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A weighted norm inequality for potentials of order (m, k)

Dedicated to Professor Yukio Kusunoki on the occasion of his 60th birthday

By

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§1. Introduction.

Let R^n be the n-dimensional Euclidean space, and for each point $x=(x_1, \cdots, x_n)$ we write $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$. If $\alpha=(\alpha_1, \cdots, \alpha_n)$ is an n-tuple of nonnegative integers α_j, we call α a multi-index and denote by x^α the monomial $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, which has degree $|\alpha|=\sum_{j=1}^n \alpha_j$. Similarly, if $D_j=\partial/\partial x_j$ for $1\leqq j\leqq n$, then

$$D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$$

denotes a differential operator of order $|\alpha|$. We also denote $\alpha!=\alpha_1!\cdots\alpha_n!$. Throughout this paper, let $1<p<\infty$, $(1/p)+(1/p')=1$ and $[r]$ denotes the integral part of a real number r. As usual we denote by L^p the class of all measurable functions for which

$$\|f\|_p=\left(\int|f(x)|^p dx\right)^{1/p}<\infty .$$

Moreover, for a nonnegative number a, $\|\cdot\|_{p,a}$ denotes the following norm:

$$\|f\|_{p,a}=\left(\int|f(x)|^p(1+|x|)^{-a}dx\right)^{1/p}.$$

Weighted norm inequalities for various integral operators on R^n have been investigated by several authors [1], [2], [4], [5]. In this paper we are concerned with a weighted norm inequality for the operators of potential type. Let m be a positive number, and let $K_m(x)$ be a homogeneous function of degree $m-n$ which is infinitely differentiable in $R^n-\{0\}$. For an integer $k<m$, we put

$$K_{m,k}(x, y)=\begin{cases} K_m(x-y)-\sum_{|\alpha|\leqq k}(x^\alpha/\alpha!)D^\alpha K_m(-y), & 0\leqq k<m, \\ K_m(x-y), & k\leqq -1. \end{cases}$$

For a locally integrable function f, $K_{m,k}^f(x)$ is defined by

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$$K_{m,k}^{f}(x)=\int K_{m,k}(x, y)f(y)dy$$

if it exists. We call $K_{m,k}^{f}$ a potential of order (m, k) of f. The potentials of order (m, k) play an important role in the study of the space of Beppo Levi type [3]. In the paper [3], we proved the integral estimate for $K_{m,k}^{f}$.

Theorem A. *Let* $m-(n/p)\neq 0, 1, \cdots, m-1$ *and* $k=[m-(n/p)]$. *Then*

$$\left(\int|x|^{-q(m-(n/p))-n}|K_{m,k}^{f}(x)|^{q}dx\right)^{1/q}\leqq C\|f\|_{p}$$

for each q *satisfying* $p\leqq q<\infty$ *in case of* $m-(n/p)>0$ *and* $p\leqq q\leqq p_{m}$ *in case of* $m-(n/p)<0$, *where* $(1/p_{m})=(1/p)-(m/n)$.

The purpose of this paper is to establish the following weighted norm inequality for $K_{m,k}^{f}$.

Theorem B. *Let* $m-(n/p)\neq 0, 1, \cdots, m-1$ *and* $k=[m-(n/p)]$. *Then*

$$\left(\int(1+|x|)^{-mp-a}|K_{m,k}^{f}(x)|^{p}dx\right)^{1/p}\leqq C\|f\|_{p,a}$$

for each a *satisfying* $0\leqq(a/p)<k+1-m+(n/p)$ *in case of* $m-(n/p)>0$ *and* $0\leqq(a/p)<(n/p)-m$ *in case of* $m-(n/p)<0$.

The proof of Theorem B is given in Section 3. Section 2 is devoted to several lemmas which we need to prove Theorem B.

Throughout this paper, we use the symbol C for generic positive constant whose value may be different at each occurrence, even on the same line.

§ 2. Lemmas.

We put $l_{x}=\{tx ; 0\leqq t\leqq 1\}$ and denote by $d(y, l_{x})$ the distance between y and l_{x}.

We begin with

Lemma 2.1. *If* $k\geqq 0$, *then for* $d(y, l_{x})\geqq|x|/2$ *we have*

$$|K_{m,k}(x, y)|\leqq C|x|^{k+1}|y|^{m-k-1-n}.$$

Proof. By Taylor's formula, for $y\notin l_{x}$ we have

$$K_{m,k}(x, y)=(k+1)\sum_{|\alpha|=k+1}\int_{0}^{|x|}((|x|-t)^{k}/\alpha !)(x')^{\alpha}D^{\alpha}K_{m}(tx'-y)dt,$$

where $x'=x/|x|$. From the homogeneity of K_{m} it follows that

$$|K_{m,k}(x, y)|\leqq C\int_{0}^{|x|}(|x|-t)^{k}|tx'-y|^{m-n-k-1}dt,$$

for $y \notin l_x$. Since $d(y, l_x) \geqq |x|/2$ implies $|y|/3 \leqq |tx' - y| \leqq 3|y|$ for $0 \leqq t \leqq |x|$, we obtain

$$|K_{m,k}(x, y)| \leqq C\int_0^{|x|} (|x| - t)^k |y|^{m-n-k-1} dt$$
$$= C|x|^{k+1}|y|^{m-n-k-1},$$

for $d(y, l_x) \geqq |x|/2$ and so the lemma is proved.

To each function $f \in L^p$ the maximal function $M(f)$ is defined by

$$M(f)(x) = \sup_{r>0} (m(B(x, r)))^{-1} \int_{B(x,r)} |f(y)| dy,$$

where $B(x, r)$ is the open ball of radius r, centered at x, and $m(B(x, r))$ denotes its Lebesgue measure. By the Hardy-Littlewood theorem the following inequality holds: For any $f \in L^p$

$$\|M(f)\|_p \leqq C\|f\|_p. \tag{2.1}$$

As consequences of the above maximal inequality we establish two lemmas.

Lemma 2.2. *If $m - (n/p) > 0$, then*

$$\left(\int |x|^{-mp} \left| \int_{d(y,l_x)<|x|/2} |x-y|^{m-n} f(y) dy \right|^p dx\right)^{1/p} \leqq C\|f\|_p.$$

Proof. From $m - (n/p) > 0$, we can choose a and q such that $a > 0$, $1 < q < p$ and $m - (n/q) - (a/p) > 0$. We note that $d(y, l_x) < |x|/2$ implies $|x-y| < (3/2)|x|$. Using Hölder's inequality we have

$$I = \int |x|^{-mp} \left| \int_{d(y,l_x)<|x|/2} |x-y|^{m-n} f(y) dy \right|^p dx$$
$$= \int |x|^{-mp+a} \left| \int_{d(y,l_x)<|x|/2} |x|^{-(a/p)} |x-y|^{m-n} f(y) dy \right|^p dx$$
$$\leqq C\int |x|^{-mp+a} \left(\int_{|x-y|<(3/2)|x|} |x-y|^{m-(a/p)-n} |f(y)| dy\right)^p dx$$
$$\leqq C\int |x|^{-mp+a} \left(\int_{|x-y|<(3/2)|x|} |x-y|^{q'(m-(a/p)-n)} dy\right)^{p/q'}$$
$$\times \left(\int_{|x-y|<(3/2)|x|} |f(y)|^q dy\right)^{p/q} dx,$$

where $(1/q) + (1/q') = 1$. From $q'(m - (a/p) - n) > -n$, it follows that

$$I \leqq C\int |x|^{-mp+a+p(m-(a/p)-n)+(p/q')n} \left(\int_{|x-y|<(3/2)|x|} |f(y)|^q dy\right)^{p/q} dx$$
$$= C\int \left(|x|^{-n} \int_{|x-y|<(3/2)|x|} |f(y)|^q dy\right)^{p/q} dx$$
$$\leqq C\int (M(|f|^q)(x))^{p/q} dx.$$

Hence from (2.1) we obtain

$$I \leqq C\int(|f(x)|^{q})^{p/q}dx = C\int|f(x)|^{p}dx\,.$$

This concludes the proof of Lemma 2.2.

Lemma 2.3 can be proved in the same way as the proof of Lemma 2.2.

Lamma 2.3. *If* $m-(n/p)>0$, *then*

$$\left(\int|x|^{-mp}\left|\int_{|y|<|x|}|y|^{m-n}f(y)dy\right|^{p}dx\right)^{1/p} \leqq C\|f\|_{p}\,.$$

The following lemma is due to G. O. Okikiolu [6], which is useful for estimates of integral operators.

Lemma 2.4. *Let* (X, m_X) *and* (Y, m_Y) *be measure spaces, let* p, q, μ_1, μ_2 *be positive numbers such that*

$$1<p\leqq q,\ (\mu_1/q)+(\mu_2/p')=1\,,$$

and let $K(x, y)$ *be a measurable function on* $X\times Y$. *Suppose that there are positive measurable functions* ϕ_1 *on* X, ϕ_2 *on* Y *and positive constants* M_1, M_2 *such that*

$$\int_Y \phi_2(y)^{p'}|K(x, y)|^{\mu_2}dm_Y(y) \leqq M_1^{p'}\phi_1(x)^{p'}, \tag{2.2}$$

$$\int_X \phi_1(x)^{q}|K(x, y)|^{\mu_1}dm_X(x) \leqq M_2^{q}\phi_2(y)^{q}. \tag{2.3}$$

If the operator K *is defined by*

$$Kf(x)=\int_Y K(x, y)f(y)dy\,,$$

then

$$\|Kf\|_q \leqq M_1M_2\|f\|_p\,.$$

As applications of Lemma 2.4 we shall show two lemmas.

Lemma 2.5. *If* $m-(n/p)>0$, $k=[m-(n/p)]$ *and* $0\leqq a/p<k+1-m+(n/p)$, *then*

$$\left(\int(1+|x|)^{-mp-a}|x|^{(k+1)p}\left|\int_{|y|\geqq|x|}|y|^{m-k-1-n}f(y)dy\right|^{p}dx\right)^{1/p} \leqq C\|f\|_{p,a}\,.$$

Proof. It suffices to show

$$\left(\int(1+|x|)^{-mp-a}|x|^{(k+1)p}\left|\int_{|y|\geqq|x|}|y|^{m-k-1-n}(1+|y|)^{a/p}g(y)dy\right|^{p}dx\right)^{1/p} \leqq C\|g\|_{p}\,.$$

In order to apply Lemma 2.4, we put

$$K(x,\ y)=\begin{cases}(1+|x|)^{-m-(a/p)}|x|^{k+1}|y|^{m-k-1-n}(1+|y|)^{a/p}, & |y|\geqq|x|,\\ 0, & |y|<|x|.\end{cases}$$

For $\phi_1(x)=\phi_2(x)=(1+|x|)^{-(n/pp')}$, $p=q$ and $\mu_1=\mu_2=1$, we shall show (2.2) and (2.3). From $(a/p)-(n/p)<0$ and $m-k-1-(n/p)+(a/p)<0$, it follows that

$$\begin{aligned}I_1&=\int\phi_2(y)^{p'}K(x,\ y)dy\\ &=(1+|x|)^{-m-(a/p)}|x|^{k+1}\int_{|y|\geqq|x|}|y|^{m-k-1-n}(1+|y|)^{(a/p)-(n/p)}dy\\ &\leqq(1+|x|)^{-m-(a/p)}|x|^{k+1}\int_{|y|\geqq|x|}|y|^{m-k-1-n+(a/p)-(n/p)}dy\\ &=C(1+|x|)^{-m-(a/p)}|x|^{m-(n/p)+(a/p)}.\end{aligned}$$

Because of $m-(n/p)+(a/p)>0$, we have

$$I_1\leqq C(1+|x|)^{-(n/p)}=C\phi_1(x)^{p'}.$$

Thus we obtain (2.2). Next, in order to show (2.3), we put

$$\begin{aligned}I_2&=\int\phi_1(x)^pK(x,\ y)dx\\ &=(1+|y|)^{a/p}|y|^{m-k-1-n}\int_{|x|\leqq|y|}(1+|x|)^{-m-(a/p)-(n/p')}|x|^{k+1}dx.\end{aligned}$$

From $-m-(a/p)-(n/p')+k+1>-n$, it follows that

$$\int_{|x|\leqq|y|}(1+|x|)^{-m-(a/p)-(n/p')}|x|^{k+1}dx\leqq C(1+|y|)^{-m-(a/p)-(n/p')}|y|^{k+n+1}.$$

Therefore

$$\begin{aligned}I_2&\leqq C(1+|y|)^{a/p}|y|^{m-k-1-n}(1+|y|)^{-m-(a/p)-(n/p')}|y|^{k+n+1}\\ &=C(|y|/(1+|y|))^m(1+|y|)^{-(n/p')}\\ &\leqq C(1+|y|)^{-(n/p')}\\ &=C\phi_2(y)^p.\end{aligned}$$

Thus we obtain (2.3), and so the lemma is proved.

Lemma 2.6. *If* $m-(n/p)<0$ *and* $0\leqq a/p<(n/p)-m$, *then*

$$\left(\int(1+|x|)^{-mp-a}\left|\int_{|y|\geqq|x|}|y|^{m-n}f(y)dy\right|^pdx\right)^{1/p}\leqq C\|f\|_{p,a}.$$

Proof. We shall show

$$\left(\int(1+|x|)^{-mp-a}\left|\int_{|y|\geqq|x|}|y|^{m-n}(1+|y|)^{a/p}g(y)dy\right|^pdx\right)^{1/p}\leqq C\|g\|_p.$$

We apply Lemma 2.4 again. To do so we put

$$K(x, y)=\begin{cases}(1+|x|)^{-m-(a/p)}|y|^{m-n}(1+|y|)^{a/p}, & |y|\geqq|x|, \\ 0, & |y|<|x|,\end{cases}$$

For $\phi_1(x)=\phi_2(x)=|x|^{-(n/pp')}$, $p=q$ and $\mu_1=\mu_2=1$, we shall verify (2.2) and (2.3). We set

$$\begin{aligned}J_1&=\int\phi_2(y)^{p'}K(x, y)dy \\ &=(1+|x|)^{-m-(a/p)}\int_{|y|\geqq|x|}|y|^{m-(n/p)-n}(1+|y|)^{a/p}dy.\end{aligned}$$

From $m-(n/p)+(a/p)<0$, it follows that

$$\int_{|y|\geqq|x|}|y|^{m-(n/p)-n}(1+|y|)^{a/p}dy\leqq C|x|^{m-(n/p)}(1+|x|)^{a/p}.$$

Therefore

$$\begin{aligned}J_1&\leqq C(1+|x|)^{-m-(a/p)}|x|^{m-(n/p)}(1+|x|)^{a/p} \\ &=C(|x|/(1+|x|))^m|x|^{-(n/p)} \\ &\leqq C|x|^{-(n/p)} \\ &=C\phi_1(x)^{p'}.\end{aligned}$$

Thus we get (2.2). Next, to show (2.3), we put

$$\begin{aligned}J_2&=\int\phi_1(x)^pK(x, y)dx \\ &=|y|^{m-n}(1+|y|)^{a/p}\int_{|x|\leqq|y|}|x|^{-(n/p')}(1+|x|)^{-m-(a/p)}dx.\end{aligned}$$

On account of $-m-(n/p')-(a/p)>-n$, we have

$$\int_{|x|\leqq|y|}|x|^{-(n/p')}(1+|x|)^{-m-(a/p)}dx\leqq C|y|^{n/p}(1+|y|)^{-m-(a/p)}.$$

Hence

$$\begin{aligned}J_2&\leqq C|y|^{m-n}(1+|y|)^{a/p}|y|^{n/p}(1+|y|)^{-m-(a/p)} \\ &=C((|y|/(1+|y|))^m|y|^{-(n/p')} \\ &\leqq C|y|^{-(n/p')} \\ &=C\phi_2(y)^p.\end{aligned}$$

Thus we obtain (2.3), and so the lemma is proved.

The final lemma is due to E. M. Stein and G. Weiss [7].

Lemma 2.7. *If* $m-(n/p)<0$, *then*

$$\left(\int|x|^{-mp}\left|\int|x-y|^{m-n}f(y)dy\right|^p dx\right)^{1/p}\leqq C\|f\|_p.$$

§3. Proof of Theorem B.

First let $m-(n/p)>0, \neq 1, 2, \cdots, m-1$, $k=[m-(n/p)]$ and $0\leqq a/p<k+1-m+(n/p)$. From the definition of $K_{m,k}$ and the homogeneity of K_m we have

$$
\begin{aligned}
I&=\Big(\int(1+|x|)^{-mp-a}|K^f_{m,k}(x)|^p dx\Big)^{1/p}\\
&\leqq C\Big(\int(1+|x|)^{-mp-a}\Big(\int_{d(y,l_x)<|x|/2}|x-y|^{m-n}|f(y)|dy\Big)^p dx\Big)^{1/p}\\
&\quad+C\sum_{|\alpha|\leqq k}\Big(\int(1+|x|)^{-mp-a}\Big(\int_{d(y,l_x)<|x|/2}|x|^{|\alpha|}|y|^{m-|\alpha|-n}|f(y)|dy\Big)^p dx\Big)^{1/p}\\
&\quad+\Big(\int(1+|x|)^{-mp-a}\Big(\int_{d(y,l_x)\geqq|x|/2}|K_{m,k}(x, y)f(y)|dy\Big)^p dx\Big)^{1/p}\\
&=CI_1+CI_2+I_3.
\end{aligned}
$$

For I_1, we have

$$
I_1\leqq\Big(\int|x|^{-mp}\Big(\int_{d(y,l_x)<|x|/2}|x-y|^{m-n}\Big(\frac{1+|y|}{1+|x|}\Big)^{a/p}\times|f(y)|(1+|y|)^{-(a/p)}dy\Big)^p dx\Big)^{1/p}.
$$

Since $d(y, l_x)<|x|/2$ implies $(1+|y|)/(1+|x|)\leqq C$, we get

$$
I_1\leqq C\Big(\int|x|^{-mp}\Big(\int_{d(y,l_x)<|x|/2}|x-y|^{m-n}|f(y)|(1+|y|)^{-(a/p)}dy\Big)^p dx\Big)^{1/p}.
$$

On account of Lemma 2.2, we have

$$
I_1\leqq C\|f\|_{p,a}.
$$

For I_2, we have

$$
I_2\leqq\sum_{|\alpha|\leqq k}\Big(\int|x|^{-(m-|\alpha|)p}\Big(\int_{d(y,l_x)<|x|/2}|y|^{m-|\alpha|-n}\Big(\frac{1+|y|}{1+|x|}\Big)^{a/p}\times|f(y)|(1+|y|)^{-(a/p)}dy\Big)^p dx\Big)^{1/p}.
$$

Since $d(y, l_x)<|x|/2$ implies $|y|<(3/2)|x|$, we get

$$
I_2\leqq C\sum_{|\alpha|\leqq k}\Big(\int|x|^{-(m-|\alpha|)p}\Big(\int_{|y|<(3/2)|x|}|y|^{m-|\alpha|-n}|f(y)|(1+|y|)^{-(a/p)}dy\Big)^p dx\Big)^{1/p}.
$$

From $m-(n/p)>0, \neq 1, 2, \cdots, m-1$ it follows that $m-|\alpha|-(n/p)>0$ for $|\alpha|\leqq k$. Hence from Lemma 2.3 we see that

$$
I_2\leqq C\sum_{|\alpha|\leqq k}\|f\|_{p,a}=C\|f\|_{p,a}.
$$

To estimate I_3, we apply Lemma 2.1. Then we get

$$I_3 \leqq C\left(\int (1+|x|)^{-mp-a}|x|^{(k+1)p}\left(\int_{d(y,l_x)\geqq |x|/2}|y|^{m-k-n-1}|f(y)|dy\right)^p dx\right)^{1/p}.$$

Since $d(y, l_x)\geqq |x|/2$ implies $|y|\geqq |x|/2$, by Lemma 2.5 we have

$$I_3 \leqq C\|f\|_{p,a}.$$

Thus we obtain

$$I \leqq C\|f\|_{p,a}.$$

Next, let $m-(n/p)<0$, $k=[m-(n/p)]$ and $0\leqq a/p<(n/p)-m$. From the definition of $K^f_{m,k}$ and the homogeneity of K_m, it follows that

$$\begin{aligned} J &= \left(\int (1+|x|)^{-mp-a}|K^f_{m,k}(x)|^p dx\right)^{1/p} \\ &= C\left(\int (1+|x|^{-mp-a}\left(\int_{|y|<2|x|}|x-y|^{m-n}|f(y)|dy\right)^p dx\right)^{1/p} \\ &\quad + C\left(\int (1+|x|)^{-mp-a}\left(\int_{|y|\geqq 2|x|}|x-y|^{m-n}|f(y)|dy\right)^p dx\right)^{1/p} \\ &= CJ_1+CJ_2. \end{aligned}$$

For J_1, we see

$$J_1 \leqq \left(\int |x|^{-mp}\left(\int_{|y|<2|x|}|x-y|^{m-n}\left(\frac{1+|y|}{1+|x|}\right)^{a/p}|f(y)|(1+|y|)^{-(a/p)}dy\right)^p dx\right)^{1/p}.$$

It follows from $|y|<2|x|$ that $1+|y|\leqq C(1+|x|)$. Hence by Lemma 2.7,

$$J_1 \leqq C\|f\|_{p,a}.$$

Since $|y|\geqq 2|x|$ implies $|x-y|\geqq |y|/2$, we have

$$J_2 \leqq C\left(\int (1+|x|)^{-mp-a}\left(\int_{|y|\geqq 2|x|}|y|^{m-n}|f(y)|dy\right)^p dx\right)^{1/p}.$$

Hence by Lemma 2.6, we obtain

$$J_2 \leqq C\|f\|_{p,a}.$$

Thus we have

$$J \leqq C\|f\|_{p,a}.$$

We complete the proof of Theorem B.

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