J. Math. Kyoto Univ. (JMKYAZ)
25-2 (1985) 255-267

Analytic families of entire functions of finite order

Dedicated to Professor Yukio Kusunoki on his 60th birthday

By

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(Received December 23, 1983)

Introduction. The classical theorem of Hadamard concerning entire functions of a complex variable is composed of the following three assertions: (i) If f is an entire function of finite order λ, then the order of the zero locus of f does not exceed λ. (ii) For a divisor A of finite order λ prescribed on the complex line $\boldsymbol{C}$, there exists an entire function F of the same order λ having A as its zero locus. F is given by a canonical product of genus q with $\lambda-1\leqq q\leqq\lambda$. (iii) In the same situation as (ii), every entire function f of finite order with zero locus A is written as $f=e^P F$ with a polynomial P. The order of f is $\max\{\lambda, \deg P\}$.

Now let Ω be a domain in the space $\boldsymbol{C}^m$ of m complex variables $t=(t^1, \cdots, t^m)$. We consider holomorphic functions f and divisors A on $\boldsymbol{C}\times\Omega$. They can be respectively regarded as families of entire functions and divisors on $\boldsymbol{C}$ depending analytically on the parameter $t\in\Omega$. Their orders are then defined as functions of t. In the present note we will investigate the problem: To what extent do the properties corresponding to the above Hadamard theorem remain valid for these analytic families?

For a function or a divisor on $\boldsymbol{C}\times\Omega$ we consider, along with the order $\lambda(t)$, the regularized order $\lambda^*(t)$ introduced by Lelong [7]. They take on the same value except on a pluripolar set in Ω. We shall find that the concept of regularized order is adequate for our investigation since $\lambda^*(t)$ bounds the rate of growth uniformly in the vicinity of the point t in Ω. Some basic properties of $\lambda^*(t)$ are resumed in § 1.

The central part of our problem concerns with the existence of a holomorphic function of finite order with prescribed divisor A. We want to obtain such a function by forming a canonical product for each $t\in\Omega$. To do this the genus q of the canonical product should be chosen. We wish to choose q independently of the parameter t, while $q+1$ cannot be smaller than the order $\lambda_A(t)$ of the divisor A in order to guarantee the convergence. This is impossible when $\lambda_A(t)$ is unbounded. So we first restrict the variability of t to a subdomain Ω' of Ω on which $\lambda_A(t)$ is bounded, and construct canonical products for $t\in\Omega'$. It is crucial to show that this construction actually yields a holomorphic function

$F(z, t)$ on $C\times\Omega'$. This is proved in §2 with the use of the Poisson-Jensen formula relative to the variable z as well as the parameter t. This result is stated in Griffiths [3] without proof.

Every holomorphic function f on $C\times\Omega'$ of finite order with zero locus A has the form $f(z, t)=e^{P(z,t)}F(z, t)$, where $P(z, t)$ is holomorphic on $C\times\Omega'$ and polynomial in z. Two problems are left: (α) Is there an f for which the equality $\lambda_f(t)=\lambda_A(t)$, or at least $\lambda_f^*(t)=\lambda_A^*(t)$, holds? ($\beta$) Is there an f which can be analytically continued to all of $C\times\Omega$? In §3 we show that the answers are negative in general, by giving an example.

§ 1. Orders and regularized orders.

1.1. Let Ω be a domain in C^m and R_+ the real half line: $0\leqq r<+\infty$. We consider a non-negative function $s(r, t)$ on $R_+\times\Omega$ which is non-decreasing relative to r for every fixed t. The order $\lambda_s(t)$ of s at t is defined by

$$\lambda_s(t):=\limsup_{r\to\infty} \log s(r, t)/\log r \quad (\leqq+\infty).$$

Generally, for a subset E of Ω, the order $\lambda_s(E)$ of s on E is defined by

$$\lambda_s(E):=\limsup_{r\to\infty} \sup_{t\in E} \log s(r, t)/\log r .$$

This is equivalent to the following definitions:

$$\lambda_s(E):=\inf\{\mu>0\,|\, s(r, t)/r^{\mu}\to 0 \text{ uniformly on } E \text{ as } r\to\infty\},$$

$$\lambda_s(E):=\inf\left\{\mu>0\,\middle|\int^{+\infty} s(r, t)/r^{\mu+1}dr \text{ is uniformly convergent on } E\right\}.$$

Their equivalence can be shown in a way parallel to the case of a single variable (see for example Nevanlinna [8]).

We define the regularized order $\lambda_s^*(t)$ of s at t by

$$\lambda_s^*(t):=\inf_U \lambda_s(U),$$

where U runs through the set of all neighborhoods U of t. The regularized order $\lambda_s^*(t)$ is upper semi-continuous and satisfies the inequality $\lambda_s(t)\leqq\lambda_s^*(t)$.

1.2. For a real-valued function $v(z, t)$ on $C\times\Omega$, its order $\lambda_v(E)$ and regularized order $\lambda_v^*(t)$ are defined to be those of the function $s(r, t)=\sup_{|z|\leqq r} v^+(z, t)$, where $v^+(z, t)=\max\{v(z, t), 0\}$.

Theorem 1 (*Lelong*). *If $v(z, t)$ is a plurisubharminic function on $C\times\Omega$, then*
(i) *$\lambda_v^*(t)$ is the upper envelope of $\lambda_v(t)$, i.e., $\lambda_v^*(t)=\limsup_{t'\to t} \lambda_v(t')$.*
(ii) *$\lambda_v^*(t)=\lambda_v(t)$ except on a negligible set in Ω.*
(iii) *$-1/\lambda_v^*(t)$ is a plurisubharmonic function on Ω.*

For the proof we refer to Lelong [7, Chap. VI, Théorème 6.6.2.] (see also Kieselman [5]). We need thereby some remarks.
(1) In [7], the relative order and the regularized relative order are treated.

We can easily modify the argument for order and regularized order.
(2) In [7], the regularized (relative) order is defined to be the upper envelope of the (relative) order, differently from our definition. Their equivalence is known by [7, Théorème 6.6.4].
(3) Negligible sets and pluripolar sets are equivalent by a theorem of Bedford-Taylor [1].

Corollary. *$\lambda_v^*(t)$ is either finite throughout Ω or identically infinite.*

Proof. $-1/\lambda_v^*(t)$ is non-positive and plurisubharmonic. Therefore, if $-1/\lambda_v^*(t_0)=0$ at some point t_0 in Ω, then $-1/\lambda_v^*(t)\equiv 0$ on Ω by the maximum principle.

For a holomorphic function $f(z, t)$ on $\boldsymbol{C}\times\Omega$, its order and regularized order are defined by $\lambda_f(t)=\lambda_v(t)$ and $\lambda_f^*(t)=\lambda_v^*(t)$, where $v(z, t)=\log|f(z, t)|$. Since $v(z, t)$ is plurisubharmonic, we have

Theorem 1′. *If f is a holomorphic function on $\boldsymbol{C}\times\Omega$, then $\lambda_f(t)$ and $\lambda_f^*(t)$ have the same properties as in Theorem* 1.

1.3. Now we are about to define the order and the regularized order for a divisor on $\boldsymbol{C}\times\Omega$. Some preliminaries are necessary. A divisor (Cousin II data) on $\boldsymbol{C}\times\Omega$ is represented by a formal sum $A=\sum m_j A_j$, where A_j are analytic sets of codimension 1 in $\boldsymbol{C}\times\Omega$ such that $\operatorname{supp} A=\bigcup A_j$ is also an analytic set in $\boldsymbol{C}\times\Omega$, and m_j are integers>0. We put

$$X=X(A)=\{t\in\Omega\,|\,\boldsymbol{C}\times\{t\}\subset\operatorname{supp} A\}.$$

Then X is an analytic set in Ω, because it is the intersection of the analytic sets $\{t\in\Omega\,|\,(z, t)\in\operatorname{supp} A\}$, $z\in\boldsymbol{C}$. We will first restrict attention to $\Omega\backslash X$.

For every point t in $\Omega\backslash X$, the divisor A cuts out on the complex line $\boldsymbol{C}\times\{t\}$ a divisor A_t, which is represented by a sequence $a_\nu(t)$, $\nu\in N_t$, of the z-coordinates of the points of A_t counted with multiplicity. The set of indices N_t depends on t and it may occur that N_t is finite or empty. Therefore the notation $a_\nu(t)$ does not mean that it is a function of t.

The cardinality of the set $\{\nu\in N_t\,|\,|a_\nu(t)|\leqq r\}$ is denoted by $n(r, t)$, $(r, t)\in\boldsymbol{R}_+\times(\Omega\backslash X)$. The (modified) counting function is defined by

$$N_\delta(R, t)=\int_\delta^R\frac{n(r, t)}{r}\,dr=\sum_{\delta<|a_\nu(t)|\leqq R}\log\frac{R}{|a_\nu(t)|}+n(\delta, t)\log\frac{R}{\delta},$$

for $(R, t)\in\{R>\delta\}\times(\Omega\backslash X)$. Here δ is a constant>0. When $n(0, t)=0$, the case $\delta=0$ is admitted. If δ is replaced by another constant, then $N_\delta(R, t)$ undergoes a change only by a continuous function of t. This subscript δ will be omitted when its choice is irrelevant.

We consider the integrals and the series

$$J_{1,\mu}(R, t)=\int_\delta^R\frac{N(r, t)}{r^{\mu+1}}\,dr,\ J_{2,\mu}(R, t)=\int_\delta^R\frac{n(r, t)}{r^{\mu+1}}\,dr,\ J_{3,\mu}(R, t)=\sum_{\delta<|a_\nu(t)|\leqq R}\frac{1}{|a_\nu(t)|^\mu}.$$

Lemma 1. *If one of the above quantities is uniformly convergent on a subset E of $\Omega\backslash X$ as $R\rightarrow\infty$, then so are the others.*

Proof. First we notice that, by partial integration

$$J_{2,\mu}(R, t)=\int_{\delta}^{R}\frac{dN(r, t)}{r^{\mu}}=\frac{N(R, t)}{R^{\mu}}+J_{1,\mu}(R, t).$$

We let $R\rightarrow\infty$ and examine uniform convergence on E. Suppose that $J_{1,\mu}(R, t)$ is convergent. Then, by the inequality

$$\int_{R}^{\infty}\frac{N(r, t)}{r^{\mu+1}}dr\geqq N(R, t)\int_{R}^{\infty}\frac{1}{r^{\mu+1}}dr=\frac{N(R, t)}{\mu R^{\mu}},$$

it follows that $N(R, t)/R^{\mu}\rightarrow 0$. Hence $J_{2,\mu}(R, t)$ is convergent. To see the converse we notice that, for $R_{0}<R$,

$$\int_{R_{0}}^{R}\frac{n(r, t)}{r^{\mu+1}}dr\geqq\frac{1}{R^{\mu}}\int_{R_{0}}^{R}\frac{n(r, t)}{r}dr=\frac{1}{R^{\mu}}[N(R, t)-N(R_{0}, t)],$$

and hence

$$N(R, t)/R^{\mu}\leqq[J_{2,\mu}(R, t)-J_{2,\mu}(R_{0}, t)]+N(R_{0}, t)/R^{\mu}.$$

Suppose that $J_{2,\mu}(R, t)$ is convergent. The first term on the right tends to 0 as R_{0}, $R\rightarrow\infty$. The second term tends to 0 when R_{0} is fixed and $R\rightarrow\infty$. Therefore $N(R, t)/R^{\mu}\rightarrow 0$ and $J_{1,\mu}(R, t)$ is convergent.

The equivalence of convergence of $J_{2,\mu}(R, t)$ and $J_{3,\mu}(R, t)$ can be shown in the same manner.

By vertue of this lemma, the orders of $n(r, t)$ and $N_{\delta}(r, t)$ coincide. For the divisor A we define its order and regularized order by

$$\lambda_{A}(t)=\begin{cases}\lambda_{n}(t)=\lambda_{N}(t) & \text{for } t\in\Omega\backslash X,\\ 0 & \text{for } t\in X;\end{cases}$$

$$\lambda_{A}^{*}(t)=\begin{cases}\lambda_{n}^{*}(t)=\lambda_{N}^{*}(t) & \text{for } t\in\Omega\backslash X,\\ \limsup\limits_{\Omega\backslash X\ni t'\rightarrow t}\lambda_{A}^{*}(t') & \text{for } t\in X.\end{cases}$$

These definitions will be justified by the following arguments.

1.4. If f is a holomorphic function on $\boldsymbol{C}\times\Omega$ and A is the zero locus of f, then we have by Jensen's formula

$$N_{\delta}(R, t)=\frac{1}{2\pi}\int_{0}^{2\pi}\log|f(Re^{i\theta}, t)|d\theta-\frac{1}{2\pi}\int_{0}^{2\pi}\log|f(\delta e^{i\theta}, t)|d\theta. \tag{1.1}$$

We notice that the both sides of (1.1) depend on the divisor A rather than f. Important consequences can be derived from this formula.

Suppose that a divisor A is given on $\boldsymbol{C}\times\Omega$. Let t_{0} be any point in $\Omega\backslash X$. We choose $\delta\geqq 0$ so that $\operatorname{supp}A\cap(\{|z|=\delta\}\times\{t_{0}\})=\emptyset$, and then choose a polydisc Δ in $\Omega\backslash X$ with center t_{0} so that $\operatorname{supp}A\cap(\{|z|=\delta\}\times\Delta)=\emptyset$.

Lemma 2. *Under this situation $N_{\delta}(|z|, t)$ is a plurisubharmonic function on*

$\{|z|>\delta\}\times\Delta$. (Cf. Kujala [6, Proposition 2.3.].)

Proof. By the classical solution to the Cousin II problem there is a holomorphic function f on $\boldsymbol{C}\times\Delta$ having A as its zero locus. We substitute $R=|z|$ into the formula (1.1). The first integral on the right is a plurisubharmonic function of z and t; the second integral is a pluri*harmonic* function of t, since $f(z, t)\neq 0$ for $|z|=\delta$. This proves Lemma 2.

Theorem 1''. *If A is a divisor on $\boldsymbol{C}\times\Omega$, then $\lambda_A(t)$ and $\lambda_A^*(t)$ have the same properties as in Theorem* 1.

Proof. For $t\in\Omega\backslash X$ the theorem is true by virtue of Lemma 2. The properties (i) and (ii) hold for all $t\in\Omega$ by the definition and the fact that X is a negligible set. To see that $-1/\lambda_A^*(t)$ is plurisubharmonic on Ω, we note that it is bounded above by 0. By a theorem of Grauert and Remmert [2] on extension of plurisubharmonic functions we obtain the required result.

Another consequence of Jensen's formula (1.1) is the following theorem. It is a trivial generalization of the classical result, i.e., the first assertion of Hadamard theorem mentioned in Introduction.

Theorem 2. *If f is a holomorphic function on $\boldsymbol{C}\times\Omega$ and A is the zero locus of f, then we have $\lambda_A(t)\leqq\lambda_f(t)$ and $\lambda_A^*(t)\leqq\lambda_f^*(t)$.*

§2. Analyticity of canonical products.

2.1. Let us first recall results of the classical theory. We consider a divisor $A=\{a_\nu\}_\nu$ on $\boldsymbol{C}$ of finite order λ_A and assume that $a_\nu\neq 0$ for simplicity. Let p_0 be the least integer such that $\sum 1/|a_\nu|^{p_0}$ is convergent. Then $p_0-1\leqq\lambda_A\leqq p_0$. The number p_0-1 is called the genus of A.

For every integer $p\geqq p_0$, the canonical product of genus $p-1$

$$F_p(z)=\prod_\nu\left(1-\frac{z}{a_\nu}\right)\exp\left[\frac{z}{a_\nu}+\frac{1}{2}\left(\frac{z}{a_\nu}\right)^2+\cdots+\frac{1}{p-1}\left(\frac{z}{a_\nu}\right)^{p-1}\right]$$

is convergent and defines an entire function with zero locus A. We note that, if we take the q-th logarithmic derivatives of both sides ($q\geqq p$), we get

$$\left(\frac{F_p'}{F_p}\right)^{(q-1)}=-(q-1)!\,\Phi_q\,,$$

where Φ_q is the meromorphic function defined by the series

$$\Phi_q(z)=\sum_\nu 1/(a_\nu-z)^q.$$

The entire function F_{p_0} has the same order as $A:\lambda_{F_{p_0}}=\lambda_A$. Every entire function f with zero locus A is written in the form $f=e^g F_{p_0}$ with an entire function g. The order λ_f of f is finite if and only if g is a polynomial; and $\lambda_f=\max\{\lambda_A, \deg g\}$. If we take the q-th logarithmic derivative of f ($q\geqq p_0$), we get

$$\left(\frac{f'}{f}\right)^{(q-1)}=g^{(q)}-(q-1)!\,\Phi_q.$$

This implies that, g is of degree$\leqq q-1$ if and only if $(f'/f)^{(q-1)}=-(q-1)!\,\Phi_q$. It follows in particular that F_p is of order$\leqq p$.

In the sequel we will first construct the meromorphic function Φ_p depending analytically on the parameters and then obtain the entire function F_p by integration.

2.2. We consider a divisor A on $\boldsymbol{C}\times\Omega$ whose regularized order $\lambda_A^*(t)$ is bounded on Ω. To be specific we take an integer $p\geqq 1$ and pose the following

Condition (C_p): *The integral* $J_{1,p}(R, t)$ $(J_{2,p}(R, t)$ *or* $J_{3,p}(R, t)$ *equivalently) is uniformly convergent as* $R\rightarrow\infty$ *on every compact set in* $\Omega\backslash X$.

If $p>\lambda_A^*(t)$ for all t in Ω then Condition (C_p) is satisfied; and (C_p) implies $(C_{p'})$ for all $p'\geqq p$.

Theorem 3. *If the divisor A satisfies Condition* (C_p), *then the expression*

$$\Phi_p(z, t)=\Phi_{A,p}(z, t)=\sum_{\nu\in N_t}\frac{1}{(a_\nu(t)-z)^p}$$

defines a meromorphic function on $\boldsymbol{C}\times\Omega$.

Strictly speaking Φ_p is defined only on $\boldsymbol{C}\times(\Omega\backslash X)$. But it is extended meromorphically to all of $\boldsymbol{C}\times\Omega$. We prove first the following lemma. The essential step of this section lies in its proof.

Lemma 3. *In the same situation, suppose further that* supp $A\cap(\{0\}\times\Omega)=\emptyset$. *Then*

$$S_p(t)=S_{A,p}(t)=\sum_{\nu\in N_t}\frac{1}{a_\nu(t)^p}$$

is a holomorphic function on Ω.

Proof. In view of Hartogs' theorem on separate analyticity, it suffices to consider the case $m=1$, i.e., $\Omega\subset\boldsymbol{C}$. By the classical solution to the Cousin II problem, there exists a holomorphic function f on $\boldsymbol{C}\times\Omega$ whose zero locus is A. For convenience we assume $f(0, t)\equiv 1$.

By the Poisson-Jensen formula, we have for $|z|<R$, $t\in\Omega$,

$$\log|f(z, t)|=\frac{1}{2\pi}\int_{|\zeta|=R}\mathrm{Re}\frac{\zeta+z}{\zeta-z}\log|f(\zeta, t)|\frac{d\zeta}{i\zeta}-\sum\log\left|\frac{R^2-\bar{a}_\nu z}{R(z-a_\nu)}\right|,$$

where the sum is taken for all $a_\nu=a_\nu(t)$ such that $|a_\nu|\leqq R$. When the operator $2\partial/\partial z=\partial/\partial x-i\partial/\partial y$ $(z=x+iy)$ is applied to both sides this yields

$$\frac{f_z}{f}(z, t)=\frac{1}{2\pi}\int_{|\zeta|=R}\frac{2\zeta}{(\zeta-z)^2}\log|f(\zeta, t)|\frac{d\zeta}{i\zeta}+\sum\left\{\frac{1}{z-a_\nu}+\frac{\bar{a}_\nu}{R^2-\bar{a}_\nu z}\right\}.$$

Further differentiating $p-1$ times relative to z, we have

$$\left(\frac{\partial}{\partial z}\right)^{p-1}\frac{f_z}{f}(z, t)$$
$$=\frac{p\,!}{2\pi}\int_{|\zeta|=R}\frac{2\zeta}{(\zeta-z)^{p+1}}\log|f(\zeta, t)|\frac{d\zeta}{i\zeta}-(p-1)\,!\sum\left\{\frac{1}{(a_\nu-z)^p}-\frac{\bar{a}_\nu{}^p}{(R^2-\bar{a}_\nu z)^p}\right\}.$$

In particular putting $z=0$, we obtain

$$\left(\frac{\partial}{\partial z}\right)^{p-1}\frac{f_z}{f}(0, t)=p\,!\,I_p(R, t)-(p-1)\,!\,S_p(R, t)\,, \tag{2.1}$$

where

$$\begin{cases} S_p(R, t):=\displaystyle\sum_{|a_\nu|\leqq R}\left\{\frac{1}{a_\nu{}^p}-\frac{\bar{a}_\nu{}^p}{R^{2p}}\right\}, \\ I_p(R, t):=\displaystyle\frac{1}{2\pi}\int_{|\zeta|=R}\frac{2}{\zeta^p}\log|f(\zeta, t)|\frac{d\zeta}{i\zeta} \end{cases} \tag{2.2}$$

The left-hand side of (2.1) is a holomorphic function of t. Both of $S_p(R, t)$ and $I_p(R, t)$ are continuous with respect to R and t. We assert that $S_p(R, t)$ converges to $S_p(t)$ as $R\to\infty$ uniformly on compact sets in Ω. Indeed we have

$$|S_p(R, t)-S_p(t)|\leqq\sum_{|a_\nu|>R}\frac{1}{|a_\nu|^p}+\sum_{|a_\nu|\leqq R}\frac{|\bar{a}_\nu|^p}{R^{2p}}\leqq\sum_{|a_\nu|>R}\frac{1}{|a_\nu|^p}+\frac{n(R, t)}{R^p},$$

which tends to 0 as $R\to\infty$ by Condition (C_p). Consequently $S_p(t)$ is continuous. It follows that $I_p(R, t)$ also converges uniformly on compact sets to a continuous function $I_p(t):=\lim I_p(R, t)$. We have thus obtained the formula

$$S_p(t)=pI_p(t)-\frac{1}{(p-1)\,!}\left(\frac{\partial}{\partial z}\right)^{p-1}\frac{f_z}{f}(0, t)\,. \tag{2.3}$$

The proof is reduced to showing the analyticity of $I_p(t)$.

Let us take a closed disc $|t-t_0|\leqq\rho$ in Ω and restrict our consideration to this disc. With no loss of generality we assume $t_0=0$. We apply the Poisson-Jensen formula with respect to the variable t:

$$\log|f(\zeta, t)|=\frac{1}{2\pi}\int_{|\tau|=\rho}\mathrm{Re}\frac{\tau+t}{\tau-t}\log|f(\zeta, \tau)|\frac{d\tau}{i\tau}-\sum\log\left|\frac{\rho^2-\bar{\alpha}_\kappa t}{\rho(t-\alpha_\kappa)}\right|, \tag{2.4}$$

where $\alpha_\kappa=\alpha_\kappa(\zeta)$ denote the t-coordinates of the intersection points of the divisor A and the disc $\{(\zeta, t)\mid|t|\leqq\rho\}$. We substitute (2.4) into the expression of $I_p(R, t)$ in (2.2):

$$I_p(R, t)=\frac{1}{2\pi}\int_{|\zeta|=R}\frac{2}{\zeta^p}\left[\frac{1}{2\pi}\int_{|\tau|=R}\mathrm{Re}\frac{\tau+t}{\tau-t}\log|f(\zeta, \tau)|\frac{d\tau}{i\tau}\right]\frac{d\zeta}{i\zeta} \tag{2.5}$$
$$-\frac{1}{2\pi}\int_{|\zeta|=R}\frac{2}{\zeta^p}\sum\log\left|\frac{\rho^2-\bar{\alpha}_\kappa t}{\rho(t-\alpha_\kappa)}\right|\frac{d\zeta}{i\zeta}.$$

Reversing the order of integration, we find that the first integral equals to

$$\frac{1}{2\pi}\int_{|\tau|=\rho}\mathrm{Re}\frac{\tau+t}{\tau-t}I_p(R, \tau)\frac{d\tau}{i\tau}.$$

This converges as $R\to\infty$ to

$$\frac{1}{2\pi}\int_{|\tau|=\rho}\mathrm{Re}\frac{\tau+t}{\tau-t}I_p(\tau)\frac{d\tau}{i\tau}, \tag{2.6}$$

which is harmonic in $|t|<\rho$.

As for the second integral in (2.5), we notice first that $\log\left|\dfrac{\rho^2-\bar{\alpha}_\kappa t}{\rho(t-\alpha_\kappa)}\right|\geqq 0$, so that the absolute value of the integral is bounded by

$$\frac{1}{2\pi}\int_{|\zeta|=R}\frac{2}{R^p}\sum\log\left|\frac{\rho^2-\bar{\alpha}_\kappa t}{\rho(t-\alpha_\kappa)}\right|\frac{d\zeta}{i\zeta}.$$

We want to show that this quantity tends to 0 as $R\to\infty$. For this purpose we consider the mean value of (2.4) over the circle $|\zeta|=R$:

$$N_0(R,\,t)=\frac{1}{2\pi}\int_{|\tau|=\rho}\mathrm{Re}\frac{\tau+t}{\tau-t}N_0(R,\,\tau)\frac{d\tau}{i\tau}-\frac{1}{2\pi}\int_{|\zeta|=R}\sum\log\left|\frac{\rho^2-\bar{\alpha}_\kappa t}{\rho(t-\alpha_\kappa)}\right|\frac{d\zeta}{i\zeta}. \tag{2.7}$$

Here we have used that

$$N_0(R,\,t)=\frac{1}{2\pi}\int_{|\zeta|=R}\log|f(\zeta,\,t)|\frac{d\zeta}{i\zeta},$$

which follows from Jensen's formula and the assumption $f(t,\,0)=1$. We divide the both sides of (2.7) by R^p. When $R\to\infty$, the quotients tend uniformly to 0 on the disc $|t|\leqq\rho$, because so is $N_0(R,\,t)/R^p$. It follows that the second integral in (2.5) tends to 0.

Thus $I_p(t)$ is represented by the Poisson integral (2.6), so that it is harmonic. By (2.3) the series $S_p(t)$ is also harmonic.

To prove that $S_p(t)$ is holomorphic, it suffices to see that, for a non-constant holomorphic function $\varphi(t)$, the product $\varphi(t)S_p(t)$ is also harmonic. Indeed, if this is the case, then from

$$0=\frac{\partial^2}{\partial t\partial\bar{t}}(\varphi S_p)=\frac{\partial}{\partial t}\left(\varphi\frac{\partial S_p}{\partial\bar{t}}\right)=\frac{d\varphi}{dt}\frac{\partial S_p}{\partial\bar{t}}$$

it will follow that $\partial S_p/\partial\bar{t}=0$.

We introduce a new coordinate system $(z',\,t)$ on $\boldsymbol{C}\times\Omega$ defined by $z'=\psi(t)z$, where $\psi(t)$ is holomorphic, non-vanishing and non-constant on Ω. Let $S'_p(t)$ denote the series corresponding to $S_p(t)$ relative to the new coordinate z':

$$S'_p(t)=\sum_{\nu\in N_t}\frac{1}{(\psi(t)a_\nu(t))^p}=\frac{1}{\psi(t)^p}S_p(t)\,.$$

Performing the argument relative to the coordinate z', we know that $S'_p(t)$ is also harmonic. Hence $S_p(t)$ is holomorphic. q. e. d.

Proof of Theorem 3. First we show the meromorphy of $\Phi_{A,p}$ on $\boldsymbol{C}\times(\Omega\backslash X)$. Let $(z_0,\,t_0)$ be any point in $\boldsymbol{C}\times(\Omega\backslash X)$. Choose $R_0>|z_0|$ such that $\operatorname{supp}A\cap(\{|z|=R_0\}\times\{t_0\})=\emptyset$, and then choose a polydisc $\varDelta$ in $\Omega\backslash X$ with center t_0 such that $\operatorname{supp}A\cap(\{|z|=R_0\}\times\bar{\varDelta})=\emptyset$. The restriction of the divisor A to $\boldsymbol{C}\times\varDelta$ is decomposed as $A|_{\boldsymbol{C}\times\varDelta}=A'+A''$ in such a way that $\operatorname{supp}A'\subset\{|z|<R_0\}\times\varDelta$, $\operatorname{supp}A''\subset\{|z|>R_0\}\times\varDelta$. Correspondingly we have the decomposition $\Phi_{A,p}=\Phi_{A',p}+\Phi_{A'',p}$, where

$$\begin{cases} \Phi_{A',p}(z,\ t)=\sum_{|a_\nu|<R_0} \dfrac{1}{(a_\nu(t)-z)^p}, \\ \Phi_{A'',p}(z,\ t)=\sum_{|a_\nu|>R_0} \dfrac{1}{(a_\nu(t)-z)^p}. \end{cases}$$

The divisor A' is the zero locus of the polynomial in z:

$$W(z,\ t)=\prod_{|a_\nu|<R_0}(z-a_\nu(t)),$$

which is holomorphic on $\boldsymbol{C}\times\Delta$ by Weierstrass preparation theorem. Therefore

$$\Phi_{A',p}(z,\ t)=\frac{1}{(p-1)\,!}\left(\frac{\partial}{\partial z}\right)^{p-1}\frac{W_z}{W}(z,\ t)$$

is meromorphic on $\boldsymbol{C}\times\Delta$. On the other hand, $\Phi_{A'',p}$ is expanded into the power series in z:

$$\Phi_{A'',p}(z,\ t)=S_{A'',p}(t)+pS_{A'',p+1}(t)z+\cdots+\binom{p+k-1}{p-1}S_{A'',p+k}(t)z^k+\cdots. \tag{2.8}$$

We apply Lemma 3 to the divisor A'' and know that $S_{A'',p+k}(t)$, $k=0, 1, 2, \cdots$, are holomorphic on Δ. Further, we have

$$|S_{A'',p+k}(t)|\leqq \sum_{|a_\nu|>R_0}\frac{1}{|a_\nu(t)|^{p+k}}\leqq\frac{1}{R_0{}^k}\sum_{|a_\nu|>R_0}\frac{1}{|a_\nu(t)|^p}.$$

Since this last series is uniformly convergent on $\bar{\Delta}$ by Condition (C_p), the sum is bounded by a constant. It follows that the series (2.8) is uniformly convergent on compact sets in $\{|z|<R_0\}\times\bar{\Delta}$, and $\Phi_{A'',p}$ is holomorphic there. Thus $\Phi_{A,p}=\Phi_{A',p}+\Phi_{A'',p}$ is meromorphic in a neighborhood of $(z_0,\ t_0)$.

Now we are to show that $\Phi_{A,p}$ can be extended to all of $\boldsymbol{C}\times\Omega$. Let X^1 be the union of all irreducible components of codimension 1 of $X=X(A)$. Then $\boldsymbol{C}\times X^1$ consists of some irreducible components of supp A. We decompose the divisor $A=\sum m_jA_j$ into the sum $A=A^0+A^1$, where the divisor A^0 is the sum over the components $A_j\not\subset\boldsymbol{C}\times X^1$ and A^1 over $A_j\subset\boldsymbol{C}\times X^1$. It is clear that $X(A^1)=X^1$ and that $X(A^0)$ is of codimension$\geqq 2$. We apply what we have shown thus far to the divisor A^0 in place of A, and obtain a meromorphic function $\Phi_{A^0,p}$ on $\boldsymbol{C}\times(\Omega\backslash X(A^0))$. Obviously $A_t=A^0{}_t$ for $t\in\Omega\backslash X$, and hence $\Phi_{A^0,p}=\Phi_{A,p}$ on $\boldsymbol{C}\times(\Omega\backslash X)$. This implies that $\Phi_{A,p}$ is extended to $\boldsymbol{C}\times(\Omega\backslash X(A^0))$. Now that $\boldsymbol{C}\times X(A^0)$ is of codimension$\geqq 2$, we can extend $\Phi_{A,p}$ to the whole $\boldsymbol{C}\times\Omega$ by Levi's theorem on analytic continuation of meromorphic functions. Thus Theorem 3 is proved.

It should be remarked that the set of poles (and indetermination points) of $\Phi_{A,p}$ is exactly supp A^0.

2.3. We continue considering a divisor A on $\boldsymbol{C}\times\Omega$ satisfying Condition (C_p).

Theorem 4. *Suppose that* supp $A\cap(\{0\}\times\Omega)=\emptyset$. *Then the cononical product of genus* $p-1$

$$F_p(z, t)=\prod_{\nu\in N_t}\Big(1-\frac{z}{a_\nu(t)}\Big)\exp\Big[\frac{z}{a_\nu(t)}+\frac{1}{2}\Big(\frac{z}{a_\nu(t)}\Big)^2+\cdots+\frac{1}{p-1}\Big(\frac{z}{a_\nu(t)}\Big)^{p-1}\Big]$$

is a holomorphic function on $\boldsymbol{C}\times\Omega$ *with zero locus* A *such that*

$$\begin{cases} \lambda_A(t)\leqq\lambda_{F_p}(t)\leqq\max\{p-1,\ \lambda_A(t)\}, \\ \lambda_A^*(t)\leqq\lambda_{F_p}^*(t)\leqq\max\{p-1,\ \lambda_A^*(t)\}. \end{cases} \tag{2.9}$$

Proof. We integrate $-(p-1)!\,\Phi_p(z, t)$ relative to z from the origin to z termwise $p-1$ times for each fixed t in Ω, and take the exponential. Then we obtain F_p. Its analyticity is obvious from this construction. The first line of (2.9) is a consequence of the one variable theory. The second is obtained by taking the upper envelopes. Thus Theorem 4 is proved.

In particular this theorem shows that the condition supp $A\cap(\{0\}\times\Omega)=\emptyset$ implies the solvability of the the Cousin II problem. In the general case we must pose the latter condition.

Theorem 5. *Suppose that the Cousin II problem is solvable for the divisor* A *on* $\boldsymbol{C}\times\Omega$. *Then there is a holomorphic function* F *on* $\boldsymbol{C}\times\Omega$ *with the zero locus* A *satisfying the inequalities* (2.9).

Proof. Let f be a holomorphic function on $\boldsymbol{C}\times\Omega$ with zero locus A. Consider its p-th logarithmic derivative $(\partial/\partial z)^{p-1}(f_z/f)$. It is defined at first only on $\boldsymbol{C}\times(\Omega\backslash X)$, but is extended meromorphically to $\boldsymbol{C}\times\Omega$. To see this we take again the decomposition of the divisor $A=A^0+A^1$ as in the proof of Theorem 3. For any t_0 in X we take a polydisc Δ in Ω containing t_0. We can factorize f into the product $f=f^0f^1$ on $\boldsymbol{C}\times\Delta$. Here f^0, f^1 are holomorphic on $\boldsymbol{C}\times\Delta$ and have the zero loci A^0, A^1 respectively; and f^1 depends only on t. Then we have $(\partial/\partial z)^{p-1}(f_z/f)=(\partial/\partial z)^{p-1}(f^0{}_z/f^0)$ on $\boldsymbol{C}\times(\Delta\backslash X)$. The right-hand side provides the extension of $(\partial/\partial z)^{p-1}(f_z/f)$ to $\boldsymbol{C}\times(\Delta\backslash X(A^0))$ and further to $\boldsymbol{C}\times\Delta$ by Levi's theorem. This shows our assertion, and also that the poles of $(\partial/\partial z)^{p-1}(f_z/f)$ lie on supp A^0.

The meromorphic functions $(\partial/\partial z)^{p-1}(f_z/f)$ and $-(p-1)!\,\Phi_p$ have the same principal part, i.e.,

$$h(z, t)=\Big(\frac{\partial}{\partial z}\Big)^{p-1}\frac{f_z}{f}(z, t)+(p-1)!\,\Phi_p(z, t)$$

is holomorphic. This is obvious on $\boldsymbol{C}\times(\Omega\backslash X)$; and since the right-hand side has the poles only on supp A^0, h is holomorphic on $\boldsymbol{C}\times\Omega$.

We integrate h relative to z from the origin to z, $p-1$ times for each fixed t in Ω and obtain a holomorphic function g on $\boldsymbol{C}\times\Omega$. We define

$$F(z, t)=f(z, t)\exp[-g(z, t)].$$

Then F has the same zero locus A. The p-th logarithmic derivative of F relative to z equals to $-(p-1)!\,\Phi_p$. From this fact follows the inequalities (2.9) as was mentioned in 2.1. q. e. d.

§ 3. Analytic continuation of canonical products.

3.1. We discuss an example to illustrate the problem.

Let Ω be the right half plane: $\mathrm{Re}\, t>0$ in $\boldsymbol{C}$. We notice that every Cousin II problem on $\boldsymbol{C}\times\Omega$ is solvable. We give on $\boldsymbol{C}\times\Omega$ a divisor $A=\sum_{n=1}^{\infty}A_n$ where $A_n=\{(z, t)\in\boldsymbol{C}\times\Omega \mid z=n^t\}$. The restriction A_t of the divisor A is represented by $a_n(t)=n^t$, $n=1, 2, \cdots$.

1) To find the order of A, it suffices to examine the series

$$\sum |a_n(t)|^{-\mu}=\sum n^{-\mu\,\mathrm{Re}\, t}, \qquad \mu>0,$$

which is convergent if $\mu\,\mathrm{Re}\, t>1$ and divergent if $\mu\,\mathrm{Re}\, t\leqq 1$. It follows that

$$\lambda_A^*(t)=\lambda_A(t)=1/\mathrm{Re}\, t.$$

2) Let $\Omega_p=\{\mathrm{Re}\, t>1/p\}$, $p=1, 2, \cdots$. On Ω_p we have $\lambda_A^*(t)<p$ and Condition (C_p) is satisfied. Hence on $\boldsymbol{C}\times\Omega_p$ we have the meromorphic function

$$\Phi_p(z, t)=\sum_{n=1}^{\infty}\frac{1}{(n^t-z)^p} \tag{3.1}$$

by Theorem 3, and the holomorphic function

$$F_p(z, t)=\prod_{n=1}^{\infty}\left(1-\frac{z}{n^t}\right)\exp\left[\frac{z}{n^t}+\cdots+\frac{1}{p-1}\left(\frac{z}{n^t}\right)^{p-1}\right] \tag{3.2}$$

by Theorem 4.

3) We examine whether analytic continuation of F_p to $\boldsymbol{C}\times\Omega$ is possible. We compare the expressions (3.2) for F_p and F_q $(p<q)$:

$$\begin{aligned}\frac{F_q(z, t)}{F_p(z, t)}&=\exp\left[\frac{1}{p}\sum\left(\frac{z}{n^t}\right)^p+\cdots+\frac{1}{q-1}\sum\left(\frac{z}{n^t}\right)^{q-1}\right]\\&=\exp\left[\frac{1}{p}\zeta(pt)z^p+\cdots+\frac{1}{q-1}\zeta((q-1)t)z^{q-1}\right] \quad \text{on} \quad \boldsymbol{C}\times\Omega_p.\end{aligned} \tag{3.3}$$

Here $\zeta(s)$ is Riemann's zeta function; it is defined by the series $\sum n^{-s}$ for $\mathrm{Re}\, s>1$, and continued to a meromorphic function on $\boldsymbol{C}$ with a simple pole at $s=1$ and holomorphic elsewhere. The right-hand side of (3.3) is holomorphic except for the essential singularities on the lines $t=1/p$, $1/(p+1)$, $\cdots$, $1/(q-1)$. By means of the relation (3.3) we can extend F_p to a holomorphic function on $\boldsymbol{C}\times(\Omega_q\backslash\{1/p, \cdots, 1/(q-1)\})$. This procedure is possible for all $q>p$. Hence F_p is extended to $\boldsymbol{C}\times(\Omega\backslash\{1/p, 1/(p+1), \cdots\})$ with essential singularities on the lines $t=1/p$, $1/(p+1)$, $\cdots$. It is also clear from (3.3) that the extended function F_p has the zero locus A.

4) The order of F_p is given by

$$\lambda_{F_p}(t)=\begin{cases}1/\mathrm{Re}\, t & \text{on} \quad \Omega\backslash\{1/p, 1/(p+1), \cdots\}\backslash\Omega_{p-1},\\ p-1, & \text{on} \quad \Omega_{p-1}.\end{cases} \tag{3.4}$$

To prove this we first notice that $\lambda_{F_p}(t)\geqq\lambda_A(t)=1/\mathrm{Re}\, t$ for all t by Theorem 2, and that

$$(3.5)_p \qquad \begin{cases} \lambda_{F_p}(t)=\lambda_A(t)=1/\mathrm{Re}\, t\,, & \text{on} \quad \Omega_p\backslash\Omega_{p-1}\,, \\ \lambda_{F_p}(t)\leqq p-1\,, & \text{on} \quad \Omega_{p-1} \end{cases}$$

by Theorem 4. Hence (3.4) is true for $t\in\Omega_p\backslash\Omega_{p-1}$. Suppose that $t\in\Omega\backslash\{1/p, 1/(p+1), \cdots\}\backslash\Omega_p$. Then $t\in\Omega_q\backslash\Omega_{q-1}$ for some $q>p$, and we have

$$F_p(z, t)=F_q(z, t)\exp\left[-\frac{1}{p}\zeta(pt)z^p-\cdots-\frac{1}{q-1}\zeta((q-1)t)z^{q-1}\right].$$

We have $\lambda_{F_q}(t)=1/\mathrm{Re}\, t\geqq q-1$ by $(3.5)_q$, and the exponential factor is of order $\leqq q-1$. Hence $\lambda_{F_p}(t)\leqq\lambda_{F_q}(t)=1/\mathrm{Re}\, t$. This shows the first case in (3.4). Now suppose that $t\in\Omega_{p-1}$. Then $t\in\Omega_{q'}\backslash\Omega_{q'-1}$ for some $q'<p$, and we have

$$F_p(z, t)=F_{q'}(z, t)\exp\left[\frac{1}{q'}\zeta(q't)z^{q'}+\cdots+\frac{1}{p-1}\zeta((p-1)t)z^{p-1}\right].$$

We have $\lambda_{F_{q'}}(z, t)\leqq q'-1<p-1$ by $(3.5)_{q'}$. The exponential factor is exactly of order $p-1$, since $\zeta((p-1)t)\neq 0$ if $\mathrm{Re}((p-1)t)>1$. Hence $\lambda_{F_p}(t)=p-1$. This proves the second case in (3.4).

Since $\lambda_{F_p}(t)$ is continuous we have $\lambda^*_{F_p}(t)=\lambda_{F_p}(t)$.

5) Now we examine analytic continuation of Φ_p. For this purpose it suffices to consider the p-th logarithmic derivative of (3.3) relative to z and recall that

$$\Phi_p(z, t)=-\frac{1}{(p-1)\,!}\left(\frac{\partial}{\partial z}\right)^{p-1}\frac{(F_p)_z}{F_p}(z, t)\,.$$

We obtain for $q>p$

$$\Phi_p(z, t)=-\frac{1}{(p-1)\,!}\left(\frac{\partial}{\partial z}\right)^{p-1}\frac{(F_q)_z}{F_q}(z, t)+\zeta(pt)+p\zeta((p+1)t)z+\cdots$$
$$+\binom{q-1}{p-1}\zeta((q-1)t)z^{q-p-1}, \qquad \text{on} \quad \boldsymbol{C}\times\Omega_p\,.$$

This implies that Φ_p is extended to $\boldsymbol{C}\times\Omega_q$ with poles on supp A and on the lines $t=1/p,\ 1/(p+1), \cdots, 1/(q-1)$. Since this is valid for all $q>p$, Φ_p can be extended to all of $\boldsymbol{C}\times\Omega$ with poles on supp A and on the lines $t=1/p,\ 1/(p+1), \cdots$.

6) *Let Δ be a (connected) subdomain of Ω containing the point $t=1/p$ and f be a holomorphic function on $\boldsymbol{C}\times\Delta$ with zero locus A. Then $\lambda^*_f(t)\geqq p$ for all t in Δ.*

We prove this by contradiction. Suppose that there is a point t_0 in Δ for which $\lambda^*_f(t_0)<p$. Then there is a neighborhood U of t_0 on which $\lambda_f(t)<p$. (Necessarily $U\subset\Delta\cap\Omega_p$.) We consider the p-th logarithmic derivative of f. As was mentioned in 2.1, we have

$$(f_z/f)^{(p-1)}=-(p-1)\,!\,\Phi_p\,.$$

on $\boldsymbol{C}\times U$. This identity should be extended to all of $\boldsymbol{C}\times\Delta$. The left-hand side is meromorphic and has the poles exactly on supp A, while the right-hand side has the poles on supp A and on the line $t=\dfrac{1}{p}$, which is a contradiction.

Thus the present example shows that the complete analogue of the second assertion of the Hadamard theorem mentioned in Introduction is no more valid, even locally, for the case of analytic families.

7) *Let f be a holomorphic function on $C\times\Omega$ whose zero locus is A. Then $\lambda_f^*(t)\equiv+\infty$ on Ω.*

This follows immediately from 6).

3.2. The above observations can be formulated in the general situation. Let Ω be a domain in C^m and A a divisor on $C\times\Omega$. Recall that A is decomposed into the sum $A=A^0+A^1$ as in the proof of Theorem 3. Suppose that A is of finite order and that Cousin II problem is solvable for A. We take an integer p and a subdomain Ω' of Ω such that $\lambda_A^*(t)<p$ for $t\in\Omega'$. Then Condition (C_p) is satisfied on Ω' and we have the meromorphic function Φ_p on $C\times\Omega'$. Under this situation we have

Theorem 6. *There exists a holomorphic function F on $C\times\Omega$ with zero locus A such that $\lambda_F^*(t)<p$ for $t\in\Omega'$ if and only if Φ_p is extended to a meromorphic function on $C\times\Omega$ whose poles lie exactly on* supp A^0.

Proof. If there is such an F, then $-\frac{1}{(p-1)!}\left(\frac{\partial}{\partial z}\right)^{p-1}\frac{F_z}{F}$ provides the desired extension of Φ_p. Conversely if Φ_p admits such an extension, then we can construct an F with the desired properties by the same procedure as in the proof of Theorem 5. q. e. d.

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