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A necessary condition for H^∞-wellposedness of the Cauchy problem for linear partial differential operators of Schrödinger type

(Schrödinger equations and generalizations, IV)

Dedicated to Professor SIGERU MIZOHATA on the occasion of his sixtieth birthday

By

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Introduction.

Consider a partial differential operator of Schrödinger type:

$$P(x, D_x, D_t)=D_t+\sum_{j=1}^{n}(D_j-a_j(x))^2+c(x), \quad (x, t)\in R^n\times R^1, \tag{1}$$

where $a_j(x)$, $c(x)$ are *complex-valued* functions in $\mathscr{B}^\infty(R^n)$.

$$D_t=\frac{1}{i}\frac{\partial}{\partial t},\ D_x=(D_1, \cdots, D_n),\ D_j=\frac{1}{i}\frac{\partial}{\partial x_j} \quad (1\leqq j\leqq n).$$

We are concerned with the Cauchy problem for $P(x, D_x, D_t)$ both for the future and for the past in H^∞-space:

$$\begin{cases} P(x, D_x, D_t)u(x, t)=f(x, t) \quad \text{in} \quad R^n\times[-T, T], \\ u(x, 0)=u_0(x). \end{cases} \tag{2}$$

Recently, W. Ichinose [4] has given a necessary condition (*) for the Cauchy problem (2) to be H^∞-wellposed:

$$\begin{cases} \text{There exist constants } M \text{ and } N \text{ such that} \\ \sup_{(x,\omega)\in R^n\times S^{n-1}}\left|\int_0^\rho \sum_{j=1}^{n}\operatorname{Im} a_j(x+\theta\omega)\omega_j d\theta\right|\leqq M\log(1+\rho)+N \quad \text{for} \quad \rho\geqq 0. \end{cases} \tag{*}$$

To prove this result, he used localization in phase space along the classical trajectories for Hamiltonian $|\xi|^2$. His method is an extension of the method developed by Mizohata [7] for hyperbolic equations.

In this paper, using asymptotic solutions, we shall give another proof of this result for higher order operators "of Schrödinger type" with distinct characteristic roots.

§ 1. Statement of the results.

Consider a linear partial differential operator

$$P(x, t, D_x, D_t)=D_t^m+a_1(x, t, D_x)D_t^{m-1}+\cdots+a_m(x, t, D_x), \tag{1.1}$$

where

$$a_j(x, t, D_x)=\sum_{|\alpha|\leq 2j} a_{\alpha j}(x, t)D_x^\alpha, \quad (1\leqq j\leqq m), \tag{1.2}$$

$$a_{\alpha j}(x, t)\in \mathcal{B}^\infty(R^n\times R^1).$$

We are concerned with the Cauchy problem for $P(x, t, D_x, D_t)$ both for the future and for the past in H^∞-space:

$$\begin{cases} P(x, t, D_x, D_t)u(x, t)=f(x, t) \quad \text{in} \quad R^n\times[-T, T], \\ D_t^j u(x, 0)=u_j(x), \quad (0\leqq j\leqq m-1). \end{cases} \tag{1.3}$$

We say that the operator $P(x, t, D_x, D_t)$ is "of Schrödinger type" if the Cauchy problem (1.3) for $P(x, t, D_x, D_t)$ is wellposed both for the future and for the past in appropriate function space. (cf. Takeuchi [12], [13].)

The first assumption is the following:

Condition (A.1). *$a_{\alpha j}(x, t)=a_{\alpha j}$ (constant) for $|\alpha|=2j$, $1\leqq j\leqq m$.*

Denote the principal symbol of $a_j(x, t, D_x)$ by $a_j^0(\xi)$ and the homogeneous part of $a_j(x, t, \xi)$ of degree $2j-k$ by $a_j^k(x, t, \xi)$, i.e.,

$$a_j^0(\xi)=\sum_{|\alpha|=2j} a_{\alpha j}\xi^\alpha, \quad a_j^k(x, t, \xi)=\sum_{|\alpha|=2j-k} a_{\alpha j}(x, t)\xi^\alpha, \tag{1.4}$$

$$(1\leqq k\leqq 2j, \ 1\leqq j\leqq m).$$

(If $k>2j$, we assume that $a_j^k(x, t, \xi)$ is identically zero.)

Denote the principal symbol of $P(x, t, D_x, D_t)$ as 2-evolution in the sense of Petrowski by $P_{2m}(\xi, \tau)$:

$$P_{2m}(\xi, \tau)=\tau^m+a_1^0(\xi)\tau^{m-1}+\cdots+a_m^0(\xi). \tag{1.5}$$

Put

$$P_{2m-k}(x, t, \xi, \tau)=a_1^k(x, t, \xi)\tau^{m-1}+\cdots+a_m^k(x, t, \xi), \quad (1\leqq k\leqq 2m), \tag{1.6}$$

and

$$P_{2m}(x, t, \xi, \tau)=P_{2m}(\xi, \tau).$$

The second assumption is as follows:

Condition (A.2). *The roots of $P_{2m}(\xi, \tau)=0$ are real, distinct for $\xi\in R^n\setminus\{0\}$, i.e.,*

$$P_{2m}(\xi, \tau)=\prod_{j=1}^m(\tau-\lambda_j(\xi)), \quad \lambda_j(\xi)\neq\lambda_k(\xi), \quad (j\neq k, \ \xi\neq 0). \tag{1.7}$$

Remark 1.1. $\lambda_j(\xi)$ is homogeneous of degree 2 in ξ. $P_{2m-k}(x, t, \xi, \tau)$ is quasi-homogenous of weight (1, 2), i.e.,

$$P_{2m-k}(x, t, \rho\xi, \rho^2\tau)=\rho^{2m-k}P_{2m-k}(x, t, \xi, \tau), \quad \text{for} \quad \rho\in R^1.$$

Remark 1.2. It is necessary for the Cauchy problem (1.3) to be H^∞-wellposed that the roots $\lambda_j(\xi)\,(1\leqq j\leqq m)$ are real for $\xi\in R^n$. (cf. Petrowski [11], Mizohata [7].)

The third and main condition is the following.

Condition (A.3). *There exist constants M and N such that*

$$\sup_{\substack{(x,\omega)\in R^n\times S^{n-1}\\ 1\leqq j\leqq m}}\left|\int_0^\rho \operatorname{Im} P_{2m-1}(x+s(\nabla_\xi\lambda_j)(\omega), 0, \omega, \lambda_j(\omega))ds\right| \leqq M\log(1+|\rho|)+N, \quad \text{for} \quad \rho\in R^1. \tag{1.8}$$

Our results are as follows.

Theorem 1. *Under the assumptions* (A.1) *and* (A.2), *it is necessary for the Cauchy problem* (1.3) *to be H^∞-wellposed that the condition* (A.3) *holds.*

Corollary 1. *Under the assumptions* (A.1) *and* (A.2), *it is necessary for the Cauchy problem* (1.3) *to be H^s-wellposed that the condition* (A.3) *with $M=0$ holds.* (cf. Takeuchi [13]. See also Mizohata [8], [9], [10], Ichinose [4], Takeuchi [14], [15].)

Corollary 2. *Assume the conditions* (A.1) *and* (A.2). *Moreover, assume that one of the roots of $P_{2m}(\xi, \tau)=0$ is identically zero. Then, it is necessary for the Cauchy problem* (1.3) *to be H^∞-wellposed that* $\operatorname{Im} a_{\alpha m}(x, 0)\equiv 0$ $(|\alpha|=2m-1)$ *holds.* (cf. Ichinose [4].)

Remark 1.3. We shall prove this theorem by constructing the asymptotic solutions. (cf. Birkhoff [1], Leray [5], Maslov [6], Mizohata [8].)

§ 2. Asymptotic solutions.

For simplicity, we put

$$P_{2m}(\xi, \tau)=(\tau-\lambda(\xi))Q(\xi, \tau), \quad Q(\xi, \lambda(\xi))\neq 0, \quad (\xi\neq 0), \tag{2.1}$$

and $\rho t=s$ where ρ is a real parameter, so that $D_t=\rho D_s$. We define the phase function $\varphi(x, s, \omega)$ by

$$\varphi(x, s, \omega)=\omega x+\lambda(\omega)s, \quad \omega\in S^{n-1}, \tag{2.2}$$

so that $\varphi(x, s, \omega)$ satisfies $P_{2m}\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial s}\right)=0$. We construct the asymptotic solutions of the following form:

$$u(x, t, \omega)=e^{i\rho\varphi(x,s,\omega)}v(x, s, \omega), \quad (s=\rho t). \tag{2.3}$$

Applying $P(x, t, D_x, D_t)$ to $u(x, t, w)$, we have

$$(2.4)\qquad e^{-i\rho\varphi(x,s,\omega)}P(x, s/\rho, D_x, \rho D_s)(e^{i\rho\varphi(x,s,\omega)}v(x, s, \omega))$$
$$=\rho^{2m-1}M_1(x, s, D_x, D_s)v(x, s, \omega)$$
$$+\rho^{2m-2}M_2(x, s, D_x, D_s)v(x, s, \omega)$$
$$+\cdots\cdots\cdots+$$
$$+M_{2m}(x, s, D_x, D_s)v(x, s, \omega).$$

Here $M_r(x, s, D_x, D_s)$ is a differential operator of order r:

$$(2.5)\qquad M_r(x, s, D_x, D_s)$$
$$=\sum_{k+|\alpha|+l+h=r}\frac{s^h}{\alpha!\,l!}P_{2m-k(0,h)}^{(\alpha,l)}(x, 0, \omega, \lambda(\omega))D_x^\alpha D_s^l,$$

where

$$(2.6)\qquad P_{2m-k(\beta,h)}^{(\alpha,l)}(x, 0, \omega, \lambda(\omega))$$
$$=\left(\frac{\partial}{\partial\xi}\right)^\alpha\left(\frac{\partial}{\partial\tau}\right)^l\left(\frac{\partial}{\partial x}\right)^\beta\left(\frac{\partial}{\partial t}\right)^h P_{2m-k}(x, t, \xi, \tau)\Big|_{(t,\xi,\tau)=(0,\omega,\lambda(\omega))}.$$

Especially, $M_1(x, s, D_x, D_s)$ and $M_2(x, s, D_x, D_s)$ have the following forms:

$$(2.7)_1\qquad M_1(x, s, D_x, D_s)=P_{2m}^{(0,1)}(\omega, \lambda(\omega))D_s$$
$$+\sum_{|\alpha|=1}P_{2m}^{(\alpha,0)}(\omega, \lambda(\omega))D_x^\alpha+P_{2m-1}(x, 0, \omega, \lambda(\omega))$$
$$=Q(\omega, \lambda(\omega))[D_s-(\nabla_\xi\lambda)(\omega)\cdot D_x]+P_{2m-1}(x, 0, \omega, \lambda(\omega)),$$

$$(2.7)_2\qquad M_2(x, s, D_x, D_s)$$
$$=\sum_{|\alpha|+l=2}\frac{1}{\alpha!\,l!}P_{2m}^{(\alpha,l)}(\omega, \lambda(\omega))D_x^\alpha D_s^l$$
$$+\sum_{|\alpha|+l=1}\frac{1}{\alpha!\,l!}P_{2m-1}^{(\alpha,l)}(x, 0, \omega, \lambda(\omega))D_x^\alpha D_s^l$$
$$+sP_{2m-1(0,1)}(x, 0, \omega, \lambda(\omega))+P_{2m-2}(x, 0, \omega, \lambda(\omega)).$$

We put $v(x, s, \omega)=\sum_{j=0}^{N}\rho^{-j}v_j(x, s, \omega)$. Then we have

$$(2.8)\qquad e^{-i\rho\varphi(x,s,\omega)}P(x, s/\rho, D_x, \rho D_s)(e^{i\rho\varphi(x,s,\omega)}v(x, s, \omega))$$
$$=\rho^{2m-1}M_1(x, s, D_x, D_s)v_0(x, s, \omega)$$
$$+\rho^{2m-2}\{M_1(x, s, D_x, D_s)v_1(x, s, \omega)+M_2(x, s, D_x, D_s)v_0(x, s, \omega)\}$$
$$+\cdots\cdots\cdots\cdots$$
$$+\rho^{2m-1-N}\{M_1v_N+M_2v_{N-1}+\cdots+M_{2m}v_{N+1-2m}\}$$
$$+\rho^{2m-2-N}\{M_2v_N+M_3v_{N-1}+\cdots+M_{2m}v_{N+2-2m}\}$$
$$+\rho^{2m-3-N}\{M_3v_N+M_4v_{N-1}+\cdots+M_{2m}v_{N+3-2m}\}$$

$$+\cdots\cdots\cdots\cdots$$

$$+\rho^{-N}M_{2m}v_N.$$

We solve the transport equations:

$$(2.9)_0 \qquad \begin{cases} M_1(x, s, D_x, D_s)v_0(x, s, \omega)=0, \\ v_0(x, 0, \omega)=g_0(x), \end{cases}$$

and

$$(2.9)_j \qquad \begin{cases} M_1(x, s, D_x, D_s)v_j(x, s, \omega)+M_2v_{j-1}+\cdots+M_{j+1}v_0=0, \\ v_j(x, 0, \omega)=0, \qquad (1\leqq j\leqq N), \end{cases}$$

where $M_k(x, s, D_x, D_s)\equiv 0$ if $k>2m$. Put

$$(2.10) \qquad v_j(x, s, \omega)=e^{i\psi(x,s,\omega)}w_j(x, s, \omega), \qquad (0\leqq j\leqq N).$$

From $(2.9)_0$, we have

$$(2.11) \qquad \psi(x, s, \omega)=-\int_0^s P_{2m-1}(x+s'(\nabla_\xi\lambda)(\omega), 0, \omega, \lambda(\omega))ds'/Q(\omega, \lambda(\omega)),$$

and

$$(2.12) \qquad w_0(x, s, \omega)=g_0(x+s(\nabla_\xi\lambda)(\omega)).$$

Lemma 2.1. *$\psi(x, s, \omega)$ and $w_0(x, s, \omega)$ have the following properties:*

$$(2.13) \qquad \psi(x+s'(\nabla_\xi\lambda)(\omega), s-s', \omega)=\psi(x, s, \omega)-\psi(x, s', \omega),$$

$$(2.14) \qquad w_0(x+s'(\nabla_\xi\lambda)(\omega), s-s', \omega)=w_0(x, s, \omega).$$

Proof. Obvious.

Substituting (2.10) into $(2.9)_j$, we have

$$(2.15)_j \qquad \begin{cases} Q(\omega, \lambda(\omega))[D_s-(\nabla_\xi\lambda)(\omega)\cdot D_x]w_j(x, s, \omega) \\ +e^{-i\psi(x,s,\omega)}M_2(x, s, D_x, D_s)e^{i\psi(x,s,\omega)}w_{j-1}(x, s, \omega) \\ +\cdots\cdots+ \\ +e^{-i\psi(x,s,\omega)}M_{j+1}(x, s, D_x, D_s)e^{i\psi(x,s,\omega)}w_0(x, s, \omega)=0, \\ w_j(x, 0, \omega)=0. \end{cases}$$

Lemma 2.2. *The solution of the equation*

$$\begin{cases} \left(\frac{\partial}{\partial s}-a\cdot\frac{\partial}{\partial x}\right)w(x, s)=g(x, s), \qquad (a\in R^n), \\ w(x, 0)=0 \end{cases}$$

is given by

$$w(x, s)=\int_0^s g(x+s'a, s-s')ds'.$$

Proof. Obvious.

Using Lemma 2.2, we can prove

Lemma 2.3. *The solution of* $(2.15)_j$ *has the following form*:

$$(2.16)_j \qquad w_j(x, s, \omega)=e^{-i\psi(x,s,\omega)}\tilde{M}_j(x, s, D_x, D_s)e^{i\psi(x,s,\omega)}w_0(x, s, \omega)$$

where

$$\begin{aligned}(2.17)_j \qquad \tilde{M}_j(x, s, D_x, D_s)=&\left[\frac{-i}{Q(\omega, \lambda(\omega))}\right]^j\int_0^s ds_j\int_{s_j}^s ds_{j-1}\cdots\int_{s_2}^s ds_1\\ &\times e^{i\psi(x,s_j,\omega)}M_2(x+s_j(\nabla_\xi\lambda)(\omega), s-s_j, D_x, D_s)e^{-i\psi(x,s_j,\omega)}\\ &\times e^{i\psi(x,s_{j-1},\omega)}M_2(x+s_{j-1}(\nabla_\xi\lambda)(\omega), s-s_{j-1}, D_x, D_s)e^{-i\psi(x,s_{j-1},\omega)}\\ &\times\cdots\cdots\cdots\times\\ &\times e^{i\psi(x,s_1,\omega)}M_2(x+s_1(\nabla_\xi\lambda)(\omega), s-s_1, D_x, D_s)e^{-i\psi(x,s_1,\omega)}\\ &+(\textit{lower order terms}).\end{aligned}$$

Proof. For simplicity, we omit the variable ω of functions $\psi(x, s, \omega)$ and $w_0(x, s, \omega)$ in the proof. It follows from $(2.15)_1$ that

$$\begin{aligned}&\left[\frac{\partial}{\partial s}-(\nabla_\xi\lambda)(\omega)\cdot\frac{\partial}{\partial x}\right]w_1(x, s)\\ &\qquad=\frac{-i}{Q(\omega, \lambda(\omega))}e^{-i\psi(x,s)}M_2(x, s, D_x, D_s)e^{i\psi(x,s)}w_0(x, s).\end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}w_1(x, s)=&\frac{-i}{Q(\omega, \lambda(\omega))}\int_0^s e^{-i\psi(x+s'(\nabla_\xi\lambda)(\omega), s-s')}\\ &\times M_2(x+s'(\nabla_\xi\lambda)(\omega), s-s', D_x, D_{s''})\\ &\times e^{i\psi(x+s'(\nabla_\xi\lambda)(\omega), s'')}w_0(x+s'(\nabla_\xi\lambda)(\omega), s'')|_{s''=s-s'}ds'\\ =&\frac{-i}{Q(\omega, \lambda(\omega))}\int_0^s e^{-i[\psi(x,s)-\psi(x,s')]}M_2(x+s'(\nabla_\xi\lambda)(\omega), s-s', D_x, D_s)\\ &\times e^{i[\psi(x,s)-\psi(x,s')]}w_0(x, s)ds' \qquad \text{(by (2.13) and (2.14))}\\ =&e^{-i\psi(x,s)}\tilde{M}_1(x, s, D_x, D_s)e^{i\psi(x,s)}w_0(x, s),\end{aligned}$$

which proves $(2.16)_1$ and $(2.17)_1$. From $(2.15)_2$ and $(2.16)_1$, we have

$$\begin{aligned}w_2(x, s)=&\left[\frac{-i}{Q(\omega, \lambda(\omega))}\right]^2\int_0^s ds_2\, e^{-i\psi(x+s_2(\nabla_\xi\lambda)(\omega), s-s_2)}\\ &\times M_2(x+s_2(\nabla_\xi\lambda)(\omega), s-s_2, D_x, D_s)\int_0^{s-s_2}ds_1\\ &\times e^{i\psi(x+s_2(\nabla_\xi\lambda)(\omega), s_1)}M_2(x+s_2(\nabla_\xi\lambda)(\omega)+s_1(\nabla_\xi\lambda)(\omega), s-s_2-s_1, D_x, D_s)\\ &\times e^{-i\psi(x+s_2(\nabla_\xi\lambda)(\omega), s_1)}e^{i\psi(x+s_2(\nabla_\xi\lambda)(\omega), s-s_2)}\\ &\times w_0(x+s_2(\nabla_\xi\lambda)(\omega), s-s_2)\end{aligned}$$

$$+\frac{-i}{Q(\omega,\ \lambda(\omega))}\int_0^s ds_2\, e^{-i\phi(x+s_2(\nabla_\xi\lambda)(\omega),\ s-s_2)} M_3(x+s_2(\nabla_\xi\lambda)(\omega),\ s-s_2,\ D_x,\ D_s)$$

$$\times e^{i\phi(x+s_2(\nabla_\xi\lambda)(\omega),\ s-s_2)} w_0(x+s_2(\nabla_\xi\lambda)(\omega),\ s-s_2)$$

$$=\left[\frac{-i}{Q(\omega,\ \lambda(\omega))}\right]^2\int_0^s ds_2\, e^{-i[\phi(x,s)-\phi(x,s_2)]} M_2(x+s_2(\nabla_\xi\lambda)(\omega),\ s-s_2,\ D_x,\ D_s)$$

$$\times\int_{s_2}^s ds_1\, e^{i[\phi(x,s_1)-\phi(x,s_2)]} M_2(x+s_1(\nabla_\xi\lambda)(\omega),\ s-s_1,D_x,\ D_s)$$

$$\times e^{-i[\phi(x,s_1)-\phi(x,s_2)]} e^{i[\phi(x,s)-\phi(x,s_2)]} w_0(x,\ s)$$

$$+\frac{-i}{Q(\omega,\ \lambda(\omega))}\int_0^s ds_2\, e^{-i[\phi(x,s)-\phi(x,s_2)]} M_3(x+s_2(\nabla_\xi\lambda)(\omega)),\ s-s_2,\ D_x,\ D_s)$$

$$\times e^{i[\phi(x,s)-\phi(x,s_2)]} w_0(x,\ s)$$

$$=e^{-i\phi(x,s)}\tilde{M}_2(x,\ s,\ D_x,\ D_s)e^{i\phi(x,s)} w_0(x,\ s).$$

By induction in j, we can prove $(2.16)_j$ and $(2.17)_j$. Q.E.D.

In view of $(2.17)_j$, we have

Lemma 2.4. *$\phi(x,\ s,\ \omega)$ and $w_j(x,\ s,\ \omega)$ have the following estimates:*

(2.18) $$|\phi(x,\ s,\ \omega)|\leqq \text{const.}(1+|s|),$$

(2.19) $$|w_j(x,\ s,\ \omega)|\leqq \text{const.}(1+|s|)^{3j}\cdot\sup|w_0(x,\ s,\ \omega)|,$$

(2.20) $$\text{supp}_x[w_j(\cdot,\ s,\ \omega)]\subset\text{supp}_x[w_0(\cdot,\ s,\ \omega)] \quad \textit{for each} \quad (s,\ \omega).$$

Summing up, we have the following asymptotic solutions;

(2.21) $$u(x,\ t,\ \omega)=e^{i[\rho\varphi(x,\rho t,\omega)+\phi(x,\rho t,\omega)]}\sum_{j=0}^{N}\rho^{-j}w_j(x,\ \rho t,\ \omega),$$

where $w_0(x,\ s,\ \omega)$ and $w_j(x,\ s,\ \omega)$ are of the form (2.12) and $(2.16)_j$ respectively. $u(x,\ t,\ \omega)$ satisfies the equation

(2.22) $$P(x,\ t,\ D_x,\ D_s)u(x,\ t,\ \omega)=f(x,\ \rho t,\ \omega),$$

where

(2.23) $$f(x,\ s,\ \omega)=e^{i[\rho\varphi(x,s,\omega)+\phi(x,s,\omega)]}[\rho^{2m-2-N}$$
$$\times\{e^{-i\phi(x,s,\omega)}M_2(x,\ s,\ D_x,\ D_s)e^{i\phi(x,s,\omega)}w_N(x,\ s,\ \omega)+\cdots$$
$$+e^{-i\phi(x,s,\omega)}M_{2m}(x,\ s,\ D_x,\ D_s)e^{i\phi(x,s,\omega)}w_{N+2-2m}(x,\ s,\ \omega)\}$$
$$+\rho^{2m-3-N}\{e^{-i\phi(x,s,\omega)}M_3(x,\ s,\ D_x,\ D_s)e^{i\phi(x,s,\omega)}w_N(x,\ s,\ \omega)$$
$$+\cdots+e^{-i\phi(x,s,\omega)}M_{2m}(x,\ s,\ D_x,\ D_s)e^{i\phi(x,s,\omega)}w_{N+3-2m}(x,\ s,\ \omega)\}$$
$$+\cdots\cdots\cdots+$$
$$+\rho^{-N}e^{-i\phi(x,s,\omega)}M_{2m}(x,\ s,\ D_x,\ D_s)e^{i\phi(x,s,\omega)}w_N(x,\ s,\ \omega)].$$

§ 3. Proof of Theorem 1.

3.1. Using the asymptotic solutions constructed in a previous section, we shall prove Theorem 1. H^∞-wellposedness of the Cauchy problem (1.3) means that there exists non-negative integer q such that

$$(3.1)\qquad |||u(\cdot,t)|||_{(0)}\leqq C(T)\left\{|||u(\cdot,0)|||_{(q)}+\left|\int_0^t\|f(\cdot,t')\|_{(q)}dt'\right|\right\},\ t\in[-T,T],$$

where

$$(3.2)\qquad |||u(\cdot,t)|||_{(q)}^2=\sum_{j=1}^m\|(1-\Delta_x)^{m-j}D_t^{j-1}u(x,t)\|_{(q)}^2,$$

$$(3.3)\qquad \|u(x,t)\|_{(q)}^2=\sum_{|\alpha|\leqq q}\|D_x^\alpha u(x,t)\|_{L^2(R_x^n)}^2.$$

Assume that the conditions (A.1) and (A.2) hold, but the condition (A.3) does not hold. Then we shall construct a sequence of solutions of the Cauchy problem (1.3) which violate the inequality (3.1).

The following Lemma is essentially due to Ichinose [4].

Lemma 3.1. *Assume that the condition* (A.3) *does not hold. Then, for any positive integer* k, *there exists* $(x^{(k)},\omega^{(k)},\rho_k)\in R^n\times S^{n-1}\times R^1$ *such that*

$$(3.4)\qquad \int_0^{\rho_k}\mathrm{Im}\,P_{2m-1}(x^{(k)}+\sigma(\nabla_\xi\lambda)(\omega^{(k)}),0,\omega^{(k)},\lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)},\lambda(\omega^{(k)}))$$
$$\geqq k\log(1+|\rho_k|)+k,$$

$$(3.5)\qquad \lim_{k\to+\infty}\rho_k=+\infty,$$

and

$$(3.6)\qquad \int_0^{\theta\rho_k}\mathrm{Im}\,P_{2m-1}(x^{(k)}+\sigma(\nabla_\xi\lambda)(\omega^{(k)}),0,\omega^{(k)},\lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)},\lambda(\omega^{(k)}))$$
$$\geqq 0\qquad \textit{for}\quad \theta\in[0,1].$$

Proof. (3.5) follows from (3.4). Thus we only prove the Lemma with (3.4) and (3.6).

(i) By the assumption, for any positive integer k, there exists $(\hat{x}^{(k)},\omega^{(k)},\hat{\rho}_k)\in R^n\times S^{n-1}\times R^1$ such that

$$(3.7)\qquad \left|\int_0^{\hat{\rho}_k}\mathrm{Im}\,P_{2m-1}(\hat{x}^{(k)}+\sigma(\nabla_\xi\lambda)(\omega^{(k)}),0,\omega^{(k)},\lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)},\lambda(\omega^{(k)}))\right|$$
$$\geqq k\log(1+|\hat{\rho}_k|)+k.$$

Set

$$\Psi_k(\theta)=\int_0^{\theta\hat{\rho}_k}\mathrm{Im}\,P_{2m-1}(\hat{x}^{(k)}+\sigma(\nabla_\xi\lambda)(\omega^{(k)}),0,\omega^{(k)},\lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)},\lambda(\omega^{(k)}))$$
$$\text{for}\quad\theta\in[0,1].$$

If $\Psi_k(1)\leqq 0$, we put $\hat{\hat{x}}^{(k)}=\hat{x}^{(k)}+\hat{\rho}_k(\nabla_\xi\lambda)(\omega^{(k)})$, $\hat{\hat{\rho}}_k=-\hat{\rho}_k$. Then we have

$$\int_0^{\hat{\hat{\rho}}_k} \mathrm{Im}\, P_{2m-1}(\hat{\hat{x}}+\sigma(\nabla_\xi\lambda)(\omega^{(k)}),\ 0,\ \omega^{(k)},\ \lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)},\ \lambda(\omega^{(k)}))$$

$$=-\int_0^{\hat{\rho}_k} \mathrm{Im}\, P_{2m-1}(\hat{x}^{(k)}+\sigma(\nabla_\xi\lambda)(\omega^{(k)}),\ 0,\ \omega^{(k)},\ \lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)},\ \lambda(\omega^{(k)}))$$

$$\geqq 0.$$

Thus we can assume that $\Psi_k(1)\geqq 0$.

(ii) We take $\theta_k\in[0,\ 1]$ such that $\min_{0\leq\theta\leq1}\Psi_k(\theta)=\Psi_k(\theta_k)$. We put

$$\begin{cases} x^{(k)}=\hat{x}^{(k)}+\theta_k\hat{\rho}_k(\nabla_\xi\lambda)(\omega^{(k)}), \\ \rho_k=\hat{\rho}_k-\theta_k\hat{\rho}_k=(1-\theta_k)\hat{\rho}_k. \end{cases}$$

Then, for $t=\theta\rho_k$ $(\theta\in[0,\ 1])$, we have

$$\int_0^t \mathrm{Im}\, P_{2m-1}(x^{(k)}+\sigma(\nabla_\xi\lambda)(\omega^{(k)}),\ 0,\ \omega^{(k)},\ \lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)},\ \lambda(\omega^{(k)}))$$

$$=\int_{\theta_k\hat{\rho}_k}^{t+\theta_k\hat{\rho}_k} \mathrm{Im}\, P_{2m-1}(\hat{x}^{(k)}+\sigma(\nabla_\xi\lambda)(\omega^{(k)}),\ 0,\ \omega^{(k)},\ \lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)},\ \lambda(\omega^{(k)}))$$

$$=\Phi_k(\theta(1-\theta_k)+\theta_k)-\Psi_k(\theta_k)\geqq 0.$$

(iii) $$\int_0^{\rho_k} \mathrm{Im}\, P_{2m-1}(x^{(k)}+\sigma(\nabla_\xi\lambda)(\omega^{(k)}),\ 0,\ \omega^{(k)},\ \lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)},\ \lambda(\omega^{(k)}))$$

$$=\int_{\theta_k\hat{\rho}_k}^{\rho_k+\theta_k\hat{\rho}_k} \mathrm{Im}\, P_{2m-1}(\hat{x}^{(k)}+\sigma(\nabla_\xi\lambda)(\omega^{(k)}),\ 0,\ \omega^{(k)},\ \lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)},\ \lambda(\omega^{(k)}))$$

$$=\int_{\theta_k\hat{\rho}_k}^{\hat{\rho}_k} \mathrm{Im}\, P_{2m-1}(\hat{x}^{(k)}+\sigma(\nabla_\xi\lambda)(\omega^{(k)}),\ 0,\ \omega^{(k)},\ \lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)},\ \lambda(\omega^{(k)}))$$

$$=\Psi_k(1)+\{-\Psi_k(\theta_k)\}$$

$$\geqq\Psi_k(1) \qquad (\because\ 0=\Psi_k(0)\geqq\min_{0\leq\theta\leq1}\Psi_k(\theta)=\Psi_k(\theta_k))$$

$$\geqq k\log(1+|\hat{\rho}_k|)+k$$

$$\geqq k\log(1+|\rho_k|)+k \qquad (\because\ |\rho_k|\leqq|\hat{\rho}_k|).$$

Q.E.D.

3.2. Let $G(x)$ be C^∞-function such that

(3.8) $$G(x)>0 \quad \text{for} \quad |x|<1, \quad \mathrm{supp}\, G(x)\subset\{x\in R^n;\ |x|\leqq 1\}.$$

In (2.12), (2.21) and (2.23), we define

(3.9) $$g_0(x)=|\rho_k|^{n/2}G(|\rho_k|(x-(x^{(k)}+\rho_k(\nabla_\xi\lambda)(\omega^{(k)})))).$$

In (2.21), (2.22) and (2.23), we define $\rho=\rho_k^4$, $t=\rho_k^{-3}$, $\omega=\omega^{(k)}$, so that $\rho t=\rho_k$. We denote $u_k(x,\ t)=u(x,\ t,\ \omega^{(k)})$, $f_k(x,\ s)=f(x,\ s,\ \omega^{(k)})$ with $g_0(x)$ in (3.9). By (3.1), we have

(3.10) $$|||u_k(\cdot,\ \rho_k^{-3})|||_{(0)}\leqq C(T)\left\{|||u_k(\cdot,\ 0)|||_{(q)}+|\rho_k^{-4}|\left|\int_0^{\rho_k}\|f_k(\cdot,\ s')\|_{(q)}ds'\right|\right\}.$$

3.3. Evaluation of $|||u_k(\cdot, \rho_k^{-3})|||_{(0)}$. $u_k(x, \rho_k^{-3})$ is the following form:

$$u_k(x, \rho_k^{-3})=e^{i\rho_k^4\varphi(x,\rho_k)+i\psi(x,\rho_k)} \times[w_0(x, \rho_k)+\rho_k^{-1}(\rho_k^{-3}w_1(x, \rho_k))+\cdots+\rho_k^{-N}(\rho_k^{-3N}w_N(x, \rho_k))]. \tag{3.11}$$

It follows from (3.11) that, for large k,

$$\begin{aligned}&|u_k(x, \rho_k^{-3})| \\ &\geqq \exp\left[\int_0^{\rho_k} \mathrm{Im}\, P_{2m-1}(x+\sigma(\nabla_\xi\lambda)(\omega^{(k)}), 0, \omega^{(k)}, \lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)}, \lambda(\omega^{(k)}))\right] \\ &\times\{|w_0(x, \rho_k)|-|\rho_k|^{-1}|\rho_k^{-3}w_1(x, \rho_k)|-\cdots-|\rho_k|^{-N}|\rho_k^{-3N}w_N(x, \rho_k)|\} \\ &\geqq\frac{1}{2}|w_0(x, \rho_k)|\exp\left[\int_0^{\rho_k} \mathrm{Im}\, P_{2m-1}(x+\sigma(\nabla_\xi\lambda)(\omega^{(k)}), 0, \omega^{(k)}, \lambda(\omega^{(k)}))d\sigma/\right. \\ &\left. \qquad Q(\omega^{(k)}, \lambda(\omega^{(k)}))\right]\end{aligned} \tag{3.12}$$

(by (2.19) in Lemma 2.4)

$$\begin{aligned}&=\frac{1}{2}\exp\left[\int_0^{\rho_k} \mathrm{Im}\, P_{2m-1}(x^{(k)}+\sigma(\nabla_\xi\lambda)(\omega^{(k)}), 0, \omega^{(k)}, \lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)}, \lambda(\omega^{(k)}))\right] \\ &\times\exp\left[\int_0^1 d\theta\int_0^1 d\sigma\, \mathrm{Im}\,(\nabla_x P_{2m-1})(x^{(k)}+\rho_k\sigma(\nabla_\xi\lambda)(\omega^{(k)})+\theta(x-x^{(k)}),\right. \\ &\left. 0, \omega^{(k)}, \lambda(\omega^{(k)}))\cdot\rho_k(x-x^{(k)})/Q(\omega^{(k)}, \lambda(\omega^{(k)})\right]|w_0(x, \rho_k)|.\end{aligned}$$

By (2.12), (3.8) and (3.9), we have

$$w_0(x, \rho_k)=|\rho_k|^{n/2}G(|\rho_k|(x-x^{(k)})), \tag{3.13}$$

$$\mathrm{supp}\, w_0(x, \rho_k)\subset\left\{x\in R^n;\ |x-x^{(k)}|\leqq\frac{1}{|\rho_k|}\right\}. \tag{3.14}$$

In view of (3.12), (3.13) and (3.14), we have, for large k,

$$\begin{aligned}|||u(\cdot, \rho_k^{-3})|||_{(0)}&\geqq\|u_k(\cdot, \rho_k^{-3})\|_{(0)} \\ &\geqq C_0\|G(x)\|_{L^2(R^n)}\exp\left[\int_0^{\rho_k} \mathrm{Im}\, P_{2m-1}(x^{(k)}+\sigma(\nabla_\xi\lambda)(\omega^{(k)}),\right. \\ &\qquad \left. 0, \omega^{(k)}, \lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)}, \lambda(\omega^{(k)}))\right] \\ &\geqq C_0\|G(x)\|_{L^2(R^n)}\cdot(1+|\rho_k|)^k, \qquad \text{(by (3.4) in Lemma 3.1)},\end{aligned} \tag{3.15}$$

where C_0 is a positive constant independent of k.

3.4. Evaluation of $|||u_k(\cdot, 0)|||_{(q)}$.

It is easy to see that, for large k,

$$\begin{aligned}&|||u_k(\cdot, 0)|||_{(q)} \\ &=\left\{\sum_{j=1}^{m}\sum_{|\alpha|\leq q}\|D_x^\alpha(1-\Delta_x)^{m-j}D_t^{j-1}u_k(x, t)\|_{(0)}^2|_{t=0}\right\}^{1/2}\end{aligned} \tag{3.16}$$

$$\leqq \text{const.}\left[\sum_{j=1}^{m}\sum_{|\alpha|\leqq q}\{|\rho_k|^{4|\alpha|}|\rho_k|^{4(m-j)+4(j-1)}\}^2\right]^{1/2}\|G(x)\|_{(0)}$$

$$\leqq \text{const.}\,|\rho_k|^{4(q+m-1)}\|G(x)\|_{(0)}.$$

3.5. Evaluation of $\int_0^{\rho_k}\|f_k(\cdot,\, s')\|_{(q)}ds'$. In view of (2.13) and (2.14) in Lemma 2.1 and (2.19) in Lemma 2.4, we have, for large k,

$$(3.17)\qquad \left|\rho_k^{-4}\int_0^{\rho_k}\|f_k(x,\, s')\|_{(q)}ds'\right|$$

$$=\left|\rho_k^{-4}\int_0^{\rho_k}\|f_k(x+s'(\nabla_\xi\lambda)(\omega^{(k)}),\ \rho_k-s')\|_{(q)}ds'\right|$$

$$\leqq \text{const.}\,|\rho_k|^{-4+4(2m-2-N)+4q}$$

$$\times\left|\int_0^{\rho_k}\|e^{-\text{Im}\,\phi(x+s'(\nabla_\xi\lambda)(\omega^{(k)}),\,\rho_k-s',\,\omega^{(k)})}\right.$$

$$\times\{e^{-i\phi(x+s'(\nabla_\xi\lambda)(\omega^{(k)}),\,\rho_k-s',\,\omega^{(k)})}M_2(x+s'(\nabla_\xi\lambda)(\omega^{(k)}),\ \rho_k-s',\ D_x,\ -D_{s'})$$

$$\times\tilde{M}_N(x+s'(\nabla_\xi\lambda)(\omega^{(k)}),\ \rho_k-s',\ D_x,\ -D_{s'})e^{i\phi(x+s'(\nabla_\xi\lambda)(\omega^{(k)}),\,\rho_k-s',\,\omega^{(k)})}\}$$

$$\left.\times w_0(x+s'(\nabla_\xi\lambda)(\omega^{(k)}),\ \rho_k-s',\ \omega^{(k)})\|_{(0)}ds'\right|$$

$$\leqq \text{const.}\,|\rho_k|^{-4+4(2m-2-N)+4q+3N+2}$$

$$\times\left|\int_0^{\rho_k}\|e^{-\text{Im}[\phi(x,\,\rho_k,\,\omega^{(k)})-\phi(x,\,s',\,\omega^{(k)})]}w_0(x,\ \rho_k,\ \omega^{(k)})\|_{(0)}ds'\right|$$

$$\leqq \text{const.}\,|\rho_k|^{4(2m+q)-10-N}\left|\int_0^{\rho_k}e^{-\text{Im}[\phi(x^{(k)},\,\rho_k,\,\omega^{(k)})-\phi(x^{(k)},\,s',\,\omega^{(k)})]}\right.$$

$$\times\|e^{-\text{Im}[\phi(x,\,\rho_k,\,\omega^{(k)})-\phi(x^{(k)},\,\rho_k,\,\omega^{(k)})]+\text{Im}[\phi(x,\,s',\,\omega^{(k)})-\phi(x^{(k)},\,s',\,\omega^{(k)})]}$$

$$\left.\times w_0(x,\ \rho_k,\ \omega^{(k)})\|_{(0)}ds'\right|.$$

$$(3.18)\qquad -\text{Im}\,[\phi(x,\ \rho_k,\ \omega^{(k)})-\phi(x^{(k)},\ \rho_k,\ \omega^{(k)})]$$

$$=-\int_0^1\text{Im}\,(\nabla_x\phi)(x^{(k)}+\theta(x-x^{(k)}),\ \rho_k,\ \omega^{(k)})\cdot(x-x^{(k)})d\theta$$

$$=\int_0^1 d\theta\int_0^{\rho_k}\text{Im}\,(\nabla_x P_{2m-1})(x^{(k)}+\theta(x-x^{(k)})+\sigma(\nabla_\xi\lambda)(\omega^{(k)}),\ 0,\ \omega^{(k)},\ \lambda(\omega^{(k)}))$$

$$\cdot(x-x^{(k)})d\sigma/Q(\omega^{(k)},\ \lambda(\omega^{(k)}))$$

$$=\int_0^1 d\theta\int_0^1 d\sigma'\,\text{Im}\,(\nabla_x P_{2m-1})(x^{(k)}+\theta(x-x^{(k)})+\rho_k\sigma'(\nabla_\xi\lambda)(\omega^{(k)}),$$

$$0,\ \omega^{(k)},\ \lambda(\omega^{(k)}))\cdot\rho_k(x-x^{(k)})/Q(\omega^{(k)},\ \lambda(\omega^{(k)})),$$

which is uniformly bounded on $\text{supp}\,w_0(x,\ \rho_k,\ \omega^{(k)})=\left\{x\in R^n;\ |x-x^{(k)}|\leqq\frac{1}{|\rho_k|}\right\}$. Similarly, $\text{Im}\,[\phi(x,\ s',\ \omega^{(k)})-\phi(x^{(k)},\ s',\ \omega^{(k)})]$ is bounded on $\text{supp}\,w_0(x,\ \rho_k,\ \omega^{(k)})$ uniformly in k and $s'(|s'|\leqq|\rho_k|)$. Thus, using (3.6) in Lemma 3.1, we have, for large k,

$$(3.19)\quad \left|\rho_k^{-4}\int_0^{\rho_k}\|f_k(x,\ s')\|_{(q)}ds'\right|$$
$$\leqq \text{const.}\,|\rho_k|^{4(2m+q)-9-N}e^{-\text{Im}\,\phi(x^{(k)},\,\rho_k,\,\omega^{(k)})}\|G(x)\|_{(0)}$$
$$=\text{const.}\,|\rho_k|^{4(2m+q)-9-N}\|G(x)\|_{(0)}$$
$$\times\exp\left[\int_0^{\rho_k}\text{Im}\,P_{2m-1}(x^{(k)}+\sigma(\nabla_\xi\lambda)(\omega^{(k)}),\ 0,\ \omega^{(k)},\ \lambda(\omega^{(k)}))d\sigma/Q(\omega^{(k)},\ \lambda(\omega^{(k)}))\right].$$

3.6. Proof of Theorem 1. In view of (3.15), (3.16) and (3.19), from (3.10) we have the following inequality.

$$(3.20)\quad C_0\|G(x)\|_{(0)}e^{-\text{Im}\,\phi(x^{(k)},\,\rho_k,\,\omega^{(k)})}$$
$$\leqq C(T)\{C_1|\rho_k|^{4(q+m-1)}\|G(x)\|_{(0)}+C_2|\rho_k|^{4(2m+q)-9-N}$$
$$\times\|G(x)\|_{(0)}e^{-\text{Im}\,\phi(x^{(k)},\,\rho_k,\,\varphi^{(k)})}\},$$

where C_0, C_1, C_2 are positive constants independent of k. If we choose $N=4(2m+q-2)$, then (3.20) is impossible as k tends to infinity. This completes the proof of Theorem 1.

§ 4. Examples.

Examepl 4.1. Consider an operator of the following form:

$$(4.1)\quad P(x,\ y,\ D_x,\ D_y,\ D_t)=D_t+\Delta_x+\sum_{j=1}^m a_j(x,\ y)D_{x_j}$$
$$+\sum_{k=1}^n b_k(x,\ y)D_{y_k}+c(x,\ y)\ \text{in}\ R^m\times R^n\times R^1.$$

Denote the dual variables to $(x,\ y)\in R^m\times R^n$ by $(\xi,\ \eta)\in R^m\times R^n$. The condition (A.3) means that there exist constants M and N such that

$$(4.2)\quad \sup_{\substack{(x,\,y)\in R^m\times R^n\\(\xi,\,\eta)\in S^{m+n-1}}}\left|\int_0^\rho\left\{\sum_{j=1}^m\text{Im}\,a_j(x+s\xi,\ y)\xi_j+\sum_{k=1}^n\text{Im}\,b_k(x+s\xi,\ y)\eta_k\right\}ds\right|$$
$$\leqq M\log(1+|\rho|)+N,\qquad \rho\in R^1.$$

We take $\xi=0$ in (4.2). Then we have from (4.2) that

$$\text{Im}\,b_k(x,\ y)\equiv 0,\qquad (1\leqq k\leqq n).$$

Thus, from (4.2), we have

$$(4.3)\quad \begin{cases}\sup\limits_{\substack{(x,\,y)\in R^m\times R^n\\ \omega\in S^{m-1}}}\left|\int_0^\rho\sum_{j=1}^m\text{Im}\,a_j(x+s\omega,\ y)\omega_j ds\right|\leqq M\log(1+|\rho|)+N,\ \rho\in R^1,\\ \text{Im}\,b_k(x,\ y)\equiv 0,\qquad (1\leqq k\leqq n).\end{cases}$$

Note that (4.2) is equivalent to the inequality in (4.2) with $\rho\geqq 0$. When (4.3) is satisfied, we would like to say that the operator (4.1) is of Schrödinger type in the direction $(\xi,\ 0)$ and of hyperbolic type in the direction $(0,\ \eta)$.

Example 4.2. Consider an operator

$$P(x, D_x, D_t)=D_t^2-\left(\sum_{j=1}^{n} D_j^2\right)^2+B(x, D_x, D_t), \tag{4.4}$$

defined on $(x, t)\in R^n\times R^1$, where

$$B(x, D_x, D_t)=b_1(x, D_x)D_t+b_2(x, D_x), \tag{4.5}$$

and

$$b_j(x, D_x)=\sum_{|\alpha|\leq 2j-1} b_{\alpha j}(x)D_x^\alpha, \qquad j=1, 2. \tag{4.6}$$

The principal part of $P(x, D_x, D_t)$ is

$$D_t^2-\left(\sum_{j=j}^{n} D_j^2\right)^2=-\left[\left(\frac{\partial}{\partial t}\right)^2+\Delta_x^2\right], \tag{4.7}$$

which is appeared in the equation of vibrating plate. (See Courant-Hilbert [2, p. 252]. See also Takeuchi [14].) The condition (A.3) means that there exist constants M and N such that

$$\begin{aligned}\sup_{(x,\omega)\in R^n\times S^{n-1}} &\left|\int_0^\rho \operatorname{Im} b^\pm(x+s\omega, \omega)\,ds\right| \\ &\leqq M\log(1+|\rho|)+N, \qquad \rho\in R^1,\end{aligned} \tag{4.8}$$

where

$$b^\pm(x, \xi)=\frac{1}{2}\{b_2^0(x, \xi/|\xi|)\pm b_1^0(x, \xi/|\xi|)\}\,|\xi|, \tag{4.9}$$

(the same sign)

and

$$b_j^0(x, \xi)=\sum_{|\alpha|=2j-1} b_{\alpha j}(x)\xi^\alpha, \qquad (j=1, 2). \tag{4.10}$$

The condition (4.8) is equivalent to the following.

$$\begin{aligned}\sup_{\substack{(x,\omega)\in R^n\times S^{n-1}\\ j=1,2}} &\left|\int_0^\rho \operatorname{Im} b_j^0(x+s\omega, \omega)\,ds\right| \\ &\leqq M\log(1+|\rho|)+N, \qquad \rho\in R^1.\end{aligned} \tag{4.11}$$

Note that (4.8) and (4.11) are equivalent to the inequalities in (4.8) and (4.11) with $\rho\geqq 0$ respectively.

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References

[1] G.D. Birkhoff, Quantum mechanics and asymptotic series, Bull. Amer. Math. Soc., **39** (1933), 681-700.

[2] R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. I, Interscience, 1953.

[3] W. Ichinose, Sufficient condition on H_∞ wellposedness for Schrödinger type equations, Comm. in Partial Differential Equations, **9** (1984), 33-48.

[4] W. Ichinose, Some remarks on the Cauchy problem for Schrödinger type equations, Osaka J. Math., **21** (1984), 565-581.

[5] J. Leray, Lagrangian Analysis and Quantum Mechanics. A mathematical structure related to asymptotic expansions and the Maslov index, MIT Press, 1981.

[6] V.P. Maslov, Theory of Perturbations and Asymptotic Methods, Moscow, 1965. (French translation from Russian, Dunod, Paris, 1970.)

[7] S. Mizohata, Some remarks on the Cauchy problem, J. Math. Kyoto Univ., **1** (1961), 109-127.

[8] S. Mizohata, On some Schrödinger type equations, Proc. Japan Acad., **57** (1981), 81-84.

[9] S. Mizohata, Sur quelques équations du type Schrödinger, Séminaire J. Vaillant (1980/81), Univ. Paris VI.

[10] S. Mizohata, Sur quelques équations du type Schrödinger, Journées "Équations aux Dérivées Partielles", Soc. Math. France, 1981.

[11] I.G. Petrowski, Uber das Cauchysche Problem für ein System linearer partieller Differentialgleichungen im Gebiete der nichtanalytischen Funktionen, Bull. Univ. Etat Moscou, **1** (1938), 1-74.

[12] J. Takeuchi, On the Cauchy problem for some non-kowalewskian equations with distinct characteristic roots, J. Math. Kyoto Univ., **20** (1980), 105-124.

[13] J. Takeuchi, Some remarks on my paper "On the Cauchy problem for some non-kowalewskian equations with distinct characteristic roots", (Schrödinger equations and generalizations; I), J. Math. Kyoto Univ., **24** (1984), 741-754.

[14] J. Takeuchi, Remarks on the Cauchy problem for linear partial differential operators with the principal part $\left(\frac{\partial}{\partial t}\right)^2+\Delta_x^2$, (Schrödinger equations and generalizations; II), Bull. Iron and Steel Technical College, **18** (1984), 15-23.

[15] J. Takeuchi, On the Cauchy problem for systems of linear partial differential equations of Schrödinger type, (Schrödinger equations and generalizations; III), Bull. Iron and Steel Technical College, **18** (1984), 25-34.