J. Math. Kyoto Univ. (JMKYAZ)
25-3 (1985) 473-488

Moment estimates for parabolic equations in the divergence form

By

Hirofumi OSADA

(Received May 29, 1984)

§ 0. Introduction and main results.

In [4], J. Nash obtained the following estimate; if $(a_{ij}(t, x))$ satisfies $a_{ij}=a_{ji}$ and

$$C_1|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}(t, x)\xi_i\xi_j \leqq C_2|\xi|^2, \ \xi=(\xi_i)\in R^n, \tag{0.1}$$

$(t, x)\in[0, \infty)\times R^n$ for some constants $0<C_1<C_2$, then there exist constants $0<C_3<C_4$ depending only on C_1 and C_2 such that

$$C_3(n(t-s))^{1/2} \leqq \int_{R^n}\Big(\sum_{i=1}^{n}(y_i-x_i)^2\Big)^{1/2} p(s, x, t, y)\,dy \leqq C_4(n(t-s))^{1/2}, \tag{0.2}$$

where $p(s, x, t, y)$ is the fundamental solution of

$$\nabla_t - A = \nabla_t - \sum_{i,j=1}^{n} \nabla_i a_{ij} \nabla_j \Big(\nabla_t = \frac{\partial}{\partial t}, \ \nabla_i = \frac{\partial}{\partial x_i}\Big).$$

An importance of the estimate (0.2) lies in the fact that C_3 and C_4 are independent of the dimension n. The purpose of this paper is to obtain similar results for general moments of $p(s, x, t, y)$.

For this we assume the following conditions on $(a_{ij}(t, x))$((a_{ij}) is not necessarily symmetric).

$$\sum_{i,j=1}^{n} a_{ij}(t, x)\xi_i\xi_j \geqq \nu|\xi|^2 \qquad \text{for every} \quad (t, x)\in[0, \infty)\times R^n. \tag{0.3}$$

(a_{ij}) is in the form; there exist a symmetric matrix $(b_{ij}(t))$ and a matrix $(c_{ij}(t, x))$ such that $a_{ij}(t, x)=b_{ij}(t)+c_{ij}(t, x)$,

$$\sup_{1\leq j\leq n} \sum_{i=1}^{n} |b_{ij}(t)| \leqq \lambda \tag{0.4}$$

and

$$\sup_{1\leq i,j\leq n} |c_{ij}(t, x)| \leqq \frac{\mu}{n} \qquad \text{for every} \quad (t, x)\in[0, \infty)\times R^n. \tag{0.5}$$

Here we assume that the constants λ, μ, ν are independent of the dimension n. The assumption $b_{ij}=b_{ji}$ is no loss of generality because, if $b_{ij}=-b_{ji}$, then we have

$$\sum_{i,j=1}^{n} \nabla_i(b_{ij}\nabla_j) = \sum_{i,j=1}^{n} (\nabla_i b_{ij})\nabla_j + \sum_{i,j=1}^{n} b_{ij}\nabla_i\nabla_j = 0.$$

Under these assumptions we obtain the following results.

Theorem 1. *Let p and q be non-negative integers. Then there exist positive constants C_5 and C_6 depending only on λ, μ, ν, p and q such that*

$$\begin{aligned} (0.6) \qquad & C_5 n^{q+1}(t-s)^{(p+q)/2} \\ & \leqq \int_{R^n} \Big(\sum_{i=1}^{n} |x_i - y_i|^q\Big)\Big(\sum_{i=1}^{n} |x_i - y_i|\Big)^q p(s, x, t, y)\mathrm{d}y \\ & \leqq C_6 n^{q+1}(t-s)^{(p+q)/2} \end{aligned}$$

for all $x \in R^n$, $0 \leqq s < t < \infty$.

In the case of $p=0$ and $q=1$ (or $p=1$ and $q=0$), we have more precisely

Theorem 2. *For any* $x \in R^n$,

$$(0.7) \qquad \varliminf_{n\to\infty} \inf_{\substack{t>s \\ x\in R^n}} \left\{ \frac{\int_{R^n} \sum_{i=1}^{n} |x_i - y_i| p(s, x, t, y)\mathrm{d}y}{n(t-s)^{1/2}} \right\} \geqq (\nu\pi e^{-3})^{1/2} e^{\iota^-}$$

$$(0.8) \qquad \varlimsup_{n\to\infty} \sup_{\substack{t>s \\ x\in R^n}} \left\{ \frac{\int_{R^n} \sum_{i=1}^{n} |x_i - y_i| p(s, x, t, y)\mathrm{d}y}{n(t-s)^{1/2}} \right\} \leqq \frac{(\mu+\lambda)(\iota^+ + 2)}{\sqrt{2\nu}}$$

where ι^- and $\iota^+ (\iota^- < \iota^+)$ are roots of the equation

$$e^x = \left(\frac{e^3}{2\pi}\right)^{1/2} (\mu+\lambda)(x+2).$$

Note that the equation $e^x = C(x+2)$ has just two real roots when $e^{-1} < C < \infty$.

Our study of such estimates for moments is motivated by the following problem of interacting diffusion processes in probability theory. Let L_n be a diffusion operator given formally by

$$(0.9) \qquad L_n = \frac{1}{2}\Delta + \frac{1}{2(n-1)} \sum_{i\neq j}^{n} \delta(x_i - x_j)\nabla_j,$$

which can be rewritten as

$$(0.10) \qquad L_n = \frac{1}{2}\Delta + \frac{1}{2(n-1)} \sum_{i,j=1}^{n} \nabla_i H(x_i - x_j)\nabla_j,$$

where $H(t) = (1/2)1_{(t>0)} - (1/2)1_{(t<0)}$. Therefore we can define the diffution process $\{X_n(t)\} = \{(X_n^1(t), \cdots, X_n^n(t))\}$ with generator L_n by the fundamental solution of $\nabla_t - L_n$. The problem posed by H. P. McKean in [3] is to show the propagation of chaos for this $\{X_n(\cdot)\}$; that is, for any fixed integer m, if $\{X_n(\cdot)\}$ starts from some identically and independently distribution, then $\{Y_n^m(\cdot)\} = \{(X_n^1(\cdot), \cdots, X_n^m(\cdot))\}$ converges in $C([0, \infty) \to R^m)$ to the m-independent copies of the Burgers processes. One of the key steps to show this fact is to establish the tightness of the marginal processes $\{Y_n^m(\cdot)\}_{n=m}^{\infty}$. However this follows immediately from Theorem 1 if we observe the symmetry of the process $\{X_n(\cdot)\}$. Actually in Theorem 1 setting $(p, q)=(3, 0)$, we have

$$E\left\{\sum_{i=1}^{n}|X_n^i(t)-X_n^i(s)|^3\right\}\leqq C_7 n|t-s|^{3/2}.$$

This together with symmetry of $\{X_n(\cdot)\}$ implies

$$E\{|X_n^i(t)-X_n^i(s)|^3\}\leqq C_7|t-s|^{3/2}$$

for every $1\leqq i\leqq n$, which assures the tightness of $\{Y_n^m(\cdot)\}_{n=m}^{\infty}$. Although it is not complete, a partial result for the convergence of $\{Y_n^m(\cdot)\}$ as $n\to\infty$ has been given in [6].

The author would like to express his profound gratitude to Profs. S. Watanabe, S. Kotani and T. Fujita for their valuable comments and encouragement.

§ 1. Preliminary lemmas and notatinos.

In this section we prepare some lemmas and notations which will be used in the subsequent sections. First, we clarify the notion of fundamental solution of $\nabla_t - A=\nabla_t-\sum_{i,j=1}^{n}\nabla_i a_{ij}\nabla_j$.

Definition 1.1. A measurable function $p(s, x, t, y)$ is said to be a fundamental solution of $\nabla_t - A$ if it satisfies the following conditions:

(i) $p(s, x, t, y)\geqq 0$ and $\int_{R^n}p(s, x, t, y)\mathrm{d}y=1$.

(ii) Let $s\geqq 0$ be fixed and φ be a continuous function on R^n with compact support. Set $u(t, y)=\int_{R^n}p(s, x, t, y)\varphi(x)dx$. Then $\nabla_i u\in L^2([a, b]\times R^n)$ for all $s<a<b<\infty$, $u(t, y)\to\varphi(y)$ uniformly on R^n as $t\to s(t>s)$ and

$$\int_s^{\infty}\int_{R^n}\left\{u\nabla_t\phi-\sum_{i,j=1}^{n}a_{ji}\nabla_i u\nabla_j\phi\right\}\mathrm{d}y\mathrm{d}t=0$$

for every continuously differentiable function $\phi(t, y)$ on $(s, \infty)\times R^n$ with compact support.

Lemma 1.1. *There exists a unique fundamental solution of* $\nabla_t - A$. *It is continuous and satisfies*

(i) $\int_{R^n}p(s, x, t, y)p(t, y, u, z)\mathrm{d}y=p(s, x, u, z)$

(ii) $C_1(t-s)^{-(1/2)n}\exp\{-C_2|x-y|^2/(t-s)\}\leqq p(s, x, t, y)$

$$\leqq C_3(t-s)^{-(1/2)n}\exp\{-C_4|x-y|^2/(t-s)\}$$

for $0\leqq s<t<u<\infty$, x, y, $z\in R^n$, *where* C_1, C_2, C_3 *and* C_4 *are positive constants depending only on* λ, ν, μ *and* n.

See Aronson [1], [2] and Nash [4] for the proof.

The next lemma is due to J. Nash [4] and plays an important role in our argument.

Lemma 1.2.

$$p(s, x, t, y) \leqq (f_n \nu \pi (t-s))^{-(1/2)n} \quad where \quad f_n = \left(\frac{8}{n+2}\right)\left(\frac{((1/2)n)!}{1+(1/n)}\right)^{2/n}.$$

Since the constants appearing in Theorem 1 and 2 do not depend on the smoothness of (a_{ij}), in order to show Theorem 1, we may assume that (a_{ij}) are smooth and hence the strong perivative $\nabla_t p(s, x, t, y)$ exists. We may also assume $(s, x)=(0, 0)$.

We introduce some notations.

$$P = p(0, 0, t, x),$$

$$X_p = \sum_{i=1}^{n} |x_i|^p \quad \text{for} \quad x=(x_i) \in R^n,$$

$$X^q = (X_1)^q = \left(\sum_{i=1}^{n} |x_i|\right)^q.$$

For integers p and q, we set

$$M(p, q) = \int_{R^n} X_p X^q P \, dx \geqq 0,$$

$$E(p, q) = -\int_{R^n} X_p X^q P \log P \, dx,$$

$$F(p, q) = E(p, q) - \frac{1}{2} n M(p, q) \log (f_n \nu \pi t) \geqq 0,$$

where f_n is the constant defined in Lemma 1.2,

$$G(p, q) = \int_{R^n} X_p X^q P \sum_{i=1}^{n} (\nabla_i \log P)^2 dx \geqq 0,$$

$$H(p, q) = \int_{R^n} X_p X^q P (\log P)^2 dx \geqq 0,$$

$$J(p, q) = \int_{R^n} X_p X^q \sum_{i=1}^{n} |\nabla_i P \log P| dx = \int_{R^n} X_p X^q \sum_{i=1}^{n} |(\nabla_i \log P) \log P| P dx \geqq 0,$$

$$K(p, q) = \int_{R^n} X_p X^q \sum_{i=1}^{n} |\nabla_i P| dx = \int_{R^n} X_p X^q \sum_{i=1}^{n} |\nabla_i \log P| P dx \geqq 0,$$

$$L(p, q) = \int_{R^n} X_p X^q |\log P| P dx \geqq 0.$$

Also we set

$$M^*(p, q) = \begin{cases} M(p, q), & p \geqq 0 \text{ and } q \geqq 0 \\ 0 \quad , & p<0 \text{ or } q<0, \end{cases}$$

and similary for $E^*(p, q)$, $F^*(p, q)$, $\cdots$, $L^*(p, q)$.

Theorem 1 is obviously concerned with the estimate of $M(p, q)$. In [4] Nash estimated $M(0, 1)$ by obtaining inequalities between the moment $M(0, 1)$ and the entropy $E(0, 0)$. Here we would like to follow this idea of Nash by obtaining similar inequalities among $M(p, q)$ and $E(p, q-1)$. In obtaining these inequalities, however, there involve the auxiliary quantities $F(p, q-1)$, $G(p, q-1)$, $\cdots$, $L(p, q-1)$; but these quantities can be managed by the induction on p and q. In the subsequent section 2, we obtain estimates which is necessary to perform the induction. Before proceeding, we state some simple properties of these quantities in the following lemma.

Lemma 1.3.

$$J(p, q) \leqq (nG(p, q)H(p, q))^{1/2},$$

$$K(p, q) \leqq (nG(p, q)M(p, q))^{1/2},$$

$$L(p, q) \leqq (H(p, q)M(p, q))^{1/2}.$$

Lemma 1.3 is obtained by the Schwarz inequality immediately.

§2. Recurrence inequalities among $M, E, G, \cdots$.

Main object of this section is to obtain Lemma 2.4 and Lemma 2.5 which are key lemmas for the induction procedure in section 3.

We denote by $I_i(x)$ the function $I(x_i)$, $i=1, 2, \cdots, n$, $x=(x_1, \cdots, x_n)$ where $I(t)=1_{\{t>0\}}-1_{\{t<0\}}$.

Lemma 2.1. *For non-negative integers p and q and for any t,*

$$|\nabla_t M(p, q)| \leqq p(p-1)\lambda M^*(p-2, q)+pq\lambda M^*(p-1, q-1) + pn^{-1}(\mu+\lambda)K^*(p-1, q)+q(\mu+\lambda)K^*(p, q-1). \tag{2.1}$$

Proof. We show (2.1) in the case $p \geqq 2$ first. By an integration by parts

$$\begin{aligned} \nabla_t M(p, q) &= -\int_{R^n} \sum_{i,j}^{n} \nabla_i (X_p X^q) a_{ij} \nabla_j P dx \\ &= -p\int_{R^n} \sum_{i,j}^{n} I_i |x_i|^{p-1} X^q a_{ij} \nabla_j P dx \\ &\quad -q\int_{R^n} \sum_{i,j}^{n} X_p I_i X^{q-1} a_{ij} \nabla_j P dx . \end{aligned} \tag{2.2}$$

Since $p \geqq 2$, $(\nabla_i I_i)|x_i|^{p-1}=0$ in the distribution sense and hence we have

$$\begin{aligned} &\left| -p\int_{R^n} \sum_{i,j}^{n} I_i |x_i|^{p-1} X^q b_{ij} \nabla_j P dx \right| \\ &= \left| p(p-1)\int_{R^n} \sum_{i=1}^{n} |x_i|^{p-2} b_{ii} X^q P dx \right. \\ &\quad \left. + pq\int_{R^n} \sum_{i,j}^{n} I_i |x_i|^{p-1} I_j X^{q-1} b_{ij} P dx \right| \end{aligned} \tag{2.3}$$

$$\leqq p(p-1)\lambda M(p-2, q)+pq\lambda M(p-1, q-1)$$

Clearly we have

$$\left|-p\int_{R^n}\sum_{i,j}^{n} I_i|x_i|^{p-1}X^q c_{ij}\nabla_j P\,\mathrm{d}x\right| \leqq pn^{-1}\mu K(p-1, q) \tag{2.4}$$

and

$$\left|q\int_{R^n}\sum_{i,j}^{n} X_p I_i X^{q-1}a_{ij}\nabla_j P\,\mathrm{d}x\right| \leqq q(\mu+\lambda)K(p, q-1). \tag{2.5}$$

By (2.2)∼(2.5), we obtain

$$\begin{aligned} |\nabla_t M(p, q)| \leqq\, & p(p-1)\lambda M(p-2, q)+pq\lambda M(p-1, q-1) \\ & +pn^{-1}\mu K(p-1, q)+q(\mu+\lambda)K(p, q-1). \end{aligned} \tag{2.6}$$

In the case $p=0$, we have

$$\begin{aligned} |\nabla_t M(0, q)| &= \left|-\sum_{i,j}^{n}\int_{R^n}\nabla_i(nX^q)a_{ij}\nabla_j P\,\mathrm{d}x\right| \\ &= \left|-q\int_{R^n}\sum_{i,j}^{n} nX^{q-1}I_i a_{ij}\nabla_j P\,\mathrm{d}x\right| \\ &\leqq q(\mu+\lambda)K(0, q-1). \end{aligned} \tag{2.7}$$

Next we consider the case $p=1$. Since $M(1, q)=n^{-1}M(0, q+1)$, we obtain by (2.7)

$$\begin{aligned} |\nabla_t M(1, q)| &\leqq n^{-1}(q+1)(\mu+\lambda)K(0, q) \\ &= n^{-1}(\mu+\lambda)K(0, q)+q(\mu+\lambda)K(1, q-1) \end{aligned} \tag{2.8}$$

By (2.6)∼(2.8), we obtain (2.1).

Lemma 2.2. *Let p and q be non-negative integers. Then, for $0<t<\infty$,*

$$\begin{aligned} \nu G(p, q) \leqq\, & \nabla_t E(p, q)+p(p-1)\lambda|E^*(p-2, q)|+pq\lambda L^*(p-1, q-1) \\ & +\lambda n|\log(f_n\nu\pi t)|M^*(p-2, q)+p(\lambda+\mu)n^{-1}K^*(p-1, q) \\ & +p(\lambda+\mu)n^{-1}J^*(p-1, q)+q(\lambda+\mu)K^*(p, q-1) \\ & +q(\lambda+\mu)n^{1/2}(H^*(p, q-1)G^*(p, q-1))^{1/2}. \end{aligned} \tag{2.9}$$

Proof. In the case $p\geqq 2$, we have

$$\begin{aligned} \nabla_t E(p, q) &= -\int_{R^n} X_p X^q(1+\log P)\nabla_t P\,\mathrm{d}x \\ &= \int_{R^n} X_p X^q\sum_{i,j}^{n}(\nabla_i\log P)(\nabla_j\log P)a_{ij}P\,\mathrm{d}x \\ &\quad +p\int_{R^n}\sum_{i,j}^{n} I_i|x_i|^{p-1}X^q a_{ij}(\nabla_j P)(1+\log P)\mathrm{d}x \\ &\quad +q\int_{R^n}\sum_{i,j}^{n} I_i X_p X^{q-1}a_{ij}(\nabla_j P)(1+\log P)\mathrm{d}x. \end{aligned} \tag{2.10}$$

First we shall estimate the second term on the right hand side of (2.10). Since

$$(\nabla_j P)(1+\log P)=\nabla_j(P\log P)$$

and

$$(\nabla_i I_i)|x_i|^{p-1}=0$$

we have

$$\left|p\int_{R^n}\sum_{i,j}^{n} I_i|x_i|^{p-1}X^q b_{ij}(\nabla_j P)(1+\log P)\mathrm{d}x\right| \tag{2.11}$$

$$=\left|-p(p-1)\int_{R^n}\sum_{i=1}^{n}|x_i|^{p-2}X^q b_{ii}P\log P\mathrm{d}x\right.$$

$$\left.-pq\int_{R^n}\sum_{i,j}^{n} I_i|x_i|^{p-1}I_jX^{q-1}b_{ij}P\log P\mathrm{d}x\right|$$

$$=\left|p(p-1)\int_{R^n}\sum_{i=1}^{n}|x_i|^{p-2}X^q b_{ii}P\Big(-\log P+\frac{1}{2}n\log(f_n\nu\pi t)\Big)\mathrm{d}x\right.$$

$$-p(p-1)\Big\{\int_{R^n}\sum_{i=1}^{n}|x_i|^{p-2}X^q b_{ii}P\mathrm{d}x\Big\}\frac{1}{2}n\log(f_n\nu\pi t)$$

$$\left.-pq\int_{R^n}\sum_{i,j}^{n} I_i|x_i|^{p-1}I_jX^{q-1}b_{ij}P\log P\mathrm{d}x\right|$$

$$\leqq p(p-1)\lambda|E(p-2,q)|+p(p-1)\lambda n|\log(f_n\nu\pi t)|M(p-2,q)$$

$$+pq\lambda L(p-1,q-1),$$

here we used $-\log P+(1/2)\log(f_n\nu\pi t)\geqq 0$, followed from Lemma 1.2. Clearly we have

$$\left|p\int_{R^n}\sum_{i,j}^{n} I_i|x_i|^{p-1}X^q c_{ij}(\nabla_j P)(1+\log P)\mathrm{d}x\right| \tag{2.12}$$

$$\leqq p\mu n^{-1}J(p-1,q)+p\mu n^{-1}K(p-1,q).$$

By (2.11) and (2.12), we obtain

$$\left|p\int_{R^n}\sum_{i,j}^{n} I_i|x_i|^{p-1}X^q a_{ij}(\nabla_j P)(1+\log P)\mathrm{d}x\right| \tag{2.13}$$

$$\leqq p(p-1)\lambda|E(p-2,q)|+p(p-1)\lambda n|\log(f_n\nu\pi t)|M(p-2,q)$$

$$+pq\lambda L(p-1,q-1)+p\mu n^{-1}J(p-1,q)+p\mu n^{-1}K(p-1,q).$$

Next we shall estimate the third term on the right-hand side of (2.10). We have

$$\left|q\int_{R^n}\sum_{i,j}^{n} I_iX_pX^{q-1}a_{ij}(\nabla_j P)\log P\mathrm{d}x\right| \tag{2.14}$$

$$\leqq q(\lambda+\mu)J(p,q-1)\leqq q(\lambda+\mu)(nH(p,q-1)G(p,q-1))^{1/2}$$

and

$$\left|q\int_{R^n}\sum_{i,j}^{n} I_iX_pX^{q-1}a_{ij}\nabla_j P\mathrm{d}x\right|\leqq q(\lambda+\mu)K(p,q-1). \tag{2.15}$$

Hence we obtain

$$(2.16)\qquad \left|q\int_{R^n}\sum_{i,j}^{n} I_i X_p X^{q-1} a_{ij}(1+\log P)\nabla_j P\,\mathrm{d}x\right|$$
$$\leqq q(\lambda+\mu)(nH(p,\ q-1)G(p,\ q-1))^{1/2}+q(\lambda+\mu)K(p,\ q-1).$$

Using (2.10), (2.13) and (2.16) and noticing

$$\int_{R^n} X_p X^q \sum_{i,j}^{n} (\nabla_i \log P)(\nabla_j \log P) a_{ij} P\,\mathrm{d}x \geqq \nu G(p,\ q)$$

we can conclude (2.9) for $p\geqq 2$. In the case $p=0$, (2.9) is clear, and when $p=1$, the similar device as in the proof of Lemma 2.1 yields (2.7).

To show the following, it is not necessary that P is a solution of the parabolic equation.

Lemma 2.3. *For non-negative integers p and q and any t, $0<t<\infty$, we have*

$$(2.17)\qquad H(p,\ q)\leqq 4M(p,\ q+2)+\left(\frac{1}{4}n^2(\log(f_n\nu\pi t))^2+1\right)M(p,\ q)$$
$$+6e^{-1}n^{q+2}(p+q+1)\,!.$$

Proof. Let

$$f_\alpha(x)=x(\log x)^2-\alpha x,\ \alpha>-1.$$

Then

$$\max_{0<x\leqq e^{-1}} f_\alpha(x)=f_\alpha(\exp(-1-(1+\alpha)^{1/2}))$$

Therefore, if $0<P\leqq e^{-1}$ we have

$$(2.18)\qquad f_\alpha(P)\leqq 2(1+(1+\alpha)^{1/2})\exp(-1-(1+\alpha)^{1/2}).$$

Putting $\alpha=4X^2-1$ and multiplying both sides of (2.18) by X_pX^q, we then inte grate them over $\{x:\ P<e^{-1}\}$. Then we obtain

$$\int_{P<e^{-1}} X_p X^q P(\log P)^2\,\mathrm{d}x-4\int_{P<e^{-1}} X_p X^{q+2} P\,\mathrm{d}x+\int_{P<e^{-1}} X_p X^q P\,\mathrm{d}x$$
$$\leqq 2e^{-1}\int_{P<e^{-1}} X_p X^q e^{-2X_1}\,\mathrm{d}x+4e^{-1}\int_{P<e^{-1}} X_p X^{q+1} e^{-2X_1}\,\mathrm{d}x$$
$$\leqq 6e^{-1}n^{q+2}(p+q+1)\,!.$$

Hence

$$(2.19)\qquad \int_{P<e^{-1}} X_p X^q P(\log P)^2\,\mathrm{d}x\leqq 4M(p,\ q+2)+6e^{-1}n^{q+2}(p+q+1)\,!.$$

Clearly we have

$$(2.20)\qquad \int_{e^{-1}\leqq P<1} X_p X^q P(\log P)^2\,\mathrm{d}x\leqq M(p,\ q).$$

On the other hand by Lemma 1.2 we have

$$(2.21)\qquad \int_{1\leqq P} X_p X^q P(\log P)^2\,\mathrm{d}x\leqq \frac{1}{4}n^2 M(p,\ q)(\log(f_n\nu\pi t))^2.$$

Combining (2.19), (2.20) and (2.21), we obtain Lemma 2.3.

Set

$$\mathcal{F}_1=p(p-1)\lambda M^*(p-2, q)+pq\lambda M^*(p-1, q-1)+pn^{-1}(\lambda+\mu)K^*(p-1, q),$$

$$\mathcal{F}_2=q(\lambda+\mu)(\nu^{-1}nM^*(p, q-1))^{1/2},$$

$$\mathcal{F}_3=p)p-1)\lambda|E^*(p-2, q-1)|+\lambda n|\log(f_n\nu\pi t)|M^*(p-2, q-1)$$
$$+p(\lambda+\mu)n^{-1}K^*(p-1, q-1)+p(q-1)\lambda L^*(p-1, q-2)$$
$$+p(\lambda+\mu)n^{-1}J^*(p-1, q-1)+(q-1)(\lambda+\mu)K^*(p, q-2),$$

$$\mathcal{F}_4=(q-1)(\lambda+\mu)(nG^*(p, q-2))^{1/2},$$

$$\mathcal{F}_5=\left(\frac{1}{4}n^2(\log(f_n\nu\pi t))^2+1\right)M^*(p, q-2)+6e^{-1}n^q(p+q-1)!.$$

Then by means of the previous three lemmas and Lemma 1.3, we conclude the following.

Lemma 2.4. *For non-negative integers p and q and for any $t(0<t<\infty)$,*

$$|\nabla_t M(p, q)|\leqq\mathcal{F}_1+\mathcal{F}_2(\nabla_t E(p, q-1)+\mathcal{F}_3+\mathcal{F}_4(4M(p, q)+\mathcal{F}_5)^{1/2})^{1/2} \tag{2.22}$$

Next we obtain an estimate on $M(p, q)$ from below as Lemma 2.5. It should be remarked that only the following three properties of P will be used:

$$P(t, x)\geqq 0, \int_{R^n}P(t, x)\,dx=1 \quad\text{and}\quad P(t, x)\leqq(f_n\nu\pi t)^{-(1/2)n}.$$

Lmma 2.5. *For non-negative integer p and q,*

(i) $C_1 n^{q+1}\exp\{(E(p, q-1)-n^{q+1}-1)/((n+p+q-1)M(p, q-1))\}\leqq M(p, q)$,

where C_1 is a positive constant depending only on p and q,

(ii) $\frac{1}{2}e^{-1}(n+p+q-1)M(p, q-1)t^{1/2}((f_n\nu\pi)^{-(1/2)n}h(p, q-1))^{-1/(n+p+q-1)}$

$$\exp\{(F(p, q-1)-t^{(p+q-1)/2})/((n+p+q-1)M(p, q-1))\}\geqq M(p, q),$$

where $h(p, q-1)=\int_{R^n}X_pX^{q-1}e^{-2X_1}\,dx$,

(iii) $C_2 nM(p, q-1)t^{1/2}\exp\{-t^{(p+q-1)/2}/((n+p+q-1)M(p, q-1))\}\leqq M(p, q)$,

where C_2 is a positive constant depending only on p, q and ν.

Proof. For any fixed α we have

$$\min_{x>0}\{x\log x+\alpha x\}=-e^{-\alpha-1}.$$

Hence

$$X_pX^{q-1}\{P\log P+P\}\geqq -X_pX^{q-1}e^{-\alpha-1}.$$

Letting $\alpha=\beta X_1+\gamma$, where β and γ are constants, and integrating by Lebesgue measure, we obtain

$$(2.23)\qquad \begin{aligned}&-E(p,\ q-1)+\beta M(p,\ q)+\gamma M(p,\ q-1)\\ &\geqq -e^{-\gamma-1}\int_{R^n} X_p X^{q-1} e^{-\beta X_1}\,\mathrm{d}x\\ &=-e^{-\gamma-1}(2/\beta)^{n+p+q-1}h(p,\ q-1).\end{aligned}$$

Here we choose β and γ such that

$$\beta=n^{q+1}/M(p,\ q)\quad\text{and}\quad e^{-\gamma-1}(2/\beta)^{n+p+q-1}h(p,\ q-1)=1.$$

Then by (2.23), we obtain

$$(2.24)\qquad \gamma M(p,\ q-1)\geqq E(p,\ q-1)-n^{q+1}-1$$

and

$$(2.25)\qquad e^{\gamma}=e^{-1}(2M(p,\ q)n^{-(q+1)})^{n+p+q-1}h(p,\ q-1).$$

Combining (2.24) and (2.25) we obtain

$$(2.26)\qquad \begin{aligned}&\exp\{E(p,\ q-1)-n^{q+1}-1)/((n+p+q-1)M(p,\ q-1))\}\\ &\leqq(e^{-1}h(p,\ q-1))^{1/(n+p+q-1)}2M(p,\ q)n^{-(q+1)}\end{aligned}$$

Since $\inf(h(p,\ q-1))^{1/(n+p+q-1)}>0$, (2.26) implies (i).

To show (ii), we choose β and γ in (2.23) such that

$$\beta=(n+p+q-1)M(p,\ q-1)/M(p,\ q)$$

and

$$e^{-\gamma-1}(2/\beta)^{n+p+q-1}h(p,\ q-1)=t^{(p+q-1)/2}.$$

Then we have

$$(2.27)\qquad e^{\gamma}=e^{-1}(2M(p,\ q)/((n+p+q-1)M(p,\ q-1)))^{n+p+q-1}h(p,\ q-1)t^{-(1/2)(p+q-1)}$$

and

$$(2.28)\qquad \begin{aligned}\gamma M(p,\ q-1)&\geqq E(p,\ q-1)-(n+p+q-1)M(p,\ q-1)-t^{(p+q-1)/2}\\ &=F(p,\ q-1)+\Big(\frac{1}{2}n\log(f_n\nu\pi t)-(n+p+q-1)\Big)M(p,\ q-1)\\ &\quad -t^{(p+q-1)/2}\end{aligned}$$

Combining (2.27) and (2.28) we obtain

$$\begin{aligned}&\exp\Big(\{(F(p,\ q-1)-t^{(p+q-1)/2})/M(p,\ q-1)\}+\frac{1}{2}n\log(f_n\nu\pi t)-(n+p+q-1)\Big)\\ &\leqq e^{-1}(2M(p,\ q)/((n+p+q-1)M(p,\ q-1))^{n+p+q-1}h(p,\ q-1)t^{-(1/2)(p+q-1)}.\end{aligned}$$

Therefore

$$(2.29)\qquad \begin{aligned}&e^{-1}(f_n\nu\pi)^{n/2(n+p+q-1)}\exp\Big\{\frac{F(p,\ q-1)-t^{(p+q-1)/2}}{(n+p+q-1)M(p,\ q-1)}\Big\}\\ &\leqq h(p,\ q-1)^{1/(n+p+q-1)}2M(p,\ q)/((n+p+q-1)M(p,\ q-1)t^{1/2}),\end{aligned}$$

which yields (ii). Since $\lim_{n\to\infty} f_n = 4/e$, $\sup_{n\geq 1}(h(p, q-1))^{1/(n+p+q-1)} < \infty$ and $F(p, q-1) \geqq 0$, (iii) follows from (2.29) immediately.

§ 3. The proof of Theorems.

In this section we shall complete the proof of Theorem 1 and 2 given in Introduction. First, we shall obtain estimates of $M(0, 1)$, $E(0, 0)$ and $G(0, 0)$.

Proposition 3.1.

(i) $$(\nu\pi e^{-3})^{1/2} e^{\iota^-} \leqq \varliminf_{n\to\infty} \inf_{t>0} \{M(0, 1)/(n^2 t^{1/2})\},$$

$$\varlimsup_{n\to\infty} \sup_{t>0} \{M(0, 1)/(n^2 t^{1/2})\} \leqq (\mu+\lambda)(\iota^+ + 2)(2\nu)^{1/2},$$

where ι^- and ι^+ are the constants defined in Theorem 2. There exist positive constants C_1, C_2 and C_3 depending only on λ, μ and ν, such that

(ii) $$\varlimsup_{n\to\infty} \sup_{t>0} \left\{\int_0^t (\nabla_s E(0, 0))^{1/2}\, ds/t^{1/2} n\right\} \leqq C_1,$$

(iii) $$\varlimsup_{n\to\infty} \sup_{t>0} \left\{\int_0^t G(0, 0)^{1/2}\, ds/t^{1/2} n\right\} \leqq C_2,$$

(iv) $$\varlimsup_{n\to\infty} \int_0^1 |E(0, 0)|\, ds/n^2 \leqq C_3,$$

Proof. We are choosing a non-Euclidean metric, hence we have to modify the argument of J. Nash [4]. By Lemma 2.4,

$$(3.1)\qquad \begin{aligned} |\nabla_t M(0. 1)| &\leqq \nu^{-(1/2)}(\mu+\lambda) n (\nabla_t E(0, 0))^{1/2} \\ &\leqq \nu^{-(1/2)}(\mu+\lambda) n (\nabla_t F(0, 0) + n^2/2t)^{1/2}. \end{aligned}$$

It should be remarked that $\nabla_t E(0, 0) \geqq 0$. Here we used the definition of F and $M(0, 0) = n$. Since $\lim_{t\to 0} M(0, 1) = 0$, we have

$$(3.2)\qquad \begin{aligned} M(0, 1) &\leqq \nu^{-(1/2)}(\mu+\lambda) n \int_0^t (\nabla_s F(0, 0) + n^2/2s)^{1/2}\, ds \\ &\leqq \nu^{-(1/2)}(\mu+\lambda) n \int_0^t \{((2s)^{1/2}/2n)\nabla_s F(0, 0) + n/(2s)^{1/2}\}\, ds \\ &\leqq \nu^{-(1/2)}(\mu+\lambda) n \{((2t)^{1/2}/2n) F(0, 0) + n(2t)^{1/2}\}. \end{aligned}$$

Here we used the fact $F \geqq 0$ and the following inequality $(a+b)^{1/2} \leqq a/(2b^{1/2}) + b^{1/2}$ for $b > 0$ and $a+b > 0$. By (ii) of Lemma 2.5,

$$(3.3)\qquad \frac{1}{2} e^{-1} (f_n \nu\pi)^{1/2} n^{-1/n} e^{F(0,0)/n^2} e^{-1/n^2} \leqq M(0, 1)/(n^2 t^{1/2})$$

Combining (3.2) and (3.3), we obtain

$$(3.4)\qquad \begin{aligned} &\frac{1}{2} e^{-1-n^{-2}} (f_n \nu\pi)^{1/2} n^{-1/n} e^{F(0,0)/n^2} \\ &\leqq M(0, 1)/(n^2 t^{1/2}) \end{aligned}$$

$$\leqq(\mu+\lambda)(2\nu)^{-(1/2)}((F(0,\ 0)/n^2)+2)$$

Since $\lim_{n\to\infty}\frac{1}{2}e^{-1-n^{-2}}(f_n\nu\pi)^{1/2}n^{-1/n}=(\nu\pi e^{-3})^{1/2}$, we obtain

$$(\nu\pi e^{-3})^{1/2}\exp\{\varliminf_{n\to\infty}\inf_{t>0}F(0,\ 0)/n^2\}$$

$$\leqq(\mu+\lambda)(2\nu)^{-(1/2)}(\varliminf_{n\to\infty}\inf_{t>0}\{F(0,\ 0)/n^2\}+2)$$

which implies

$$(3.5)\qquad \iota^-\leqq\varliminf_{n\to\infty}\inf_{t>0}F(0,\ 0)/n^2\leqq\iota^+$$

and similary we have

$$(3.6)\qquad \iota^-\leqq\varlimsup_{n\to\infty}\sup_{t>0}F(0,\ 0)/n^2\leqq\iota^+.$$

By (3.4), (3.5) and (3.6), we obtain (i). Since (3.1) and (3.2) imply

$$\int_0^t(\nabla_s E(0,\ 0))^{1/2}\,ds\leqq\frac{(2t)^{1/2}}{2n}F(0,\ 0)+n(2t)^{1/2},$$

we obtain (ii). (iii) follows from Lemma 2.2 and (ii). By the definition of $F(0,\ 0)$, we have

$$(3.7)\qquad |E(0,\ 0)|\leqq F(0,\ 0)+\frac{1}{2}n^2|\log(f_n\nu\pi t)|.$$

By (3.6), (3.7) and $\lim_{n\to\infty}f_n=4/e$, we obtain (iv).

Now Theorem 2 is a consequence of (i) of Proposition 3.1. Next we shall proceed to the proof of Theorem 1 through showing one proposition.

Proposition 3.2. *Let p and q be non-negative intgers. Then*

(i) $\qquad M(p,\ q)\leqq C_4 n^{q+1}t^{(p+q)/2},\qquad 0<t<\infty,$

(ii) $\qquad \int_0^1 s^{(p+q-1)/2}|E^*(p,\ q-1)|\,ds\leqq C_5 n^{q+1},$

(iii) $\qquad \int_0^1(G^*(p,\ q-1))^{1/2}\,ds\leqq C_6 n^{(q+1)/2},$

(iv) $\qquad \int_0^1 H^*(p,\ q-2)\,ds\leqq C_7 n^{q+1},$

(v) $\qquad \int_0^1 K^*(p,\ q-1)\,ds\leqq C_8 n^{q+1},$

where $C_4,\ \cdots,\ C_8$ are positive constants depending only on λ, μ, ν, p and q.

Proof. We first remark that it is sufficient to show

(i) $\qquad \sup_{0<t\leq 1}M(p,\ q)\leqq C_4 n^{q+1}$

in order to prove (i). For $A^\varepsilon=\sum_{i,j}^{n}\nabla_i a_{ij}(t/\varepsilon^2,\ x/\varepsilon)\nabla_j$ also satisfies the conditions (0.3), (0.4) and (0.5) with the same λ, μ and ν, and C_4 does not depend on the smoothness of the coefficients. Hence (i) and (i)′ are equivalent.

In the following C_i always denote poitive constants depending only on λ, μ, ν, p and q.

We introduce the following order $\rightarrow$ on the set $\{(p, q): p\geqq 0, q\geqq 0\}$:

$$(p, q) \longrightarrow (p', q')$$

if and only if

$$p+q<p'+q'$$

or

$$p+q=p'+q' \quad \text{and} \quad p<p'.$$

We shall show Proposition 3.2 by induction on (p, q) with respect to this order.

If $(p, q)=(0, 0)$, then, every estimate is trivial. In the case $(p, q)=(0, 1)$, (iv) is clear and the other statements follow from Proposition 3.1 and Lemma 1.3.

Choose (p, q) and assume that the statements (i)$\cdots$(v) have been already proven up to (p, q). Then we have to show all the statements also for this (p, q). First we consider the case $q=0$. In this case (ii)$\cdots$(v) are trivial. To prove (i), we use Lemma 2.4; since $\mathcal{F}_2=0$, we have

$$||\nabla_t M(p, 0)|\leqq \mathcal{F}_1=p(p-1)M^*(p-2, 0)+p(\lambda+\mu)n^{-1}K^*(p-1, 0).$$

Since $(p-2, 0)\rightarrow(p, 0)$ and $(p-1, 1)\rightarrow(p, 0)$, the hypothesis of the induction implies

$$\int_0^1 \mathcal{F}_1\, \mathrm{d}s\leqq C_9 n$$

Therefore, noting $\lim_{t\rightarrow 0} M(p, 0)=0$, we have

$$M(p, 0)\leqq\int_0^t |\nabla_t M(p, 0)|\, \mathrm{d}s\leqq C_9 n, \qquad 0<t\leqq 1,$$

which proves (i)′.

Secondly we consider the case $q\geqq 1$ and $(p, q)\neq(0, 1)$. By the hypothesis of the induction and Lemma 1.3, we have

$$\int_0^1 \mathcal{F}_1\, \mathrm{d}s\leqq C_{10} n^{q+1}, \tag{3.8}$$

$$\mathcal{F}_2\leqq C_{11} t^{(1/4)(p+q-1)} n^{(1/2)(q+1)}, \qquad 0<t<\infty, \tag{3.9}$$

$$\begin{aligned}\int_0^1 s^{(1/2)(p+q-1)} n^{(q+1)}(\mathcal{F}_3+\mathcal{F}_4(4M(p, q)+\mathcal{F}_5)^{1/2})\, \mathrm{d}s \\ \leqq C_{12} n^{2(q+1)}+C_{13} n^{3(q+1)/2}(\sup_{0<t\leqq 1} M(p, q))^{1/2},\end{aligned} \tag{3.10}$$

here we used $p+q\geqq 2$. Combining (3.8)$\sim$(3.10) with Lemma 2.4 and noting $\lim_{t\rightarrow 0} M(p, q)=0$, we obtain

$$\begin{aligned}M(p, q)\leqq C_{10} n^{q+1}+C_{14}(n^{q+1}\int_0^t s^{(1/2)(p+q-1)}\nabla_s E(p, q-1)\, \mathrm{d}s \\ +C_{15} n^{2(q+1)}+C_{16} n^{3(q+1)/2}(\sup_{0<t\leqq 1} M(p, q))^{1/2})^{1/2},\end{aligned} \tag{3.11}$$

for $0<t\leqq 1$. Now,

$$\int_0^t s^{(p+q-1)/2}\nabla_s E(p,\ q-1)\,\mathrm{d}s$$

$$=[s^{(p+q-1)/2}E(p,\ q-1)]_0^t-\frac{1}{2}(p+q-1)\int_0^t E(p,\ q-1)s^{(p+q-3)/2}\,\mathrm{d}s,$$

By Lemma 1.1. and $\frac{1}{2}(p+q-1)>0$, we see

$$\lim_{t\to 0} t^{(p+q-1)/2}E(p,\ q-1)=0.$$

Moreover we have by Lemma 1.2 and the hypothesis of the induction

$$E(p,\ q-1)\geqq\frac{1}{2}nM(p,\ q-1)\log(f_n\nu\pi t)\geqq -C_{17}n^{q+1}, \qquad \text{for} \quad 0<t\leqq 1.$$

Hence

$$\text{(3.12)} \qquad \sup_{0<t\leqq 1}\int_0^t s^{(p+q-1)/2}\nabla_s E(p,\ q-1)\,\mathrm{d}s$$

$$\leqq \sup_{0<t\leqq 1}(t^{(p+q-1)/2}(E(p,\ q-1)+C_{17}n^{q+1})-t^{(p+q-1)/2}C_{17}n^{q+1}$$

$$+t^{(p+q-1)/2}C_{17}n^{q+1})$$

$$\leqq \sup_{0<t\leqq 1}(E(p,\ q-1)+C_{17}n^{q+1})$$

$$\leqq \sup_{0<t\leqq 1}(E(p,\ q-1)-n^{q+1}-1)+C_{18}n^{q+1}.$$

By (3.11) and (3.12), we obtain

$$\text{(3.13)} \qquad \sup_{0<t\leqq 1}M(p,\ q)\leqq C_{10}n^{q+1}+C_{14}(n^{q+1}\sup_{0<t\leqq 1}(E(p,\ q-1)-n^{q+1}-1)$$

$$+C_{19}n^{2(q+1)})^{1/2}+C_{20}n^{3(q+1)/4}(\sup_{0<t\leqq 1}M(p,\ q))^{1/4}.$$

To prove (i)′ and (ii), we divide the situation into two cases. First suppose $\sup_{0<t\leqq 1}(E(p,\ q-1)-n^{q+1}-1)\leqq 0$. Then (3.13) gives

$$\sup_{0<t\leqq 1}M(p,\ q)/n^{q+1}\leqq C_{21}+C_{20}(\sup_{0<t\leqq 1}M(p,\ q)/n^{q+1})^{1/2},$$

which yields (i)′. Lemma 1.3. combined with $E(p,\ q-1)\leqq n^{q+1}+1$ implies

$$|E(p,\ q-1)|\leqq n^{q+1}+1+\frac{1}{2}nM(p,\ q-1)|\log(f_n\nu\pi t)|, \qquad 0<t\leqq 1,$$

which shows (ii). On the other hand when $\sup_{0<t\leqq 1}(E(p,\ q-1)-n^{q+1}-1)>0$, we have by (i) of Lemma 2.5 and $\sup_{0<t\leqq 1}M(p,\ q-1)(n+p+q-1)\leqq C_{22}n^{q+1}$ (this comes from the hypothesis of the induction),

$$\text{(3.14)} \qquad C_{23}n^{q+1}\exp(\sup_{0<t\leqq 1}(E(p,\ q-1)-n^{q+1}-1)/(n^{q+1}C_{22}))\leqq \sup_{0<t\leqq 1}M(p,\ q)$$

By (3.13) and (3.14), we obtain

$$\text{(3.15)} \qquad C_{23}\exp(\sup_{0<t\leqq 1}(E(p,\ q-1)-n^{q+1}-1)/(n^{q+1}C_{22}))$$

$$\leqq \sup_{0<t\leqq 1} M(p,\ q)/n^{q+1}$$

$$\leqq C_{10}+C_{14}(\sup_{0<t\leqq 1}(E(p,\ q-1)-n^{q+1}-1)/n^{q+1}+C_{19})^{1/2}$$

$$+C_{20}(\sup_{0<t\leqq 1} M(p,\ q)/n^{q+1})^{1/4}$$

However, it is easy to conclude from the inequality

$$\begin{cases} C_{23}\exp(y/C_{22})\leqq x\leqq C_{10}+C_{14}(y+C_{19})^{1/2}+C_{20}x^{1/4} \\ 0\leqq y \end{cases}$$

that $x\leqq C_{24}$ and $y\leqq C_{25}$. Hence we obtain

$$\sup_{0<t\leqq 1} M(p,\ q)\leqq C_{24}, \tag{3.16}$$

$$\sup_{0<t\leqq 1}(E(p,\ q-1)-n^{q+1}-1)\leqq C_{25}n^{q+1}. \tag{3.17}$$

(3.16) implies (i)$'$. (3.17) together with Lemma 1.2. gives (ii).

As for the proof of (iii)$\sim$(v) we proceed as follows. (iv) is an immediate consequence of Lemma 2.3 and (i). (iii) follows from Lemma 2.2, (ii) and (iv). Since $K(p,\ q-1)\leqq(nG(p,\ q-1)M(p,\ q-1))^{1/2}$, we obtain (v) from (iii). Now the proof of Proposition 3.2 is complete.

The final step of the proof of Theorem 1.

In the statement of Theorem 1, the estimate from above can be obtained by applying Proposition 3.2. The lower estimate is shown without any difficulty by using (iii) of Lemma 2.5 inductively. There readers should remark inequalities $nM(p,\ -1)\geqq M(p-1,\ 0)$ for $p=1,\ 2,\ \cdots$.

Remark 1. In the case $p=0$, Theorem 1 can be obtained under a weaker assumption (0.6)$'$ or (0.6)$''$:

$$\sup_{1\leq j\leq n}\sum_{i=1}^{n}|c_{ij}|\leqq\mu, \tag{0.6$'$}$$

$$\Big|\sum_{i,j}^{n}c_{ij}\xi_i\eta_j\Big|\leqq\mu|\xi||\eta|, \qquad \text{for any} \quad \xi=(\xi_i),\ \eta=(\eta_i)\in R^n. \tag{0.6$''$}$$

Remark 2. Theorem 1 is obtainable for a slight more general class of parabolic equations; let

$$B=\sum_{i,j=1}^{n}\nabla_i a_{ij}(t,\ x)\nabla_j+\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\nabla_j d_{ij}(t,\ x)\Big)\nabla_i$$

where (a_{ij}) satisfies the same conditions as in Section 0 and (d_{ij}) satisfies

$$|d_{ij}(t,\ x)|\leqq\lambda/n$$

and

$$\int_{R^n}\sum_{i,j=1}^{n}d_{ij}(t,\ x)\nabla_i\nabla_j\varphi(x)\,dx=0 \qquad \text{for any} \quad \varphi\in C_0^2(R^n).$$

Then there exists a fundamental solution satisfying Theorem 1. See [5] for the precise meaning of a fundamental solution and its existence.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

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