of the equation $d\tilde{\xi}_t=dX_t^c(\tilde{\xi}_{t-})$ starting at (s, x). Then it is a G-valued Brownian motion. The solution of equation (2.33) is written as (2.21). Since $\phi_{p_{\sigma_k}}$ are all homeomorphisms by hypothesis (C, IV), the solution $\xi_{s,t}$ defines the homeomorphisms a.s. Therfore, it is a G-valued Lévy process.

In case that the intensity measure ν is σ-finite, we require additional regularity conditions to ν. Let us introduce a Lipschitz norm:

$$\|f\|=\sup_x\frac{|f(x)|}{1+|x|}+\sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|}$$

and introduce an assumption to ν.

(C, V) $\phi_f\equiv f+id$ are homeomorphisms a.s. ν. ν satisfies

$$\int_C\frac{\|f\|^2}{1+\|f\|^2}\nu(df)<\infty. \tag{2.34}$$

Note that (C, V) implies (C, III$_p$) for $p\geqq 2$. Indeed, setting $U=\{f; \|f\|<1\}$

we have $\nu(C-U)<\infty$ and $\int_U \|f\|^p \nu(df)<\infty$ for any $p\geqq 2$ by (2.34). The latter property implies $(\mathrm{C}, \mathrm{III}_p)$.

Theorem 2.5. *Assume* (C, I), (C, II) *and* (C, V). *The solution of equation* (2.7) *defines a G-valued Lévy process.*

The proof is based on the following L^p-estimates.

Theorem 2.6. *In addition to* (C, I), (C, II) *and* (C, V), *assume further that*

$$\operatorname*{ess\text{-}sup}_f (\|f\|+\|\hat{f}\|)<+\infty \quad \textit{with respect to } \nu, \tag{2.35}$$

where $\hat{f}=\phi_f^{-1}-id$. *Then, for any* $p\geqq 2$, *there is a positive constant* M_p *such that*

$$E[\sup_{s\leqq r\leqq t} |\xi_{s,r}(x)-\xi_{s,r}(y)|^{-p}]\leqq M_p|x-y|^{-p}, \qquad \forall x, y\in R^d, \tag{2.36}$$

$$E[\sup_{s\leqq r\leqq t} (1+|\xi_{s,r}(x)|)^{-p}]\leqq M_p(1+|x|)^{-p}, \qquad \forall x\in R^d. \tag{2.37}$$

Remark. In case of continuous SDE, inequalities (2.36) and (2.37) without "sup" in the expectations have been obtained in [6].

We give the proof of (2.36) only, since the proof of (2.37) is similar. For the proof of (2.36), we require a lemma.

Lemma 2.4. *Suppose conditions* (C, V) *and* (2.35). *Let* δ *be any positive number. There are positive constants* C_1 *and* C_2 *not depending on* δ *such that*

$$\begin{aligned}\int_C |(\delta+|\phi_f(x)-\phi_f(y)|^2)^{-1}-(\delta+|x-y|^2)^{-1} \\ +\sum_i (f^i(x)-f^i(y))(x^i-y^i)(\delta+|x-y|^2)^{-2}|\nu(df) \\ \leqq C_1(\delta+|x-y|^2)^{-1}.\end{aligned} \tag{2.38}$$

$$\int_C |(\delta+|\phi_f(x)-\phi_f(y)|^2)^{-1}-(\delta+|x-y|^2)^{-1}|^{p/2}\nu(df)\leqq C_2(\delta+|x-y|^2)^{-p/2}. \tag{2.39}$$

Proof. We will prove (2.38). Let $0<\varepsilon<1$ and set $U_\varepsilon=\{f\,|\,\|f\|<\varepsilon\}$. Consider the integrand of (2.38) in case $f\notin U_\varepsilon$. Since $|\phi_f^{-1}(x)-\phi_f^{-1}(y)|\leqq\|\phi_f^{-1}\|\,|x-y|$, we have $|x-y|\leqq\|\phi_f^{-1}\|\,|\phi_f(x)-\phi_f(y)|$. Therefore,

$$(\delta+|\phi_f(x)-\phi_f(y)|^2)^{-1}\leqq(1+\|\phi_f^{-1}\|^2)(\delta+|x-y|^2)^{-1}.$$

Also, note that $|f(x)-f(y)|\leqq\|f\|\,|x-y|$. Then the integrand of (2.38) is dominated by

$$(1+\|\phi_f^{-1}\|^2+\|f\|)(\delta+|x-y|^2)^{-1}, \qquad f\in C-U_\varepsilon. \tag{2.40}$$

We next estimate the case $f\in U_\varepsilon$. By the mean value theorem, the integrand of (2.38) is dominated by

$$C_3\{|x-y|^2|f(x)-f(y)|^2(\delta+|(x-y)+\theta(f(x)-f(y))|^2)^{-3}\}\,, \tag{2.41}$$

where θ is a number such that $|\theta|\leqq 1$. Since $f\in U_\varepsilon$, we have

$$|(x-y)+\theta(f(x)-f(y))|\geqq(1-\|f\|)|x-y|\geqq(1-\varepsilon)|x-y|\,.$$

Therefore (2.41) is dominated by

$$C_4(1-\varepsilon)^{-6}\|f\|^2(\delta+|x-y|^2)^{-1}\,,\qquad f\in U_\varepsilon\,. \tag{2.42}$$

Integrating (2.40) on $C-U_\varepsilon$ and (2.42) on U_ε by the measure ν, we get (2.38). The proof of (2.39) is done similarly.

The inequality (2.36) of Theorem 2.6 follows from the following lemma.

Lemma 2.5. *Suppose* (C, V) *and* (2.35). *Then it holds*

$$E[\sup_t(\delta+|\xi_{s,t}(x)-\xi_{s,t}(y)|^2)^{-p/2}]\leqq C_5(\delta+|x-y|^2)^{-p/2}\,,\qquad \forall x,\ y\in R^d \tag{2.43}$$

where the constant C_5 *does not depend on* δ.

Proof. Set $\eta_t=\xi_t(x)-\xi_t(y)$, Then by Itô's formula,

$$\begin{aligned}(\delta+|\eta_t|^2)^{-1}&=(\delta+|x-y|^2)^{-1}\\
&\quad-2\sum_i\int_s^t(\delta+|\eta_r|^2)^{-2}\eta_r^i\{dY_r^{c,i}(\xi_r(r))-dY_r^{c,i}(\xi_r(y))\}\\
&\quad+\int_s^t\int_C\{(\delta+|\eta_{r-}+g_f(r)|^2)^{-1}-(\delta+|\eta_{r-}|^2)^{-1}\}\tilde N_p(dr,\,df)\\
&\quad-2\sum_i\int_s^t(\delta+|\eta_r|^2)^{-2}\eta_r^i(b^i(\xi_r(x))-b^i(\xi_r(y)))dr\\
&\quad-\sum_{i,j}\int_s^t(\delta+|\eta_r|^2)^{-3}\{(\delta+|\eta_r|^2)\delta_{ij}-4\eta_r^i\eta_r^j\}\\
&\quad\times\{a^{ij}(\xi_r(x),\xi_r(x))-2a^{ij}(\xi_r(x),\xi_r(y))+a^{ij}(\xi_r(y),\xi_r(y))\}\,dr\\
&\quad+\int_s^t\int_C\{(\delta+|\eta_{r-}+g_f(r)|^2)^{-1}-(\delta+|\eta_{r-}|^2)^{-1}\\
&\quad+2\sum_i g_f^i(r)(\delta+|\eta_{r-}|^2)^{-2}\eta_{r-}^i\}\,\nu(df)dr\\
&=(\delta+|x-y|^2)^{-1}+M_t^c+M_t^d+A_t^1+A_t^2+A_t^3\,.\end{aligned}$$

We shall compute the $p/2$-th moments of $\sup_t|M_t^c|$ etc. Similarly as the estimate of (2.13), we can easily show

$$E[\sup_r|A_r^1|^{p/2}+\sup_r|A_r^2|^{p/2}]\leqq C_6\int_s^t E[\sup_{s\leq v\leq r}(\delta+|\eta_v|^2)^{-p/2}]dr\,. \tag{2.44}$$

For the estimate of A_t^3, we shall apply Lemma 2.4. By (2.38), we have

$$\sup_{s\leq r\leq t}|A_r^3|\leqq C_1\int_s^t(\delta+|\eta_r|^2)^{-1}dr\,.$$

Therefore we have

$$E[\sup_{s\leq r\leq t}|A_r^3|^{p/2}]\leqq C_1'\int_s^t E[\sup_{s\leq v\leq r}(\delta+|\eta_v|^2)^{-p/2}]dr. \tag{2.45}$$

Next, by Burkholder's inequality etc.,

$$\begin{aligned} E[\sup_r|M_r^c|^{p/2}] &\leqq (q)^{p/2}E[|M_t^c|^{p/2}]\leqq C_7E[\langle M^c\rangle_t^{p/4}] \\ &\leqq C_7E\Big[\Big(\int_s^t(\delta+|\eta_r|^2)^{-4}|\eta_r|^2\|a(\xi_r(x),\ \xi_r(x)) \\ &\quad -2a(\xi_r(x),\ \xi_r(y))+a(\xi_r(y),\ \xi_r(y))\|dr\Big)^{p/4}\Big] \\ &\leqq C_8\int_s^t E[\sup_{s\leq v\leq r}(\delta+|\eta_v|^2)^{-p/2}]dr. \end{aligned} \tag{2.46}$$

where q is the conjugate of $p/2$.

For the computation of $|M_t^d|^{p/2}$, we apply Itô's formula. It holds

$$\begin{aligned} |M_t^d|^{p/2} &= \text{a martingale with mean } 0 \\ &\quad +\int_s^t\int_C\Big\{|M_r^d+h_f(r)|^{p/2}-|M_r^d|^{p/2} \\ &\quad -\frac{p}{2}h_f(r)|M_r^d|^{(p/2)-1}\operatorname{sign}(M_r^d)\Big\}\nu(df)dr, \end{aligned} \tag{2.47}$$

where $h_f(r)=(\delta+|\eta_r+g_f(r)|^2)^{-1}-(\delta+|\eta_r|^2)^{-1}$. The integrant of the above is dominated by $C_9(|h_f(r)|^{p/2}+|h_f(r)|^2|M_r^d|^{(p/2)-2})$. From (2.39), we have

$$\int|h_f(r)|^{p'/2}\nu(df)\leqq C_2(\delta+|\eta_r|^2)^{-p'/2}$$

for $p'/2\in[2,\ p]$. Therefore we get

$$\begin{aligned} E[\sup_{s\leq r\leq t}|M^d|^{p/2}] &\leqq q^{p/2}E[|M_t^d|^{p/2}] \\ &\leqq C_{10}\int_s^t\{E[(\delta+|\eta_r|^2)^{-p/2}]+E[(\delta+|\eta_r|^2)^{-2}|M_r^d|^{(p/2)-2}]\}\,dr. \end{aligned}$$

Substitute $M_r^d=(\delta+|\eta_r|^2)^{-1}-(\delta+|x-y|^2)^{-1}-M_r^c-A_r^1-A_r^2-A_r^3$ to the last member of the above and then apply (2.44)—(2.46). Then

$$\begin{aligned} E[\sup_r|M_r^d|^{p/2}] &\leqq C_{11}\int_s^t E[\sup_{s\leq v\leq r}(\delta+|\eta_v|^2)^{-p/2}]dr \\ &\quad +C_{12}(\delta+|x-y|^2)^{(-p/2)+2}\int_s^t E[\sup_{s\leq v\leq r}(\delta+|\eta_v|^2)^{-2}]dr. \end{aligned} \tag{2.48}$$

Summing up (2.44), (2.45), (2.46) and (2.48), we obtain

$$\begin{aligned} E[\sup_{s\leq r\leq t}(\delta+|\eta_r|^2)^{-p/2}] &\leqq C_{13}(\delta+|x-y|^2)^{-p/2} \\ &\quad +C_{14}(\delta+|x-y|^2)^{(-p/2)+2}\int_s^t E[\sup_{s\leq v\leq r}(\delta+|\eta_v|^2)^{-2}]dr \end{aligned}$$

$$+C_{15}\int_s^t E[\sup_{s\le v\le r}(\delta+|\eta_v|^2)^{-p/2}]dr\,.$$

Then we get the desired inequality (2.43), similarly as the proof of (2.11).

Proof of Theorem 2.5. Assume first that conditions of Theorem 2.6 is satisfied. The one to one and onto property of the map $\xi_{s,t}$ for each $s<t$ can be proved similary as [6]. Indeed, inequalities (2.11) and (2.36) imply that $\sup_{r\in[s,t]}|\xi_{s,r}(x)-\xi_{s,r}(y)|^{-1}$ is continuous in $(x, y)\in R^d\times R^d-\{(x, x)|x\in R^d\}$. This proves that the map $\xi_{s,t}\,;\,R^d\to R^d$ is one to one for all $t\in[s, T]$. Next, inequality (2.37) implies $\varliminf_{|x|\to\infty}\inf_{t\in[s,T]}|\xi_{s,t}(x)|=\infty$ a.s. Then $\xi_{s,t}$ is extended to a continuous map from $\hat{R}^d$ (=one point compactification of R^d) onto itself all for all $t\in[s, T]$, such that ∞ is the invariant point. Then we see that $\xi_{s,t}\,;\,R^d\to R^d$ is onto and the inverse $\xi_{s,t}^{-1}(x)$ is continuous in x for all $t\in[s, T]$. a.s.

We next consider the general case. Set $U=\{f\,|\,\|f\|<1/2\}$ and consider the SDE

$$d\tilde{\xi}_t=dX_t^c(\tilde{\xi}_{t-})+\int_U f(\tilde{\xi}_{t-})\tilde{N}_p(dt, df)\,. \tag{2.49}$$

It satisfies condition (C, III_p) for $p\geqq 2$. Furthermore, since $|\phi_f^{-1}(x)-\phi_f^{-1}(y)-x+y|\leqq\|f\|\,|\phi_f^{-1}(x)-\phi_f^{-1}(y)|$, we have $|\phi_f^{-1}(x)-\phi_f^{-1}(y)|\leqq 2|x-y|$ if $f\in U$. Similarly, we have $|\phi_f^{-1}(x)|\leqq 2(1+|x|)$ if $f\in U$. Therefore condition (2.35) is satisfied. Consequently the solution of (2.49) defines for each s, a right continuous G-valued process $\xi_{s,t}$, $t\in[s, T]$. Now let $s<\sigma_1<\sigma_2<\cdots$ be jumping times of Poisson process $N_p(t, C-U)$, $t\geqq s$. Define $\xi_{s,t}(x)$ by

$$\xi_{s,t}(x)=\tilde{\xi}_{\sigma_n,t}\circ\phi_{p(\sigma_{n-1})}\circ\cdots\circ\phi_{p(\sigma_1)}\circ\tilde{\xi}_{s,\sigma_1}(x) \qquad \text{if} \quad \sigma_n\leqq t<\sigma_{n+1}\,. \tag{2.50}$$

Then we can prove similarly as Theorem 2.1 that for each s, $\xi_{s,t}$, $t\in[s, T]$ is a solution of equation (2.7) and defines a right continuous G-valued process with the left hand limits. The proof is complete.

§ 3. Representation of Lévy process with values in the semigroup of continuous maps.

In the previous section, we have seen that the solution of a stochastic differential equation defined by C-valued Lévy process defines a G_+ (or G)-valued Lévy process provided that its characteristics satisfy (C, I), (C, II) and (C, III_p), $p>d$ ((C, IV) or (C, V), resp.). In this section, we shall show conversely that a Lévy process with values in G_+ or G can be represented as a solution of a stochastic differential equation defined by C-valued Lévy processes. Our problem is thus to find the infinitesimal generator of G_+-valued Lévy process.

3.1. Main result. We begin with introducing some hypotheses to G_+-valued Lévy process $\xi_{s,t}$. We always assume that $\xi_{s,t}$ is stationary.

(ξ, I) $\xi_{s,t}(x)$ is square integrable for any $s<t$ and $x\in R^d$. The limit

(3.1) $$A^{ij}(x, y)=\lim_{h\to 0}\frac{1}{h}E[(\xi_{s,s+h}^{i}(x)-x_{i})(\xi_{s,s+h}^{j}(y)-y_{j})]$$

exists for any $x, y\in R^{d}$ and $i, j=1, \cdots, d$.

(ξ, II) The limit

(3.2) $$b^{i}(x)=\lim_{h\to 0}\frac{1}{h}\{E[\xi_{s,s+h}^{i}(x)]-x_{i}\}$$

exists for any $x\in R^{d}$ and $i=1, \cdots, d$.

Let p be a positive number greater than or equal to 2.

(ξ, III_{p}) There is a positive constant M such that

(3.3) $$|E[\xi_{s,t}(x)-x-(\xi_{s,t}(y)-y)]|\leqq M(t-s)|x-y|,$$

(3.4) $$E[|\xi_{s,t}(x)-x-(\xi_{s,t}(y)-y)|^{p'}]\leqq M(t-s)|x-y|^{p'},$$

(3.5) $$E[|\xi_{s,t}(x)-x|^{p'}]\leqq M(t-s)(1+|x|)^{p'}$$

holds for any $x, y\in R^{d}$ and $p'\in[2, p]$.

Remark. Condition (ξ, III_{2}) $(p=2)$ implies that $A^{ij}(x, y)$ is *bi*-Lipschitz continuous and $b^{i}(x)$ is Lipschitz continuous.

Theorem 3.1. *Let $\xi_{s,t}(x)$ be a stationary Lévy process with values in G_{+}. Suppose that it satisfies (ξ, I), (ξ, II) and (ξ, III_{p}) for some $p>d$. Then it is generated by a unique C-valued Lévy process X_{t} satisfying conditions* (1.17) *and* (1.18) *of Theorem* 1.2.

The proof will be given at Section 3.2.

It should be noted that hypotheses (ξ, I), (ξ, II) and (ξ, III_{p}), $p>d$ are satisfied for any G_{+}-valued Lévy process which is generated by a C-valued Lévy process satisfying (1.17) and (1.18). Indeed, let X_{t} be a C-valued Lévy process satisfying (1.17) and (1.18). Then it satisfies (C, I), $(\mathrm{C}, \mathrm{II})$ and $(\mathrm{C}, \mathrm{III}_{p})$ with $U=C$ by Theorem 1.2. Let $X_{t}(x)=Y_{t}(x)+b(x)t$ be the decomposition such that $Y_{t}(x)$ is of zero-mean. Then the G_{+}-valued Lévy process generated by X_{t} satisfies

$$E[\xi_{s,t}(x)]-x=E\left[\int_{s}^{t}b(\xi_{s,r}(x))dr\right].$$

The property (ξ, II) and (3.3) follow from the above. Furthermore, the relation

$$E\left[\left(\xi_{s,s+h}^{i}(x)-x_{i}-\int_{s}^{s+h}b^{i}(\xi_{s,r}(x))dr\right)\left(\xi_{s,s+h}^{j}(y)-y_{j}-\int_{s}^{s+h}b^{j}(\xi_{s,r}(y))dr\right)\right]$$

$$=E\left[\int_{s}^{s+h}dY_{r}^{i}(\xi_{s,r-}(x))\int_{s}^{s+h}dY_{r}^{j}(\xi_{s,r-}(y))\right]$$

$$=E\left[\int_{s}^{s+h}A^{ij}(\xi_{s,r}(x), \xi_{s,r}(y))dr\right]$$

implies (ξ, I). Hypothesis (ξ, III_{p}) is verified in Lemma 2.1. As a consequence, we have the following.

Corollary. *A necessary and sufficient condition that $\xi_{s,t}$ is a G_+-valued Lévy process satisfying hypotheses* (ξ, I), (ξ, II) *and* (ξ, III_p), $p>d$ *is that it is generated by a C-valued Lévy process satisfying* (1.17) *and* (1.18).

Remark. There are some gaps between the class of G-valued Lévy process constructed in Theorem 2.5 and the class of G-valued Lévy process treated in Theorem 3.1. In particular, the condition (C, V) seems to be stringent, since it is hardly checked whether the generator of a given G-valued Lévy process has the property (C, V) or not. The problem of filling out these gaps will be discussed elsewhere.

We will next consider the case of smoothness property. Let $\xi_{s,t}$ be a G_+^m-valued Lévy process. We introduce the following hypotheses.

(ξ^m, I) $A^{ij}(x, y)$ defined by (3.1) is m-times differentiable in both x and y. Furthermore, $D_x^k D_y^k A(x, y)$ is *bi*-Lipschitz continuous for any k with $|k| \leqq m$.

(ξ^m, II) $b^i(x)$ defined by (3.2) is a C^m-function and $D^k b^i(x)$ is Lipschitz continuous for any k with $|k| \leqq m$.

(ξ^m, III_p) There is a positive constant M such that (3.5) and the following inequalities hold for any k with $1 \leqq |k| \leqq m$ and $p' \in [2, p]$.

(3.6) $$E[|D^k(\xi_{s,t}(x)-x)-D^k(\xi_{s,t}(y)-y)|^{p'}] \leqq M(t-s)|x-y|^{p'}, \qquad \forall x, y \in R^d$$

(3.7) $$E[|D^k(\xi_{s,t}(x)-x)|^{p'}] \leqq M(t-s), \qquad \forall x \in R^d$$

Theorem 3.2. *Suppose that $\xi_{s,t}$ is a G_+^m-valued Lévy process satisfying* (ξ^m, I), (ξ^m, II) *and* (ξ^m, III_p) *for some* $p>(m+1)^2 d$. *Then it is generated by a uniqne C^m-valued Lévy process $X_t(x)$ whose characteristics satisfy* $(\mathrm{C}^m, \mathrm{I})$, $(\mathrm{C}^m, \mathrm{II})$ *and* $(\mathrm{C}^m, \mathrm{III}_{p'})$ *with* $U=C$ *for* $p' \leqq p/(m+1)^2$.

The proof will be given at Section 3.3.

Combining the above theorem with Theorem 1.3 and Theorem 2.3 we obtain the following corollary.

Corollary. *A necessary and sufficient condition that $\xi_{s,t}$ is a G_+^m-valued Lévy process satisfying* (ξ^m, I), (ξ^m, II) *and* (ξ^m, III_p), *for any* $p>d$ *is that it is generated by a C^m-valued Lévy prscess whsse characteristics satisfy* $(\mathrm{C}^m, \mathrm{I})$, $(\mathrm{C}^m, \mathrm{II})$ *and* $(\mathrm{C}^m, \mathrm{III}_p)$ *with* $U=C$ *for any* $p>d$.

Remark. There is a one to one correspondence between the following three.

(a) G_+-valued (resp. G_+^m-valued) Lévy process satisfying (ξ, I), (ξ, II), (ξ, III_p), for some $p>d$ (resp. (ξ^m, I), (ξ^m, II), (ξ^m, III_p) for any $p>d$).

(b) C-valued (resp. C^m-valued) Lévy process satisfying (1.17) and (1.18) (resp. (1.24) and (1.25) for any k with $|k|\leqq m$) for some (any) $p>d$.

(c) The system (a, b, ν, U) with $U=C$ satisfying (1.1), (1.3), (1.4) and (C, I), (C, II), (C, III_p), $p>d$ (resp. (C^m, I), (C^m, II) and (C^m, III_p), for any $p>d$).

Finally we restrict our attention to G_+-valued Brownian motion. In this case, we can replace the condition (ξ, III_p), $p>(m+1)^2d$ to a weaker condition (ξ, III_2).

Theorem 3.3. *Let $\xi_{s,t}$ be a G_+-valued Brownian motion satisfying* (ξ, I), (ξ, II) *and* (ξ, III_2). *Then it is generated by a unique C-valued Brownian motion X_t, whose characteristics satisfy* (C, I) *and* (C, II).

Suppose further that $\xi_{s,t}$ satisfies (ξ^m, I), (ξ^m, II) *and* (ξ^m, III_2). *Then X_t is a C^m-valued Brownian motion whose characteristics satisfy* (C^m, I) *and* (C^m, II).

Note that the G_+-valued Brownian motion of Theorem 3.3 is actually a G-valued Brownian motion, owing to the above theorem and Theorem 2.4. Therefore we have the following.

Corollary. *A necessary and sufficient condition that $\xi_{s,t}$ is a G-valued (resp. G^m-valued) Brownian motion satisfying* (ξ, I), (ξ, II) *and* (ξ, III_2) *(resp.* (ξ^m, I), (ξ^m, II), (ξ^m, III_2)) *is that it is generated by a unique C-valued (resp. C^m-valued) Brownian motion whose characteristics satisfy* (C, I) *and* (C, II) *(resp.* (C^m, I) *and* (C^m, II)).

Remark. There is a one to one correspondence between the following three.

(a) G-valued (resp. G^m-valued) Brownian motion satisfying (ξ, I), (ξ, II), (ξ, III_2) (resp. (ξ^m, I), (ξ^m, II), (ξ^m, III_2)).

(b) C-valued (resp. C^m-valued) Brownian motion satisfying (1.17′) and (1.18′) (resp. (1.24′) and (1.25′)).

(c) The pair (a, b) satisfying (1.1), (1.3) and (C, I), (C, II) (resp. (C^m, I), (C^m, II)).

Remark. The construction of C-valued Brownian motion from G-valued Brownian motion was shown by Baxendale [1] and Le Jan [9] in different contexts. The reproducing kernel Hilbert space associated with $a(x, y)$ studied in [1] is isomorphic to the Gaussian space generated by Gaussian random variables $X_t(x)-b(x)t$, $x\in R^d$ where t is fixed. The Gaussian random field $W(x, t)$ inroduced in [9] has the same law as that of $X_t(x)-b(x)t$. However, $W(x, t)$ and $\xi_{s,t}(x)$ are not directly related by SDE, but the solution governed by $W(x, t)$ has the same law as that of the given $\xi_{s,t}$.

A result analogous to Theorem 3.3 was obtained by Kunita [7] assuming an additional condition to the σ-field $\mathcal{F}_{s,t}=\sigma(\xi_{u,v}\,;\, s\leqq u\leqq v\leqq t)$.

3.2. Proof of Theorem 3.1. Since the proof of Theorem 3.1 is long, we shall first present a loose idea of the proof. The principal part is the construction of C-valued Lévy process. Suppose for a moment that we could have constructed such X_t. Then it holds

$$\xi_{s,t}(x)-x-\int_s^t b(\xi_{s,r}(x))dr=\int_s^t dY_r(\xi_{s,r-}(x)),$$

where $Y_t(x)=X_t(x)-b(x)t$. Both of the above are martingales, which we denote by $M_{s,t}(x)$. Now in case where $\xi_{s,t}$ has the inverse map $\xi_{s,t}^{-1}$, $Y_t(x)$ could be obtained by

$$\begin{aligned} Y_t(x)&=\int_s^t dM_{s,r}(\xi_{s,r-}^{-1}(x)) \\ &=\lim_{|\Delta|\to 0}\sum_k \{M_{s,t_{k+1}}(\xi_{s,t_k}^{-1}(x))-M_{s,t_k}(\xi_{s,t_k}^{-1}(x))\}, \end{aligned}$$

where $\Delta=\{s=t_0<\cdots<t_n=t\}$ are partitions and $|\Delta|=\max(t_{i+1}-t_i)$. Note that $M_{s,t}(x)$ has the additive property $M_{s,t_{k+1}}(x)=M_{s,t_k}(x)+M_{t_k,t_{k+1}}(\xi_{s,t_k}(x))$. Then $Y_t(x)$ could be written as

$$Y_t(x)=\lim_{|\Delta|\to 0}\sum_k M_{t_k,t_{k+1}}(x).$$

We shall carry out the idea rigorously. We first consider the martingale $M_{s,t}$. Set $\mathcal{F}_{s,t}=\sigma(\xi_{u,v}; s\leqq u\leqq v\leqq t)$ and define

$$M_{s,t}^i(x)\equiv\xi_{s,t}^i(x)-x_i-\int_s^t b^i(\xi_{s,r}(x))dr. \tag{3.8}$$

Lemma 3.1. *Suppose* (ξ, I), (ξ, II) *and* (ξ, III_2). *Then for each* s *and* x, $(M_{s,t}^i(x), \mathcal{F}_{s,t})$, $t\in[s, T]$ *is an* L^2*-martingale with zero-mean. Furthermore,*

$$\langle M_{s,t}^i(x), M_{s,t}^j(y)\rangle=\int_s^t A^{ij}(\xi_{s,r}(x), \xi_{s,r}(y))dr. \tag{3.9}$$

Proof. Set $m_{s,t}(x)=E[\xi_{s,t}(x)]-x$. By the multiplicative property $\xi_{s,u}=\xi_{t,u}\circ\xi_{s,t}$ for $s<t<u$, we have $m_{s,u}(x)=m_{s,t}(x)+T_{s,t}m_{t,u}(x)$, where $T_{s,t}f(x)=E[f(\xi_{s,t}(x))]$. By (3.3), it holds $|(1/h)m_{t,t+h}(x)|\leqq M(1+|x|)$ and $(1+|x|)$ is integrable relative to $T_{s,t}(\cdot, dx)=P(\xi_{s,t}(\cdot)\in dx)$. Therefore

$$\frac{\partial}{\partial t}m_{s,t}(x)=\lim_{h\to 0}T_{s,t}\Big(\frac{m_{t,t+h}}{h}\Big)(x)=T_{s,t}b(x).$$

Integrating the above by t, we get $m_{s,t}(x)=\int_s^t T_{s,r}b(x)dr$. Note that $\xi_{t,u}$ and $\mathcal{F}_{s,t}$ are independent. Then we have

$$\begin{aligned} E[M_{s,u}^i(x)-M_{s,t}^i(x)\,|\,\mathcal{F}_{s,t}]&=E\Big[\xi_{t,u}^i\circ\xi_{s,t}(x)-\xi_{s,t}^i(x)-\int_t^u b^i(\xi_{t,r}\circ\xi_{s,t}(x))dr\,|\,\mathcal{F}_{s,t}\Big] \\ &=m_{t,u}^i(\xi_{s,t}(x))-\int_t^u T_{t,r}b^i(\xi_{s,t}(x))dr \\ &=0. \end{aligned}$$

Therefore $M_{s,t}^i(x)$ is a martingale with zero-mean.

Consider next

$$V_{s,t}^{ij}(x, y)=E[M_{s,t}^{i}(x)M_{s,t}^{j}(y)].$$

Using the relation $M_{s,u}(x)=M_{s,t}(x)+M_{t,u}(\xi_{s,t}(x))$ and the martingale propery, we have

$$V_{s,u}^{ij}(x, y)=V_{s,t}^{ij}(x, y)+T_{s,t}^{(2)}V_{t,u}^{ij}(x, y),$$

where $T_{s,t}^{(2)}f(x, y)=E[f(\xi_{s,t}(x), \xi_{s,t}(y))]$. Then we have

$$\frac{\partial}{\partial t}V_{s,t}^{ij}(x, y)=\lim_{h\to 0}T_{s,t}^{(2)}\Big(\frac{V_{t,t+h}^{ij}}{h}\Big)(x, y)=T_{s,t}^{(2)}A^{ij}(x, y).$$

The change of the order of lim and integration by $T_{s,t}^{(2)}$ is verified by (3.4). Integrating the above by t, we get $V_{s,t}^{ij}(x, y)=\int_s^t T_{s,r}^{(2)}A^{ij}(x, y)dr$. Then we can prove that

$$M_{s,t}^{i}(x)M_{s,t}^{j}(y)-\int_s^t A^{ij}(\xi_{s,r}(x), \xi_{s,r}(y))dr$$

is a martingale with zero-mean, similarly as the case of $M_{s,t}^{i}(x)$. This proves the second assertion of the lemma.

Now, given a partition $\delta=\{0=t_0<t_1<\cdots<t_n=T\}$, we define

$$Y_t^{\delta}(x)=\sum_{k=0}^{n-1}M_{t_k\wedge t,\,t_{k+1}\wedge t}(x). \tag{3.10}$$

Lemma 3.2. *Suppose the same condition as in Lemma 3.1. For each s and x, $Y_{s,t}^{\delta}(x)=Y_t^{\delta}(x)-Y_s^{\delta}(x)$ is a martingale adapted to $\mathcal{F}_{s,t}$. It converges in L^2-sense as $|\delta|\to 0$. Furthermore, the limiting process denoted by $Y_{s,t}(x)=(Y_{s,t}^1(x), \cdots, Y_{s,t}^d(x))$ satisfies*

$$\langle Y_{s,t}^i(x), Y_{s,t}^j(y)\rangle=A^{ij}(x, y)(t-s), \tag{3.11}$$

$$\langle Y_{s,t}^i(x), M_{s,t}^j(y)\rangle=\int_s^t A^{ij}(x, \xi_{s,r}(y))dr. \tag{3.12}$$

Proof. The martingale property of $Y_{s,t}^{\delta}(x)$ is immediate from that of $M_{s,t}(x)$. Let $Y_{s,t}^{\delta,i}(x)$ be the i-th component of $Y_t^{\delta}(x)$. Then we have from (3.9)

$$\langle Y_{s,t}^{\delta,i}(x), Y_{s,t}^{\delta,j}(y)\rangle=\int_s^t A^{ij}(\xi_{\delta(r),r}(x), \xi_{\delta(r),r}(y))dr,$$

where $\delta(r)$ is a function such that $\delta(t)=t_i$ if $t_i\leqq t<t_{i+1}$. Now let δ_n, $n=1, 2, \cdots$ be a sequence of partitions such that $|\delta_n|\to 0$. Then

$$\begin{aligned}&\langle Y_{s,t}^{\delta_n,i}(x)-Y_{s,t}^{\delta_m,i}(x)\rangle\\ &=\langle Y_{s,t}^{\delta_n,i}(x)\rangle-2\langle Y_{s,t}^{\delta_n,i}(x), Y_{s,t}^{\delta_m,i}(x)\rangle+\langle Y_{s,t}^{\delta_m,i}(x)\rangle\\ &=\int_s^t\{A^{ii}(\xi_{\delta_n(r),r}(x), \xi_{\delta_n(r),r}(x))-2A^{ii}(\xi_{\delta_n(r),r}(x), \xi_{\delta_m(r),r}(x))\\ &\quad+A^{ii}(\xi_{\delta_m(r),r}(x), \xi_{\delta_m(r),r}(x))\}\,dr.\end{aligned}$$

Since $A^{ii}(x, y)$ is *bi*-Lipschitz continuous, we have

$$E[|Y_{s,t}^{\delta_n,i}(x)-Y_{s,t}^{\delta_m,i}(x)|^2]\leqq M\int_s^t E[|\xi_{\delta_n(r),r}(x)-\xi_{\delta_m(r),r}(x)|^2]dr\,.$$

The right hand side is dominated by

$$2M^2\Big(\int_s^t \{|\delta_n(r)-r|+|\delta_m(r)-r|\}\,dr\Big)(1+|x|)^2$$

because of (ξ, III$_2$). Therefore, $Y_t^{\delta_n,i}(x)$ converges to an L^2-martingale $Y_{s,t}^i(x)$. It satisfies

$$\langle Y_{s,t}^i(x),\ Y_{s,\cdot}^j(y)\rangle=\lim_{n,m\to\infty}\langle Y_{s,t}^{\delta_n,i}(x),\ Y_{s,t}^{\delta_m,j}(y)\rangle=A^{ij}(x,\ y)(t-s)$$

and

$$\begin{aligned}\langle Y_{s,t}^i(x),\ M_{s,t}^j(y)\rangle&=\lim\langle Y_{s,t}^{\delta_n,i}(x),\ M_{s,t}^j(y)\rangle\\&=\lim_{n\to\infty}\int_s^t A^{ij}(\xi_{\delta_n(r),r}(x),\ \xi_{s,r}(y))dr\\&=\int_s^t A^{ij}(x,\ \xi_{s,r}(y))dr\,.\end{aligned}$$

The proof is complete.

The process $Y_{s,t}^\delta(x)=(Y_{s,t}^{\delta,1}(x),\ \cdots,\ Y_{s,t}^{\delta,d}(x))$ has the additive property $Y_{s,u}^\delta=Y_{s,t}^\delta+Y_{t,u}^\delta$ for $s<t<u$ of δ. Thus the limiting process $Y_{s,t}$ has also the additive property for any $s,\ t,\ u\in[0,\ T]$. Set $Y_t(x)\equiv Y_{0,t}(x)$. It holds $Y_{s,t}(x)=Y_t(x)-Y_s(x)$. Then $Y_t(x)$ has independent increments, since $Y_{s,t}$ is independent of $\mathcal{F}_{0,s}$. Therefore, n-point process $Y_t(\boldsymbol{x})=(Y_t(x_1),\ \cdots,\ Y_t(x_n))$ is an nd-dimensional Lévy process. We shall prove that $Y_t(x)$, $x\in R^d$ is a C-valued Lévy process.

Lemma 3.3. *Suppose* (ξ. I), (ξ, II) *and* (ξ, III$_2$).

(i) *If* $\xi_{s,t}$ *is a* G_+*-valued Brownian motion, then* $Y_t(x)$ *is a* C*-valued Brownian motion.*

(ii) *Suppose further* (ξ, III$_p$) *for some* $p>d$. *Then* $Y_t(x)$ *is a* C*-valued Lévy process satisfying* (1.17) *and* (1.18).

Proof. In case that $\xi_{s,t}$ is continuous $M_{s,t}(x)$, $t\in[s,\ T]$ is a continuous martingale for each $s,\ x$. Therefore, $Y_t(x)$ is also continuous in t. Then the n-point process $Y_t(\boldsymbol{x})$ is an nd-dimensional Brownian motion such that $E[Y_t(x)]=0$ and $E[Y_t(x)Y_t(y)']=tA(x,\ y)$. Since $A^{ij}(x,\ y)$ is *bi*-Lipschitz continuous, $Y_t(x)$ has a modification such that it is a C-valued Brownian motion by Theorem 1.1.

We shall next consider the discontinuous case. We shall suppress the index i from $Y_t^{\delta,i}(x)$ and write it as $Y_t^\delta(x)$. Constants in the followings do not depend on partitions δ. Let $p'\in[2,\ p]$. By Burkholder's inequality, we have

$$E[|Y_t^\delta(x)-Y_t^\delta(y)|^{p'}]\leqq C_{p'}E[[Y^\delta(x)-Y^\delta(y)]_t^{p'/2}]\,,$$

where

$$[Y^{\delta}(x)-Y^{\delta}(y)]_t=\sum_{k=0}^{n-1}(M_{t_k\wedge t,t_{k+1}\wedge t}(x)-M_{t_{k+1}\wedge t}(y))^2.$$

For simplicity, we assume $t=t_n$. Then

$$\begin{aligned}(3.13)\quad [Y^{\delta}(x)-Y^{\delta}(y)]_t^{p'/2}&=\sum_{k=0}^{n-1}\{[Y^{\delta}(x)-Y^{\delta}(y)]_{t_{k+1}}^{p'/2}-[Y^{\delta}(x)-Y^{\delta}(y)]_{t_k}^{p'/2}\}\\ &\leqq\frac{p'}{2}\sum_{k=0}^{n-1}(M_{t_k,t_{k+1}}(x)-M_{t_k,t_{k+1}}(y))^2[Y^{\delta}(x)-Y^{\delta}(y)]_{t_{k+1}}^{(p'/2)-1}\\ &\leqq\frac{p'}{2}2^{(p'/2)-1}\Big\{\sum_{k=0}^{n-1}|M_{t_k,t_{k+1}}(x)-M_{t_k,t_{k+1}}(y)|^{p'}\\ &\quad+\sum_{k=0}^{n-1}(M_{t_k,t_{k+1}}(x)-M_{t_k,t_{k+1}}(y))^2[Y^{\delta}(x)-Y^{\delta}(y)]_{t_k}^{(p'/2)-1}\Big\}.\end{aligned}$$

Since we have

$$\begin{aligned}E[|M_{t_k,t_{k+1}}(x)-M_{t_k,t_{k+1}}(y)|^{p'}]&\leqq 2^{p'}\Big\{E[|\xi^i_{t_k,t_{k+1}}(x)-x_i-\xi^i_{t_k,t_{k+1}}(y)+y_i|^{p'}]\\ &\quad+E\Big[\Big|\int_{t_k}^{t_{k+1}}|b^i(\xi_{t_k,r}(x))-b^i(\xi_{t_k,r}(y))dr|^{p'}\Big]\Big\}\\ &\leqq 2^{p'}M(1+|t_{k+1}-t_k|^{p'-1})(t_{k+1}-t_k)|x-y|^{p'}\end{aligned}$$

by hypothesis (ξ,III_p), the expectation of the first member of the right hand side of (3.13) is dominated by $C_1t|x-y|^{p'}$. Next, note that the last member of (3.13) is of the form $\sum_i X_iY_i$, where X_i and Y_i are independent random variables. Then the expectation of the last member of (3.13) is dominated by $C_2|x-y|^2\int_0^t E[[Y^{\delta}(x)-Y^{\delta}(y)]_r^{(p'/2)-1}]dr$. We have thus obtained

$$\begin{aligned}(3.14)\qquad E[[Y^{\delta}(x)-Y^{\delta}(y)]_t^{p'/2}]&\leqq C_1t|x-y|^{p'}\\ &\quad+C_2|x-y|^2\int_0^t E[[Y^{\delta}(x)-Y^{\delta}(y)]_r^{(p'/2)-1}]dr.\end{aligned}$$

We now prove

$$(3.15)\qquad E[[Y^{\delta}(x)-Y^{\delta}(y)]_t^{p'/2}]\leqq C_3t|x-y|^{p'}.$$

In case $p'=2$, (3.14) is nothing but (3.15). Then by Hölder's inequality, we have

$$E[[Y^{\delta}(x)-Y^{\delta}(y)]_t^{p'/2}]\leqq C_4|x-y|^{p'}$$

for any $0<p'<2$. Substitute this to the right hand side of (3.14). Then we can see that (3.15) is valid for any $2\leqq p'\leqq 4$. Repeating this argument inductively, we obtain (3.15).

Now apply Burkholder's inequality to (3.15). Then we get

$$(3.16)\qquad E[|Y_t^{\delta}(x)-Y_t^{\delta}(y)|^{p'}]\leqq C_5t|x-y|^{p'}.$$

Let $|\delta|$ tend to 0 in (3.16). Then we get the inequality (1.17). The inequality

1.18) can be proved similarly. Therefore $Y_t(x)$ is a C-valued Lévy process whose characteristics satisfy (C, I), (C, II) and (C, III_p) with $p>d$. The proof is complete.

Proof of Theorem 3.1. Let Y_t be the C-valued Lévy process of Lemma 3.3. We prove that $\xi_{s,t}$ is generated by the Lévy process $X_t=Y_t+tb$. It is enough to prove

$$M_{s,t}(x)=\int_s^t dX_r(\xi_{s,r-}(x))-\int_s^t b(\xi_{s,r-}(x))dr=\int_s^t dY_r(\xi_{s,r-}(x)).$$

Denote the right hand side by $\tilde{M}_{s,t}(x)$. Then, since $\langle Y^i_{s,t}(x), Y^j_{s,t}(y)\rangle=(t-s)A^{ij}(x, y)$, we have

$$\langle\tilde{M}^i_{s,t}(x), \tilde{M}^j_{s,t}(y)\rangle=\int_s^t A^{ij}(\xi_{s,r}(x), \xi_{s,r}(y))dr$$

by (2.4). On the other hand, from (3.12) we obtain

$$\langle\tilde{M}^i_{s,t}(x), M^j_{s,t}(y)\rangle=\int_s^t A^{ij}(\xi_{s,r}(x), \xi_{s,r}(y))dr.$$

Using the above two equalities and (3.9), we get

$$\langle M^i_{s,t}(x)-\tilde{M}^i_{s,t}(x)\rangle=\langle M^i_{s,t}(x)\rangle-2\langle M^i_{s,t}(x), M^i_{s,t}(x)\rangle +\langle\tilde{M}^i_{s,t}(x), \tilde{M}^i_{s,t}(x)\rangle=0.$$

This proves $M^i_{s,t}(x)=\tilde{M}^i_{s,t}(x)$. The uniqueness of X_t is obvious from the method of the construction of Y_t.

3.3. Proof of Theorems 3.2 and 3.3. In this section, we assume that G^m_+-valued Lévy process $\xi_{s,t}$ satisfies (ξ^m, I), (ξ^m, II) and (ξ^m, III_p). We have to prove that C-valued Lévy process constructed at Lemma 3.2 is a C^m-valued Lévy process. Let $Y^\delta_t(x)$ be the martingale defined by (3.10). It is m-times continuously differentiable in x.

Lemma 3.4. *There is a positive constant C independent of the partitions δ such that*

(3.17) $$E[|D^kY^\delta_t(x)-D^kY^\delta_t(y)|^{p'}]\leqq Ct|x-y|^{p'}, \qquad \forall x, y\in R^d,$$

(3.18) $$E[|D^kY^\delta_t(x)|^{p'}]\leqq Ct, \qquad \forall x\in R^d$$

hold for any k with $1\leqq|k|\leqq m$ and $p'\in[2, p/(m+1)^2]$.

Proof. We prove (3.17) only. We fix indices k, l and write $D^kY^{\delta,l}_t(x)=Z^\delta_t(x)$. It is a discrete martingale with parameter δ. Let $[Z^\delta(x)-Z^\delta(y)]_t$ be the quadratic variation of $Z^\delta_t(x)-Z^\delta_t(y)$, i.e.,

$$[Z^\delta(x)-Z^\delta(y)]_t=\sum_{t_{i+1}\leqq t}|D^kM^l_{t_i,t_{i+1}}(x)-D^kM^l_{t_i,t_{i+1}}(y)|^2.$$

Then, similarly as in the proof of Lemma 3.3 we have

$$[Z^{\delta}(x)-Z^{\delta}(y)]_t^{p'/2}=\frac{p'}{2}2^{(p'/2)-1}\{\sum_{t_{i+1}\leq t}|D^kM^l_{t_i,t_{i+1}}(x)-D^kM^l_{t_i,t_{i+1}}(y)|^{p'}$$
$$+\sum_{t_{i+1}\leq t}[Z^{\delta}(x)-Z^{\delta}(y)]_{t_i}^{(p'/2)-1}|D^kM^l_{t_i,t_{i+1}}(x)-D^kM^l_{t_i,t_{i+1}}(y)|^2\}.$$

Taking the expectations,

$$E[[Z^{\delta}(x)-Z^{\delta}(y)]_t^{p'/2}]\leqq\frac{p'}{2}2^{(p'/2)-1}\{E[\sum_{t_{i+1}\leq t}|D^kM^l_{t_i,t_{i+1}}(x)-D^kM^l_{t,t_{i+1}}(y)|^{p'}]$$
$$+\sum_{t_{i+1}\leq t}E[[Z^{\delta}(x)-Z^{\delta}(y)]_t^{(p'/2)-1}]E[|D^kM^l_{t_i,t_{i+1}}(x)-D^kM^l_{t_i,t_{i+1}}(y)|^2]\}. \tag{3.19}$$

Note that derivatives of b^l are all bounded up to k. Then, we get

$$E\left[\left|\int_{t_i}^{t_{i+1}}\{D^k(b^l(\xi_{t_i,r}(x)))-D^k(b^l(\xi_{t_i,r}(y)))\}\,dr\right|^{p'}\right]\leqq C_6(t_{i+1}-t_i)|x-y|^{p'}$$

making use of (ξ^m, III_p). This and (3.6) imply that

$$E[|D^kM^l_{t_i,t_{i+1}}(x)-D^kM^l_{t,t_{i+1}}(y)|^{p'}]\leqq C_7(t_{i+1}-t_i)|x-y|^{p'}.$$

Then (3.19) is written by

$$E[[Z^{\delta}(x)-Z^{\delta}(y)]_t^{p'/2}]\leqq C_8t|x-y|^{p'}+C_9|x-y|^2\int_0^t E[[Z^{\delta}(x)-Z^{\delta}(y)]_r^{(p'/2)-1}]dr$$

Then Burkholder's inequality implies

$$E[|Z_t^{\delta}(x)-Z_t^{\delta}(y)|^{p'}]\leqq C_8't|x-y|^{p'}+C_9'|x-y|^2\int_0^t E[|Z_r^{\delta}(x)-Z_r^{\delta}(y)|^{p'}]dr.$$

This proves (3.17) as in the proof of Lemma 3.3.

Proof of Theorem 3.2. Let δ_n be a sequence of partitions of $[0, T]$ such that $|\delta_n|\to 0$. We first show that for each x $D^kY_t^{\delta_n}(x)$ converges in L^2-sense as $n\to\infty$. Consider the bracket process of $D^kY_t^{\delta_n}(x)-D^kY_t^{\delta_{n'}}(x)$. It holds

$$\langle D^kY^{\delta_n}(x)-D^kY^{\delta_{n'}}(x)\rangle_t=\int_s^t D_x^kD_y^k\{A^{ii}(\xi_{\delta_n(r),r}(x),\ \xi_{\delta_n(r),r}(y))$$
$$-2A^{ii}(\xi_{\delta_n(r),r}(x),\ \xi_{\delta_{n'}(r),r}(y))+A^{ii}(\xi_{\delta_{n'}(r),r}(x),\ \xi_{\delta_{n'}(r),r}(y))\}|_{y=x}dr$$

which converges to 0 a.s. Since $\{|D^{k'}\xi_{\delta_n(r),r}(x)|^{p'},\ n\in N\}$ is uniformly integrable for each x, $k'(1\leqq|k'|\leqq|k|)$ and $p'\leqq p$ by (ξ^m, III_p), the expectation of the above converges to 0 as $n, n'\to\infty$. Therefore for each k, t and x, $D^kY_t^{\delta_n}(x)$ converges to an L^2-random variable $\tilde{Y}_t^k(x)$. It is a right continuous L^2-martingale for each k and x. Let $|\delta_n|$ tend to 0 in (3.17) and (3.18). Then we obtain by Doob's inequality for L^p-martingales and by Fatou's lemma,

$$E[\sup_{0\leq s\leq t}|\tilde{Y}_s^k(x)-\tilde{Y}_s^k(y)|^{p'}]\leqq C't|x-y|^{p'},$$

$$E[\sup_{0\leq s\leq t}|\tilde{Y}_s^k(x)|^{p'}]\leqq C't.$$

Therefore, by Kolmogorov's criterion (Lemma 1.1), we see that $\tilde{Y}_t^k$ is a C-valued

Lévy process.

We show that Y_t is a C^m-valued process and satisfies $D^k Y_t = \tilde{Y}_t$. Consider first the case where $k=(0, \cdots, 1, 0, \cdots, 0)$ (1 is at i-th component). It holds

$$\frac{1}{y}\{Y_t^\delta(x+ye_i)-Y_t^\delta(x)\}=\int_0^1 \partial_i Y_t^\delta(x+vye_i)dv .$$

Let $|\delta|$ tend to 0. Then we get for each t, x and y,

$$\frac{1}{y}\{Y_t(x+ye_i)-Y_t(x)\}=\int_0^1 \tilde{Y}_t^k(x+vye_i)dv \qquad \text{a, s. } P.$$

However, each of the above is a right continuous C-valued function. Hence the above holds for all t, x and y. Then, the derivative $\partial_i Y_t(x)$ exists and equals $\tilde{Y}_t^k(x)$ for all t, x a.s. Repeating this argument inductively, we see that $Y_t(x)$ is a right continuous C^m-valued process and satisfies $D^k Y_t(x)=\tilde{Y}_t^k(x)$ for any k with $|k|\leqq m$. Therefore, both $Y_t(x)$ and $X_t(x)=Y_t(x)+b(x)t$ are C^m-valued Lévy processes satisfying (1.24) and (1.25) for any k with $|k|\leqq m$ and $p'\in[2, p/(m+1)^2]$. The assertion of the thoerem follows from Theorem 1.3.

Proof of Theorem 3.3. The first assertion of the theorem is immediate from Lemma 3.3 (i). For the proof of the second statement, we proceed as follows. Since the infinitesimal generator X_t is a C-valued Brownian motion, the pair (A, b) defined by (3.1) and (3.2) is the characteristics of X_t. It satisfies (C^m, I) and (C^m, II) because of our assumption (ξ^m, I) and (ξ^m, II). Therefore X_t is a C^m-valued Brownian motion.

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