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On some limit theorems for occupation times of one dimensional Brownian motion and its continuous additive functionals locally of zero energy

By

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Introduction.

Let B_t be a one dimensional Brownian motion and f be a continuous function with compact support. Then the following limit theorems are well known (see e.g. (4), (8), (12) or (14)).

Theorem I (First order limit theorem). *If* $\int_{-\infty}^{\infty} f(x)dx \neq 0$ *the family of stochastic processes* $t \mapsto \frac{1}{\sqrt{\lambda}}\int_0^{\lambda t} f(B_s)ds$, $\lambda > 0$ *converges in the sense of law on the space of continuous functions to the process* $t \mapsto \left(\int_{-\infty}^{\infty} f(x)dx\right)L_t^0$, *as* $\lambda \to \infty$ *where* L_t^0 *is the local time of* B_t *at* 0 *defined by Tanaka formula* $\frac{1}{2}L_t^0 = (B_t)^+ - (B_0)^+ - \int_0^t H(B_s)dB_s$, *where*

$$H(x) = \begin{cases} 1, & x \geqq 0 \\ 0, & x < 0 \end{cases}$$

Theorem II (Second order limit theorem). *If* $\int_{-\infty}^{\infty} f(x)dx = 0$ *but* f *is not identically zero, the family of stochastic process* $t \mapsto \frac{1}{\lambda^{1/4}}\int_0^{\lambda t} f(B_s)ds$, $\lambda > 0$, *converges in the sens of law on the space of continuous functions to the process* $t \mapsto \sqrt{\langle f, f \rangle}\tilde{B}(L_t^0)$ *as* $\lambda \to \infty$, *where* $\tilde{B}(t)$ *is another Brownian motion independent of* B_t *with* $\tilde{B}(0) = 0$ *and* L_t^0 *is the local time of* B_t *at* 0, *and* $\langle f, f \rangle = 2\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} f(u)du\right)^2 dx$.

In the last ten years, many studies have been produced by Kasahara, Kotani and other authors to generalize these limit theorems for more general processes than one dimensional Brownian motion. (see e.g. (6), (7), (8), (9) and (10)).

In the present paper, we are interested in the problem of finding some new limit theorems concerning one dimensional Brownian motion in the case

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where f does not belong to $L^1(R^1)$, or in the case where $\int_{-\infty}^{\infty} f(x)dx=0$ but $\langle f, f\rangle$ diverges. In our study some examples in the class of continuous additive functionals locally of zero energy will play important roles.

1. Continuous additive functionals locally of zero energy.

In this section we shall introduce some continuous additive functionals locally of zero energy and will explain quickly their properties. Some of them has been proved elsewhere. For more precise informations of the class of these additive functionals, one can get them consulting (1) and (11) on the general theory or consulting (16), (17) and (18) for concrete examples.

Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ be a complete probability space with right continuous increasing family $(\mathcal{F}_t)_{t>0}$ of σ-field $\mathcal{F}$. Let B_t be a continuous $\mathcal{F}_t$-martingale such that

(i) $E[(B_t-B_s)^2/\mathcal{F}_s]=t-s$ for $t\geqq s\geqq 0$,

(ii) $B_0=0$.

That is to say, B_t is a one dimensional Brownian motion.

Let L_t^a be local times of the Brownian motion B_t, characterized by Tanaka formula

$$\frac{1}{2}L_t^a=(B_t-a)^+-(B_0-a)^+-\int_0^t H(B_s-a)dB_s. \tag{1.1}$$

Hereafter we mean by $(t, a)\mapsto L_t^a$ a jointly continuous version of Brownian local times.

A. First, we shall introduce continuous additive functionals which correspond to Cauchy's principal value.

Consider the function $F_a(x)=(x-a)\log|x-a|-(x-a)$, $a\in R^1$. Then the derivative of the function

$$\frac{dF_a(x)}{dx}=F'_a(x)=\log|x-a|$$

belongs to $L^2_{\text{loc}}(R^1)$.

The second derivative of $F_a(x)$ in the sense of Schwartz's distribution is

$$\frac{d^2F_a(x)}{dx^2}=F''_a(x)=\text{v. p.}\frac{1}{(x-a)}$$

where v. p. stands for Cauchy's principal value.

Definition 1.1 (Continuous additive functionals corresponding to Cauchy's principal value). Put

$$\frac{1}{2}C_t^a=F_a(B_t)-F_a(B_0)-\int_0^t F'_a(B_s)dB_s, \tag{1.2}$$

where the stochastic integral is understood in the sense of Ito integral.

The right hand side of (1.2) is well defined, because $F'_a(x)$ belongs to $L^2_{\mathrm{loc}}(R^1)$. We call C_t^a continuous additive functional of B_t defined by Cauchy's principal value $\mathrm{v.p.}\frac{1}{(x-a)}$ (cf. (5), (16) and (18)).

The following lemma is proved in (16) and (18).

Lemma 1.1. *One can choose a version of* $(t, a)\mapsto C_t^a$ *such that the following* (i), (ii) *and* (iii) *hold.*

(i) *Let* g *belong to* $L^2(R^1)$.

Then

$$\int_0^t g(B_s)ds=\frac{1}{\pi}\int_{R^1}C_t^a(\mathcal{H}^{-1}g)(a)da\,, \tag{1.3}$$

where $\mathcal{H}$ *stands for Hilbert transform*; $\mathcal{H}f=\frac{1}{\pi}\mathrm{v.p.}\frac{1}{x}*f$ *and* $\mathcal{H}^{-1}$ *means its inverse transform* $\mathcal{H}^{-1}=-\mathcal{H}$ (cf. (15)).

(ii)

$$C_t^a=\pi\mathcal{H}^{-1}(L_t^{\cdot})(a)\,. \tag{1.4}$$

(iii) *For any* $T>0$, $M>0$ *and* $\varepsilon\in\left(0, \frac{1}{2}\right)$, *there exist finite-valued variables* $H_{T,M,\varepsilon}(\omega)$, $K_{T,M,\varepsilon}(\omega)$ *such that*

$$\sup_{t\leq T}|C_t^a(\omega)-C_t^b(\omega)|\leqq H_{T,M,\varepsilon}(\omega)|a-b|^{1/2-\varepsilon}, \qquad \forall a, b\in[-M, M], \quad a.s., \tag{1.5}$$

and

$$\sup_{|a|\leq M}|C_t^a(\omega)-C_s^a(\omega)|\leqq K_{T,M,\varepsilon}(\omega)|t-s|^{1/2-\varepsilon}, \qquad \forall t, s\in[0, T], \quad a.s. \tag{1.6}$$

Remark 1.1. It is known that the following formula holds. (cf. (16) and (18)).

$$C_t^a=\lim_{\varepsilon\downarrow 0}\int_0^t\{I_{(-\infty, a-\varepsilon)}(B_s)+I_{(a+\varepsilon,\infty)}(B_s)\}\frac{ds}{B_s-a}, \qquad a.s. \tag{1.7}$$

B. Second, we shall introduce continuous additive functionals which correspond to Hadamard's finite part.

Consider the function

$$G_a(x)=\frac{(x-a)_+^{1-\alpha}}{(-\alpha)(1-\alpha)}=\begin{cases}0 & x<a\\ \dfrac{(x-a)^{1-\alpha}}{(-\alpha)(1-\alpha)} & x\geqq a\end{cases}$$

where $0<\alpha<\frac{1}{2}$. Then the derivative of the function $G_a(x)$

$$\frac{dG_a(x)}{dx}=G'_a(x)=\frac{(x-a)_+^{-\alpha}}{(-\alpha)}$$

belongs to $L^2_{loc}(R^1)$.

The second derivative of the function $G_a(x)$ in the sense of Distribution is

$$\frac{d^2G_a(x)}{dx^2}=G''_a(x)=\text{p. f.}(x-a)_+^{-1-\alpha}$$

where p. f. stands for Hadamard's finite part.

Definition 1.2 (Continuous additive functionals corresponding to p.f.$(x-a)_+^{-1-\alpha}$). Let $0<\alpha<\frac{1}{2}$. Put

$$\frac{1}{2}H^a(-1-\alpha,\, t)=G_a(B_t)-G_a(B_0)-\int_0^t G'_a(B_s)dB_s \tag{1.8}$$

where the stochastic integral is understood in the sense of Ito integral.

Since $G'_a(x)$ belongs to $L^2_{loc}(R^1)$, the right hand side of (1.8) is well defined. We call $H^a(-1-\alpha, t)$ continuous additive functional defined by Hadamard's finite part p. f.$(x-a)_+^{-1-\alpha}$ (cf. (16) and (17)).

To state some properties of the continuous additive functionals $H^a(-1-\alpha, t)$ $a\in R^1$, we need to introduce following definition.

Definition 1.3 (fractional derivative) (cf. (2) and (13)). Let g be a locally integrable function with a left compact support; i. e. there exists a number L such that g vanishes on $(-\infty, L)$.

We put

$$D^\beta g=\frac{1}{\Gamma(-\beta)}\text{p. f.}(x_+^{-1-\beta})*g, \qquad \beta\in R^1$$

where Γ stands for the Gamma function and $*$ means the convolution operator in the sense of Schwartz's distribution. We call $D^\beta g$ the fractional derivative of order β of the function g.

The following lemma is due to Hardy and Littlewood ((3)).

Lemma 1.2. *Suppose that* $0<\alpha<\beta\leqq 1$.

Let g be a continuous function with compact support and satisfy Hölder's condition of order β; i.e. there exists a constant $K_1>0$ such that $|g(x+h)-g(x)|\leqq K_1|h|$, $\forall x, h\in R^1$.

Then,

(i) $(D^\alpha g)(x)\in L^2(R^1)\cap L^1(R^1)$ *holds,*

and

(ii) *there exists a constant $K_2>0$ such that* $|D^\alpha g(x+h)-D^\alpha g(x)|\leqq K_2|h|^{\beta-\alpha}$, *i.e. $D^\alpha g$ satisfies Hölder's condition of order $\beta-\alpha$.*

We now come back to the additive functionals $H^a(-1-\alpha, t)$, $a\in R^1$.

Lemma 1.3. *Let* $0<\alpha<\frac{1}{2}$ *and* $\alpha<\beta\leqq 1$.

One can choose a version of $(t, a)\mapsto H^{a}(-1-\alpha, t)$ *such that the following* (i), (ii) *and* (iii) *hold.*

(i) *If the function g satisfies the same condition as in Lemma* 1.2, *then*

$$(1.9)\qquad \int_0^t (D^{\alpha}g)(B_s)ds=\frac{1}{\Gamma(-\alpha)}\int_{R^1} H^{a}(-1-\alpha, t)g(a)da, \qquad \textit{holds.}$$

(ii) *The formula*

$$(1.10)\qquad H^{a}(-1-\alpha, t)=-\cos(\pi(1+\alpha))(D^{\alpha}L_t^{\cdot})(a)-\sin(\pi(1+\alpha))\mathcal{H}((D^{\alpha}L_t^{\cdot})(\cdot))(a),$$

holds.

(iii) *For any* $T>0$, $M>0$ *and* $\varepsilon\in\left(0, \frac{1}{2}-\alpha\right)$ *there exist finite-valued variables* $D_{T,M,\varepsilon}(\omega)$ *and* $E_{T,M,\varepsilon}(\omega)$ *such that*

$$(1.11)\qquad \sup_{t\leq T}|H^{a}(-1-\alpha, t)-H^{b}(-1-\alpha, t)|\leqq D_{T,M,\varepsilon}(\omega)|a-b|^{1/2-\alpha-\varepsilon} \qquad \forall a, b\in[-M, M], \quad a.s.,$$

and

$$(1.12)\qquad \sup_{|a|\leq M}|H^{a}(-1-\alpha, t)-H^{a}(-1-\alpha, s)|\leqq E_{T,M,\varepsilon}(\omega)|t-s|^{1/2-\alpha-\varepsilon} \qquad \forall s, t\in[0, T], \quad a.s.,$$

hold.

Proof. (i) and (ii) are proved in (17), so we will give only the proof of (iii).

The proof of the inequality (1.11) will be devided in several steps. (1°) First we shall show that for any $T>0$ and $\varepsilon\in\left(0, \frac{1}{2}-\varepsilon\right)$ there exists a finite-valued variable $D_{T,\varepsilon}^{(1)}(\omega)$ such that

$$(1.13)\qquad \sup_{t\leq T}|(D^{\alpha}L_t^{\cdot})(a)-(D^{\alpha}L_t^{\cdot})(b)|\leqq D_{T,\varepsilon}^{(1)}(\omega)|a-b|^{1/2-\alpha-\varepsilon} \qquad \forall a, b\in R^1, \quad \text{a.s.}$$

We owe the proof of (1.13) much to the method employed in the proof of theorem 20 in (3). It is well known that there exists a finite-valued variable $H_{T,\varepsilon}(\omega)$ such that

$$(1.14)\qquad \sup_{t\leq T}|L_t^a-L_t^b|\leqq H_{T,\varepsilon}(\omega)|a-b|^{1/2-\varepsilon} \qquad \forall a, b\in R^1, \quad \text{a.s.,}$$

holds (cf. e.g. (5) and (18)).

Since $a\mapsto L_t^a$ satisfies the Hölder's condition of order $\frac{1}{2}-\varepsilon$ and $\alpha<\frac{1}{2}-\varepsilon$, we have

$$\begin{aligned}
&\Gamma(-\alpha)\{(D^{\alpha}L_{t}^{\cdot})(x)-(D^{\alpha}L_{t}^{\cdot})(x-h)\} \\
&=\int_{-\infty}^{x}\frac{L_{t}^{x}-L_{t}^{a}}{(x-a)^{1+\alpha}}da-\int_{-\infty}^{x-h}\frac{L_{t}^{x-h}-L_{t}^{a}}{(x-h-a)^{1+\alpha}}da \\
&=\int_{0}^{\infty}\frac{L_{t}^{x}-L_{t}^{x-u}}{u^{1+\alpha}}du-\int_{h}^{\infty}\frac{L_{t}^{x-h}-L_{t}^{x-u}}{(u-h)^{1+\alpha}}du \\
&=-\int_{h}^{\infty}\{L_{t}^{x-h}-L_{t}^{x-u}\}\{(u-h)^{-(1+\alpha)}-u^{-(1+\alpha)}\}du \\
&\quad+\int_{h}^{\infty}\frac{L_{t}^{x}-L_{t}^{x-h}}{u^{1+\alpha}}du+\int_{0}^{h}\frac{L_{t}^{x}-L_{t}^{x-u}}{u^{1+\alpha}}du \\
&=I_1+I_2+I_3 \quad \text{say.}
\end{aligned}$$

Here by (1.14),

$$\begin{aligned}
|I_1|&\leqq\int_{h}^{\infty}H_{T,\varepsilon}(u-h)^{1/2-\varepsilon}\{(u-h)^{-(1+\alpha)}-u^{-(1+\alpha)}\}du \\
&=H_{T,\varepsilon}h^{1/2-\varepsilon-\alpha}\int_{1}^{\infty}(s-1)^{1/2-\varepsilon}\{(s-1)^{-1-\alpha}-s^{-1-\alpha}\}ds\,.
\end{aligned}$$

Since $1+\alpha-\frac{1}{2}+\varepsilon<1$ and $\alpha+2-\frac{1}{2}+\varepsilon>1$ hold, we can see that

$$0<\int_{1}^{\infty}(s-1)^{1/2-\varepsilon}\{(s-1)^{-1-\alpha}-s^{-1-\alpha}\}ds<+\infty\,.$$

For I_2, we observe that

$$|I_2|\leqq\int_{h}^{\infty}H_{T,\varepsilon}\frac{h^{1/2-\varepsilon}}{u^{1+\alpha}}du=\frac{H_{T,\varepsilon}}{\alpha}h^{1/2-\varepsilon-\alpha}\,.$$

Also,

$$|I_3|=\int_{0}^{h}H_{T,\varepsilon}\frac{u^{1/2-\varepsilon}}{u^{1+\alpha}}du=\frac{H_{T,\varepsilon}}{1/2-\varepsilon-\alpha}h^{1/2-\varepsilon-\alpha}$$

put

$$D_{T,\varepsilon}^{1}(\omega)=\frac{H_{T,\varepsilon}(\omega)}{\Gamma(-\alpha)}\left\{\int_{1}^{\infty}(s-1)^{1/2-\varepsilon}\{(s-1)^{-1-\alpha}-s^{-1-\alpha}\}ds+\frac{1}{\alpha}+\frac{1}{1/2-\varepsilon-\alpha}\right\}.$$

Then the inequality (1.13) holds.

(2°) For any $T>0$, $M>0$ and $\varepsilon\in\left(0,\frac{1}{2}-\varepsilon\right)$ there exists a finite-valued variable $D_{T,M,\varepsilon}^{(2)}(\omega)$ such that

(1.15) $$\sup_{t\leq T}|\mathscr{H}((D^{\alpha}L_{t}^{\cdot})(\cdot))(a)-\mathscr{H}((D_{\alpha}L_{t}^{\cdot})(\cdot))(b)|\leqq D_{T,M,\varepsilon}^{(2)}(\omega)|a-b|^{1/2-\varepsilon-\alpha}$$
$$\forall a,\, b\in[-M,\, M],\quad \text{a.s.}$$

As we know the inequality (1.13), we can show (1.15) by the same argument employed by Yor in the proof of Theorem 2.1 in (18).

(3°) Put

$$D_{T,M,\varepsilon}(\omega)=|\cos(\pi(1+\alpha))|D^{(1)}_{T,M,\varepsilon}(\omega)+|\sin(\pi(1+\alpha))|D^{(2)}_{T,M,\varepsilon}(\omega).$$

Note that the inequalities (1.13) and (1.15) hold. Then by the formula (1.10), we can prove the inequality (1.11).

Finally we shall give the proof of the inequality (1.12).

It is known that for any $T>0$ there exists a finite-valued variable $K_{T,\varepsilon}(\omega)$ such that

$$(1.16)\qquad \sup_a |L_t^a-L_s^a|\leqq K_{T,\varepsilon}|t-s|^{1/2-\varepsilon}\qquad \forall s,\ t\in[0,\ T],\quad \text{a.s.},$$

holds (cf. e.g. (18)).

Note that

$$\begin{aligned}&\Gamma(-\alpha)\{D^\alpha(L_t^{\cdot})(a)-D^\alpha(L_s^{\cdot})(a)\}\\&=\int_0^\infty\frac{L_t^a-L_t^{a-u}}{u^{1+\alpha}}du-\int_0^\infty\frac{L_s^a-L_s^{a-u}}{u^{1+\alpha}}du\\&=\int_0^{|t-s|}\frac{(L_t^a-L_t^{a-u})-(L_s^a-L_s^{a-u})}{u^{1+\alpha}}du\\&\quad+\int_{|t-s|}^\infty\frac{(L_t^a-L_s^a)-(L_t^{a-u}-L_s^{a-u})}{u^{1+\alpha}}du\\&=K_1+K_2,\quad \text{say.}\end{aligned}$$

Here by (1.14) we have

$$|K_1|\leqq\int_0^{|t-s|}\frac{2H_{T,\varepsilon}u^{1/2-\varepsilon}}{u^{1+\alpha}}du=2H_{T,\varepsilon}\frac{1}{1/2-\varepsilon-\alpha}|t-s|^{1/2-\varepsilon-\alpha}.$$

Also by (1.16),

$$|K_2|\leqq\int_{|t-s|}^\infty\frac{2K_{T,\varepsilon}|t-s|^{1/2-\varepsilon}}{u^{1+\alpha}}du=2K_{T,\varepsilon}\frac{1}{\alpha}|t-s|^{1/2-\varepsilon-\alpha}.$$

Thus we have

$$(1.17)\qquad \sup_a|(D^\alpha L_t^{\cdot})(a)-(D^\alpha L_s^{\cdot})(a)|\leqq\frac{1}{\Gamma(-\alpha)}\left\{\frac{2H_{T,\varepsilon}}{1/2-\varepsilon-\alpha}+\frac{2K_{T,\varepsilon}}{\alpha}\right\}|t-s|^{1/2-\varepsilon-\alpha}$$
$$\forall t,\ s\in[0,\ T],\quad \text{a.s.}$$

Then by the same argument employed in the proof of Theorem 2.1 in (18), the inequality (1.17) implies that for any $T>0$, $M>0$ and $\varepsilon\in\left(0,\ \frac{1}{2}-\alpha\right)$ there exists a finite-valued variable $E^{(1)}_{T,M,\varepsilon}(\omega)$ such that

$$(1.18)\qquad \sup_{|a|\leqq M}|\mathcal{H}((D_\alpha L_t^{\cdot})(\cdot))(a)-\mathcal{H}((D_\alpha L_s^{\cdot})(\cdot))(a)|\leqq E^{(1)}_{T,M,\varepsilon}(\omega)|t-s|^{1/2-\varepsilon-\alpha}$$
$$\forall t,\ s\in[0,\ T],\quad \text{a.s.}$$

Put

$$\begin{aligned}E_{T,M,\varepsilon}(\omega)=&\frac{1}{\Gamma(-\alpha)}|\cos(\pi(1+\alpha))|\left\{\frac{2H_{T,\varepsilon}(\omega)}{1/2-\varepsilon-\alpha}+\frac{2K_{T,\varepsilon}(\omega)}{\alpha}\right\}\\&+|\sin(\pi(1+\alpha))|E^{(1)}_{T,M,\varepsilon}(\omega).\end{aligned}$$

Then by the formula (1.10), the inequalities (1.17) and (1.18) imply the desired result (1.12). Q.E.D.

Remark 1.2. It is known that the following formula holds ((16)).

$$H^a(-1-\alpha,\, t)=\lim_{\varepsilon\downarrow 0}\left\{\frac{\varepsilon^{-\alpha}}{(-\alpha)}L_t^a+\int_0^t I_{[a+\varepsilon,\infty)}(B_s)(B_s-a)^{-1-\alpha}ds\right\},\quad \text{a.s.} \tag{1.19}$$

C. At the end of this section we shall treat continuous additive functionals which correspond to the function $(x-a)_+^{\beta-1}$, $a\in R^1$ where $0<\beta<1$.

Definition 1.4 (Continuous additive functional corresponding to the functions $(x-a)_+^{\beta-1}$, $a\in R^1$, $0<\beta<1$). Put

$$\frac{1}{2}H^a(-1+\beta,\, t)=\frac{(B_t-a)_+^{1+\beta}}{\beta(\beta+1)}-\frac{(B_0-a)_+^{1+\beta}}{\beta(\beta+1)}-\int_0^t\frac{(B_s-a)_+^{\beta}}{\beta}dB_s\,. \tag{1.20}$$

Since the function $(x-a)_+^{\beta}$ belongs to $L^2_{\text{loc}}(R^1)$, the right hand side of (1.20) is well defined. We call $H^a(-1+\beta,\, t)$ continuous additive functional defined by the function $(x-a)_+^{\beta-1}$.

By the similar way as in Theorem 1 in (17), we have the following.

Lemma 1.4. *Suppose that* $0<\beta<1$. *Let* g *be a continuous function with compact support. Let* $(I^\beta g)(x)=\left(\frac{1}{\Gamma(\beta)}x_+^{\beta-1}*g\right)(x)$ *be the* β*-th integral of the function* g. (*cf.* (2) *and* (13)).
Then

$$\int_0^t(I^\beta g)(B_s)ds=\frac{1}{\Gamma(\beta)}\int_{R^1}H^a(-1+\beta,\, t)g(a)da \tag{1.21}$$

holds.

Concerning the problem of continuity of additive functionals $H^a(-1+\beta,\, t)$ we can obtain similar results as in Lemma 1.3 for functionals $H^a(-1-\alpha,\, t)$, but here we only note that we can choose a version of $H^a(-1+\beta,\, t)$, such that $(t,\, a)\mapsto H^a(-1+\beta,\, t)$ is jointly continuous, a.s.

Remark 1.3. Since the function $(x-a)_+^{\beta-1}$ belongs to $L^1_{\text{loc}}(R^1)$, we get

$$H^a(-1+\beta,\, t)=\int_0^t(B_s-a)_+^{\beta-1}ds\,. \tag{1.22}$$

Remark 1.4. The continuous additive functionals introduced in this section, C_t^a, $H^a(-1-\alpha,\, t)$ and $H^a(-1+\beta,\, t)$ belong to the class of continuous additive functionals of B_t locally of zero energy. (cf. (1) and (11)).

2. Limit theorems.

The first topic of this section is the limit theorem where the additive functional C_t^0 and Hilbert transform play a role.

Theorem 2.1. *Let f be a function which belongs to $L^2(R^1)$. Suppose that the Hilbert transform of the function f, $(\mathcal{H}f)(x)=\frac{1}{\pi}\left(\mathrm{v.p.}\frac{1}{x}*f\right)(x)$ vanishes outside a compact set in R^1.*

Then the family of continuous stochastic processes

$$t\mapsto\frac{1}{\sqrt{\lambda}}\int_0^{\lambda t}f(B_s)ds,\qquad \lambda>0, \tag{2.1}$$

converges in the sense of law on the space of continuous functions to the continuous process

$$t\mapsto\frac{1}{\pi}\left(\int_{R^1}(\mathcal{H}^{-1}f)(x)dx\right)\cdot C_t^0 \tag{2.2}$$

as $\lambda\to\infty$ where $\mathcal{H}^{-1}$ means the inverse transform of $\mathcal{H}$; $\mathcal{H}^{-1}f=-\mathcal{H}f$.

Proof. Let $\underset{(\mathcal{L})}{\sim}$ mean the equivalence in law on the space of continuous functions.

By the scaling property of Brownian motion $\{B(t)\}\underset{(\mathcal{L})}{\sim}\left\{\frac{1}{\sqrt{\lambda}}B(\lambda t)\right\}$ for each $\lambda>0$, it is known that $\{L_t^a\}\underset{(\mathcal{L})}{\sim}\left\{\frac{1}{\sqrt{\lambda}}L_{\lambda t}^{\sqrt{\lambda}a}\right\}$ for each $\lambda>0$ (cf. (4)). Then we have

$$\frac{1}{\sqrt{\lambda}}\int_0^{\lambda t}f(B_s)ds=\frac{1}{\sqrt{\lambda}}\int_{R^1}f(a)L_{\lambda t}^a da\underset{(\mathcal{L})}{\sim}\int_{R^1}L_t^{a/\sqrt{\lambda}}f(a)da. \tag{2.3}$$

On the other hand we will show that

$$\int_{R^1}L_t^{a/\sqrt{\lambda}}f(a)da=\frac{1}{\pi}\int_{R^1}C_t^{a/\sqrt{\lambda}}(\mathcal{H}^{-1}f)(a)da \tag{2.4}$$

holds.

Indeed, by Lemma 1.1

$$\begin{aligned}\int_{R^1}L_t^{a/\sqrt{\lambda}}f(a)da&=\sqrt{\lambda}\int_{R^1}L_t^b(\sqrt{\lambda}\,b)db=\sqrt{\lambda}\int_0^t f(\sqrt{\lambda}\,B_s)ds\\&=\frac{\sqrt{\lambda}}{\pi}\int_{R^1}C_t^b(\mathcal{H}^{-1}f(\sqrt{\lambda}\,\cdot))(b)db=\frac{\sqrt{\lambda}}{\pi}\int_{R^1}C_t^b(\mathcal{H}^{-1}f)(\sqrt{\lambda}\,b)db\\&=\frac{1}{\pi}\int_{R^1}C_t^{a/\sqrt{\lambda}}(\mathcal{H}^{-1}f)(a)da\,.\end{aligned}$$

Combine (2.3) and (2.4). Then we have

(2.5) $$t \mapsto \frac{1}{\sqrt{\lambda}}\int_0^t f(B_s)ds \underset{(\mathcal{L})}{\sim} t \mapsto \frac{1}{\pi}\left(\int_{R^1} C_t^{a/\sqrt{\lambda}} (\mathcal{H}^{-1}f)(a)da\right)$$

for each $\lambda>0$.

Note that $\mathcal{H}^{-1}f$ belongs to $L^2(R^1)$ and vanishes outside of a compact set. Then $\mathcal{H}^{-1}f$ belongs to $L^1(R^1)$. On the other hand we know by Lemma 1.1 that $(t, a)\mapsto C_t^a$ is jointly continuous (a.s.).

Then $\int_{R^1} C_t^{a/\sqrt{\lambda}}(\mathcal{H}^{-1}f)(a)da$ converges to $\left(\int_{R^1}(\mathcal{H}^{-1}f)(a)da\right)C_t^0$ as $\lambda\to\infty$, uniformly in t on each bounded interval (a.s.). Thus the Theorem 2.1 is proved.

Q.E.D.

Remark 2.1. If there exists a function f which satisfies both the following conditions;

(i) $f\in L^2(R^1)\cap L^1(R^1)$ and $\int_{R^1} f(x)dx\neq 0$,

(ii) The Hilbert transform of the function f vanishes outside of a compact set,

then the family of processes $t\mapsto \frac{1}{\sqrt{\lambda}}\int_0^{\lambda t} f(B_s)ds$, $\lambda>0$ converges as $\lambda\to\infty$ to the process $t\mapsto\left(\int_{R^1} f(x)dx\right)L_t^0$ (by Theorem I in the introduction) and on the other hand converges to the process $t\mapsto \frac{1}{\pi}\left(\int_{R^1}(\mathcal{H}^{-1}f)(x)dx\right)\cdot C_t^0$ as $\lambda\to\infty$ (by Theorem 2.1). But there exists no such a function.

Here we give a direct proof of non-existence of such a function.

Let f satisfy both the conditions (i) and (ii). Since the Hilbert transform $\mathcal{H}$ maps an L^2-function to an L^2-function, the function $\mathcal{H}^{-1}f=-\mathcal{H}f$ belongs to $L^2(R^1)$. Moreover the function $\mathcal{H}^{-1}f$ vanishes outside an interval $[-M, M]$. Then $\mathcal{H}^{-1}f$ belongs to $L^2(R^1)\cap L^1(R^1)$.

Note that $\mathcal{H}(I_{[-n,n]}(\cdot))(x)=\frac{1}{\pi}\log\left|\frac{n-x}{n+x}\right|$.

Then we have

$$\int_{R^1} f(x)I_{[-n,n]}(x)dx=\frac{1}{\pi}\int_{R^1}(\mathcal{H}f)(x)\log\left|\frac{n-x}{n+x}\right|dx$$
$$=\frac{1}{\pi}\int_{-M}^{M}(\mathcal{H}f)(x)\log\left|\frac{n-x}{n+x}\right|dx\,.$$

Note that the family of functions $\log\left|\frac{n-x}{n+x}\right|$, $n\geqq M+1$ is uniformly bounded and converges to zero on the interval $[-M, M]$ as $n\to\infty$.

Then

$$0\neq\int_{R^1} f(x)dx=\lim_{n\to\infty}\int_{R^1} f(x)I_{[-n,n]}(x)dx$$
$$=\lim_{n\to\infty}\frac{1}{\pi}\int_{R^1}(\mathcal{H}f)(x)\log\left|\frac{n-x}{n+x}\right|dx=0\,.$$

Thus we obtain a contradiction. Q.E.D.

The second limit theorem is related to the continuous additive functional $H^0(-1-\alpha, t)$ and the fractional derivative of order α.

Theorem 2.2. *Suppose that* $0<\alpha<\frac{1}{2}$, $\alpha<\beta\leqq 1$.

Let g be a function with compact support satisfying the following condition (H, β). *(Hölder's condition of order* β*).*
(H, β) *there exists a constant* $K>0$ *such that*

$$|g(x)-g(y)|\leqq K|x-y|^{\beta}, \qquad \forall x, y\in R^1.$$

Put $(D^{\alpha}g)(x)=f(x)$
Then the family of continuous stochastic processes

$$t\mapsto \frac{1}{\lambda^{1/2-\alpha/2}}\int_0^{\lambda t} f(B_s)ds, \qquad \lambda>0 \tag{2.6}$$

converges in the sense of law on the space of continuous functions to the continuous process

$$t\mapsto \left(\int_{R^1} g(x)dx\right) H^0(-1-\alpha, t) \qquad as \quad \lambda\to\infty. \tag{2.7}$$

Proof. As in the proof of Theorem 2.1, we note that

$$\{L_t^{a/\sqrt{\lambda}}\} \underset{(\mathcal{L})}{\simeq} \left\{\frac{1}{\sqrt{\lambda}} L_{\lambda t}^{a}\right\}$$

for any $\lambda>0$.
Then we have

$$\begin{aligned}
\frac{1}{\lambda^{1/2-\alpha/2}}\int_0^{\lambda t} f(B_s)ds &= \frac{1}{\lambda^{1/2-\alpha/2}}\int_0^{\lambda t}(D^{\alpha}g)(B_s)ds \\
&= \frac{1}{\lambda^{1/2-\alpha/2}}\int_{R^1}(D^{\alpha}g)(a)L_{\lambda t}^{a}da \underset{(\mathcal{L})}{\simeq} \lambda^{\alpha/2}\int_{R^1}(D^{\alpha}g)(a)L_t^{a/\sqrt{\lambda}}da \\
&= \lambda^{1/2+\alpha/2}\int_{R^1}(D^{\alpha}g)(\sqrt{\lambda}\,u)L_t^u du \\
&= \lambda^{1/2+\alpha/2}\int_0^t (D^{\alpha}g)(\sqrt{\lambda}\,B_s)ds\,.
\end{aligned} \tag{2.8}$$

By the definition of the fractional derivative, we have

$$\begin{aligned}
(D^{\alpha}g)(\sqrt{\lambda}\,x) &= \frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{\sqrt{\lambda}x}\frac{\{g(\sqrt{\lambda}\,x)-g(a)\}}{\{\sqrt{\lambda}\,x-a\}^{1+\alpha}}da \\
&= \frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{x}\frac{\{g(\sqrt{\lambda}\,x)-g(\sqrt{\lambda}\,u)\}}{(x-u)^{1+\alpha}}\lambda^{-\alpha/2}du \\
&= D^{\alpha}(\lambda^{-\alpha/2}g(\sqrt{\lambda}\,\cdot))(x)\,.
\end{aligned} \tag{2.9}$$

Combine (2.8) and (2.9). Then we get

(2.10) $$\frac{1}{\lambda^{1/2-\alpha/2}}\int_0^{\lambda t} f(B_s)ds \underset{(\mathcal{I})}{\sim} \lambda^{1/2+\alpha/2}\int_{R^1} D^\alpha(\lambda^{-\alpha/2}g(\sqrt{\lambda}\,\cdot))(a)L_t^a da$$

$$=\lambda^{1/2}\int_{R^1} D^\alpha(g(\sqrt{\lambda}\,\cdot))(a)L_t^a da=\lambda^{1/2}\int_0^t D^\alpha(g(\sqrt{\lambda}\,\cdot))(B_s)ds\,.$$

On the other hand by Lemma 1.3, we have

(2.11) $$\lambda^{1/2}\int_0^t D^\alpha(g(\sqrt{\lambda}\,\cdot))(B_s)ds=\lambda^{1/2}\int_{R^1} H^a(-1-\alpha,\, t)g(\sqrt{\lambda}\,a)da$$

$$=\int_{R^1} H^{u/\sqrt{\lambda}}(-1-\alpha,\, t)g(u)du\,.$$

The relations (2.10) and (2.11) imply that

(2.12) $$t\mapsto\frac{1}{\lambda^{1/2-\alpha/2}}\int_0^{\lambda t} f(B_s)ds \underset{(\mathcal{I})}{\sim} t\mapsto\int_{R^1} H^{u/\sqrt{\lambda}}(-1-\alpha,\, t)g(u)du$$ holds.

By Lemmal 1.3 it is known that $(t,\, a)\mapsto H^a(-1-\alpha,\, t)$ is jointly continuous, a.s. Note that the function g is continuous with compact support. Then

$$\int_{R^1} H^{u/\sqrt{\lambda}}(-1-\alpha,\, t)g(u)du$$

converges as $\lambda\to\infty$ to the process $\left(\int_{R^1} g(a)da\right)H_t^0(-1-\alpha,\, t)$ uniformly in t on each bounded interval a.s. Thus the Theorem 2.2 is proved. Q.E.D.

Remark 2.2. Let f and g satisfy the same condition as in Theorem 2.2. Then,

(A) $\int_{R^1} f(x)dx=\int_{R^1}(D^\alpha g)(x)dx=0$ holds,

(B) If $\int_{R^1} g(x)dx\neq 0$, then $\langle f,\, f\rangle=\langle D^\alpha g,\, D^\alpha g\rangle=+\infty$ holds

Proof of (A). Let M be a positive number such that the function g vanishes outside of the interval $[-M,\, M]$. Let $I^\beta g(x)=\left(\frac{1}{\Gamma(\beta)}x_+^{\beta-1}*g\right)(x)$ be the fractional integral of order β of the function $g(x)$. It is well known that $I^1(D^\alpha g)(x)=I^{1-\alpha}g(x)$ (cf. e.g. (2)).

Here

$$I^{1-\alpha}g(x)=\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{x}\frac{g(a)}{(x-a)^\alpha}da\,.$$

Note that $I^{1-\alpha}g(x)=0$ for $x<-M$, and

$$I^{1-\alpha}g(x)=\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{M}\frac{g(a)}{(x-a)^\alpha}da\sim O(x^{-\alpha})\,.$$

Then, we get

$$\int_{R^1} f(x)dx = \int_{R^1} (D^\alpha g)(x)dx = 0. \qquad \text{Q.E.D.}$$

Proof of (*B*). Let $g_+(x)=g(x)\vee 0$ and $g_-(x)=(-g(x))\vee 0$.
We consider the case where

$$\int_{R^1} g(x)dx = \int_{R^1} g_+(x)dx - \int_{R^1} g_-(x)dx > 0.$$

Choose a number M such that the function g vanishes outside of the interval $[-M, M]$.

Note that, for $x>M$

$$\int_{-\infty}^{x} (D^\alpha g)(u)du = (I^{1-\alpha}g)(x) - (I^{1-\alpha}g)(-\infty) = (I^{1-\alpha}g)(x)$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_{-M}^{x} \frac{g(a)}{(x-a)^\alpha} da = \frac{1}{\Gamma(1-\alpha)} \int_{-M}^{M} \frac{g(a)}{(x-a)^\alpha} da$$

$$= \frac{1}{\Gamma(1-\alpha)} \left(\int_{-M}^{M} \frac{g_+(a)}{(x-a)^\alpha} da - \int_{-M}^{M} \frac{g_-(a)}{(x-a)^\alpha} da \right)$$

$$\geqq \frac{1}{\Gamma(1-\alpha)} \left(\frac{1}{(x+M)^\alpha} \int_{-M}^{M} g_+(a)da - \frac{1}{(x-M)^\alpha} \int_{-M}^{M} g(a)da \right).$$

These inequalities imply

$$\int_{-\infty}^{x} (D^\alpha g)(u)du \sim O(x^{-\alpha}) \qquad 0<\alpha<\frac{1}{2}.$$

Thus we get

$$\langle D^\alpha g, D^\alpha g \rangle = \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{x} (D^\alpha g)(u)du \right)^2 dx \right) = +\infty.$$

In the case where $\int_{R^1} g(x)dx<0$, we can also see by the similar way that $\langle D^\alpha g, D^\alpha g \rangle = +\infty$ holds. Q.E.D.

The third limit theorem is related to the continuous additive functional $H^0(-1-\alpha, t)$, $0<\beta<1$ and the fractional integral of order β.

The proof of the theorem can be done following a similar way as in the proof of preceding theorems, so we will omit it.

Theorem 2.3. *Suppose that* $0<\beta<1$. *Let* g *be a continuous function with compact support.*

Put $f(x)=I^\beta g(x)=\left(\frac{1}{\Gamma(\beta)} x_+^{\beta-1} * g\right)(x)$; *the fractional integral of order* β *of the function* g.

Then the family of continuous stochastic processes

$$t \mapsto \frac{1}{\lambda^{1/2+\beta/2}} \int_0^{\lambda t} f(B_s)ds, \qquad \lambda>0, \tag{2.13}$$

converges in the sense of law on the space of continuous functions to the continuous process

$$t \mapsto \left(\int_{R^1} g(x)dx\right) \cdot H^0(-1+\beta, t) \qquad as \quad \lambda \to \infty . \tag{2.14}$$

Remark 2.3. Let g be a continuous function with compact support. If $\int_{R^1} g(x)dx \neq 0$ holds, then $f(x) = I^\beta g(x)$ does not belong to $L^1(R^1)$.

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