J. Math. Kyoto Univ. (JMKYAZ)
25-4 (1985) 757-766

Extreme Pick-Nevanlinna interpolating functions

Dedicated to Professor Yukio Kusunoki on the occasion of his sixtieth birthday

By

Maurice HEINS

(Communicated by Prof. Y. Kusunoki, October 1, 1984)

1. In this paper we treat an aspect of Pick-Nevanlinna interpolation theory [13-17] which finds its setting in the theory of convex sets. Specifically, we consider the class I of analytic functions f on Δ, the open unit disk in $\boldsymbol{C}$, which satisfy the following conditions: (i) $\operatorname{Re} f>0$, (ii) $f(0)=1$, (iii) $f(z_k)=w_k$, $k=1,\dots, n$, where the z_k are given distinct points of $\Delta-\{0\}$ and the w_k are given points of $\{\operatorname{Re} z>0\}$, $k=1,\dots, n$, n a nonnegative integer. In other words, we are concerned with a harmlessly normalized version of the finite Pick-Nevanlinna interpolation problem where the value 1 is assigned to 0. [For the sake of simplicity of exposition we confine our attention to 0 order interpolation. To be sure, the results obtained will be seen to extend readily.] We suppose that *the class I contains more than one member*. The class I is a compact convex subset of the space of analytic functions on Δ. We seek *to characterize the extreme points of I*, i.e. the members of I *not* admitting a representation of the form $(1-t)f_1+tf_2$, where f_1 and f_2 are distinct members of I and $0<t<1$. It is to be noted that the extreme points associated with a non-normalized finite Pick-Nevalinna problem correspond directly to those associated with a simply related normalized problem as we see with the aid of the map $f\mapsto Af\circ\alpha+iB$, $A>0$, $B\in\boldsymbol{R}$, α a conformal automorphism of Δ. The map in question is a bijection of the space of analytic functions on Δ with positive real part onto itself.

We have the following theorem.

Theorem 1. *The extreme points of I are precisely the members of I having constant valence on* $\{\operatorname{Re} z>0\}$, *the value v of the valence satisfying* $1+n\leqq v\leqq 1+2n$.

The proof of the theorem (§2) will be based on the Poisson-Stieltjes representation for analytic functions on Δ with non-negative real part [10, 18] and an elementary fact from Pick-Nevanlinna interpolation theory.

In §3 the extreme points of I will be given a simple representation based on a Nevanlinna representation for the members of I. As a consequence, the extreme points of I will be given a parametric representation the domain of which is the frontier of a convex body in $\boldsymbol{C}^{n+1}$ specified in the manner of the Carathéodory theory of coefficient bodies [2, 3]. Each frontier point of the convex body in question will be seen to be an extreme point of the body. Using this latter representation for the extreme points of I we shall conclude that there is a class of simple extremal problems for I the solutions of which are precisely the extreme points of I.

The corresponding extreme point problems in the setting of the unit ball of $H^{\infty}(\Delta)$ yield a large class of functions that are far less tractable to consider than the class of extreme functions in I.

It is appropriate to cite instances of convexity considerations related to the present paper. The pioneer work of Carathéodory [2, 3] on coefficient problems for analytic functions with positive real part is, as far as I am aware, the first bringing together of the Minkowski theory of convex sets and complex function theory. Extreme points are present in the fundamental work of R. S. Martin [12] on the representation of positive harmonic functions as normalized minimal positive harmonic functions. My paper [7] showed the existence of minimal positive harmonic functions on Riemann surfaces using elementary standard normal family results without the intervention of the Krein-Milman theorem and gave applications to qualitative aspects of Pick-Nevanlinna interpolation on Riemann surfaces with finite topological characteristics and nonpointlike boundary components. Such Riemann surfaces will be termed *finite* Riemann surfaces henceforth. In [8] the Carathéodory theory cited above was extended to the setting of finite Riemann surfaces for interpolation problems subsuming those of Pick-Nevanlinna type. Forelli [5] has studied the extreme points of the family of analytic functions with positive real part on a given finite Riemann surface S normalized to take the value 1 at a given point of S. In my paper [9] the results of Forelli were supplemented by precise characterizing results for the case where the genus of S is positive. The problem in question is, of course, the one of this paper with $n=0$, Δ replaced by S and 0 by the point of normalization.

The results of Pick-Nevanlinna interpolation theory which will be wanted will be given in the course of the exposition. An elementary approach to Pick-Nevanlinna interpolation theory has been given by Marshall [11].

2. Proof of Theorem 1. It will be based upon a simple standard result of Pick-Nevanlinna interpolation theory, to be given as Lemma 2, and the Poisson-Stieltjes representation for analytic functions on Δ having nonnegative real part.

Lemma 2. *Let n be a positive integer. Let $(z_k)_1^n$ be an injection into Δ and let $(w_k)_1^n$ satisfy $|w_k|\leqq 1$, $k=1,\dots,n$. If there exists an analytic function F on Δ of modulus at most one satisfying $F(z_k)=w_k$, $k=1,\dots,n$ then there exists a finite Blaschke product b of degree $\leqq n$ satisfying the interpolation condition: $b(z_k)=w_k$, $k=1,\dots,n$.*

There exists exactly one such F if and only if there exists a finite Blaschke product b of degree $\leqq n-1$ satisfying: $b(z_k)=w_k$, $k=1,\dots,n$.

Proof of Lemma 2. Use will be made of the standard Schur-Nevanlinna algorithm [13, 14, 19]. Given $a \in \Delta$, let L_a denote the Möbius transformation $z \mapsto (a-z)/(1-\bar{a}z)$. We note that L_a is an involution.

First assertion. We proceed by induction on n. For $n=1$ the assertion is immediate. Indeed, if $|w_1|=1$, the constant value w_1 serves, while if $|w_1|<1$, the function $L_{w_1} \circ L_{z_1}$ serves. Suppose that the first assertion holds for a given n and that F is a function of the stated type where n is replaced by $n+1$. If $\max |w_k|=1$, the assertion is immediate. If $\max |w_k|<1$, we consider $g=(L_{w_{n+1}} \circ F)/L_{z_{n+1}}$ (Schur-Nevanlinna algorithm) and note that g is an analytic function on Δ taking values of modulus $\leqq 1$ and satisfying $g(z_k)=L_{w_{n+1}}(w_k)/L_{z_{n+1}}(z_k)$, $k=1,\dots,n$. The inductive hypothesis permits us to replace g by G, a finite Blaschke product of degree $\leqq n$ satisfying $G(z_k)=g(z_k)$, $k=1,\dots,n$. The function $b=L_{w_{n+1}} \circ (GL_{z_{n+1}})$ is a finite Blaschke product of degree $\leqq n+1$ satisfying $b(z_k)=w_k$, $k=1,\dots,n+1$.

Second assertion. Suppose that there is exactly one such F. We proceed by induction on n. If $n=1$, then $|w_1|=1$. Otherwise there would not be a unique such F. Consequently a Blaschke product of degree 0 satisfies the interpolation condition. Passing from n to $n+1$, we see that we may put aside the trivial case where $\max |w_k|=1$ and that in the remaining case there is a unique analytic function g on Δ taking values of modulus at most one and satisfying $g(z_k)=L_{w_{n+1}}(w_k)/L_{z_{n+1}}(z_k)$, $k=1,\dots,n$. Indeed, $L_{w_{n+1}} \circ F/L_{z_{n+1}}$ is such a g and for each such g we have $F=L_{w_{n+1}} \circ (gL_{z_{n+1}})$. There is exactly one such g. By the inductive hypothesis g is a finite Blaschke product of degree $\leqq n-1$. Consequently, F is a finite Blaschke product of positive degree $\leqq n$.

The converse part of the second assertion is immediate for $n=1$. To pass inductively from n to $n+1$ we put aside the trivial case where $\max |w_k|=1$ and note that it suffices to apply the induction hypothesis to $L_{w_{n+1}} \circ b/L_{z_{n+1}}$ and $L_{w_{n+1}} \circ F/L_{z_{n+1}}$ where F satisfies the stated conditions.

For the application of Lemma 2 to the proof of Theorem 1, to which we now turn, we shall employ a fixed mediating Möbius transformation, μ: $z \mapsto (z-1)/(z+1)$, which maps $\{\mathrm{Re}\, z>0\}$ bijectively onto Δ. The notation "μ" is to be understood in this sense for the remainder of the paper.

Let f be an extreme point of I. We introduce its Poisson-Stieltjes respesentation

$$f(z)=\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\gamma(\theta), \tag{2.1}$$

where γ is nondecreasing on $\boldsymbol{R}$ and satisfies the following conditions: $\gamma(0)=0$; $\gamma(\theta+2\pi)=\gamma(\theta)+1$, $\gamma(\theta)=[\gamma(\theta+)+\gamma(\theta-)]/2$, $\theta \in \boldsymbol{R}$. Let $(\theta_k)_0^{m+1}$ be a partitioning of $[\alpha, \alpha+2\pi]$ where $m \geqq 1+2n$ and γ is continuous at the θ_k. We show that γ *is constant on one of the segments* $[\theta_k, \theta_{k+1}]$. To that end we introduce the functions

$$g_k\colon z \longrightarrow \int_{\theta_k}^{\theta_{k+1}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\gamma(\theta), \quad |z|<1, \quad k=0,\dots,m. \tag{2.2}$$

Let $g=\Sigma\, \alpha_k g_k$, where the α_k are real but not all 0 such that $g(z)=0$ for $z=0, z_1,\dots,z_n$. For t small and real $f+tg \in I$. Since f is extreme, we infer from $2f=(f+tg)+(f-tg)$ that g is identically 0. Let $l \in \{0,\dots,m\}$ be such that $\alpha_l \neq 0$. Since g is identically 0, we see that $\operatorname{Re} g_l$ vanishes continuously on the open arc $\{e^{i\theta}, \theta_l<\theta<\theta_{l+1}\}$. Using the continuity hypothesis on the θ_k we infer that $\operatorname{Re} g_l=0$ and thereupon that γ is constant on the segment $[\theta_l, \theta_{l+1}]$. Let γ be continuous at $\beta\in \boldsymbol{R}$. We conclude that there are *at most* $1+2n$ points in the open interval $(\beta, \beta+2\pi)$ in all the neighborhoods of which γ is not constant. Of course, there is at least one such point. We are led to the conclusion that f is a map of constant valence ν of Δ onto $\{\operatorname{Re} z>0\}$ where $1\leqq\nu\leqq 1+2n$.

To show that $1+n$ is a lower bound on the valence of an extreme f we apply Lemma 2 and conclude with aid of μ specified above that if there existed an extreme f of degree $\leqq n$, the set I would reduce to a singleton contrary to hypothesis.

To complete the proof of Theorem 1 it remains to show that if $f\in I$ is a map of constant valence ν of Δ onto $\{\operatorname{Re} z>0\}$ where $1+n\leqq\nu\leqq 1+2n$, then f is an extreme point of I. Suppose that $f=(1-t)g+th$ where $0<t<1$ and $g, h\in I$. On considering $\operatorname{Re} g$ and $\operatorname{Re} h$ we see that g and h are each maps of constant finite valence of Δ onto $\{\operatorname{Re} z>0\}$ and that the poles of each are contained in the set of poles of f. Hence $g-h$ is a rational function having at most ν poles, all simple, and taking the value 0 at $0, z_1,\dots, z_n$ and at $\infty, \bar{z}_1^{-1},\dots, \bar{z}_n^{-1}$. It follows that $g-h=0$. Consequently, f is an extreme point of I as we wished to show.

3. A representation formula for the extreme members of *I*. We return to the study of finite Pick-Nevanlinna interpolation problems where there is more than one solution and recall a standard representation for the totality of solutions. Cf. [14]. It will be seen that the part of the Nevanlinna interpolation theory to be used is easily established with the aid of the Schur-Nevanlinna algorithm employed in Lemma 2.

We suppose that n, $(z_k)_1^n$, $(w_k)_1^n$, satisfy the conditions of the first two sentences of Lemma 2 and that the family of analytic functions F on Δ of modulus <1 satisfying $F(z_k)=w_k$, $k=1,\dots, n$, *contains more than one member.* We have

Lemma 3. *There exist rational functions A, B, C, where (i) B is a Blaschke product of degree n, (ii) $|A|$, $|C|<1$ for $|z|\leqq 1$, and (iii) $C(z)=\bar{A}(z)B(z)$, $|z|=1$, such that the totality of functions satisfying the stated interpolation condition is exactly the set of functions*

$$F_g\colon z\longmapsto \frac{A(z)+B(z)g(z)}{1+C(z)g(z)},\quad |z|<1, \tag{3.1}$$

where g is in the closed unit ball of $H^\infty(\Delta)$.

Proof. For $n=1$ we see from $F=L_{w_1}\circ(gL_{z_1})$ that with $A=w_1$, $B=-L_{z_1}$, $C=-\bar{w}_1L_{z_1}$ the requirements of the lemma are fulfilled. To treat the case of index $n+1$ we introduce a representation

$$z\longmapsto \frac{A_n(z)+B_n(z)g(z)}{1+C_n(z)g(z)} \tag{3.2}$$

for the interpolating functions corresponding to the truncated condition: $F(z_k)=w_k$, $k=1,\dots, n$, and note that the map

$$M: \zeta \longmapsto [A_n(z_{n+1}) + B_n(z_{n+1})\zeta]/[1 + C_n(z_{n+1})\zeta] \tag{3.3}$$

is not constant since the points z_k, $k=1,\ldots, n$, are zeros of $B_n - A_nC_n$ and there are no other zeros in Δ by the inductive assumption on A_n, B_n, C_n and the theorem of Rouché. Since there is more than one interpolating function for the problem of index $n+1$, we see with the aid of (3.2) that $|\operatorname{inv} M(w_{n+1})| < 1$, "inv" denoting "inverse". The interpolating functions for the case of index $n+1$ are exactly the functions (3.2) where the g are the members of the closed unit ball of $H^\infty(\Delta)$ satisfying the interpolation condition: $g(z_{n+1}) = \operatorname{inv} M(w_{n+1})$. Using the coefficients given at the beginning of this proof for the case, $n=1$, we obtain the desired result by composition and normalization.

Of course, the discussion just given is simply a reduced qualitative version of the Nevanlinna developments [14] combined with the Walsh normalization [20, p. 299] intended for our present purposes.

Our object in introducing Lemma 3 is to obtain a representation for the extreme members of I. By the results of §2 the map $f \mapsto \mu \circ f$ is a bijection of the set of extreme members of I onto the set of finite Blaschke products b satisfying $b(0)=0$, $b(z_k) = \mu(w_k)$, $k=1,\ldots, n$, and having degree $v(b)$ satisfying $1+n \leqq v(b) \leqq 1+2n$. We are thus led to inquire under what circumstances F_g of (3.1) is a finite Blaschke product of given degree v. We have

Lemma 4. *The function F_g of* (3.1) *is a finite Blaschke product of degree v if and only if g is a finite Blaschke product of degree $v-n$. The set of realized degrees v is the set of positive integers at least as large as n.*

The proof of Lemma 4 follows on noting that F_g is a finite Blaschke product if and only if g is and that, since B_n is a finite Blaschke product of degree n, by the theorem of Rouché when g is a finite Blaschke product of degree $v(g)$, the degree of F_g is $n+v(g)$.

The extreme members of I are the functions $\operatorname{inv} \mu \circ F_b$ where b is a finite Blaschke product of degree $\leqq n$ and (3.1) is taken relative to the interpolation data: $0 \mapsto 0$, $z_k \mapsto \mu(w_k)$, $k=1,\ldots, n$.

We now apply this result to obtain a parametric representation of the extreme points of I in terms of a simply described convex body in $\boldsymbol{C}^{n+1}$. To that end, let $\zeta_1,\ldots, \zeta_{n+1}$ be $n+1$ distinct points of $\Delta - \{0, z_1,\ldots, z_n\}$. We introduce

$$K = \{(f(\zeta_1),\ldots, f(\zeta_{n+1})), f \in I\} \tag{3.4}$$

and observe that K is a compact convex body ($\operatorname{int} K \neq \emptyset$) in $\boldsymbol{C}^{n+1}$. The fact that K is compact and convex is routine to verify. The fact that $\operatorname{int} K \neq \emptyset$ is a consequence of (3.1).

We next observe that by Lemma 2 the map

$$b \longmapsto (b(\zeta_1),\ldots, b(\zeta_{n+1})), \tag{3.5}$$

b a finite Blaschke product of degree $\leqq n$, is injective. The image $\mathscr{B}$ of this map is the frontier of

$$(3.6)\qquad K_1=\{(g(\zeta_1),\ldots,g(\zeta_{n+1})) : g\in H^\infty(\Delta),\ \sup|g|\leqq 1\},$$

which is a compact convex subset of C^{n+1}. Indeed, if g is a point of the closed unit ball of $H^\infty(\Delta)$ such that $(g(\zeta_1),\ldots,g(\zeta_{n+1}))\in frK_1$, by Lemma 3 the function g is the only member of the closed unit ball of $H^\infty(\Delta)$ taking the value $g(\zeta_k)$ at ζ_k, $k=1,\ldots,n+1$. By Lemma 2 we see that g is a finite Blaschke product of degree $\leqq n$. Consequently $frK_1\subset\mathscr{B}$.

By a second application of Lemma 2 we show that $\mathscr{B}\subset frK_1$. To that end we note that a point of $\operatorname{int}K_1$ is attained by some $g\in H^\infty(\Delta)$ satisfying $\sup|g|<1$ as we see with the aid of a homothetic contraction $(c_1,\ldots,c_{n+1})\mapsto(\lambda c_1,\ldots,\lambda c_{n+1})$, $0<\lambda<1$, λ near one. Hence a point of $\operatorname{int}K_1$ is attained by more than one member of the open unit ball of $H^\infty(\Delta)$. Using Lemma 2 we conclude that $\mathscr{B}\subset frK_1$. It follows that $\mathscr{B}=frK_1$.

With the aid of the equality

$$(3.7)\qquad K=\{(\operatorname{inv}\mu\circ F_g(\zeta_1),\ldots,\operatorname{inv}\mu\circ F_g(\zeta_{n+1})\},$$

g ranging over the closed unit ball of $H^\infty(\Delta)$, we conclude that frK is the set of elements of K having a unique antecedent with respect to

$$(3.8)\qquad \theta: f\longmapsto(f(\zeta_1),\ldots,f(\zeta_{n+1})),\quad f\in I,$$

and that $\theta^{-1}(frK)$ is the set of extreme elements of I. It suffices to refer to the representation $\operatorname{inv}\mu\circ F_g$ of f. The map $\operatorname{inv}[\theta|\theta^{-1}(frK)]$ is a continuous bijection of frK onto the set of extreme points of I.

As consequences of the results just stated we have (i) each point of frK is an extreme point of K, (ii) each extreme member of I is the unique maximizer on I of some continuous real linear functional defined on the space of analytic functions on Δ.

(i) Given $\theta(f)\in frK$, $f\in I$. Suppose that $\theta(f)=(1-t)\theta(f_1)+t\theta(f_2)$, where $0<t<1$ and $f_1, f_2\in I$. From $\theta(f)=\theta[(1-t)f_1+tf_2]\in frK$ we conclude that $f=(1-t)f_1+tf_2$. Since f is an extreme point of I, we have $f_1=f_2$ and so $\theta(f_1)=\theta(f_2)$ as we wished to show.

(ii) Let $\operatorname{Re}\sum_1^{n+1} c_k w_k=d$ define a supporting plane for K passing through $(f_0(\zeta_1),\ldots,f_0(\zeta_{n+1}))$, f_0 an extreme member of I. It is supposed that the c_k and d are so normalized that

$$(3.9)\qquad \max\{\operatorname{Re}\sum_1^{n+1} c_k u_k,\ (u_1,\ldots,u_{n+1})\in K\}=d.$$

The point $(f_0(\zeta_1),\ldots,f_0(\zeta_{n+1}))$ is the only point common to K and the supporting plane since each point of frK is an extreme point of K. Hence the real continuous linear function $f\mapsto\operatorname{Re}\sum_1^{n+1} c_k f(\zeta_k)$ restricted to I attains its maximum exactly at f_0.

By the Krein-Milman theorem every real linear continuous functional on the space of analytic functions on Δ is maximized on I by some extreme member of I.

The result (ii) shows that every extreme member of I appears as such a maximizer. Hence the set of the extreme points of I is the minimal set of such maximizers.

One may also consider corresponding convex bodies of the form $\{(f(\zeta_1),\dots,f(\zeta_m)): f\in I\}$ where $m\geqq n+1$ and $\zeta_1,\dots,\zeta_m$ are distinct points of $\Delta-\{0, z_1,\dots,z_n\}$. We remark that the extreme points are exactly the images of the extreme functions of I with respect to $f\mapsto(f(\zeta_1),\dots,f(\zeta_m))$.

4. A quantitative specification of the extreme members of I via Pick theory. We recall some basic facts of the Pick theory [15–17]. Cf. [6, pp. 6–10]. Here we are concerned with n and $(z_k)_1^n$ as in Lemma 2 and $(w_k)_1^n$, $w_k\in \boldsymbol{C}$, $k=1,\dots,n$. We seek a necessary and sufficient condition for the existence of a function f analytic on Δ having nonnegative real part and satisfying $f(z_k)=w_k$, $k=1,\dots,n$. Let $H(s)$ denote the Hermitian form

$$\sum \frac{w_j+\bar{w}_k}{1-z_j\bar{z}_k} s_j\bar{s}_k. \tag{4.1}$$

The theorem of Pick may be stated as follows: *A necessary and sufficient condition for the existence of an allowed f is that $H(s)\geqq 0$, all s. In this case there is exactly one solution if and only if there exists $s^0\neq 0$ such that $H(s^0)=0$.*

We sketch a proof of Pick's theorem and show that a very simple modification of a remark of Pick [15, pp. 12–18] permits the exhibiting of the solution in the case of uniquenesss. Our object is to describe the extreme functions of I explicitly in terms of the associated points of frK as well as to characterize the latter points in a simple quantitative way in terms of the data.

The starting point of Pick's necessity considerations is the study of the Hermitian form

$$\frac{1}{2\pi}\int_0^{2\pi}[f(re^{i\theta})+\overline{f(re^{i\theta})}]\left|\sum_1^n \frac{s_k}{re^{i\theta}-z_k}\right|^2 d\theta, \tag{4.2}$$

where f is a function satisfying the imposed requirements and $\max|z_k|<r<1$. On evaluating (4.2) with the aid of the Cauchy integral formula for a disk and letting $r\mapsto 1$ we obtain $H(s)$. Hence if an allowed f exists, $H(s)\geqq 0$. On introducing the nonnegative linear functional

$$l_r\colon X\longmapsto \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} f(re^{i\theta})X(\theta)d\theta \tag{4.3}$$

on the space of real-valued periodic continuous functions X on $\boldsymbol{R}$ with period 2π, we see that

$$\lim_{r\to 1} l_r(X)=\int_0^{2\pi} X d\gamma, \tag{4.4}$$

where γ is the normalized generating function in the Poisson-Stieltjes representation of $\operatorname{Re} f$ and conclude, specializing $X(\theta)$ to $|\sum_1^n s_k(e^{i\theta}-z)^{-1}|$, that

$$H(s)=2\int_0^{2\pi}\left|\sum_1^n \frac{s_k}{e^{i\theta}-z_k}\right|^2 d\gamma. \tag{4.5}$$

Cf. [1]. Suppose that there is exactly one allowed f. Putting aside the trivial case where $\operatorname{Re} f=0$, we see by Lemma 2 that there exists a polynomial Q of positive degree $<n$ having zeros at the poles of f. From the partial fraction decomposition

$$\frac{Q(z)}{\prod_1^n (z-z_k)} = \sum_1^n \frac{s_k^0}{z-z_k} \tag{4.6}$$

we obtain a vector $s^0 \neq 0$ such that $H(s^0)=0$.

Sufficiency. We put aside the trivial cases where $n=1$ or $n>1$ and w_k is independent of k and approach the question with the aid of minimum considerations using Lemmas 2 and 3. Let $\alpha(\in \boldsymbol{R})$ be such that there exists an analytic function f on Δ with positive real part satisfying $f(z_k)=w_k+\alpha$, $k=1,\ldots,n$. Such α exist. They satisfy $\alpha>-\min \operatorname{Re} w_k$. Using Lemma 2 we see that for such α there exists an analytic function f_α of the form

$$z \longmapsto \sum_1^n \gamma_k \frac{\eta_k+z}{\eta_k-z} + i\delta, \tag{4.7}$$

$|\eta_k|=1$, $\gamma_k \geqq 0$, $\delta \in \boldsymbol{R}$, satisfying $f_\alpha(z_k)=\alpha+w_k$. We have

$$\left(\sum_1^n \gamma_k\right)\left(\frac{1-|z_1|}{1+|z_1|}\right) \leqq \alpha + \operatorname{Re} w_1, \tag{4.8}$$

and

$$|\delta| \leqq |\operatorname{Im} w_1| + (\alpha+\operatorname{Re} w_1)\left[\left(\frac{1+|z_1|}{1-|z_1|}\right)^2\right]. \tag{4.9}$$

There exists a sequence of allowed α tending to β, the infimum of the allowed α, such that the associated η_k, γ_k and δ converge. Thus there exists a function f_β of the form (4.7) satisfying $f_\beta(z_k)=\beta+w_k$, $k=1,\ldots,n$. By Lemma 2 if the degree of f_β were n, the function A of Lemma 3 associated with the interpolation requirement $z_k \mapsto \mu(\beta+w_k)$, $k=1,\ldots,n$, would have the property that $\inf_\Delta \operatorname{Re} \operatorname{inv} \mu \circ A>0$, so that β would not be the infimum of allowed α. Hence f_β is of degree $\leqq n-1$ and it is the unique analytic function F on Δ with positive real part satisfying $F(z_k)=\beta+w_k$, $k=1,\ldots,n$.

The remainder of the Pick theorem is now readily established. The Hermitian form corresponding to the interpolation requirement $z_k \mapsto \beta+w_k$ is

$$H(s)+\frac{\beta}{\pi}\int_0^{2\pi}\left|\sum_1^n \frac{s_k}{e^{i\theta}-z_k}\right|^2 d\theta. \tag{4.10}$$

Since a necessary and sufficient condition for there to exist a function satisfying the requirements of the Pick interpolation problem is that $\beta \leqq 0$ we conclude that also the Pick condition is necessary and sufficient. If there exists $s^0 \neq 0$ such that $H(s^0)=0$, then

$$\frac{\beta}{\pi}\int_0^{2\pi}\left|\sum_1^n \frac{s_k^0}{e^{i\theta}-z_k}\right|^2 d\theta \geqq 0. \tag{4.11}$$

We conclude that $\beta=0$. There is a unique interpolating function.

A remark of Pick. Pick essentially noted (loc. cit.) that if there existed $s^0 \neq 0$ at which the nonnegative H vanished, one could make explicit the only possible interpolating function. At that point of the exposition sufficiency is not established. Pick uses arguments involving the rank of a matrix associated with the interpolation data.

The object of the following observation is to note that one may operate directly with H for an augmented problem to obtain the only possible interpolating function. We merely consider the augmented problem with the interpolation requirement: $z_k \mapsto w_k$, $k=1,\ldots, n+1$, $z_{n+1} \neq z_k$, $k<n+1$, and note that the value of the corresponding H at $(s_1^0,\ldots, s_n^0, \eta\sigma)$ is nonnegative. Here $|\eta|=1$, $\sigma>0$. We obtain the inequality

$$\text{(4.12)} \qquad \frac{2\operatorname{Re} w_{n+1}}{1-|z_{n+1}|^2}\sigma^2 + 2\operatorname{Re}\left[\sum_1^n \frac{w_{n+1}+\bar{w}_k}{1-z_{n+1}\bar{z}_k}\eta\sigma\bar{s}_k^0\right] \geq 0.$$

Dividing by σ and thereupon taking the limit as $\sigma \to 0$, we conclude that

$$\text{(4.13)} \qquad \sum_1^n \frac{w_{n+1}+\bar{w}_k}{1-z_{n+1}\bar{z}_k}\bar{s}_k^0 = 0.$$

It follows that the only possibility for the interpolating function is

$$\text{(4.14)} \qquad z \longmapsto \left(\sum_1^n \frac{\bar{w}_k \bar{s}_k^0}{1-z\bar{z}_k}\right) \Big/ \left(\sum_1^n \frac{\bar{s}_k^0}{1-z\bar{z}_k}\right).$$

Extreme members of I given in terms of the Pick theory. For convenience of notation we write z_{n+k} for ζ_k, $k=1,\ldots, n+1$, and denote a point of $\boldsymbol{C}^{n+1}$ by $(w_{n+1},\ldots, w_{2n+1})$. Further, in accord with the normalization for I we set $z_0=0$, $w_0=1$. We see that the points of frK are the points $(w_{n+1},\ldots, w_{2n+1})$ for which

$$\text{(4.15)} \qquad \sum_{j,k=0}^{2n+1} \frac{w_j+\bar{w}_k}{1-z_j\bar{z}_k} s_j \bar{s}_k$$

is positive semidefinite or, equivalently, such that the least root of

$$\text{(4.16)} \qquad \det\left(\frac{w_j+\bar{w}_k}{1-z_j\bar{z}_k} - \lambda\delta_{jk}\right) = 0$$

is 0. The so obtained $(w_k)_0^{2n+1}$ and an associated eigenvector $(s_k^0)_0^{2n+1}$ yield the extreme function associated with $(w_{n+1},\ldots, w_{2n+1})$ with the aid of (4.14).

Remark. Using both the qualitative results of §3 and the modified Pick approach, which led to (4.14), we may obtain in the case where H is *strictly* positive for $s \neq 0$ the corresponding coefficients A, B, C for a suitably normalized representation of Lemma 3 without recourse to the customary recursive algorithms. It suffices to introduce $z_{n+1} \in \Delta - \{z_1,\ldots, z_n\}$ and to note that the vanishing of the determinant of the augmented matrix determines exactly the values at z_{n+1} for which the augmented problem is unique. On calculating the interpolating functions corresponding to three such values with the aid of (4.14) and reverting to the situation of §3 by use of μ we see that the qualitative facts of §3 permit the calculation of A, B, C on normalizing suitably the correspondence at z_{n+1} (Denjoy normalization, cf. [4]).

UNIVERSITY OF MARYLAND
DEPARTMENT OF MATHEMATICS
COLLEGE PARK, MARYLAND 20742
U. S. A.

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