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Formation of singularities for Hamilton-Jacobi equation II

Dedicated to Professor Sigeru MIZOHATA on his sixtieth birthday

By

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§1. Introduction.

Consider the Cauchy problem for Hamilton-Jacobi equation in two space dimensions:

$$\frac{\partial u}{\partial t}+f\left(\frac{\partial u}{\partial x}\right)=0 \qquad \text{in } \{t>0,\ x\in R^2\}, \tag{1}$$

$$u(0,\ x)=\phi(x)\in \mathscr{S}(R^2)\,. \tag{2}$$

We assume that f is C^∞ and uniformly convex, i.e., there exists a constant C such that

$$f''(p)\underset{\text{def}}{=}\left[\frac{\partial^2 f}{\partial p_i \partial p_j}(p)\right]_{1\leq i,\,j\leq 2}\geqq C>0\,.$$

It is well known that, even for smooth initial data, the Cauchy problem (1) and (2) does not admit a smooth solution for all t. Therefore we consider a generalized solution whose definition will be given a little latter. The existence of global generalized solution for (1) and (2) is already established (for example [7], [8]). For detailed bibliography, refer to [1]. This paper is concerned with the singularities of global generalized solutions.

For a single conservation law in one dimensional space, a solution satisfying the entropy condition is piecewise smooth for any smooth initial data in $\mathscr{S}(R^2)$ except for initial data in a certain subset of the first category ([4], [5], [6] and [12]). T. Debeneix [2] treated certain systems of conservation laws which is essentially equivalent to Hamilton-Jacobi equation (1) in R^n ($n\leqq 4$), and proved the similar results to [12] by the same method as [12]. The aim of this paper is to make clear the situation how the singularities appear.

One of the classical method to solve first order non-linear equations is the characteristic one. The weak point is that it is the local theory. The reason is due to the fact that a smooth mapping can not have its smooth inverse uniquely in a neighborhood of a point where the Jacobian vanishes, i.e., that the inverse mapping takes many values in a neighborhood of a critical point of

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the mapping. Therefore the solution also takes many values there. As the following definition says, we look for one-valued and continuous solution. Our aim is to show that we can uniquely choose one value from many values of solution so that the solution becomes one-valued and continuous. Then the condition of semi-concavity is automatically satisfied. Here we give the definition of generalized solutions.

Definition. A lipschitz continuous function $u(t, x)$ defined on $R^1\times R^2$ is called a generalized solution of (1) and (2) if and only if i) $u(t, x)$ satisfies (1) almost everywhere in $R^1\times R^2$ and (2) on $\{t=0,\ x\in R^2\}$, ii) $u(t, x)$ is semi-concave, i.e., there exists a constant $K>0$ such that

$$u(t,\ x+y)+u(t,\ x-y)-2u(t,\ x)\leqq K|y|^2 \quad \text{for any } x,\ y\in R^2 \text{ and } t>0. \tag{3}$$

Remark. Put $v_i=\partial u/\partial x_i$ $(i=1, 2)$, then the equation (1) is written down as a system of conservation laws:

$$\frac{\partial}{\partial t}v_i+\frac{\partial}{\partial x_i}f(v)=0 \qquad (i=1,\ 2). \tag{4}$$

Then the inequality (3) turns into the entropy condition for (4). See a Remark in §3.

§2. Construction of solutions.

The characteristic lines corresponding to (1) and (2) are determined by the equations:

$$\dot{x}_i=\frac{\partial f}{\partial p_i}(p), \qquad \dot{p}_i=0 \qquad (i=1,\ 2)$$

with initial data

$$x_i(0)=y_i, \qquad p_i(0)=\frac{\partial \phi}{\partial y_i}(y) \qquad (i=1,\ 2).$$

On the characteristic line $x=x(t, y)$, the value $v(t, y)$ of the solution for (1) and (2) satisfies the equation:

$$\dot{v}=-f(p)+\langle p,\ f'(p)\rangle, \qquad v(0)=\phi(y)$$

where $f'(p)=(\partial f/\partial p_1,\ \partial f/\partial p_2)$ and $\langle p, q\rangle$ is scalar product of vectors p and q. Solving these equations, we have

$$x=y+tf'(\phi'(y))\underset{\text{def}}{=}H_t(y) \tag{5}$$

$$v(t,\ y)=\phi(y)+t\{-f(\phi'(y))+\langle\phi'(y),\ f'(\phi'(y))\rangle\}. \tag{6}$$

Then H_t is a smooth mapping from R^2 to R^2 and its Jacobian is given by

$$\frac{\partial x}{\partial y}(t,\ y)=\det\left[I+tf''(\phi'(y))\phi''(y)\right].$$

We write $A(y)=f''(\phi'(y))\phi''(y)$ and the eigenvalues of $A(y)$ by $\lambda_1(y)\leqq\lambda_2(y)$. When the space dimension is one, $\lambda_1(y)=f''(\phi'(y))\phi''(y)$. Since $f''(\phi'(y))>0$ and $\phi(y)\in\mathcal{S}(R^1)$, $\lambda_1(y)$ takes necessarily negative values at some points. In this case also, we can prove

$$\min_y \lambda_1(y)=\lambda_1(y^0)=-M<0\,,$$

and put $t^0=1/M$. Since we have for $t<t^0$

$$\frac{\partial x}{\partial y}(t,\,y)\neq 0 \qquad \text{for any } y\in R^2\,,$$

we can uniquely solve the equation (5) with respect to y and denote it by $y=y(t,\,x)$. Then $u(t,\,x)=v(t,\,y(t,\,x))$ is a unique solution of (1) and (2) for $t<t^0$. Our problem is to construct the solution for $t>t^0$.

Suppose that $t-t^0$ is positive and sufficiently small, and consider the equation (5) in a neighborhood of $(t^0,\,y^0)$. The Jacobian of H_t vanishes on $\Sigma_t=\{y\in R^2;\ 1+t\lambda_1(y)=0\}$. Assume the condition

(A.1) $\lambda_1(y)\neq\lambda_2(y)$, $\operatorname{grad}\lambda_1(y)\neq 0$ *on* Σ_t *and* Σ_t *is a simple closed curve.*

In this case, Σ_t is parametrized as $\Sigma_t=\{{}^t(y_1(s),\,y_2(s))\,;\ s\in I\}$ where I is a closed interval and $y_i(s)\in C^\infty(I)$ $(i=1,\,2)$. Put

$$\Sigma_t^e=\left\{y(s^0)\in\Sigma_t\,;\ \frac{d}{ds}v(t,\,y(s))=0 \ \text{ at } s=s^0\right\}.$$

By the definition of H. Whitney [15], a point in $\Sigma_t-\Sigma_t^e$ is a fold point of the mapping H_t, i.e.,

$$\frac{d}{ds}x(t,\,y(s))\neq 0 \qquad \text{on } \Sigma_t-\Sigma_t^e\,.$$

Lemma 1. *Assume that the number of elements of* Σ_t^e *is two, then it follows*

$$\frac{d}{ds}x(t,\,y(s))=(I+tA(y))\frac{dy}{ds}=0 \qquad \textit{on } \Sigma_t^e\,.$$

Proof. Put

$$I+tA(y)=\begin{bmatrix} a_1(t,\,y) \\ a_2(t,\,y)\end{bmatrix}, \qquad a_i(t,\,y)\in R^2 \quad (i=1,\,2)\,,$$

then $a_1(t,\,y)$ and $a_2(t,\,y)$ are linearly dependent on Σ_t. As they are smooth in the interior of Σ_t, they do not take any direction of R^2. Especially, when $t-t^0$ is sufficiently small, $a_i(t,\,y)$ $(i=1,\,2)$ are almost constant, i.e., they move in a small neighborhood of $a_i(t^0,\,y^0)$ $(i=1,\,2)$ where $a_1(t^0,\,y^0)$ and $a_2(t^0,\,y^0)$ are linearly dependent. Contrary, when the point $y=y(s)$ makes round of Σ_t, $dy/ds(s)$ takes every direction. Therefore $d/ds\,x(t,\,y(s))$ vanishes at least at two point. But, the points where it vanishes are contained in Σ_t^e, because

$$\frac{d}{ds}v(t, y(s))=\left\langle\frac{\partial v}{\partial y}, \frac{dy}{ds}\right\rangle$$

$$=\left\langle\phi'(y), \frac{d}{ds}x(t, y(s))\right\rangle. \tag{8}$$

Hence we get this Lemma. Q. E. D.

Assume here the following condition:

(A.2) $\Sigma_t^e=\{Y_1, Y_2\}$, *i.e., the number of elements of* Σ_t^e *is two, and* Y_i $(i=1, 2)$ *are cusp points of* H_t, *i.e.,*

$$\frac{d^2}{ds^2}x(t, y(s))\neq 0 \quad \text{at } y(s)=Y_i\ (i=1, 2).$$

Concerning the above assumptions (A.1) and (A.2), we give the following

Remark. Assume

(C.1) the singularities of $\lambda_1(y)$ are non-degenerate, i.e., if grad $\lambda_1(y)=0$, the hessian of $\lambda_1(y)$ is regular at the point,

(C.2) $\partial v/\partial y(t^0, y^0)\neq 0$,

then, for $t>t^0$ where $t-t^0$ is small, Σ_t becomes a simple closed curve and the number of elements of Σ_t^e is two.

We denote the restriction of $v(t, y)$ on Σ_t by $v_\Sigma(t, y)$. By (8), we see that $v_\Sigma(t, y)$ takes its extremum on Σ_t^e. Especially, if we put $v(t, Y_i)=c_i$ $(i=1, 2)$ and suppose $c_1<c_2$, then v_Σ takes the minimum at $y=Y_1$ and the maximum at $y=Y_2$. Denote by D_t the interior of the curve Σ_t and by Ω_t the interior of $H_t(\Sigma_t)$. Then the curve $\{y\in R^2; v(t, y)=c_i\}$ is tangent to D_t at $y=Y_i$ $(i=1, 2)$. Here we apply the results of H. Whitney [15]. According to his theorem, the canonical forms of a fold and cusp points are expressed respectively as follows:

$$x_1=y_1^2, \quad x_2=y_2 \quad \text{in a neighborhood of a fold point} \tag{9$_1$}$$

$$x_1=y_1y_2-y_1^3, \quad x_2=y_2 \quad \text{in a neighborhood of a cusp point.} \tag{9$_2$}$$

This means that the mapping H_t can be regarded as the mappings $(9)_1$ and $(9)_2$ in a neighborhood of a fold and cusp point respectively. Moreover he proved that any smooth mapping from R^2 to R^2 can be approximated by smooth mappings whose singularities are fold and cusp points only. By this result we see that, when we put $H_t(Y_i)=X_i$ $(i=1, 2)$, the curve $H_t(\Sigma_t)$ has the cusps at $x=X_i$ $(i=1, 2)$. When we solve the equation (5) with respect to y for $x\in\Omega_t$, the expressions $(9)_1$ and $(9)_2$ mean that the solution $y=y(t, x)$ becomes three-valued. Write these values by $y=g_1(t, x)$, $g_2(t, x)$ and $g_3(t, x)$ where $g_2(t, x)\in D_t$ for any $x\in\Omega_t$. When we write $u_i(t, x)=v(t, g_i(t, x))$ $(i=1, 2, 3)$, the solution of (1) and (2) takes three values $u_i(t, x)$ $(i=1, 2, 3)$ on Ω_t. Concerning these situations, see Figure 1.

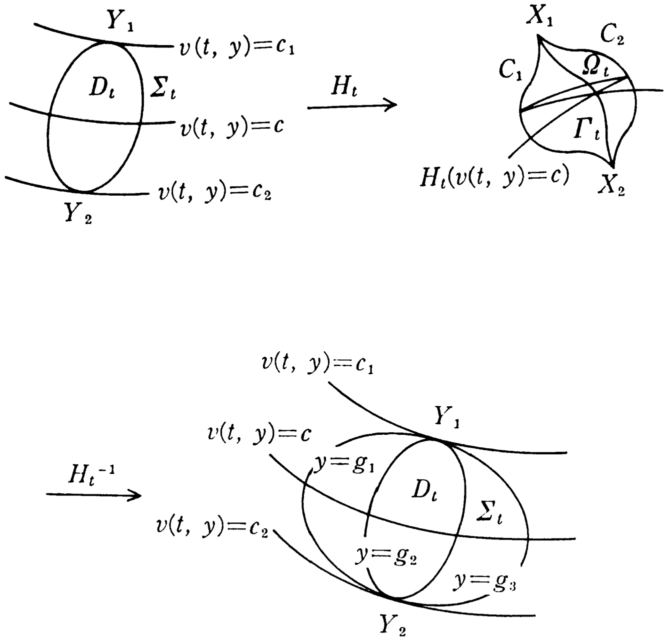

Figure 1. Curves Σ_t, $H_t(\Sigma_t)$ and $H_t^{-1}(H_t(\Sigma_t))$.

Lemma 2.

i) $\frac{\partial}{\partial x} u_i(t, x)=\frac{\partial \phi}{\partial y}(g_i(t, x))$ *for* $x \in \Omega_t$ $(i=1, 2, 3)$,

ii) $\left\langle g_i(t, x)-g_j(t, x), \frac{\partial u_i}{\partial x}-\frac{\partial u_j}{\partial x}\right\rangle<0$ *for* $x \in \Omega_t$, $i \neq j$,

iii) $u_1(t, x)<u_2(t, x)$ *and* $u_3(t, x)<u_2(t, x)$ *for* $x \in \Omega_t$.

Proof. i) This is obtained by simple calculation. ii) From the definition of $g_i(t, x)$, we have

$$x=g_i(t, x)+tf'\left(\frac{\partial u_i}{\partial x}(t, x)\right), \quad x \in \Omega_t.$$

As $g_i(t, x) \neq g_j(t, x)$ for $i \neq j$, it follows $\partial u_i/\partial x(t, x) \neq \partial u_j/\partial x(t, x)$ for $i \neq j$. Using the convexity of $f(p)$, we get the inequality ii). iii) We prove the first inequality. Divide the simple closed curve $\partial \Omega_t$ into two parts joining two cusp points X_1 and X_2 of Ω_t, and write them C_1 and C_2. Here we introduce the family of solution curves of following differential equation

$$\frac{dx}{dr}=g_1(t, x)-g_2(t, x).$$

Then these curves start from C_1 (or from C_2) and end at C_2 (or at C_1 respectively), and the family of these curves covers the domain Ω_t. On each curve it holds

$$\frac{d}{dr}(u_1(t, x)-u_2(t, x))=\left\langle\frac{\partial u_1}{\partial x}-\frac{\partial u_2}{\partial x}, g_1-g_2\right\rangle<0.$$

Since $u_1(t, x)=u_2(t, x)$ on C_1 (or on C_2), we get $u_1(t, x)<u_2(t, x)$ in Ω_t.

We are looking for a continuous solution. The iii) of Lemma 2 means that we can not attain our aim by advancing from the first branch to the second and also from the second to the third. The last choice is to pass from the first branch to the third.

Lemma 3. *Put* $I(t, x)=u_1(t, x)-u_3(t, x)$. *Then* $\Gamma_t=\{x\in\bar{\Omega}_t ; I(t, x)=0\}$ *is a smooth curve in* Ω_t *joining two cusp poihts of* $\partial\Omega_t$.

Proof. In this case we introduce the family of curves defined by

$$\frac{dx}{dr}=g_1(t, x)-g_3(t, x). \tag{10}$$

Then these curves also start from C_1 (or from C_2) and end at C_2 (or C_1) and the family of the curves covers the domain Ω_t. On each curve it holds

$$\frac{d}{dr}I(t, x)=\left\langle\frac{\partial u_1}{\partial x}-\frac{\partial u_3}{\partial x}, g_1-g_3\right\rangle<0.$$

By Lemma 2, we have

$$I(t, x)|_{C_1}=u_1(t, x)-u_2(t, x)|_{C_1}<0,$$

$$I(t, x)|_{C_2}=u_2(t, x)-u_3(t, x)|_{C_2}>0.$$

Therefore on each curve of (10), $I(t, x)=0$ has a unique solution. Obviously $I(t, x)=0$ at the cusps X_1 and X_2, and we have by ii) of Lemma 2

$$\mathrm{grad}_x I(t, x)\neq 0 \quad \text{in } \Omega_t.$$

Hence we see that $\Gamma_t=\{x\in\bar{\Omega}_t ; I(t, x)=0\}$ is the smooth curve joining the points X_1 and X_2 in Ω_t. Q. E. D.

Since we seek for one-valued and continuous solution, we define the solution $u(t, x)$ in Ω_t as follows: Writing $\Omega_{t,\pm}=\{x\in\Omega_t ; u_3(t, x)-u_1(t, x)\gtrless 0\}$, we define

$$u(t, x)=\begin{cases} u_1(t, x) & \text{in } \Omega_{t,+} \\ u_3(t, x) & \text{in } \Omega_{t,-}. \end{cases}$$

As Γ_t is smooth, it can be parametrized as $\Gamma_t=\{x=x(s)\}$. Then

$$\frac{d}{ds}I(t,\ x(s))=\left\langle\frac{\partial u_1}{\partial x}-\frac{\partial u_3}{\partial x},\ \frac{dx}{ds}\right\rangle=0\,.$$

This means that, though the first derivative of the solution $u(t,\ x)$ has jump discontinuity along the curve Γ_t, it is continuous with respect to the tangential direction of Γ_t.

§ 3. Semi-concavity of $u(x,\ t)$.

Let $n(t,\ x)$ be a unit normal of Γ_t advancing from $\Omega_{t,-}$ to $\Omega_{t,+}$, and define at $x\in\Gamma_t$

$$\frac{\partial u}{\partial x}(t,\ x\pm 0)=\lim_{\varepsilon\to+0}\frac{\partial u}{\partial x}(t,\ x\pm\varepsilon n)\,.$$

Any C^2-function satisfies the semi-concavity condition (3). Therefore, for the proof of (3), it suffices to treat the case where $x\in\Gamma_t$ and $y=\varepsilon n$ ($\varepsilon>0$). Then we have

$$\begin{aligned} &u(t,\ x+y)+u(t,\ x-y)-2u(t,\ x)\\ &=\int_0^1\left\langle\frac{\partial u}{\partial x}(t,\ x+sy)-\frac{\partial u}{\partial x}(t,\ x+0),\ y\right\rangle ds\\ &+\int_0^1\left\langle\frac{\partial u}{\partial x}(t,\ x-0)-\frac{\partial u}{\partial x}(t,\ x-sy),\ y\right\rangle ds\\ &+\left\langle\frac{\partial u}{\partial x}(t,\ x+0)-\frac{\partial u}{\partial x}(t,\ x-0),\ y\right\rangle. \end{aligned}$$

The first and second terms are easily estimated by $K|y|^2$. To get the inequality (3), it must be

$$\left\langle\frac{\partial u}{\partial x}(t,\ x+0)-\frac{\partial u}{\partial x}(t,\ x-0),\ n(t,\ x)\right\rangle\leqq 0\,. \tag{11}$$

Contrary, if (11) is true, then we can get (3). Hence (11) is equivalent to the semi-concavity property.

On the other hand, we have by the definition

$$u_3(t,\ x)-u_1(t,\ x)\gtrless 0 \qquad \text{in } \Omega_{t,\pm}$$

which means

$$\frac{d}{ds}\{u_3(t,\ x+sn)-u_1(t,\ x+sn)\}\Big|_{s=0}\geqq 0\,,$$

that is to say,

$$\left\langle\frac{\partial u_3}{\partial x}(t,\ x)-\frac{\partial u_1}{\partial x}(t,\ x),\ n\right\rangle\geqq 0 \qquad \text{on } \Gamma_t\,. \tag{12}$$

From the definition of $u(t,\ x)$ in Ω_t, it holds

$$\frac{\partial u}{\partial x}(t,\ x+0)=\frac{\partial u_1}{\partial x}(t,\ x) \quad \text{and} \quad \frac{\partial u}{\partial x}(t,\ x-0)=\frac{\partial u_3}{\partial x}(t,\ x)\,.$$

Substituting these relations into (12), we get (11), i.e., $u(t, x)$ is semi-concave. Summing up the above results, we have the following

Theorem 1. *After the time t^0 where the Jacobian of the mapping H_t vanishes at first, the solution takes many values. But we can uniquely pick up one value from them so that the solution becomes one-valued and continuous. Then the condition of semi-concavity is automatically satisfied.*

Remark. Putting $v=\partial u/\partial x$ in (11), we get the condition on the jump discontinuity of $v(t, x)$:

$$\langle v(t, x+0)-v(t, x-0), n\rangle \leqq 0 \qquad \text{on } \Gamma_t .$$

This is the entropy condition for the system of conservation laws (4) given in Remark in § 1.

§ 4. Collision of singularities.

In this section we consider the collision of two singularities $\Gamma_{1,t}$ and $\Gamma_{2,t}$ constructed in § 2, assuming the hypotheses (A.1) and (A.2). Here we use the notations $\Sigma_{i,t}$, $\Omega_{i,t}$, $D_{i,t}$, $\cdots$, for $\Gamma_{i,t}$ $(i=1, 2)$ which correspond to Σ_t, Ω_t, D_t, $\cdots$, for Γ_t introduced in § 2. We see that there exist three kinds of collision as described in Figure 2.

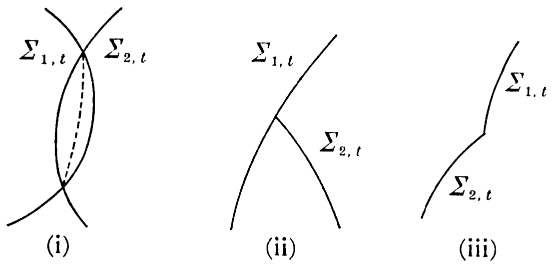

Figure 2. Collision of singularities.

Case (i). Consider the case where $\Gamma_{1,t}$ and $\Gamma_{2,t}$ collide as (i) of Figure 2. Then the solution becomes two-valued on a domain bounded by $\Gamma_{1,t}$ and $\Gamma_{2,t}$. By the almost same discussions as in § 2, we can uniquely pick up one from two values so that the solution is one-valued and continuous. Then we can prove that the solution is semi-concave. In this case the new singularity appears as a smooth curve joining two points where $\Gamma_{1,t}$ and $\Gamma_{2,t}$ intersect each other. It is described as a dotted curve in (i) of Figure 2.

Case (ii). Consider the collision (ii) of Figure 2. We put $\Sigma^e_{i,t}=\{Y_{i,1}, Y_{i,2}\}$ and

$$\Lambda_{i,t}=\{y\,;\, y\in H_t^{-1}(\Omega_{i,t})-D_{i,t} \text{ and } H_t(y)\in\Gamma_{i,t}\} \qquad (i=1, 2),$$

then $\Lambda_{i,t}$ is a smooth simple closed curve which is tangent to $\Sigma_{i,t}$ at $y=Y_{i,1}$ and $Y_{i,2}$ $(i=1, 2)$. When the end point of $\Gamma_{2,t}$ is on $\Gamma_{1,t}$, $\Lambda_{2,t}$ is tangent to $\Lambda_{1,t}$ at the point $y=A$ where $A=Y_{2,1}$ or $Y_{2,2}$. See Figure 3.

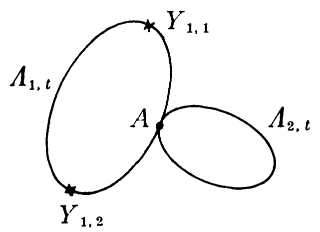

Figure 3. Relation between $\Lambda_{1,t}$ and $\Lambda_{2,t}$.

As $v(t, y)$ restricted on $\Lambda_{1,t}$ does not take an extremum at $y=A$, we get $\partial v/\partial y(t, y)\neq 0$ at $y=A$. i.e., the curve $C_A=\{y\in R^2; v(t, y)=v|_A\}$ is smooth in a neighborhood of $y=A$, and it intersects $\Lambda_{1,t}$ at $y=A$ transversally. On the other hand, as $v(t, y)$ restricted on $\Lambda_{2,t}$ takes extremum at $y=A$, the curve C_A is tangent to $\Lambda_{2,t}$ at $y=A$. This is in contradiction with the above. Hence this case (ii) does not happen.

Case (iii). When $\Gamma_{1,t}$ and $\Gamma_{2,t}$ meet first at a time $t=t^0$ as (iii) of Figure 2, $\Sigma_{1,t^0}\cup\Sigma_{2,t^0}$ is drawn as (i) of Figure 4. But, as the interior domain of the curve $\Sigma_t=\{y\in R^2; 1+t\lambda_1(y)=0\}$ is monotonely increasing, $\Sigma_{1,t}\cup\Sigma_{2,t}$ is described as (ii) of Figure 4 for $t>t^0$. When it satisfies the conditions (A.1) and (A.2), we can construct the singularity of solution by the just same way as in § 2.

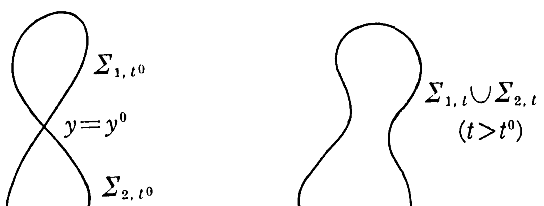

Figure 4. Changement of $\Sigma_{1,t}\cup\Sigma_{2,t}$ with respect to the time.

Remark on Figure 4. Assume that Σ_{1,t^0} and Σ_{2,t^0} meet at $y=y^0=(a, b)$ and that the singularities of $\lambda_1(y)$ are non-degenerate. As $\lambda_1(y)$ does not take minimum and maximum at $y=y^0$, we can suppose by Morse's lemma

$$\lambda_1(y)=\lambda_1(y^0)+(y_1-a)^2-(y_2-b)^2, \qquad 1+t^0\lambda_1(y^0)=0.$$

Therefore Σ_{i,t^0} $(i=1, 2)$ have the singularities at $y=y^0$. But, for $t>t^0$, the curve $\{y\in R^2; 1+t\lambda_1(y)=0\}$ is smooth in a neighborhood of $y=y^0$.

Summing up the above results, we get

Theorem 2. *Assume that the assumptions* (A.1) *and* (A.2) *are conserved. Then, even if two singularities collide each other, we can uniquely pick up one reasonable value from two values of solution so that the solution becomes one-valued and continuous. In this case also the condition of semi-concavity is naturally satisfied.*

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