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Approximate solutions for the Cauchy problem for a semilinear hyperbolic system

By

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1. Introduction.

We consider the following system of semi-linear partial differential equations

$$\frac{\partial u}{\partial t}-\lambda\frac{\partial u}{\partial x}=au-uv,$$

$$\frac{\partial v}{\partial t}-\mu\frac{\partial v}{\partial x}=bv+uv,\quad t>0,\quad -\infty<x<\infty,$$

with the initial data $u(x, 0)=\phi(x)$ and $v(x, 0)=\psi(x)$. When $a>0$ and $b<0$ the above system has been considered in ([7]) as a model for the development in time of a prey $u(x, t)$ and predator $v(x, t)$ running on a straight line with speeds λ and μ respectively. The constants a and b are considered as rates of natural multiplication of prey without predator and rate of natural extinction of predator without prey respectively.

By setting $\frac{b}{a}=\gamma$ and $a=\varepsilon$ the above system can be rewritten as

$$\begin{aligned}\frac{\partial u}{\partial t}-\lambda\frac{\partial u}{\partial x}&=\varepsilon u-uv,\\ \frac{\partial v}{\partial t}-\mu\frac{\partial v}{\partial x}&=\gamma\varepsilon v+uv\end{aligned}\tag{1.1}$$

and it is in this form that we will consider the problem throughout the paper.

In earlier papers [5, 6] representations for the exact solutions of (1.1) when $\varepsilon=0$ were obtained. Motivated by these, in [7] the authors consider problem (1.1) as a perturbation of the problem when $\varepsilon=0$. Assuming $u_0(x, t)$ and $v_0(x, t)$ to be the exact solutions when $\varepsilon=0$, problem (1.1) is then studied by a perturbation procedure and a solution is sought in the form $\left(\sum_0^\infty u_n(x, t)\varepsilon^n, \sum_0^\infty v_n(x, t)\varepsilon^n\right)$. The main theorem of their paper derives sufficient conditions on ε in terms of T in order that solutions of the above form exist over $(-\infty, \infty)\times[0, T]$. Uniform convergence of the series is also discussed.

In this paper we study problem (1.1) by following a "Peano-Arzela" type constructive approximation scheme in the spirit of [2]. This approach enables us to obtain global existence results for (1.1) and since uniqueness holds for the

ε-problem, it allows us to view the unique solution as limit of successive approximations. A further advantage is that the asymptotic behaviour as $t\to\infty$ may be studied directly without considering a differential equation which characterises the limiting behaviour.

Some of the generalizations of the above approach are as follows: a) the regularity hypotheses on the initial data can be weakened considerably; b) when "global existence" in t is not possible as in the Boltzmann equation [4], we can obtain quantitative estimates on the interval of existence; c) in our approach the nature of the linear part is not critical, thereby enabling us to consider more general nonlinear problems. These discussions and ideas will be developed elsewhere.

2. The nonlinear problem and an approximating scheme.

Let $\phi(x)$ and $\psi(x)$ be continuous non-negative functions from $\boldsymbol{R}\to\boldsymbol{R}$ with continuous first derivatives. Further let ϕ, ψ, ϕ' and ψ' be bounded. With λ, μ, ε and γ being real parameters we consider the semilinear hyperbolic system

$$\begin{aligned} u_t-\lambda u_x&=\varepsilon u-uv, \\ v_t-\mu v_x&=\gamma\varepsilon v+uv \end{aligned} \tag{2.1}$$

with initial data

and
$$\begin{aligned} u(0, x)&=\phi(x), \\ v(0, x)&=\psi(x), \qquad -\infty<x<\infty. \end{aligned}$$

We are looking for functions U and V in $C^1([0, \infty)\times\boldsymbol{R})$ satisfying (2.1). It can be seen by an application of Haar's inequality ([7], also see Section 5 here) that if (2.1) has a solution it must be unique.

For any positive number T let P_m be a partition of $[0, T]$ given by $P_m:\{t_0=0<t_1<t_2<\cdots<t_m=T\}$. Let $\delta_i=t_{i+1}-t_i$. Then we define the functions $\{\phi_i\}$, $\{\psi_i\}$ as follows:

$$\phi_0=\phi, \qquad \psi_0=\psi,$$

and
$$\begin{aligned} \phi_{i+1}(x)&=\phi_i(x+\lambda\delta_1)[1+(\varepsilon-\psi_i(x+\lambda\delta_i))\delta_i], \\ \psi_{i+1}(x)&=\psi_i(x+\mu\delta_i)[1+(\gamma\varepsilon+\phi_i(x+\mu\delta_i))\delta_i]. \end{aligned}$$

We now consider the following approximating sequences for (2.1):

$$\begin{aligned} U(t, x)&=\phi_i[x+\lambda(t-t_i)][1+(\varepsilon-\psi_i(x+\lambda(t-t_i)))(t-t_i)] \\ V(t, x)&=\psi_i[x+\mu(t-t_i)][1+(\gamma\varepsilon+\phi_i(x+\mu(t-t_i)))(t-t_i)] \end{aligned}$$

for $(t, x)\in[t_i, t_{i+1}]\times\boldsymbol{R}$.

3. Estimates on ϕ_i, ψ_i, $|\phi_i'|$, $|\psi_i'|$, $|\phi_i'(x)-\phi_i'(y)|$, $|\psi_i'(x)-\psi_i'(y)|$, $|\phi_{i+h}'(x)-\phi_i'(x)|$ and $|\psi_{i+h}'(x)-\psi_i'(x)|$.

Let T be a positive number and P_n be a partition of $[0, T]$ given by $P_n=$

$\{t_0=0<t_1<t_2<\cdots<t_n=T\}$, $n\geqq 1$. For this partition P_n, let $\delta_i=t_{i+1}-t_i$, $i=0, 1, \cdots, n-1$ and let $\|P_n\|=\max\{\delta_i, i=0, 1, \cdots, n-1\}$. We now assume that the partition P_n has been chosen to meet the following hypotheses:

(3.1) $\|P_n\|\leqq\eta n^{-1}$ where η is an arbitrary but fixed number such that $\eta\geqq T$;

(3.2) $\|P_n\|<\min\{[|\varepsilon|+M\exp([|\gamma\varepsilon|+M\exp(|\varepsilon|\eta)]\eta)]^{-1}, |\gamma\varepsilon|^{-1}\}$, where M is an upper bound for $|\phi|$, $|\psi|$, $|\phi'|$ and $|\psi'|$.

Note. If either γ or ε is zero, in (3.2) $|\gamma\varepsilon|^{-1}$ is replaced by $+\infty$.

The sequences $\{\phi_i(x)\}$, $\{\psi_i(x)\}$ are now defined as follows: for $i=0, 1, \cdots, n-1$, let

$$\phi_0=\phi, \qquad \psi_0=\psi;$$

$$\phi_{i+1}(x)=\phi_i(x+\lambda\delta_i)[1+(\varepsilon-\psi_i(x+\lambda\delta_i))\delta_i], \tag{3.3}$$

$$\psi_{i+1}(x)=\psi_i(x+\mu\delta_i)[1+(\gamma\varepsilon+\phi_i(x+\mu\delta_i))\delta_i]. \tag{3.4}$$

Step I. We first show, by induction, that ϕ_i, ψ_i are nonnegative and bounded. More precisely, we show that

$$0\leqq\phi_i(x)\leqq M\prod_{j=0}^{i-1}(1+|\varepsilon|\delta_j); \tag{3.5}$$

$$0\leqq\psi_i(x)\leqq M\prod_{j=0}^{i-1}[1+(|\gamma\varepsilon|+M\exp(|\varepsilon|\eta))\delta_j] \qquad \text{for} \quad i=1, 2, \cdots, n. \tag{3.6}$$

Assuming that (3.5) and (3.6) hold for ϕ_i and ψ_i, we verify them for ϕ_{i+1} and ψ_{i+1}. Thus

$$\begin{aligned} 0\leqq\psi_i(x+\lambda\delta_i)&\leqq M\prod_{j=0}^{i-1}[1+(|\gamma\varepsilon|+M\exp(|\varepsilon|\eta))\delta_j] \\ &\leqq M\Big[1+(|\gamma\varepsilon|+M\exp(|\varepsilon|\eta))\frac{\eta}{n}\Big]^n \\ &<M\exp\{[|\gamma\varepsilon|+M\exp(|\varepsilon|\eta)]\eta\}. \end{aligned}$$

Hence

$$\begin{aligned} |\varepsilon-\psi_i(x+\lambda\delta_i)\delta_i|&\leqq[|\varepsilon|+M\exp\{[|\gamma\varepsilon|+M\exp(|\varepsilon|\eta)]\eta\}]\|P\| \\ &<1, \quad \text{by (3.2)}. \end{aligned}$$

This implies, from (3.3), that

$$0\leqq\phi_{i+1}(x).$$

Hence, from (3.3),

$$\phi_{i+1}(x)\leqq M\Big[\prod_{j=0}^{i-1}(1+|\varepsilon|\delta_j)\Big](1+|\varepsilon|\delta_i)$$

thus verifying (3.5).

Further

$$\psi_{i+1}(x)\leqq M\Big[\prod_{j=0}^{i-1}(1+(|\gamma\varepsilon|+M\exp(|\varepsilon|\eta))\delta_j)\Big]$$

$$\times\Big[1+\Big(|\gamma\varepsilon|+M\prod_{j=0}^{i-1}(1+|\varepsilon|\delta_j)\Big)\delta_i\Big]$$

$$\leqq M\Big[\prod_{j=0}^{i-1}\{1+(|\gamma\varepsilon|+M\exp(|\varepsilon|\eta))\delta_j\}\Big]$$

$$\times\Big[1+\Big\{|\gamma\varepsilon|+M\Big(1+\frac{|\varepsilon|\eta}{n}\Big)^n\Big\}\delta_i\Big]$$

$$<M\Big[\prod_{j=0}^{i-1}\{1+(|\gamma\varepsilon|+M\exp(|\varepsilon|\eta))\delta_j\}\Big]$$

$$\times[1+\{|\gamma\varepsilon|+M\exp(|\varepsilon|\eta)\}\delta_i],$$

thereby establishing (3.6).

It must be noted that $\psi_{i+1}(x)\geqq 0$ because ϕ_i and ψ_i are nonnegative and $|\gamma\varepsilon\delta_i|\leqq|\gamma\varepsilon|\ \|P_n\|<1$, by (3.2). It is easy to verify that (3.5) and (3.6) hold for $i=1$. Hence, inductively, we have proved (3.5) and (3.6) for all $i=1, \cdots, n$.

From (3.5) and (3.6) it also follows that, for $x\in\boldsymbol{R}$ and for $i=1, 2, \cdots, n$,

$$0\leqq\phi_i(x)\leqq M\exp(|\varepsilon|\eta)=M_1;$$

$$0\leqq\psi_i(x)\leqq M\exp\{\eta(|\gamma\varepsilon|+M\exp(|\varepsilon|\eta))\}=M_2;$$

(we have used (3.1) again).

Choosing $\overline{M}=\max(M_1, M_2)$ we conclude that

$$0\leqq\phi_i(x),\quad \psi_i(x)\leqq\overline{M},\quad x\in\boldsymbol{R}\quad\text{and}\quad i=1, \cdots, n. \tag{3.7}$$

Step II. We now show that there exists $M^*>0$ such that

$$|\phi_i'(x)|,\ |\psi_i'(x)|\leqq M^*$$

for $x\in\boldsymbol{R}$ and $i=1, 2, \cdots, n$. It must be recalled here that by the hypotheses on the initial data we have

$$|\phi_0'(x)|=|\phi'(x)|,\qquad |\psi_0'(x)|=|\psi'(x)|. \tag{3.8}$$

The proof of (3.8) proceeds once again inductively. Thus

$$\phi_{i+1}'(x)=\phi_i'(x+\lambda\delta_i)[1+(\varepsilon-\psi_i(x+\lambda\delta_i))\delta_i]-\phi_i(x+\lambda\delta_i)\psi_i'(x+\lambda\delta_i)\delta_i$$

$$=\phi_i'(x+\lambda\delta_i)(1+\varepsilon\delta_i)-\delta_i[\phi_i'(x+\lambda\delta_i)\psi_i(x+\lambda\delta_i)+\phi_i(x+\lambda\delta_i)\psi_i'(x+\lambda\delta_i)].$$

Also

$$\psi_{i+1}'(x)=\psi_i'(x+\mu\delta_i)[1+(\gamma\varepsilon+\phi_i(x+\mu\delta_i))\delta_i]+\psi_i(x+\mu\delta_i)\phi_i'(x+\mu\delta_i)\delta_i$$

$$=\psi_i'(x+\mu\delta_i)(1+\gamma\varepsilon\delta_i)+[\phi_i(x+\mu\delta_i)\psi_i'(x+\mu\delta_i)+\phi_i'(x+\mu\delta_i)\psi_i(x+\mu\delta_i)]\delta_i.$$

We now define the sequence $\{N_i\}$ of real numbers as follows:

$$N_0=M\quad\text{and}\quad N_i=\sup\{|\phi_i'(x)|,\ |\psi_i'(x)|: x\in\boldsymbol{R}\},\qquad i=1, \cdots, n.$$

Now, from (3.7), for $i=1, \cdots, n$

$$|\phi_{i+1}'(x)|\leqq N_i(1+|\varepsilon|\delta_i)+2\overline{M}N_i\delta_i,$$

$$|\psi'_{i+1}(x)|\leqq N_i(1+|\gamma\varepsilon|\delta_i)+2\bar{M}N_i\delta_i.$$

Denoting by $\hat{\varepsilon}$ the quantity $\max\{|\varepsilon|, |\gamma\varepsilon|\}$ the above estimates can be written as

$$N_{i+1}\leqq N_i(1+[\hat{\varepsilon}+2\bar{M}]\delta_i),$$
$$N_0=M.$$

Thus, for $i=1, \cdots, n$,

$$\begin{aligned} N_i &\leqq M\prod_{j=0}^{i-1}(1+[\hat{\varepsilon}+2\bar{M}]\delta_j) \\ &\leqq M\left(1+\frac{\eta(\hat{\varepsilon}+2\bar{M})}{n}\right)^n \\ &< M\exp[\eta(\hat{\varepsilon}+2\bar{M})]=M^*, \quad \text{say.} \end{aligned}$$

Hence

$$|\phi'_i(x)|, |\psi'_i(x)|\leqq M^*$$

for $x\in\boldsymbol{R}$ and $i=1, \cdots, n$.

(3.9) We now set $\tilde{M}=\max\{\bar{M}, M^*\}$.

Step. III. In this step we show that for each $A>0$ there exists a modulus of continuity function Ω which does not depend on i such that

$$|\phi'_i(x)-\phi'_i(y)|, |\psi'_i(x)-\psi'_i(y)|\leqq\Omega(|x-y|) \tag{3.10}$$

for $i=1, \cdots, n$ and all x, y such that $|x|, |y|\leqq A$.

Now

$$\begin{aligned} |\psi'_{i+1}(x)-\psi'_{i+1}(y)| &\leqq[1+(\hat{\varepsilon}+\tilde{M})\delta_i]|\psi'_i(x+\mu\delta_i)-\psi'_i(y+\mu\delta_i)| \\ &\quad+\tilde{M}\delta_i|\phi'_i(x+\mu\delta_i)-\phi'_i(y+\mu\delta_i)| \\ &\quad+\tilde{M}\delta_i|\psi_i(x+\mu\delta_i)-\psi_i(y+\mu\delta_i)| \\ &\quad+\tilde{M}\delta_i|\phi_i(x+\mu\delta_i)-\phi_i(y+\mu\delta_i)| \\ &\leqq[1+(\hat{\varepsilon}+\tilde{M})\delta_i]|\psi'_i(x+\mu\delta_i)-\psi'_i(y+\mu\delta_i)| \\ &\quad+\tilde{M}\delta_i|\phi'_i(x+\mu\delta_i)-\phi'_i(y+\mu\delta_i)| \\ &\quad+2\tilde{M}^2\delta_i|x-y|. \end{aligned} \tag{3.11}$$

A similar inequality holds for $|\phi'_{i+1}(x)-\phi'_{i+1}(y)|$ where in the right hand side we interchange ψ_i and ϕ_i and replace μ by λ.

Let $A^*=A+\max\{|\lambda|, |\mu|\}T$ and let Ω_0 be the modulus of continuity function such that

$$|\phi'(x)-\phi'(y)|, |\psi'(x)-\psi'(y)|\leqq\Omega_0(|x-y|), \qquad \text{for} \quad |x|, |y|\leqq A^*. \tag{3.12}$$

It can then be proved by induction that for $i=1, 2, \cdots$

$$|\phi'_i(x)-\phi'_i(y)|, |\psi'_i(x)-\psi'_i(y)|\leqq\left(\prod_{j=0}^{i-1}[1+(\hat{\varepsilon}+2\tilde{M})\delta_j]\right)\Omega_0(|x-y|) \tag{3.13}$$

$$+2\tilde{M}^2\Big[\|P_n\|\Big(\sum_0^{i-1}(1+(\hat{\varepsilon}+2\tilde{M})\|P_n\|)^j\Big)\Big]|x-y|$$

for all $|x|,\ |y|\leqq A^*-\max\{|\lambda|,\ |\mu|\}(\delta_0+\cdots+\delta_{i-1})$.

Note that (assuming the induction hypothesis) from (3.11)

$$|\psi'_{i+1}(x)-\psi'_{i+1}(y)|\leqq[1+(\hat{\varepsilon}+2\tilde{M})\delta_i]\Big[\prod_{j=0}^{i-1}(1+(\hat{\varepsilon}+2\tilde{M})\delta_j)\Big]\Omega_0(|x-y|)$$

$$+2\tilde{M}^2\Big[\|P_n\|\Big(\sum_0^{i-1}(1+(\hat{\varepsilon}+2\tilde{M})\|P_n\|)^{j+1}\Big)\Big]|x-y|$$

$$+2\tilde{M}^2\delta_i|x-y|$$

because

$$|x+\mu\delta_i|\leqq A^*-\max\{|\lambda|,\ |\mu|\}(\delta_0+\cdots+\delta_i)+|\mu|\delta_i$$

$$\leqq A^*-\max\{|\lambda|,\ |\mu|\}(\delta_0+\cdots+\delta_{i-1}).$$

Thus

$$|\psi'_{i+1}(x)-\psi'_{i+1}(y)|\leqq\Big[\prod_{j=0}^{i}(1+(\hat{\varepsilon}+2\tilde{M})\delta_j\Big]\Omega_0(|x-y|)$$

$$+2\tilde{M}^2\Big[\|P_n\|\Big(\sum_0^{i-1}(1+(\hat{\varepsilon}+2\tilde{M})\|P_n\|\Big)^{j+1}+\|P_n\|\Big]|x-y|.$$

We can now obtain from (3.13) that

$$|\phi'_i(x)-\phi'_i(y),\ |\psi'_i(x)-\psi'_i(y)|\leqq\exp[(\hat{\varepsilon}+2\tilde{M})\eta]\Omega_0(|x-y|)$$

$$+2\tilde{M}^2(\hat{\varepsilon}+2\tilde{M})^{-1}(\exp[\hat{\varepsilon}+2\tilde{M})\eta]-1)|x-y|$$

for all $|x|,\ |y|\leqq A$.

Hence (3.10) holds where

$$\Omega(\xi)=\exp[(\hat{\varepsilon}+2\tilde{M})\eta]\Omega_0(\xi)+2\tilde{M}^2(\hat{\varepsilon}+2\tilde{M})^{-1}(\exp[(\hat{\varepsilon}+2\hat{M})\eta]-1)\xi.$$

Step. IV. We now show that for each $B>0$ there exists $\omega:[0,\infty)\to\boldsymbol{R}$ with $\lim_{\xi\to 0+}\omega(\xi)=0$ such that

$$(3.14)\qquad |\phi'_{i+h}(x)-\phi'_i(x)|,\ |\psi'_{i+h}(x)-\psi'_i(y)|\leqq\omega(\Delta_{i,i+h})$$

for $1\leqq i\leqq n,\ h\geqq 1,\ |x|\leqq B$ and $\Delta_{i,i+h}=\sum_{j=1}^{i+h-1}\delta_j$.

Note that, from the calculations on $\phi'_{i+1}(x)$ in Step II,

$$|\phi'_{i+1}(x)-\phi'_i(x+\lambda\delta_i)|\leqq\tilde{M}(\hat{\varepsilon}+2\tilde{M})\delta_i.$$

Thus

$$|\phi'_{i+h}(x)-\phi'_i(x+\lambda\Delta_{i,i+h})|\leqq\tilde{M}(\hat{\varepsilon}+2\tilde{M})\Delta_{i,i+h},$$

and

$$|\phi'_{i+h}(x)-\phi'_i(x)|\leqq\tilde{M}(\hat{\varepsilon}+2\tilde{M})\Delta_{i,i+h}+\Omega(\max\{|\lambda|,\ |\mu|\}\Delta_{i,i+h}),$$

where Ω is the modulus of continuity function arising from Step III for appropriate A. A similar inequality holds for ψ' and thus (3.14) has been proved with

$$\omega(\xi)=\tilde{M}(\hat{\varepsilon}+2\tilde{M})\xi+\Omega(\max\{|\lambda|,\ |\mu|\}\xi).$$

4. Convergence of Approximations.

We first observe that for every m if the partition P_m is chosen so as to satisfy (3.1) and (3.2) then the corresponding approximations U_m and V_m are uniformly bounded by virtue of Step I in Section 3. It also follows from the Lipschitz nature of ϕ_i and ψ_i that there exists N such that

$$|U_m(t, x)-U_m(t^*, x^*)|,\ |V_m(t, x)-V_m(t^*, x^*)|\leqq N|t-t^*|+N|x-x^*|$$

for $x, x^*\in \boldsymbol{R}$ and $t, t^*\in[t_i^m, t_{i+1}^m]$.

However if t, t^* in $[0, T]$ do not belong to the same strip, then let $t_{i-1}^m<t \leqq t_i^m \leqq t_j^m \leqq t^* < t_{j+1}^m$. We now have

$$\begin{aligned}|U_m(t, x)-U_m(t^*, x^*)| &\leqq |U_m(t, x)-U_m(t_i^m, x)|+\sum_{s=i}^{j-1}|U_m(t_{s+1}^m, x)-U_m(t_s^m, x)| \\ &\quad +|U_m(t_j^m, x)-U_m(t^*, x^*)| \\ &\leqq N|t-t_i^m|+N\sum_{s=i}^{j-1}|t_{s+1}^m-t_s^m|+N|t_j^m-t^*|+N|x-x^*| \\ &\leqq N|t-t^*|+N|x-x^*|.\end{aligned}$$

A similar inequality holds for V_m also.

Hence we can extract convergent subsequences U_k, V_k which converge uniformly on bounded subsets of $[0, T]\times \boldsymbol{R}$. Also

$$\begin{aligned}(4.1)\qquad \frac{\partial U_m}{\partial x} &= \phi_i'[x+\lambda(t-t_i^m)][1+(\varepsilon-\psi_i(x+\lambda(t-t_i^m))(t-t_i^m)] \\ &\quad +\phi_i[x+\lambda(t-t_i^m)][-\psi_i'(x+\lambda(t-t_i^m))(t-t_i^m)],\quad t_i^m<t<t_{i+1}^m \\ &= \phi_i'(x), \qquad t=t_i^m.\end{aligned}$$

Hence, from Step IV of Section 3, we have

$$\left|\frac{\partial U_m}{\partial x}(t_i^m, x)-\frac{\partial U_m}{\partial x}(t_{i+h}^m, x)\right|\leqq \omega(\varDelta_{i, i+h}) \qquad \text{for} \quad |x|\leqq \eta$$

Also, from Step III of Section 3

$$\left|\frac{\partial U_m}{\partial x}(t, x)-\frac{\partial U_m}{\partial x}(t^*, x^*)\right|\leqq \Omega(|t-t^*|+|x-x^*|)$$

if $t, t^*\in[t_i^m, t_{i+1}^m]$ and $|x|, |x^*|\leqq \eta$. Hence for t, t^* such that $t_{i-1}<t\leqq t_i \leqq t_j \leqq t^* < t_{j+1}$

$$\begin{aligned}\left|\frac{\partial U_m}{\partial x}(t, x)-\frac{\partial U_m}{\partial x}(t^*, x^*)\right| &\leqq \left|\frac{\partial U_m}{\partial x}(t, x)-\frac{\partial U_m}{\partial x}(t_i^m, x)\right|+\left|\frac{\partial U_m}{\partial x}(t_i^m, x)-\frac{\partial U_m}{\partial x}(t_j^m, x)\right| \\ &\quad +\left|\frac{\partial U_m}{\partial x}(t_j^m, x)-\frac{\partial U_m}{\partial x}(t^*, x^*)\right| \\ &\leqq \Omega(|t-t_i^m|)+\omega(\varDelta_{i, j})+\Omega(|t_j^m-t^*|+|x-x^*|) \\ &\leqq \Omega\left(\frac{\eta}{m}\right)+\omega(|t-t^*|)+\Omega\left(\frac{\eta}{m}+|x-x^*|\right).\end{aligned}$$

Hence by Arzela's theorem ([1, 3]) there is a subsequence of U_m (also denoted by U_m) such that

$$\frac{\partial U_m}{\partial x} \text{ converges uniformly on } [-\eta, \eta]$$

and the limit function U has a continuous partial derivative with respect to x and

$$\frac{\partial U_m}{\partial x} \longrightarrow \frac{\partial U}{\partial x} \text{ on } [-\eta, \eta].$$

Finally letting

(4.2) $$P_m(t, x)=\lambda\phi_i'[x+\lambda(t-t_i^m)][1+\{\varepsilon-\psi_i(x+\lambda(t-t_i^m))\}(t-t_i^m)]$$
$$+\phi_i[x+\lambda(t-t_i^m)][\varepsilon-\psi_i(x+\lambda(t-t_i^m))-\lambda\psi_i'(x+\lambda(t-t_i^m))(t-t_i^m)]$$

Thus, from (4.1) and (4.2),

$$P_m(t, x)-\lambda\frac{\partial U_m}{\partial x}(t, x)=\phi_i[x+\lambda(t-t_i^m)][\varepsilon-\psi_i(x+\lambda(t-t_i^m))].$$

Hence

$$P_m(t, x)-\lambda\frac{\partial U_m}{\partial x}(t, x)-\varepsilon U_m(t, x)+U_m(t, x)V_m(t, x)$$
$$=\phi_i[x+\lambda(t-t_i^m)][\varepsilon-\psi_i(x+\lambda(t-t_i^m))]$$
$$-\varepsilon\phi_i[x+\lambda(t-t_i^m)][1+(\varepsilon-\psi_i(x+\lambda(t-t_i^m))(t-t_i^m)]+U_m(t, x)V_m(t, x).$$

Noting the fact that $|\phi_i|$ and $|\psi_i|$ are bounded independent of m and further that $|t-t_i^m|\leqq\frac{\eta}{m}$, it follows that

$$P_m(t, x)-\lambda\frac{\partial U_m}{\partial x}-\varepsilon U_m(t, x)+U_m(t, x)V_m(t, x)\longrightarrow 0.$$

Thus $P_m(t, x)$ converges uniformly to a continuous function. Passing to the limits, it follows that the limit functions U, (and proceeding analogously) V satisfy (1.1) in $[-\eta, \eta]\times[0, T]$.

5. Error Estimates.

We denote by (U, V) the exact solution to (1.1) and let (U_m, V_m) be the approximation corresponding to a partition P_m satisfying (3.1) and (3.2). Finally let $a_m=U-U_m$ and $b_m=V-V_m$. Then

$$\frac{\partial a_m}{\partial t}-\lambda\frac{\partial a_m}{\partial t}=\varepsilon[U-\phi_i[x+\lambda(t-t_i^m)]]-UV+\phi_i[x+\lambda(t-t_i^m)]\psi_i[x+\lambda(t-t_i^m)],$$

and

$$\frac{\partial b_m}{\partial t}-\mu\frac{\partial b_m}{\partial x}=\gamma\varepsilon[V-\psi_i[x+\mu(t-t_i^m)]]+UV-\phi_i[x+\mu(t-t_i^m)]\psi_i[x+\mu(t-t_i^m)].$$

where $-\infty<x<\infty$ and $t_i^m\leqq t\leqq t_{i+1}^m$. We recall here that ϕ_i, ψ_i, ϕ_i' and ψ_i' are bounded. Further if we let

$$\phi_i[x+\lambda(t-t_i^m)]=U_m(t, x)+R_i(t, x),$$
$$\phi_i[x+\mu(t-t_i^m)]=U_m(t, x)+R_i^*(t, x),$$

$$\phi_i[x+\mu(t-t_i^m)]=V_m(t,\ x)+S_i(t,\ x),$$

and

$$\phi_i[x+\lambda(t-t_i^m)]=V_m(t,\ x)+S_i^*(t,\ x)$$

then

$$\frac{\partial a_m}{\partial t}-\lambda\frac{\partial a_m}{\partial x}=\varepsilon a_m+\alpha_{11}^i a_m+\alpha_{12}^i b_m+\rho_i^{**},$$

$$\frac{\partial b_m}{\partial t}-\mu\frac{\partial b_m}{\partial x}=\gamma\varepsilon b_m+\alpha_{21}^i a_m+\alpha_{22}^i b_m+\rho_i^*.$$

We now observe that $|R_m|$, $|S_m|$, $|R_m^*|$ and $|S_m^*|\leqq C\|P_m\|$ where $C=\hat{M}[\hat{\varepsilon}+\hat{M}+|\mu-\lambda|]$. Further $|\alpha_{h,k}^i|\leqq K$ where K is a bound on U_m, V_m and finally $|\rho_i^{**}|$, $|\rho_i^*|\leqq h\|P_m\|$ where $h\geqq C(\hat{\varepsilon}+2K+C\|P_m\|)$.

Also $a_0(0,\ x)=b_0(0,\ x)=0$. Then denoting by ρ_i the quantity $\max\{|a_m(t,\ x)|, |b_m(t,\ x)|\}$ for $t\in[t_i^m,\ t_i^{m+1}]$ we have by an application of Haar's lemma that

$$\rho_0\leqq\frac{h\|P_m\|}{2(\hat{\varepsilon}+K)}(\exp[2(\hat{\varepsilon}+K)\|P_m\|]-1)$$

and

$$\rho_{i+1}\leqq\rho_i\exp[2(\hat{\varepsilon}+K)\|P_m\|]+\frac{h\|P_m\|}{2(\hat{\varepsilon}+K)}(\exp[2(\hat{\varepsilon}+K)\|P_m\|]-1)$$

The last inequality thus leads to

$$\rho_i\leqq\frac{h\eta}{2(K+\hat{\varepsilon})}\left[\exp\left(2(K+\hat{\varepsilon})\left(1+\frac{1}{m}\right)\eta\right)-1\right]\frac{1}{m}.$$

We conclude this section by stating Haar's lemma from [7] for the sake of completeness:

Denote by D the triangle with vertices $(x_0+c_1T,\ 0)$, $(x_0,\ T)$ and $(x_0+c_2T,\ 0)$ and further let $[x_0+c_1T,\ x_0+c_2T]$ be denoted by D_0. Suppose that the following system of linear partial differential equations is given:

$$\begin{cases}\dfrac{\partial u_1}{\partial t}-c_1\dfrac{\partial u_1}{\partial x}=a_{11}(x,\ t)u_1+a_{12}(x,\ t)u_2+b_1(x,\ t),\\[2mm] \dfrac{\partial u_2}{\partial t}-c_2\dfrac{\partial u_2}{\partial x}=a_{22}(x,\ t)u_1+a_{22}(x,\ t)u_2+b_2(x,\ t),\end{cases}$$

with the initial data:

$$\begin{cases}u_1(x,\ 0)=\phi_1(x),\\ u_2(x,\ 0)=\phi_2(x),\end{cases}$$

where c_1 and c_2 are constants such that $c_1<c_2$, $a_{ij}(x,\ t)$ and $b_{ij}(x,\ t)$ $(1\leqq i,\ j\leqq 2)$ are supposed to be continuous over $(-\infty,\ +\infty)\times[0,\ +\infty)$. Furthermore we suppose that $\phi_1(x)$ and $\phi_2(x)$ are continuous together with their first derivatives over $(-\infty,\ +\infty)$. By setting

$$a=\max_{1\leq i,j\leq 2}\{\sup_{(x,t)\in D}|a_{ij}(x,\ t)|\},$$

$$b=\max_{1\leq i\leq 2}\{\sup_{(x,t)\in D}|b_i(x,\ t)|\},$$

$$h=\max_{1\leq i\leq 2}\{\sup_{x\in D_0}|\phi_i(x)|\},$$

Haar's inequality can be written in such a way that

$$|u_1(x, t)|,\ |u_2(x, t)|\leqq he^{2at}+\frac{b}{2a}(e^{2at}-1)$$

$$\leqq he^{2aT}+\frac{b}{2a}(e^{2aT}-1), \qquad (x, t)\in D.$$

6. Remarks.

i) It is clear from Sections 3 and 4 that we have demonstrated in the process a global existence result for (1.1). More precisely, it has been proved that (1.1) has a solution for all time t.

ii) The regularity assumptions on the initial data can be relaxed and details of applicability of this scheme with weaker initial data for more general hyperbolic systems will be published elsewhere.

iii) An alternate scheme to study (1.1) is (we define ϕ_i and ψ_i only here; U_i, V_i can be defined analogous to Section 2) given by

$$\phi_0=\phi, \qquad \psi_0=\psi$$

$$\phi_{i+1}(x)=\phi_i(x+\lambda\delta_i)\exp\{[\varepsilon-\psi_i(x+\lambda\delta_i)]\delta_i\}$$

$$\psi_{i+1}(x)=\psi_i(x+\mu\delta_i)\exp\{[\gamma\varepsilon+\phi_i(x+\mu\delta_i)]\delta_i\}.$$

In fact, the scheme of Section 2 may be seen as a linear approximation to the above. This scheme can also be utilised to study the asymptotic behaviour as $t\to\infty$ and some of the results in [3] can be seen to follow easily.

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