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Kodaira dimension of embeddings of the line in the plane

By

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Introduction

Let i: $C=A^1 \hookrightarrow A^2=X$ be a closed embedding of the line in the affine plane. Already in 1956, B. Segre realized that the embedding may be far from trivial, i.e., may not be just a "linear" embedding followed by an automorphism of the plane. Considerable effort has gone into determining the nature and extent of this departure from the trivial: A conjecture has formed (phrased variously, and originating with various people independently), to the effect that all such embeddings *are* obtained, if one allows in addition certain "relative Frobeniuses" of the plane (regarding X as a variety fibred over A^1 by the parallel translates of the curve C).

In this note will be found the solution of a considerably simpler problem, posed by M. P. Murthy in 1980, namely, that of classifying all embeddings of the line in the plane by Kodaira dimension (Theorem 2.4). The classification is achieved by purely numerical means, reminiscent of those in Nagata's article [8], but using more heavily the machinery of characteristic pairs.

It is a pleasure for me to thank Peter Russell for many stimulating discussions and constant encouragement.

Let k denote an algebraically closed field of characteristic $p \geqslant 0$, and R be a polynomial ring in two variables over k. "Surface" means reduced, irreducible, nonsingular surface. If X is a complete surface, K_X denotes a canonical divisor on X.

1. Lines in the plane

Let $f \in R$. Then $R \to R/fR$ defines an embedding i: (zero locus of f) $= C \hookrightarrow A^2$. One calls C a *line* if R/fR is a polynomial ring over k in one variable. In the following I will often make statements about C, or about f, when the reference is really to the embedding i.

Call C a *coordinate line* if there is a $g \in R$ such that $R=k[f, g]$. It is well known (theorem of Abhyankar-Moh — see [2], [3], [6], [9],...) that a line in the plane is

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a coordinate line under certain tameness hypotheses (e.g., $d_h \not\equiv 0 \pmod p$), where d_h is the last characteristic multiplicity of C at infinity — see below). Hence it is natural to call noncoordinate lines *wild*. B. Segre [10, p. 18] gave examples of wild embeddings of the line in the plane for $p>0$. Later a larger class of embeddings was given by Abhyankar and Moh ([2, p. 148], in parametric form). It is common knowledge among the specialists that there abound examples of wild lines which do not fit the Abhyankar-Moh mold.

Let $f \in R$ define a line. It is well known, and easy to see, that

there exist variables x, y for R such that
$$d = \deg f = \deg_y f > \deg_x f = e.$$

moreover

if f defines a *wild* line, then
$e>1$, p divides both d and e, and
one may choose x, y so that the bidegree $\{d, e\}$ is "nonprincipal",
i.e., neither of d, e is a multiple of the other.

In fact, given f and arbitrary variables for R, by successive automorphisms of R which fix one variable, one arrives at suitable x, y in a finite number of steps. Moreover, given a plane curve f with only one place at infinity (e.g., a line),

a nonprincipal bidegree $\{d, e\}$ is unique, if it exists.

(I thank Avinash Sathaye for calling my attention to this fact. The elementary proof is omitted, since this point is not essential to the subsequent discussion.)

Embed Spec R in the projective plane $\boldsymbol{P}^2$ in standard fashion: Let $X_0 : X_1 : X_2$ be homogeneous coordinates, with $x = X_1/X_0$, $y = X_2/X_0$. Denote again by C the closure of $\{f=0\}$ in $\boldsymbol{P}^2$. Then C has a unibranch point $P_1 = (0: 1: 0)$ at infinity. Note that

(multiplicity of C at P_1) $= d - e$ and
the intersection multiplicity $(C, L_\infty)_{P_1}$
of C at P_1 with the line at infinity is d.

In this situation one has available the machinery of characteristic pairs for an irreducible plane algebroid curve (see [9], [7], [1],...). Out of familiarity with them, I choose the approach and notation of [9]. Choosing the parameters $\xi = 1/x$, $\eta = y/x$ at P_1 and the local equation $f_\infty = f/x^d$ for C, we consider the *Hamburger-Noether tableau*

$$HN(f_\infty, \xi, \eta) = \begin{bmatrix} p_i \\ c_i \\ \theta_i \end{bmatrix}_{i>0}.$$

Here $c_1 = d$, $p_1 = d - e$, in general $c_{i+1} = \gcd(p_i, c_i)$ for $i>0$, and $c_i = 1$ for i large. Note that if f is a *line*, the formula for the length of the conductor at P_1 is

1.1 $$(p_1-1)(c_1-1)+\sum_{i>1} p_i(c_i-1)=(c_1-1)(c_1-2).$$

(See e.g. [9], 6.1.)

If $d>1$, then the *characteristic numbers* d_i of f relative to ξ, η at P_1 are those c_i which are greater than one, say $c_1=d_1>d_2>\cdots>d_h$ $(h>0)$. Let $i_1=1$, and for $j>1$ let i_j be the greatest s such that $c_s=d_j$. Let $q_1=p_1$, and for $j>1$ let $q_j=\sum_{i_{j-1}<i\leqslant i_j} p_i$. The data $(q_j, d_j)_{j>0}$ are the *characteristic pairs* of the singularity. Equation 1.1 may be rewritten as

1.2 $$\sum_{j=2}^{h} q_j(d_j-1)=(d_1-1)(d_1-q_1-1).$$

1.3 Let a_0, a_1 be positive integers, and let

$$\begin{aligned} a_0&=Q_1a_1+a_2\\ a_1&=Q_2a_2+a_3\\ &\ \vdots\\ a_{\alpha-1}&=Q_\alpha a_\alpha \end{aligned}$$

be the Euclidean algorithm. ($Q_1=0$ if $a_0<a_1$.) One has

1.4 $$\sum_{i=1}^{\alpha} Q_i a_i^2=a_0a_1 \quad\text{and}\quad \sum_{i=1}^{\alpha} Q_i a_i=a_0+a_1-a_\alpha.$$

(See [8], 1.1.)

1.5 Lemma. 1) *With notation as in* 1.3,

$$\sum_{i=1}^{\alpha} Q_i \begin{cases} \leqslant a_0/a_\alpha & \text{if } a_0\geqslant a_1,\\ < a_0/a_1+a_1/a_\alpha & \text{in any case.}\end{cases}$$

2) *Let $h>1$, and let $d_2,\ldots,d_h$ be integers such that $d_j>d_{j+1}$ and d_{j+1} divides d_j for $j>1$. ($d_{h+1}:=1$.) Then $\sum_{j=2}^{h} d_j/d_{j+1}\leqslant d_2$.*

Proof. First note that $Q_i\geqslant 1$ for $1<i\leqslant\alpha$, and $Q_1\geqslant 1\Leftrightarrow a_0\geqslant a_1$.
Now for 1), one may assume $\alpha\geqslant 2$. Write a_0/a_α as

$$\frac{1}{a_\alpha}(((Q_1-1)a_1+a_2)+((Q_2-1)a_2+a_3)+\cdots+((Q_{\alpha-1}-1)a_{\alpha-1}+a_\alpha)+Q_\alpha a_\alpha).$$

From this it is clear that $Q_1\geqslant 1$ implies $a_0/a_\alpha\geqslant Q_1+Q_2+\cdots+Q_\alpha$. Applying this to the Euclidean algorithm on a_1 and a_2, one has $a_1/a_\alpha\geqslant Q_2+\cdots+Q_\alpha$, whence follows the remaining statement in 1).

2) is clear for $h=2$. For $h=3$, $d_2/d_3+d_3\leqslant d_2/2+2\leqslant d_2$, the second inequality holding since $d_2\geqslant 4$, the first by calculus on the interval $2\leqslant d_3\leqslant d_2/2$. The case $h>3$ reduces to the above case by induction.

2. Kodaira dimension

Recall the following variant of Kodaira dimension, whose roots are classical.

Let L be a function field in two variables over k, and let V be a prime divisor of L/k. Clearly one can find a complete nonsingular model X of L, on which the centre of V is a nonsingular curve D. Define $\kappa(V)$ to be the logarithmic Kodaira dimension $\bar{\kappa}(X\backslash D)$ (see [4]). Using the structure of birational maps of complete surfaces, and the invariance of $l(n(D+K_X))$ under blowing up a point of multiplicity <2 on D, one sees that $\kappa(V)$ is well defined. (That one is really measuring a prime divisor here was pointed out to me by Peter Russell, as was the next fact, which is mentioned here in order to elucidate the significance of nonnegative Kodaira dimension.)

2.1 Proposition. *Let V be a prime divisor of a rational function field L in two variables. Then $\kappa(V)\geqslant 0$ if and only if V has a one-dimensional centre on any complete nonsingular model of L.*

Proof. If the centre of V on X is a point P, then $\bar{\kappa}(V)=\kappa(X\backslash P)=-\infty$. For the converse, suppose the centre of V on X is a nonsingular curve D. Since $|K+D|=\emptyset$, D is rational. If the self-intersection $n=(D^2)\geqslant -1$ one blows up on D $n+1$ times and is done. Otherwise, use [5], Theorem 2.1 ($d)\Rightarrow a)$) to replace X by the minimal ruled surface $\boldsymbol{F}_{-n}$ and assume D is the directix. Then $-n-1$ suitably chosen elementary transformations yield a birational map $\boldsymbol{F}_{-n}\rightarrow\boldsymbol{F}_1$, under which the proper transform of D is exceptional.

As mentioned in the introduction, M. P. Murthy suggested classifying lines in the plane using Kodaira dimension. This will be done after a few preliminary observations.

Let $f\in R$ define a line in the plane. Define $\kappa(f)$ to be the Kodaira dimension of the prime divisor $V=R_{fR}$ of the field of fractions of R. This will be computed by choosing suitable variables for R, embedding Spec R in $\boldsymbol{P}^2$ (all as in section 1), closing up $\{f=0\}$ to obtain the curve $C\subset\boldsymbol{P}^2$, resolving minimally the singularity of C at infinity to get a birational morphism $X\rightarrow\boldsymbol{P}^2$, under which the proper transform of C is a normal rational curve D, and computing the logarithmic Kodaira dimension of the surface $Y=X\backslash D$.

In all but the most trivial cases ($\deg f<3$), C will be singular at infinity. For use in the proof of 2.4, denote by $P_1,\ldots, P_r$ the successive infinitely near multiple points of C at infinity, and by $m_1,\ldots, m_r$ their multiplicities.

2.2 If f defines a wild line then $(D^2)<0$.

In fact, the *equivalence* of wildness and $(D^2)<0$ is well known. (See e.g. [8], [6], or [3].)

2.3 Theorem. (*Kumar-Murthy*) *Let $\boldsymbol{P}^1\simeq D\hookrightarrow X$ be a closed embedding, X a complete rational surface, $Y=X\backslash D$.*

1) *$\bar{\kappa}(Y)$ is negative $\Leftrightarrow$ $|2K_X+D|$ is empty.*

Suppose $(D^2)<0$. Then

2) *$\bar{\kappa}(Y)$ is* 0 *or* $1 \Leftrightarrow l\,(2K_X+D)=1$.
3) *Suppose* $(D^2)=-4$, $(K_X^2)=-1$, *and* $|-K_X|$ *is empty. Then* $l(2K_X+D)=1$, *and* $\bar{\kappa}(Y)=0 \Leftrightarrow |3K_X+D|$ *is empty.*

See [5], Corollaries 2.4 and 3.2, Theorems 3.1 and 3.3. (Note that $(D^2)<0$ is not needed in Corollary 2.4, since the general case follows from the case of negative self-intersection.)

2.4 Theorem. *Let $f \in R$ define an embedding of the line in the plane, with Kodaira dimension $\kappa(f)$. Then*

1) $\kappa(f)<0 \Leftrightarrow f$ *defines a coordinate line.*
2) $\kappa(f)=0 \Leftrightarrow p=2$ *and there exist generators x, y for R such that* $\deg_y f=3p$ *and* $\deg_x f=2p$.
3) $\kappa(f)=1 \Leftrightarrow p=3$ *and there exist generators x, y for R such that* $\deg_y f=3p$ *and* $\deg_x f=2p$.
4) $\kappa(f)=2$ *for all other embeddings f.*

The group $\mathrm{Aut}_k R$ *acts on the set L_i of embeddings $A^1 \hookrightarrow A^2$ of Kodaira dimension i $(i=0, 1)$. The quotient is k^*/U, where U is the group of cube [resp. square] roots of unity if $p=2$ [resp. $p=3$]. More specifically, f defines such an embedding if and only if f may be written as the Segre line* $(y^3-x^2)^p-\begin{cases} cy & \text{if } p=2 \\ cx & \text{if } p=3, \end{cases}$ *for suitable x, y. The scalar $c \in k^*$ is unique up to a cube [square] root of unity.*

Remark. There is an abundance of embeddings of general type as in 4) above, in all positive characteristics. (E. g., the Segre lines $f=(y^q-x^p)^p+y$, with q large and not a multiple of p.)

Proof of 2.4. The bidegree $(\deg_y f, \deg_x f)$ will be determined below. Once this has been done, it is just a matter of brute force (using e.g. the techniques of [3], 2.2–2.9) to check that every term $x^i y^j$ occurring in f with nonzero coefficient f_{ij} satisfies $i+j=1$ or i, j divisible by p. After some further reduction, f is easily brought to the given canonical form. (We indicate here how to start. The fact that f is unibranch at infinity forces $f_{ij}=0$ for $3i+2j>6p$, as one sees by considering the local equation $f_\infty=f/x^{3p}$ at infinity and blowing up twice. Moreover, unibranchness at P_3 requires that $\sum_{3i+2j=6p} f_{ij}x^i y^j=(ay^3-bx^2)^p$ for suitable a, $b \in k^*$. The requirement that our line be unibranch with multiplicity p at P_4 already forces $f_{ij}=0$ for $3i+2j=6p-1$, etc. It should be remarked here that T–t. Moh has more streamlined techniques for determining the coefficients of f; here these would involve "the (ay^3-bx^2)-adic expansion of f".)

Next, recall that the sufficiency in 1) is well known: Embed Spec R in $\boldsymbol{P}^2$ as in section 1 and for $\lambda \in k$, let $\Lambda_\lambda \subset \boldsymbol{P}^2$ be the closure of $\{f=\lambda\}$. After blowing up the base locus of Λ one gets a surface Z with a $\boldsymbol{P}^1$-fibration $Z \to \boldsymbol{P}^1$ ([3], [6],...). Letting ' denote proper transform on Z, one has that $Z \backslash \Lambda_0'$ contains a "cylinderlike open set", so $\kappa(f)<0$.

For the remainder of the proof, assume that f defines a wild line, and that Spec R

is embedded in $\boldsymbol{P}^2$ in standard fashion relative to generators x, y for R chosen to give f nonprincipal bidegree. Then

$$(\dagger) \qquad d_1 - q_1 \geqslant 2d_2.$$

In earlier notation, the linear system $|2K_X + D|$ on X is in natural bijection with the linear system $|(d-6)H - \sum (m_i - 2)P_i|$ of effective divisors on $\boldsymbol{P}^2$ of degree $d-6$ which pass through each P_i with multiplicity at least $m_i - 2$. (Classically this is "the system of special adjoints of C of index two".) Clearly

$$l(2K_X + D) \geqslant \frac{1}{2}[(d-5)(d-4) - \sum (m_i - 1)(m_i - 2)].$$

Let $\Sigma := \sum_{i=1}^{r} (m_i - 1)$. From the conductor formula $\sum m_i(m_i - 1) = (d-1)(d-2)$, one concludes that

$$(*) \qquad l(2K_X + D) \geqslant \Sigma - (3d_1 - 9).$$

For a wild line f, one knows that the number h of characteristic pairs is at least 2. (Both the x- and y-degree are divisible by p.)

(A) Suppose $h = 2$.

(Note in passing that in this case, it is unnecessary to *assume* that d is not a multiple of e. This follows easily from the fact that $\gcd(d, e)$ is the residue field degree over $k(t)$ of the place at infinity of the generic member Λ_t of the pencil Λ, hence is independent of the choice of generators for R.) By 1.2, one has

$$(\text{A1}) \qquad q_2(d_2 - 1) = (d_1 - 1)(d_1 - q_1 - 1),$$

whence $q_2 \equiv -1 \pmod{d_2}$. The multiplicity sequence $\langle m_i \rangle$ begins with

Q_1 points of multiplicity $q_1, \ldots, Q_\alpha$ points of multiplicity d_2 (see 1.3 and [8], 1.2), then continues with

K (say) points of multiplicity d_2.

There follow one point of multiplicity $d_2 - 1$, and simple points.

By the conductor formula and 1.4 (with a_0, a_1 replaced by d_1, q_1), one has

$$(d_1 - 1)(d_1 - 2) = (d_1 q_1 - d_1 - q_1 + d_2) + K d_2(d_2 - 1) + (d_2 - 1)(d_2 - 2),$$

hence $\qquad K(d_2 - 1) = \frac{1}{d_2}(d_1^2 - q_1 d_1 - 2d_1 + q_1 + 2d_2 - d_2^2).$

Let $Q = Q_1 + \cdots + Q_\alpha$, in the notation of 1.3. Then Σ is given by the following function δ of d_1, q_1:

$$(\text{A2}) \qquad \Sigma = (q_1 + d_1 - d_2 - Q) + K(d_2 - 1) + (d_2 - 2)$$

$$= q_1 + d_1 - d_2 - Q + \frac{1}{d_2}(d_1^2 - q_1 d_1 - 2d_1 + q_1) =: \delta. \quad \text{Hence}$$

$$(\text{A3}) \qquad \Sigma = 3d_1 + q_1 - d_2 - Q + q_1/d_2 + d_1 \Delta / d_2,$$

where by $(\dagger)$, $\Delta = d_1 - q_1 - 2d_2 - 2 \geqslant -2$.

Case 1. $\Delta>0$. Then $d_1\Delta/d_2 \geqslant d_1/d_2 \geqslant Q$ by 1.5 1), so $\Sigma>3d_1$, hence $\kappa(f)=2$ by (*) and 2.3.

Case 2. $\Delta \leqslant 0$. $\Delta+2$ is a nonnegative multiple of $d_2>1$, hence $=0$ or 2.

(a) $\Delta=0$. Then d_2 divides 2, so $d_2=2$, so $d_1-q_1=3d_2$, and by (A3) and 1.5 1) one has
$\Sigma \geqslant 3d_1+(q_1-d_2)-(d_1-q_1)/d_2 \geqslant 3d_1-3$, so $\kappa(f)=2$ again.

(b) $\Delta+2=0$. Then $q_1=rd_2$, $d_1=(r+2)d_2$, r odd. It is easily checked that $Q=(r+5)/2$, hence by (A3) and a quick computation,
$\Sigma=3d_1-8+(r-1)(d_2-3/2)$.

(b1) Suppose $r \geqslant 3$. Then $\Sigma>3d_1-8$ and $\kappa(f)$ is again 2.

(b2) Suppose $r=1$. Then $\Sigma=3d_1-8$. From (b) and (A1) one sees at once that d_2-1 divides 2. So d_2 is 2 or 3. But f is by assumption wild. So by [3, 1.3 1)], e.g., and the observation in (A), the conclusion is that char $k=p$ is 2 or 3. In either case, one verifies easily that $m_i=p$ for $i<10$, $m_{10}=2$, $(K_X^2)=-1$ and $(D^2)=-4$.

Looking further at case (b2), one sees easily that in the case of the sextic with ten double points (i.e., $p=2$), $2K_X+D\sim 0$, so $|-K_X|=|K_X+D|$ is empty since D is an irreducible nonsingular rational curve on a rational surface. Moreover, $|3K_X+D| =|K_X|$ is empty. By 2.3, $\kappa(f)=0$.

If $p=3$ in (b2), one checks that $3K_X+D\sim E_{10}=$last exceptional fibre in the minimal desingularization of C. If X has an effective anticanonical divisor then from $D\cdot K_X=2$ one has that D is a fixed component of $|-K_X|$, hence $-(2K_X+D)$ is effective. But by direct calculation $|2K_X+D|$ contains a *positive* linear combination of L_∞, $E_1,\dots, E_8$. So $|-K_X|$ is empty. By 2.3, $\kappa(f)=1$.

(B) Suppose $h>2$.

For $1\leqslant j\leqslant h$, let $Q(j)$ be the sum of the coefficients in the Euclidean algorithm on q_j, d_j. Repeated application of 1.4 gives

$$\begin{aligned}\Sigma&=(d_1+q_1-d_2-Q(1))+\cdots+(d_h+q_h-1-Q(h))\\&=d_1+\sum_{j=1}^{h} q_j-\sum_{j=1}^{h} Q(j)-1.\end{aligned}$$

By 1.5 1), for $2\leqslant j<h$ one has $Q(j)<d_j/d_{j+1}+q_j/d_j$. Also, by 1.2 one has $q_h\equiv -1 \pmod{d_h}$, whence it follows that $Q(h)=d_h+\frac{1}{d_h}(q_h-d_h+1)$. From this expression and 1.5 2) it follows that (with $d_{h+1}:=1$)

$$\begin{aligned}\Sigma&>q_1+d_1-Q(1)+\sum_{j=2}^{h} q_j(1-1/d_j)-\sum_{j=2}^{h} d_j/d_{j+1}-1/d_h\\&\geqslant q_1+d_1-d_2-Q(1)+\sum_{j=2}^{h} q_j(1-1/d_j)-1/d_h=:\sigma.\end{aligned}$$

Now regard q_1, $d_1>\cdots>d_h$ as constants and $q_2,\dots, q_h$ as nonnegative variables satisfying 1.2. Since $\dfrac{1-1/d_i}{1-1/d_{i+1}}<\dfrac{d_i-1}{d_{i+1}-1}$, a trivial application of linear programming shows that for $q_3+\cdots+q_h>0$, $\sigma=\sigma(q_2,\dots, q_h)>\sigma(q_2, 0,\dots, 0)=$

$q_1+d_1-d_2-Q(1)+q_2(d_2-1)/d_2-1/d_h=:\varepsilon$, where, using 1.2 and comparing with δ in (A2), one has $\varepsilon=\delta+1/d_2-1/d_h>\delta\geqslant 3d_1-8$. So $\Sigma>3d_1-8$, and by (*) and 2.3, $\kappa(f)=2$. This completes the proof of 2.4.

Remarks. 1) Let $\phi: Z\to \boldsymbol{P}^1$ be the morphism, mentioned in the proof of 2.4, which results upon minimally ridding Λ of its base points. Then, as Bogomolov pointed out to me, the cases 2), 3) of the theorem are precisely those in which ϕ is a quasielliptic fibration. The exceptional curve E_9 is in each case a pseudo-section of ϕ, i.e., it meets the general fibre physically once with intersection number p, "normally" if $p=2$, trangentially if $p=3$.

2) A well-known theorem in Coolidge's book on algebraic plane curves says that an irreducible rational projective plane curve C has $\kappa(C)$ negative in our sense $\Leftrightarrow$ there is a Cremona transformation, the proper transform of C under which is a straight line. In view of this, part of 2.4 says that if C is the closure in $\boldsymbol{P}^2$ of a line in $\boldsymbol{A}^2$ and a Cremona "straightening" exists, then the straightening can in fact be achieved by an automorphism of $\boldsymbol{A}^2$.

3) One might also classify lines $C\subset \boldsymbol{A}^2$ defined by f, by $\dot{\kappa}(f):=\kappa((\boldsymbol{A}^2\backslash C)$. Obviously $\dot{\kappa}(f)\geqslant\kappa(f)$ in our sense. It can also be seen directly (or via 2.4) that $\dot{\kappa}(f)$ is negative $\Leftrightarrow f$ is coordinate. Making a lengthy calculation in cases 2), 3), one finds that $\dot{\kappa}=2$ for all wild lines.

Note also that, for nonconstant $f\in R$ and any embedding of $\boldsymbol{A}^2$ in $\boldsymbol{P}^2$, with $C=$closure of $\{f=0\}$, one has $\kappa(f)\leqslant\bar{\kappa}(\boldsymbol{P}^2\backslash C)\leqslant\dot{\kappa}(f)$. Hence f coordinate line $\Rightarrow\bar{\kappa}(\boldsymbol{P}^2\backslash C)$ negative, and the converse follows from 2.4. Can it be seen more directly?

Let C be an irreducible rational projective plane curve. A lot is known about conditions on Sing C which ensure that $\bar{\kappa}(\boldsymbol{P}^2\backslash C)$ is nonnegative. (See papers in the Proceedings of the Japan Academy by Wakabayashi (1978), Yoshihara (1979), and Tsunoda (1981).) Call a unibranch curve sigularity a *cusp* of the curve. Theorem 2.4 shows that wild lines provide examples of such curves C, having Sing $C=\{\text{cusp}\}$. (Tsunoda found examples of such curves — with singular locus consisting of a solitary cusp — over the complex ground field.) As far as I know, the following is an open

Question. Let k have characteristic zero. Let $C\subset\boldsymbol{P}^2$ be an irreducible rational curve with only cusps as singularities, and let $\kappa(C)$ be the Kodaira dimension in the sense of the present note. Is $\kappa(C)$ negative?

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References

[1] S. Abhyankar, Lectures on Expansion Techniques in Algebraic Geometry (Notes by B. Singh), Tata Lecture Notes 57, Tata Institute of Fundamental Research, Bombay (1977).

[2] S. Abhyankar and T-t. Moh, Embeddings of the line in the plane, J. reine angew. Math., **276** (1975), 148–166.

[3] R. Ganong, On plane curves with one place at infinity, J. reine angew. Math., **307/308**

(1979), 173–193.

[4] S. Iitaka, On logarithmic Kodaira dimension of algebraic varieties, Complex Analysis and Algebraic Geometry, Iwanami (1977), 175–189.

[5] N. Mohan Kumar and M. Pavaman Murthy, Curves with negative self intersection on rational surfaces, J. Math. Kyoto Univ., **22**–4 (1983), 767–777.

[6] M. Miyanishi, Lectures on curves on rational and unirational surfaces, Tata Lecture Notes 60, Tata Institute of Fundamental Research, Bombay (1978).

[7] T-t. Moh, On characteristic pairs of algebroid plane curves for characteristic p, Academia Sinica, **1** (1973), 75–91.

[8] M. Nagata, A theorem of Gutwirth, J. Math. Kyoto Univ., **11**–**1** (1971), 149–154.

[9] P. Russell, Hamburger-Noether expansions and approximate roots of polynomials, Manuscripta math., **31** (1980), 25–95.

[10] B. Segre, Corrispondenze di Möbius e trasformazioni cremoniane intere, Atti Accademia delle Scienze di Torino, **91** (1956–57), 1–19.