J. Math. Kyoto Univ. (JMKYAZ)
26-3 (1986) 513–545

On the conjugacy of nilpotent elements in the classical Lie algebras in relation to their representations

By

Tôru UMEDA

Introduction.

Let G be an algebraic group over an algebraically closed field K, and $\mathfrak{g}$ the Lie algebra of G. The subject of the present paper is on the conjugacy of nilpotent elements in $\mathfrak{g}$ in relation to representations of G. Let us explain our problems. Let (σ, V) be a finite dimensional rational representation of G. We use the same notation for the corresponding representation of $\mathfrak{g}$. For nilpotent elements X, X' in $\mathfrak{g}$, we consider the following two types of conjugacy:

$$(C) \qquad X \sim X' \qquad \text{under} \quad \mathrm{Ad}(G)\,,$$

$$(C_\sigma) \qquad \sigma(X) \sim \sigma(X') \qquad \text{under} \quad GL(V)\,.$$

It is easy to see that (C) implies (C_σ). Then there arise naturally the converse problems:

Problem I $(G; \sigma)$. Does it hold that (C_σ) implies (C)?

And more generally,

Problem $\tilde{\mathrm{I}}$ $(G; \sigma)$. Find all the conjugacy classes in the sense (C) which are glued up in (C_σ).

In the present paper, we are mainly interested in a special case where σ is the adjoint representation σ_1. We call these problems Problem $\mathrm{I}_1(G)$, Problem $\tilde{\mathrm{I}}_1(G)$, for this special case.

For the case where $\mathfrak{g}$ is the simple Lie algebra of exceptional type E_6, E_7, E_8, T. Hirai announced in [3, Th. 6] the affirmative answer to Problem $\mathrm{I}_1(G)$ when the characteristic of K is good for $\mathfrak{g}$. This result is the first motive for our study, and we are concerned with the classical cases.

Let us specify our object and state the main results. In this paper K is always assumed to be of characteristic zero. To Problem $\mathrm{I}_1(G)$ with $G = GL(n, K)$, $Sp(n, K)$, $O(n, K)$, we obtain the following answers both in the affirmative and the negative.

Theorem A (*Corollary to Theorem* 1). *Let G be one of the groups $GL(n, K)$,*

Received April 26, 1985

$Sp(n,K)$, $O(n,K)$. If G is not $O(2^{2^r+1},K)$ $(r=1,2,\cdots)$, then (C_{σ_1}) implies (C). Here σ_1 is the adjoint representation of G.

Theorem B (*Theorem* 4 (*ii*)). *For $r\geqq 1$ except for $r=2,4$, there exist nilpotent elements X, $X'\in\mathfrak{o}(2^{2^r+1},K)$ such that $X\not\sim X'$ under $O(2^{2^r+1},K)$ but* $\mathrm{ad}(X)\sim\mathrm{ad}(X')$ *under $GL(\mathfrak{o}(2^{2^r+1},K))$.*

Thus among the classical groups, a particular series of groups $O(2^{2^r+1},K)$ of type D_{4^r} behaves exceptionally for Problem $\tilde{\mathrm{I}}_1$. Problem I_1 for these groups turns out to be difficult and of much interest. In this paper, we reduced Problem $\tilde{\mathrm{I}}_1(O(n,K))$ to looking for certain polynomials (§4). And by giving explicit examples of such polynomials, we approached Problem $\tilde{\mathrm{I}}_1(O(n,K))$. To establish Theorem B, our examples are sufficient. But they are not complete and fall short of Problem I_1 for the two cases $r=2,4$ excluded in Theorem B.

Now we explain the contents of this paper in more detail. In §1, we develop preliminary reductions for Problem $\mathrm{I}_1(G)$ with $G=GL(n,K)$, $Sp(n,K)$, $O(n,K)$ as follows. In the first place, we settle Problem $\mathrm{I}(G;\sigma_0)$ affirmatively with σ_0 the natural identical representation. Therefore we can replace (C) by its equivalent (C_{σ_0}) in Problem $\mathrm{I}_1(G)$ for these groups. In the second place, using the Jacobson-Morozov theorem, we rewrite Problem $\mathrm{I}_1(G)$ into the following problems of equivalence between representations of $\mathfrak{sl}(2,K)$.

Problem II. Let π_1 and π_2 be finite dimensional representations of $\mathfrak{sl}(2,K)$. Then

(i) Does $\otimes^2\pi_1\sim\otimes^2\pi_2$ imply $\pi_1\sim\pi_2$?

(ii) Does $S^2\pi_1\sim S^2\pi_2$ imply $\pi_1\sim\pi_2$?

(iii) Does $\Lambda^2\pi_1\sim\Lambda^2\pi_2$ imply $\pi_1\sim\pi_2$?

Here (i), (ii), (iii) correspond to $G=GL(n,K)$, $Sp(n,K)$, $O(n,K)$, respectively. (To be more precise, certain restrictions should be put on (ii) and (iii) for the exact correspondence between Problems I_1 and II.) Thus our major concern is transferred to the problems on representations of $\mathfrak{sl}(2,K)$. Then we can utilize their characters. In §2, along this line we prove Theorem 1, which asserts the affirmative answers to Problem II, hence to Problem $\mathrm{I}_1(G)$ with $G=GL(n,K)$, $Sp(n,K)$, $O(n,K)$.

In §3, we study some generalizations of Problem II, which treat representations on tensor spaces of higher degrees instead of 2nd degrees. Let Γ be a group, and π a representation of Γ. We can decompose the N-th tensor representation $\otimes^N\pi$ under the action of the symmetric group $\mathfrak{S}_N$, and denote by $S_\lambda\pi$ its component corresponding to the irreducible representation of $\mathfrak{S}_N$ labelled by a partition λ of N. For representations π_1 and π_2 of Γ, we consider

Problem III $(\Gamma;\lambda)$. Does it hold that $S_\lambda\pi_1\sim S_\lambda\pi_2$ implies $\pi_1\sim\pi_2$?

The main result in §3 is the following theorem, which gives some affirmative answers to Problem III$(\Gamma;\lambda)$.

Theorem 2. *Let Γ be a connected algebraic group, and π_i $(i=1, 2)$ be completely reducible rational representations of Γ, of finite dimensions. Assume that a partition λ of N is a hook, and*

($\sharp$) *the dimensions of π_i $(i=1, 2)$ do not divide $(N-1)!\, N^k$ for any k.*

Then $S_\lambda\pi_1 \sim S_\lambda\pi_2$ implies $\pi_1 \sim \pi_2$. Moreover, in case $\lambda=(N)$, i.e., $S_\lambda=S^N$ the N-th symmetric tensor product, the conclusion holds without the assumption ($\sharp$).

From Theorem 2, we can also deduce Theorem 3 asserting affirmative results for Problem $\mathrm{I}(G; \sigma)$.

In §4, we return to Problem II(iii) and Problem $\mathrm{I}_1(O(n, K))$, and investigate the counterexamples to these problems. We reduce the problem of such counterexamples to that of finding certain polynomials with integral coefficients, which we call distinguished (Theorem 5). Briefly speaking, these polynomials $P(T)$ come from the differences of the characters of representations, and are characterized by two properties: (a) $P(T)$ divides $P(T^2)$; (b) certain unimodality conditions (see Definition 4.2 for the precise meaning). Theorem B follows from the following explicit examples of distinguished polynomials: $P(T)=(T-1)^{2k}$ $(k\geqq 1)$, and $P(T)=(T^2-1)^2(T-1)^{12+2k}$ $(k\geqq 0)$.

In §5, we proceed to discuss distinguished polynomials. By virtue of the property (a), we can obtain a scope for their explicit form, and some fundamental properties as well. Here our problems come across an elementary theory of cyclotomic fields. To exhaust the distinguished polynomials is, however, rather complicated because of the property (b), and we have not yet attained to a general rule.

In §6, we prove the following

Theorem 6. *Let Γ be a group, and π a finite dimensional representation of Γ. Put*

$$\pi_1 = \bigoplus_{i\,:\,\text{even}} \wedge^i \pi\,, \quad \text{and} \quad \pi_2 = \bigoplus_{i\,:\,\text{odd}} \wedge^i \pi\,.$$

Assume that π_1 and π_2 are completely reducible. Then $\wedge^2\pi_1 \sim \wedge^2\pi_2$.

Theorem 6 provides us with counterexamples to Problem III $(\Gamma; (1^2))$ for Γ of fairly broad classes of groups. Especially, applying Theorem 6 to $\Gamma=SL(2, K)$ and π the $2l$-dimensional irreducible representation, we see that the polynomial $P(T)=\prod_{i=1}^{l}(T^{2i-1}-1)^2$ is distinguished. For this fact, we give another calculative proof in connection with a q-analogue of the binomial theorem and the Gaussian polynomials.

In §7, we study more closely a special case of Theorem 2, and obtain the precise result as follows.

Theorem 7. *For $\Gamma=SL(2, K)$, Problem* $\mathrm{III}(\Gamma; (2, 1))$ *is solved affirmatively without the assumption* ($\sharp$).

To prove this, we make use of some facts on cyclotomic polynomials over

$\boldsymbol{Z}[\omega]$, where ω is a cubic root of unity.

Our method is elementary, and is naturally connected with other branches of mathematics. For example, with cyclotomic field (§§5 & 7), and with a q-analogue of the binomial theorem (§6). We think this worth observing.

The author wishes to express his gratitude to Professor T. Hirai for valuable advices and comments in this presentation. The author also expresses his thanks to Dr. S. Matsumoto and Mr. H. Yamashita for fruitful discussions.

Notations.

We denote by $\boldsymbol{Q}$ and $\boldsymbol{C}$ the field of rational numbers and of complex numbers, respectively. Moreover $\boldsymbol{Z}$ denotes the ring of integers, and $\boldsymbol{Z}_+$ the set of non-negative integers. Throughout this paper, K denotes an algebraically closed field of characteristic zero, and a vector space is assumed to be finite dimensional over K. Also a representation is assumed to be finite dimensional.

Let R be a ring with unit, and T an indeterminate. We denote by $R^\times$ the group of invertible elements in R. And $R[T]$, $R[T, T^{-1}]$, or $R[[T]]$ denotes the ring of polynomials, of Laurent polynomials, or of formal power series with coefficients in R, respectively.

By $GL(n, K)$, $SL(n, K)$, $Sp(n, K)$, and $O(n, K)$, we denote respectively the general linear group, the special linear group, the symplectic group, and the orthogonal group of size $n\times n$. Their Lie algebras are denoted by $\mathfrak{gl}(n, K)$, $\mathfrak{sl}(n, K)$, $\mathfrak{sp}(n, K)$, and $\mathfrak{o}(n, K)$, respectively. The total matric algebra of rank n is denoted by $M_n(K)$.

For a set M, the cardinality of M is denoted by $\sharp M$.

§1. Problems and preliminary observation.

1.1. Throughout this paper K denotes an algebraically closed field of characteristic zero. Let G be an algebraic group over K, and $\mathfrak{g}$ its Lie algebra. Let (σ, V) be a finite dimensional rational representation of G. We use the same notation σ for the corresponding representation of $\mathfrak{g}$. For nilpotent elements X, X' in $\mathfrak{g}$, we consider the following two types fo conjugacy:

$$(C) \qquad X \sim X' \quad \text{under} \quad \mathrm{Ad}(G)\,,$$

$$(C_\sigma) \quad \sigma(X) \sim \sigma(X') \quad \text{under} \quad GL(V)\,.$$

It is easy to see that (C) implies (C_σ). So we ask for the converse implication, Problem I for $(G; \sigma)$:

Problem I $(G; \sigma)$. Does it hold that (C_σ) implies (C)?

In this paper we study this problem mainly for the classical groups and their adjoint representations. Especially by Problem $\mathrm{I}_1(G)$ we denote Problem $\mathrm{I}(G; \sigma)$ with σ the adjoint representation.

1.2. Now we take the groups $G=GL(n, K)$, $Sp(n, K)$, $O(n, K)$, and remark some preliminary results for $\sigma=\sigma_0$ the natural identical representation of G. In general, the following fact is known.

Proposition 1.1 (*due to Iwahori*). *Let R be a K-algebra of finite degree, and τ an involutive anti-automorphism of R. Put $G=R^{\times}$, $G_{\tau}=\{a\in G;\ a^{\tau}=a^{-1}\}$, and $\mathfrak{g}_{\tau}=\{a\in R;\ a^{\tau}=-a\}$. Then for $x, x'\in G_{\tau}$ (resp. $x, x'\in\mathfrak{g}_{\tau}$), x and x' are conjugate under G_{τ} if and only if they are conjugate under G.*

We apply this proposition to the case with $R=M_n(K)$ and $a^{\tau}=S{}^t aS^{-1}$, where $S\in GL(n, K)$ is an alternating or a symmetric matrix, and ${}^t a$ denotes the transposed of a. Then for $G=Sp(n. K)$ or $O(n, K)$, the conjugacies (C) and (C_{σ_0}) are mutually equivalent. In other words, Problem $\mathrm{I}(G;\sigma_0)$ is affirmative for these groups. From this, we see that Problem $\mathrm{I}(G;\sigma)$ for $G=GL(n, K)$, $Sp(n, K)$, $O(n, K)$ is reduced to the comparison of the conjugacies (C_{σ_0}) and (C_{σ}).

Remark 1. (1) For a simple algebraic group G of type A_l (resp. B_l, C_l), the adjoint group is identical to that of $GL(l+1, K)$ (resp. $O(2l+1, K)$, $Sp(2l, K)$). So, for G of these types, the study of Problem $\mathrm{I}_1(G)$ is equivalent to that for the latter groups.

(2) On the other hand, for G of type D_l, the circumstances are slightly complicated. In fact, there are nilpotent classes in $\mathfrak{o}(2l, K)$ under $O(2l, K)$, each of which splits into two classes under $SO(2l, K)$. More precisely we have the following.

For $X\in\mathfrak{gl}(n, K)$, we call X is of Jordan type $(p_1,\cdots,p_s)$ if X is conjugate to $J(p_1)\oplus\cdots\oplus J(p_s)$ under $GL(n, K)$, where $J(p)$ is the Jordan matrix of size $p\times p$:

$$J(p)=\begin{pmatrix} 0 & 1 & & & \text{\Large 0} \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ \text{\Large 0} & & & & 0 \end{pmatrix}.$$

Proposition 1.2 (c.f. [10, pp. 263–264]). *Let $X\in\mathfrak{o}(2l, K)$ be of Jordan type $(p_1,\cdots,p_s)$. Then $O(2l, K)$-class of X splits into two $SO(2l, K)$-classes if and only if p_i's are all even.*

By virtue of this proposition, for the groups of type D_l, it is sufficient to consider $O(2l, K)$ instead of $SO(2l, K)$.

1.3. For convenience of the readers, we give here a proof of Proposition 1.1, from which we see that the proposition is valid for K of any characteristic other than 2.

Proof of Proposition 1.1. Take a $g\in G$ such that $x'=gxg^{-1}$. Operate τ on both sides of this, and one gets $x'=(g^{\tau})^{-1}xg^{\tau}$. Therefore $g^{\tau}g$ commutes with x. If there exists an $h\in G$ such that $g^{\tau}g=h^{\tau}h$ and $hx=xh$, then we obtain $x'=gxg^{-1}=$

$(gh^{-1})x(gh^{-1})^{-1}$ and $gh^{-1}\in G_\tau$, so the assertion. For such an h, we can take a square root of $g^\tau g$ which is written in the form of a polynomial in $g^\tau g$ with coefficients in K. The existence of such a square root is assured by the following

Lemma 1.3. *Let $a\in R$ be invertible, and m a positive integer. Then there exists an m-th root of a which is written in the form of a polynomial in a with coefficients in K.*

Proof. This is known for $R=M_n(K)$ (see Wedderburn [11, 8.05]), and we can easily reduce our case to matrix case by way of the regular representation of R. The following discussion is pararell to the case of $M_n(K)$.

Let $f(T)\in K[T]$ be the monic minimal polynomial of a, and factorize it into

$$f(T)=\prod_i (T-\alpha_i)^{m_i} \qquad (\alpha_i\neq\alpha_j \quad \text{if} \quad i\neq j).$$

Put $f_i(T)=f(T)/(T-\alpha_i)^{m_i}$, and take $g_i(T)\in K[T]$ such that

$$1=\sum_i f_i(T)g_i(T).$$

Put $p_i(T)=f_i(T)g_i(T)$. Then we have $1=\sum_i p_i(T)$; $p_i(a)p_j(a)=0$ for $i\neq j$; $p_i(a)^2=p_i(a)$; and $(a-\alpha_i)^{m_i}p_i(a)=0$. Thus putting $b=\sum_i \alpha_i p_i(a)$ and $c=\sum_i(a-\alpha_i)p_i(a)$, we get $a=c+b$ (Jordan decomposition). Note that b is invertible, and that $b^{-1}=\sum_i \alpha_i^{-1}p_i(a)$ is also a polynomial in a. Now put

$$h=\Big(\sum_i \alpha_i^{1/m}p_i(a)\Big)\cdot\Big(\sum_{j\geqq 0}\binom{1/m}{j}(b^{-1}c)^j\Big).$$

Since $b^{-1}c$ is nilpotent, h is written in a form of polynomial in a. And we have clearly $h^m=a$ by the binomial theorem. This completes the proof of Lemma 1.3, hence of Proposition 1.1.

Remark 2. The denominator of the binomial coefficient $\binom{1/m}{j}$ is divisible only by the primes dividing m. So that Proposition 1.1 is valid for K of characteristic other than 2.

1.4. Taking $G=GL(n, K)$, $Sp(n, K)$, $O(n, K)$, we reformulate Problem $\mathrm{I}_1(G)$ in terms of representations of $\mathfrak{sl}(2, K)$. We apply the Jacobson-Morozov theorem (see Bourbaki [1], Jacobson [4]).

Let $\mathfrak{g}$ be a linear reductive Lie algebra over K acting on the n-dimensional vector space $V=K^n$. For a non-zero nilpotent element $X\in\mathfrak{g}$, there exists an $\mathfrak{sl}_2$-triplet $\mathfrak{s}=\{X, H, Y\}: [H, X]=2X$, $[H, Y]=-2Y$, $[X, Y]=H$. The existence of $\mathfrak{sl}_2$-triplet is assured by the Jacobson-Morozov theorem, and such a triplet is unique up to conjugation leaving X invariant. Thus for any non-zero nilpotent element X, we get an n-dimensional representation π_X of $\mathfrak{sl}(2, K)$ on V through $\mathfrak{s}$. For $X=0$, we define π_X as the trivial representation of $\mathfrak{sl}(2, K)$ on V. Note that π_X is unique up to equivalence, not depending on the choice of $\mathfrak{s}$. On the other hand, for nilpotent elements $X, X'\in\mathfrak{gl}(n, K)$, we see easily that X is conjugate to X' under

$GL(n, K)$ if and only if π_X is equivalent to $\pi_{X'}$.

For example, if X is the Jordan matrix $J(n)$, then π_X is the n-dimensional irreducible representation $\pi^{(n)}$ of $\mathfrak{sl}(2, K)$. In general, for X of Jordan type $(p_1, \cdots, p_s)$, we have $\pi_X \sim \pi^{(p_1)} \oplus \cdots \oplus \pi^{(p_s)}$ (equivalent).

1.5. Let X be a nilpotent element in $\mathfrak{gl}(n, K)$. We give here a criterion, by means of π_X, whether X belongs to $\mathfrak{sp}(n, K)$ or $\mathfrak{o}(n, K)$ up to conjugation under $GL(n, K)$.

Let π be a representation of $\mathfrak{sl}(2, K)$ with irreducible decomposition $\pi \sim \oplus_i m_i \cdot \pi^{(i)}$, where m_i is the multiplicity of $\pi^{(i)}$ in π. We consider the following conditions on multiplicities:

(M1) The multiplicity m_i is even for any odd integer i.

(M2) The multiplicity m_i is even for any even integer i.

Then we have

Lemma 1.4. *Let $X \in \mathfrak{gl}(n, K)$ be nilpotent. Then*

(i) *There exists a $g \in GL(n, K)$ such that* $(\mathrm{Ad}\, g)X \in \mathfrak{sp}(n, K)$ *if and only if* π_X *satisfies* (M1).

(ii) *There exists a $g \in GL(n, K)$ such that* $(\mathrm{Ad}\, g)X \in \mathfrak{o}(n, K)$ *if and only if* π_X *satisfies* $(M2)$.

This is an immediate translation of the theorem on nilpotent elements in $\mathfrak{sp}(n, K)$ and $\mathfrak{o}(n, K)$ (see e.g. Hirai [2, Th. 5.2]).

1.6. As we have transferred the conjugacy between X's to the equivalence between π_X's, it remains for Problem $\mathrm{I}_1(G)$ to find the relation between π_X and $\pi_{\mathrm{ad}\, X}$.

Let $\mathfrak{g} = \mathfrak{gl}(n, K)$, $\mathfrak{sp}(n, K)$, or $\mathfrak{o}(n, K)$, and let σ_0 be the identical representation of $\mathfrak{g}$ on the n-dimensional vector space V. Then we see

Lemma 1.5. (i) *The adjoint representation of* $\mathfrak{gl}(n, K)$ *is equivalent to* $\sigma_0 \otimes \sigma_0^*$. *Here* σ_0^* *is the contragredient representation of* σ_0.

(ii) *The adjoint representation of* $\mathfrak{sp}(n, K)$ *is equivalent to* $\mathrm{S}^2\sigma_0$. *Here* $\mathrm{S}^2\sigma_0$ *is the symmetric tensor product representation of* σ_0 *of degree* 2.

(iii) *The adjoint representation of* $\mathfrak{o}(n, K)$ *is equivalent to* $\wedge^2 \sigma_0$. *Here* $\wedge^2 \sigma_0$ *is the alternating tensor product representation of* σ_0 *of degree* 2.

Proof. The assertion (i) is clear from the fact $\mathrm{End}\, V \cong V \otimes V^*$. For (ii) and (iii), see the exercises 13.8 and 13.18 of Bourbaki [1, Ch. VIII].

From the lemma above, restricting the representation σ_0 to the $\mathfrak{sl}_2$-triplet $\mathfrak{s}$, we get

Lemma 1.6. *For a nilpotent element* $X \in \mathfrak{g}$,

(i) $\pi_{\mathrm{ad}\, X} \sim \otimes^2 \pi_X \quad$ *if* $\quad \mathfrak{g} = \mathfrak{gl}(n, K)$,

(ii) $\pi_{\mathrm{ad}\,X}\sim S^2\pi_X$ *if* $\mathfrak{g}=\mathfrak{sp}(n, K)$,

(iii) $\pi_{\mathrm{ad}\,X}\sim \wedge^2\pi_X$ *if* $\mathfrak{g}=\mathfrak{o}(n, K)$.

Here $\otimes^2\pi_X$ *denotes the tensor product representation of* π_X *of degree* 2.

Proof. Note the fact that $\pi\sim\pi^*$ for a representation of $\mathfrak{sl}(2, K)$. Then the assertions are obvious from Lemma 1.5.

1.7. Now we are led to

Problem II. Let π_1, π_2 be finite dimensional representations of $\mathfrak{sl}(2, K)$. Then

(i) Does $\otimes^2\pi_1\sim\otimes^2\pi_2$ imply $\pi_1\sim\pi_2$?

(ii) Does $S^2\pi_1\sim S^2\pi_2$ imply $\pi_1\sim\pi_2$?

(iii) Does $\wedge^2\pi_1\sim\wedge^2\pi_2$ imply $\pi_1\sim\pi_2$?

Moreover let us call (ii′) (resp. (iii′)) the problem (ii) (resp. (iii)) considered for π_1, π_2 both satisfying (M1) (resp. (M2)).

Our deduction in this section shows that Problem $\mathrm{I}_1(G)$ for $G=GL(n, K)$ (resp. $Sp(n, K)$, or $O(n, K)$) is equivalent to Problem II(i) (resp. II(ii′), or II(iii′)).

§2. Affirmative results.

2.1. In this section we show that Problems II(i) and II(ii) are affirmative. As for Problem II(iii), we prove it affirmative when the dimensions of the representations π_1 and π_2 are not equal to odd times power of 2. Thus Problem $\mathrm{I}_1(G)$ turns out to be affirmative if G is a simple algebraic group of type A_l, B_l, C_l (any l), and if $G=O(2l, K)$ (l not power of 4). More precisely, we get the following.

Theorem 1. *Let* π_1 *and* π_2 *be finite dimensional rational representations of* $SL(2, K)$. *Then*

(i) $\otimes^2\pi_1\sim\otimes^2\pi_2$ *implies* $\pi_1\sim\pi_2$.

(ii) $S^2\pi_1\sim S^2\pi_2$ *implies* $\pi_1\sim\pi_2$.

(iii) $\wedge^2\pi_1\sim\wedge^2\pi_2$ *implies* $\pi_1\sim\pi_2$, *if the dimensions of* π_1 *and* π_2 *are not equal to odd times power of* 2.

Corollary. *Let* G *be one of the groups* $GL(n, K)$, $Sp(n, K)$, $O(n, K)$, *and* $\mathfrak{g}$ *the Lie algebra of* G. *Assume that* G *is not of type* D_l *with* $l=4^r$ $(r=1, 2, \cdots)$. *Then for nilpotent elements* X, X' *in* $\mathfrak{g}$, *the following three are equivalent.*

(C) $X\sim X'$ *under* G,

(C_0) $X\sim X'$ *under* $GL(n, K)$,

(C_1) $\mathrm{ad}(X)\sim\mathrm{ad}(X')$ *under* $GL(\mathfrak{g})$.

This corollary follows from Theorem 1 and the reduction in §1. In fact, we know that the linear representations of $SL(2, K)$ correspond canonically to those of

$\mathfrak{sl}(2, K)$ (see Bourbaki [1, Ch. VIII, 1, n°V]).

Proof of Theorem 1. Recall that every finite dimensional rational representation π of $SL(2, K)$ is completely reducible and is determined by its trace $\mathrm{Tr}\,\pi$ up to equivalence. Moreover, $\mathrm{Tr}\,\pi$ is determined by the values on the Cartan subgroup

$$A = \left\{ h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} ; \ t \in K^\times \right\} .$$

Let u denote the one dimensional representation of A defined by $h(t) \mapsto t$. Then an irreducible rational representation of A is of the form $u^j : h(t) \mapsto t^j$ for some $j \in \boldsymbol{Z}$. Therefore $\pi|_A$ is written in the form $\bigoplus_j a_j \cdot u^j$, where $a_j \in \boldsymbol{Z}_+$ is the multiplicity of u^j in $\pi|_A$. Now associate π with a Laurent polynomial $c_\pi(T) = \sum_j a_j T^j$, where T is an indeterminate. Then $c_\pi(t) = (\mathrm{Tr}\,\pi)(h(t))$ for $t \in K^\times$. Since K is an infinite field (in fact K is assumed to be algebraically closed), the function $c_\pi(t)$ on $K^\times$ determines uniquely the Laurent polynomial $c_\pi(T)$. The fact that $h(t)$ and $h(t^{-1})$ are conjugate in $SL(2, K)$ shows $c_\pi(t) = c_\pi(t^{-1})$. So that $c_\pi(T)$ has the symmetry $c_\pi(T) = c_\pi(T^{-1})$.

We recall here the fundamental relations:

$$(\mathrm{Tr} \otimes^2 \pi)(g) = (\mathrm{Tr}\,\pi)(g)^2 , \tag{2.1}$$

$$(\mathrm{Tr}\, \mathrm{S}^2 \pi)(g) = ((\mathrm{Tr}\,\pi)(g)^2 + (\mathrm{Tr}\,\pi)(g^2))/2 , \tag{2.2}$$

$$(\mathrm{Tr} \wedge^2 \pi)(g) = ((\mathrm{Tr}\,\pi)(g)^2 - (\mathrm{Tr}\,\pi)(g^2))/2 . \tag{2.3}$$

(C.f. Serre [9, Prop. 3] for (2.2) and (2.3).)

Proof of (i). By (2.1), it suffices to show that $c_{\pi_1}(T)^2 = c_{\pi_2}(T)^2$ implies $c_{\pi_1}(T) = c_{\pi_2}(T)$. From $c_{\pi_1}(T)^2 = c_{\pi_2}(T)^2$, we get $(c_{\pi_1}(T) + c_{\pi_2}(T))(c_{\pi_1}(T) - c_{\pi_2}(T)) = 0$. Since $c_{\pi_1}(T) + c_{\pi_2}(T) = c_{\pi_1 \oplus \pi_2}(T)$ is not zero as a Laurent polynomial, $c_{\pi_1}(T) - c_{\pi_2}(T)$ must be zero.

Proof of (ii). By (2.2), it suffices to show that

$$c_{\pi_1}(T)^2 + c_{\pi_1}(T^2) = c_{\pi_2}(T)^2 + c_{\pi_2}(T^2) \quad \text{implies} \quad c_{\pi_1}(T) = c_{\pi_2}(T) .$$

Put

$$p(T) = c_{\pi_1}(T) - c_{\pi_2}(T) , \quad q(T) = c_{\pi_1}(T) + c_{\pi_2}(T) . \tag{2.4}$$

Then our assumption amounts to $p(T)q(T) = -p(T^2)$. Let us prove $p(T) = 0$ by contradiction. Assume that $p(T) \neq 0$, and factorize $p(T) = \mathrm{const} \times T^{-L} \times \prod_\alpha (T - \alpha)^{r_\alpha}$, where $L \in \boldsymbol{Z}_+$, and α's are the roots in $\boldsymbol{C}$ of $p(T)$ with multiplicity r_α. Then we see

$$q(1) = -\lim_{T \to 1} p(T^2)/p(T) = -2^{r_1} < 0 .$$

But $q(1) = c_{\pi_1}(1) + c_{\pi_2}(1)$ represents the dimension of the representation $\pi_1 \oplus \pi_2$, which is clearly positive. This give a contradiction. Hence $p(T) = 0$.

Proof of (iii). Let $p(T)$ and $q(T)$ be as in (2.4). Then by (2.3), the assumption $\wedge^2\pi_1 \sim \wedge^2\pi_2$ amounts to

$$(*) \qquad p(T)q(T) = p(T^2)\,.$$

Assume $p(T) \neq 0$, and write $p(T) = \text{const} \times T^{-L} \times \prod_\alpha (T-\alpha)^{r_\alpha}$ as above. Then $q(1) = \lim_{T\to 1} p(T^2)/p(T) = 2^{r_1}$.

Let n_1 and n_2 be the dimensions of π_1 and π_2 respectively, so that $n_i = c_{\pi_i}(1)$ $(i=1, 2)$, $n_1 - n_2 = p(1)$, and $n_1 + n_2 = q(1)$. Putting $T=1$ in (*), we obtain $(n_1-n_2) \times (n_1+n_2-1) = 0$. Since $n_1+n_2 \geqq 2$, we see $n_1 = n_2$, and so $n_1 = n_2 = 2^{r_1-1}$. Thus it remains to prove that r_1 is even.

Now from the symmetry $p(T) = p(T^{-1})$, we see

$$\prod_\alpha (T-\alpha)^{r_\alpha} = \prod_\alpha (1-\alpha T)^{r_\alpha}\,,$$

so that

$$\text{(a)} \qquad \prod_\alpha (-\alpha)^{r_\alpha} = 1\,; \qquad\qquad \text{(b)} \quad r_\alpha = r_{\alpha^{-1}}\,.$$

Then from (a), we have $\prod_{\alpha\neq\pm1}(-\alpha)^{r_\alpha} \times (-1)^{r_1} \times 1^{r_{-1}} = 1$. Applying (b) to this, we obtain $(-1)^{r_1} = 1$. Therefore r_1 is even, as desired. Q.E.D.

2.2. Let $p(T)$ be as in the proof of Theorem 1 (iii). Then we observe the following two properties of $p(T)$.

Proposition 2.1. (i) $p(1) = 0$.
(ii) *Write* $p(T) = \sum_{-L\leqq j\leqq L} b_j T^j$ *with* $b_L \neq 0$. *Then* $b_L = \pm 1$.

Proof. As (i) is already seen, we have only to prove (ii). Write

$$c_{\pi_i}(T) = \sum_{-L_i\leqq j\leqq L_i} a_j^{(i)} T^j \quad \text{with} \quad a_{L_i}^{(i)} \neq 0 \qquad (i=1, 2)\,.$$

We may assume $L_1 \geqq L_2$. Moreover we write $q(T) = \sum_{-L'\leqq j\leqq L'} b'_j T^j$ with $b'_{L'} \neq 0$. Then we see clearly $L_1 = L' \geqq L$. Comparing the terms of the highest degrees in (*), we get $L' = L$ and $b_L \cdot b'_L = b_L$, whence $b'_L = 1$. On the other hand, $b'_L = a_L^{(1)} + a_L^{(2)} \geqq 2$ in case $L_1 = L_2$. Therefore we have $L_1 > L_2$. Then we see $b_L = a_L^{(1)} = b'_L = 1$. Q.E.D.

From the above proposition, we can deduce a modification of Theorem 1 (iii).

Proposition 2.2. *Let π_1 and π_2 be finite dimensional rational representations of $SL(2, K)$. Let $m > 1$ be an integer. Then $\wedge^2(m\cdot\pi_1) \sim \wedge^2(m\cdot\pi_2)$ implies, $\pi_1 \sim \pi_2$, where $m\cdot\pi = \pi\oplus\cdots\oplus\pi$ the m-times multiple of π.*

Corollary. *For nilpotent elements X, X' in $\mathfrak{o}(n, K)$, the following two conjugacies are equivalent.*

(C) $\quad X \sim X'$ *under* $O(n, K)$,

(C_m) $\quad \mathrm{ad}([m]X) \sim \mathrm{ad}([m]X')$ *under* $GL(\mathfrak{o}(mn, K))$.

Here $m>1$, *and* $[m]X=X\oplus\cdots\oplus X\in\mathfrak{o}(mn, K)$ *is the m-multiple of* X.

Proof. Define $p(T)$ similarly as in the proof of Theorem 1 (iii) for the pair of representations $m\cdot\pi_1$ and $m\cdot\pi_2$. Then the coefficients of $p(T)$ are divisible by m. But by Proposition 2.1 (ii) above, the coefficient of the highest degree in $p(T)$ is ± 1 in case $p(T)\neq 0$. Consequently our assumption $m>1$ shows $p(T)=0$. Namely $m\cdot\pi_1\sim m\cdot\pi_2$, whence $\pi_1\sim\pi_2$. Q.E.D.

§3. Representations on tensor spaces.

In this section we given some generalizations of Theorem 1. More precisely, we generalize Problem II in two directions and study them. On one hand, we treat not only $SL(2, K)$ but also connected algebraic groups or connected Lie groups. On the other hand, the functors $\otimes^2$, S^2, $\wedge^2$ are generalized to those of tensors of higher degrees. The latter direction is again related to Problem I through $\mathfrak{sl}_2$-triplet.

3.1. First of all, we give an immediate generalization of Theorem 1 (i).

Proposition 3.1. *Let* Γ *be a connected algebraic group over* K. *Let* π_1 *and* π_2 *be completely reducible rational representations of* Γ, *of finite dimension, and* N *a positive integer. Then* $\otimes^N\pi_1\sim\otimes^N\pi_2$ *implies* $\pi_1\sim\pi_2$. *Here* $\otimes^N\pi$ *denotes the N-times tensor product of* π.

Proposition 3.1′. *Let* Γ *be a connected Lie group. Let* π_1 *and* π_2 *be completely reducible finite dimensional continuous representations of* Γ, *and* N *a positive integer. Then* $\otimes^N\pi_1\sim\otimes^N\pi_2$ *implies* $\pi_1\sim\pi_2$.

Proof. Let R be the ring of regular rational functions on Γ (for Proposition 3.1), or the ring of real analytic complex-valued functions on Γ (for Proposition 3.1′), respectively. Sinec Γ is connected, R is an integral domain. So an algebraic equation $x^N-a^N=0$ has just N solutions in R as $x=a, a\zeta, \cdots, a\zeta^{N-1}$, where ζ is a primitive N-th root of unity. We apply this fact to the case $a=\mathrm{Tr}\,\pi_1$. Then from $\otimes^N\pi_1\sim\otimes^N\pi_2$ we have $(\mathrm{Tr}\,\pi_1)(g)=\zeta^j\cdot(\mathrm{Tr}\,\pi_2)(g)$ for some j $(g\in\Gamma)$. Putting $g=1$, we see $\zeta^j=1$, because $(\mathrm{Tr}\,\pi_1)(1)$ and $(\mathrm{Tr}\,\pi_2)(1)$ are positive integers. Therefore we have $\mathrm{Tr}\,\pi_1=\mathrm{Tr}\,\pi_2$, whence $\pi_1\sim\pi_2$. Q.E.D.

3.2. In order to formulate a generalization of Theorem 1 (ii) and (iii), we recall some definitions and notations on partition (c.f. Macdonald [8]).

A *partition* λ of an integer N is a set of positive integers $\lambda=(\lambda_1, \cdots, \lambda_l)$ with $N=\lambda_1+\cdots+\lambda_l$. We denote $|\lambda|=\lambda_1+\cdots+\lambda_l$ and call $l=l(\lambda)$ the *length* of λ.

For a pr partition $\lambda=(\lambda_1, \cdots, \lambda_l)$, we use as well the notation $\lambda=(1^{m_1}2^{m_2}\cdots r^{m_r})$, where $m_i=\#\{j; \lambda_j=i\}$ is the *multiplicity* of i in λ. Note then $l(\lambda)=m_1+\cdots+m_r$.

The *conjugate* of a partition λ is by definition the partition $\lambda'=(\lambda_1', \lambda_2', \cdots)$ with $\lambda_i'=\#\{j; \lambda_j\geqq i\}$. For a partition $\lambda=(\lambda_1, \cdots, \lambda_l)$ with $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_l$, the *diagram* (λ) is defined as

$$(\lambda) = \{(i, j) \in \boldsymbol{Z}^2;\ 1 \leqq j \leqq \lambda_i\}\ .$$

For each $x=(i,j)\in(\lambda)$, the *content* of x is defined to be $c(x)=j-i$, and the *hook length* of λ at x is

$$h(x) = (\lambda_i - i) + (\lambda'_j - j) + 1\ .$$

A partition λ is called a *hook* when it is of the form $\lambda=(1^{m_1}r^{m_r})$ with $m_1\geqq 0$ and $0\leqq m_r\leqq 1$ for some r.

3.3. Now we proceed to a generalization of Theorem 1 (ii) and (iii).

Let Γ be a group, and π a representation of Γ on a vector space V. For a positive integer N, consider the representation $\otimes^N \pi$ on $\otimes^N V$. Then the symmetric group $\mathfrak{S}_N$ of degree N acts on $\otimes^N V$ by $v_1\otimes\cdots\otimes v_N \mapsto v_{s^{-1}(1)}\otimes\cdots\otimes v_{s^{-1}(N)}$ for $s\in\mathfrak{S}_N$ and $v_i\in V\ (i=1,2,\cdots,N)$. This action of $\mathfrak{S}_N$ commutes with $\otimes^N\pi(g)\ (g\in\Gamma)$. So the representation $\otimes^N\pi$ decomposes according to irreducible representations of $\mathfrak{S}_N$. It is well known that irreducible representations of $\mathfrak{S}_N$ are parametrized by partitions of N (see e.g. Macdonald [8, Ch. I.7] or Weyl [12, Ch. IV]). For a partition λ of N, we denote by $\mathrm{S}_\lambda\pi$ the component of $\otimes^N\pi$ corresponding to λ.

For example, in case $N=2$, we have $\mathrm{S}_{(2)}\pi=\mathrm{S}^2\pi$ and $\mathrm{S}_{(1^2)}\pi=\wedge^2\pi$. In general, $\mathrm{S}_{(N)}\pi=\mathrm{S}^N\pi$ is the N-th symmetric tensor product representation, and $\mathrm{S}_{(1^N)}\pi=\wedge^N\pi$ is the N-th alternating tensor product presentation of π.

As a generalization of Problem II (ii) and (iii), we consider

Problem III $(\Gamma;\lambda)$. Let π_1 and π_2 be representations of Γ, and λ a partition of N. Then does it hold that $\mathrm{S}_\lambda\pi_1\sim\mathrm{S}_\lambda\pi_2$ implies $\pi_1\sim\pi_2$?

As we shall see in §§4 & 6, there are some cases where Problem III is negative. But we have affirmative results as follows.

Theorem 2. *Let Γ be a connected algebraic group, and π_i $(i=1,2)$ be completely reducible rational representations of Γ, of finite dimension. Assume that a partition λ of N is a hook, and that*

(♯) *the dimensions of π_i $(i=1,2)$ do not divide $(N-1)!\,N^k$ for any k.*

Then $\mathrm{S}_\lambda\pi_1\sim\mathrm{S}_\lambda\pi_2$ implies $\pi_1\sim\pi_2$.

Moreover, if $\lambda=(N)$, i.e., S_λ is the N-th symmetric tensor product, then the conclusion holds without the assumption (♯).

Remark 3. One can obtain a similar result to Theorem 2 in parallel with Proposition 3.1$'$ to Proposition 3.1, replacing π in Theorem 2 by a connected Lie group and π_i $(i=1,2)$ by completely reducible continuous representations of finite dimensions.

3.4. Let π be an n-dimensional representation of Γ. We present here a formula expressing $\mathrm{Tr}(\mathrm{S}_\lambda\pi)$ in terms of $\mathrm{Tr}\,\pi$, which is a generalization of (2.2) and (2.3) in the proof of Theorem 1. In case $\Gamma=GL(n,K)$ and π is the identical representation, such a formula is obtained from the properties of the Schur function. The general case is immediately deduced from it.

First of all, we note that for $g\in\Gamma$,

$$\mathrm{Tr}\,(S_\lambda\pi)(g)=d_\lambda\cdot s_\lambda(x_1,\cdots,x_n)\,, \tag{3.1}$$

with $x_1,\cdots,x_n$, the eigenvalues of $\pi(g)$. Here $s_\lambda(x_1,\cdots,x_n)$ is the Schur function (see Macdonald [8, p. 24 (3.1)]) defined as

$$s_\lambda(x_1,\cdots,x_n)=\frac{\det(x_i^{\lambda_j+n-j})_{1\leq i,j\leq n}}{\det(x_i^{n-j})_{1\leq i,j\leq n}}\,,$$

and d_λ is the dimenison of the irreducible representation of $\mathfrak{S}_N$ corresponding to λ. The formula (3.1) is seen to hold from the fact that the Schur function $s_\lambda(x_1,\cdots,x_n)$ gives the trace of the irreducible representation of $GL(n,K)$, corresponding to λ, at $\mathrm{diag}(x_1,\cdots,x_n)$ (see Weyl [12, Ch. VII §6] and Macdonald [8, Appendix to Ch. I]).

In order to state an equality on the Schur function, we introduce some notations. Let $\rho=(\rho_1,\cdots,\rho_l)=(1^{m_1}2^{m_2}\cdots r^{m_r})$ be a partition of N. Put

$$z_\rho=\prod_{i\geq 1}(m_i!\,i^{m_i})\,, \tag{3.2}$$

which is the cardinality of the centralizer of an element in the conjugacy class of $\mathfrak{S}_N$ corresponding to ρ. Moreover we put

$$p_\rho(x_1,\cdots,x_n)=\prod_{i\geq 1}(x_1^{\rho_i}+\cdots+x_n^{\rho_i})\,. \tag{3.3}$$

Let χ^λ be the irreducible character of $\mathfrak{S}_N$ corresponding to λ, and χ_ρ^λ the value of χ^λ at the conjugacy class in $\mathfrak{S}_N$ of type ρ. The Schur function s_λ is rewritten in terms of p_ρ's:

$$s_\lambda=\sum_{|\rho|=N}z_\rho^{-1}\chi_\rho^\lambda p_\rho\,, \tag{3.4}$$

(see [8, p. 63 (7.10)], or [12, p. 215 (7.19)]).

Let us put $\theta(g)=\mathrm{Tr}\,\pi(g)$ for $g\in\Gamma$, and $\theta_\rho(g)=\prod_{i\geq 1}\theta(g^{\rho_i})$ for a partition $\rho=(\rho_1,\cdots,\rho_l)$. Then combining (3.1) and (3.4), we get

$$\mathrm{Tr}(S_\lambda\pi)(g)=d_\lambda\sum_{|\rho|=N}z_\rho^{-1}\chi_\rho^\lambda\theta_\rho(g)\,. \tag{3.5}$$

It should be noted that (2.2) and (2.3) are obtained from (3.5) by putting $N=2$.

3.5. To establish Theorem 2, it is sufficient to prove that for a given function $\Theta(g)$, the solution $\theta(g)$ of the functional equation

$$\Theta(g)=\sum_{|\rho|=N}z_\rho^{-1}\chi_\rho^\lambda\theta_\rho(g)\qquad(g\in\Gamma) \tag{3.6}$$

is unique under a suitable condition.

From now on Γ is assumed to be a connected algebraic group or a connected Lie group. Then $\theta(g)=\mathrm{Tr}\,\pi(g)$ is a rational or real analytic function on Γ according as Γ is algebraic or Lie group, where π is a rational or continuous representation, respectively. We note that since Γ is connected, $\theta(g)$ is determined uniquely by its derivatives of all orders at $g=1$.

Let $\mathrm{Lie}(\Gamma)$ be the Lie algebra of Γ, which is naturally identified with the Lie algebra of right invariant vector fields on Γ. For a $\delta\in\mathrm{Lie}(\Gamma)$, fixed once for all, we denote by $f^{(k)}$ the k-th derivative of a function $f(g)$ by δ at $g=1$:

$$f^{(k)} = (\delta^k f)(1)\,.$$

For $k=0$, we put $f^{(0)}=f(1)$.

Our aim is to look for a sufficient condition that for any fixed $\delta\in\mathrm{Lie}(\Gamma)$, $\theta^{(k)}$'s ($k=0, 1, 2, \cdots$) are recovered uniquely from $\Theta^{(k)}$'s ($k=0, 1, 2, \cdots$), where $\theta(g)$ and $\Theta(g)$ are related by the equation (3.6).

For the first stage, we note that $\theta^{(0)}=\theta(1)=n$, the dimension of the representation π, is uniquely determined by $\Theta^{(0)}$ under the condition $n\geqq l(\lambda)-1$. This follows from the formula (Weyl's dimnesion formula)

$$s_\lambda(1, \cdots, 1) = \prod_{x\in(\lambda)} (n+c(x))/h(x)$$

(see Macdonald [8, Ch. 1.3, Ex. 4]). Because the polynomial $\prod_{x\in(\lambda)}(n+c(x))$ is monotone increasing in n for $n\geqq l(\lambda)-1$.

For the second stage, we compute the derivatives of $\theta_\rho(g)=\prod_{i=1}^{l(\rho)}\theta(g^{\rho_i})$, for $\rho=(\rho_1, \rho_2, \cdots)$. By Leibniz rule we get

$$\theta_\rho^{(k)} = \sum_{i_1+\cdots+i_{l(\rho)}=k} \frac{N!}{i_1!\,i_2!\cdots i_{l(\rho)}!} \prod_{j=1}^{l(\rho)} (\rho_j^{i_j}\cdot\theta^{(i_j)})\,. \tag{3.7}$$

Collecting together the terms containing the highest derivative $\theta^{(k)}$ in (3.7), we see

$$\theta_\rho^{(k)} = \theta^{(k)} n^{l(\rho)-1}\Big(\sum_{i=1}^{l(\rho)} \rho_i^k\Big) + \text{terms of lower derivatives.}$$

From this, taking the k-th derivative of (3.6), we obtain

$$\Theta^{(k)} = \theta^{(k)}\psi_k(n) + (\text{terms of } \theta^{(j)}\text{'s with } 0\leqq j\leqq k-1)\,, \tag{3.8}$$

where $\psi_k(n)$ is the polynomial in n of degree $N-1$ defined by

$$\psi_k(n) = \sum_{|\rho|=N} z_\rho^{-1}\chi_\rho^\lambda\Big(\sum_{i=1}^{l(\rho)}\rho_i^k\Big) n^{l(\rho)-1}\,. \tag{3.9}$$

Lemma 3.2. *Let π be an n-dimensional completely reducible representation of* Γ. *Suppose $n\geqq l(\lambda)-1$ and $\psi_k(n)\neq 0$ for all $k\geqq 1$. Then π is recovered uniquely from* $S_\lambda\pi$.

Proof. As we have seen in the first stage, $n=\theta^{(0)}$ is determined uniquely from $\Theta^{(0)}$ under the assumption $n \geqq l(\lambda)-1$. For $k \geqq 1$, $\theta^{(k)}$ is determined inductively by (3.8) from $\Theta^{(k)}$. Q.E.D.

3.6. Proof of Theorem 2. It remains to show that Lemma 3.2 is applicable to Theorem 2.

For the case $\lambda=(N)$, i.e., the case of N-th symmetric tensor product, we see easily $\psi_k(n) \neq 0$ for all $k \geqq 1$. In fact, since $\chi^{(N)}$ is the trivial character, we see $\chi_\rho^{(N)}=1$ for all ρ, so that all the coefficients of $\psi_k(n)$ in (3.9) are positive. Thus $\psi_k(n)>0$ for any $n>0$. Note that $l((N))=1$, so the condition $n \geqq l(\lambda)-1$ is automatic for $\lambda=(N)$. Thus we have proved the latter half of Theorem 2.

Now we proceed to general case. It is well known that the values of characters χ_ρ^λ of $\mathfrak{S}_N$ are all integers. Therefore from the definition (3.2) of z_ρ, we see that the polynomial $N!\psi_k(n)$ in n has integral coefficients. Note that the constant term of $N!\psi_k(n)$ is $N!\cdot 1/N \cdot N^k \cdot \chi_{(N)}^\lambda$, because $l(\rho)=1$ if and only if $\rho=(N)$. We know that $\chi_{(N)}^\lambda=\pm 1$ or 0 and that $\chi_{(N)}^\lambda \neq 0$ if and only if λ is a hook. (This is seen from Weyl [12, pp. 213–214]. See also Kostant [6].) It follows from these facts that the polynomial $N!\psi_k(n)$ has integral coefficients and has the constant term $\pm(N-1)!N^k$ if λ is a hook. Consequently, when λ is a hook, we have $\psi_k(n) \neq 0$ for n not dividing $(N-1)!N^k$. We note that $n \geqq l(\lambda)-1$ is implied in the assumption (♯). Thus by Lemma 3.2, Theorem 2 is proved. Q.E.D.

3.7. Remark 4. Here we present the polynomials $\psi_k(n)$ for two simple λ.

(1) Let $N=2$ and $\lambda=(1^2)$. We treated this case in Theorem 1 (iii) for $\Gamma=SL(2, K)$. Now from the definition (3.9), we have

$$\psi_k(n) = n-2^{k-1}.$$

(2) Let $N=3$ and $\lambda=(1^1 2^1)=(2,1)$. Then we have

$$\mathrm{Tr}\, S_{(2,1)}\pi(g) = \frac{2}{3}(\theta(g)^3-\theta(g^3)) \tag{3.10}$$

and

$$\psi_k(n) = n^2-3^{k-1}.$$

Note that in this case the condition (♯) can be weakened into the condition that n does not divide 3^{k-1} instead of $2\cdot 3^k$.

In the proof of Theorem 2, we did not utilize in full detail the fact that $\theta(g)$ is a trace of representation, but used only the fact that $\theta(1)$ is a positive integer. So Theorem 2 is a general but rough result in some sense, and may be sharpened in each special case. For example, we shall see in §7 that for $\Gamma=SL(2, K)$ and $\lambda=(2,1)$ the condition (♯) in Theorem 2 can be dropped.

3.8. Let us return to Problem I. As seen in §1.3, by Jacobson-Morozov theorem, the problem of conjugacy of nilpotent elements is translated to that of equivalence of representations of $\mathfrak{sl}(2, K)$, whence of $SL(2, K)$. Then we see from

Proposition 3.1 and Theorem 2 for $\Gamma=SL(2, K)$ the following

Theorem 3. (i) *Let G be an algebraic group with the Lie algebra $\mathfrak{g}$. Let (σ, V) be an n-dimensional rational representation of G. Then for nilpotent elements $X, X'\in\mathfrak{g}$, the following statements are mutually equivalent.*

(C_σ) $\quad \sigma(X)\sim\sigma(X')$ *under* $GL(V)$;

$(C_{\otimes^N\sigma})$ $\quad \otimes^N\sigma(X)\sim\otimes^N\sigma(X')$ *under* $GL(\otimes^N V)$;

$(C_{S^N\sigma})$ $\quad S^N\sigma(X)\sim S^N\sigma(X')$ *under* $GL(S^N V)$;

$(C_{S_\lambda\sigma})$ $\quad S_\lambda\sigma(X)\sim S_\lambda\sigma(X')$ *under* $GL(S_\lambda V)$,

where a partition λ of N is a hook, and n does not divide $(N-1)!\,N^k$ for any k.

(ii) *Further, for $G=GL(n, K)$, $Sp(n, K)$, or $O(n, K)$, and for $\sigma=\sigma_0$ the identical representation, each of the above statements (C_σ)—$(C_{S_\lambda\sigma})$ with $\sigma=\sigma_0$ is equivalent to the following one:*

(C) $\quad X\sim X'$ *under* G.

§4. Negative results.

In this section, we return to Problem II (iii) and Problem $I_1(O(n, K))$, and investigate the counterexamples for these problems.

4.1. For a rational representation π of $SL(2, K)$, we defined a Laurent polynomial $c_\pi(T)$ in §2:

$$c_\pi(t)=\operatorname{Tr}\pi(h(t)) \qquad \text{for} \quad h(t)=\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}.$$

The following lemma characterizes $c_\pi(T)$ by its coefficients.

Lemma 4.1. *Let $f(T)=\sum_{j\in\boldsymbol{Z}} a_jT^j\in\boldsymbol{Z}[T, T^{-1}]$. Then $f(T)=c_\pi(T)$ for some rational representation π of $SL(2, K)$ if and only if $f(T)$ satisfies the following $(R0)$—$(R2)$:*

$(R0)$ $\quad a_j\geqq 0$ *for all* $j\in\boldsymbol{Z}$;

$(R1)$ $\quad f(T)=f(T^{-1})$, *i.e.,* $a_j=a_{-j}$ *for all* $j\in\boldsymbol{Z}$;

$(R2)$ $\quad a_j\geqq a_{j+2}$ *for* $j\geqq 0$.

Moreover, π satisfies the condition $(M2)$ in §1.3 if and only if

(M2′) $\quad a_j$ *is even for any odd* j.

Proof. We know for the i-dimensional irreducible representation $\pi^{(i)}$ of $SL(2, K)$,

$$c_{\pi^{(i)}}(T)=(T^{i}-T^{-i})/(T-T^{-1})=T^{i-1}+T^{i-3}+\cdots+T^{-i+1}\,.$$

So that if $\pi\sim\bigoplus_i m_i\cdot\pi^{(i)}$ and $c_\pi(T)=\sum_j a_j T^j$, then we have

$$m_i=a_{i-1}-a_{i+1}\qquad (i\geqq 1)\,.$$

The assertions can be easily deduced from these. Q.E.D.

4.2. Let π_1, π_2 be rational representations of $SL(2, K)$. We put

$$\text{(4.0)}\qquad \begin{cases} p(T)=p_{\pi_1,\pi_2}(T)=c_{\pi_1}(T)-c_{\pi_2}(T)\,,\\ q(T)=q_{\pi_1,\pi_2}(T)=c_{\pi_1}(T)+c_{\pi_2}(T)\,.\end{cases}$$

Then as seen in §2, $\wedge^2\pi_1\sim\wedge^2\pi_2$ isequivalent to

$$(*)\qquad p(T)q(T)=p(T^2)\,,$$

and $\pi_1\not\sim\pi_2$ is equivalent to $p(T)\neq 0$.

Conversely, let $p(T)\in \boldsymbol{Z}[T, T^{-1}]$ be a non-zero Laurent polynomial satisfying

$$(*1)\qquad p(T) \text{ divides } p(T^2) \text{ in } \boldsymbol{Z}[T, T^{-1}]\,.$$

Put

$$(*4.1)\qquad q(T)=p(T^2)/p(T)\,,$$

$$(*4.2)\qquad \begin{cases} f_1(T)=(q(T)+p(T))/2\,,\\ f_2(T)=(q(T)-p(T))/2\,.\end{cases}$$

Suppose moreover

(*2) both $f_1(T)$ and $f_2(T)$ have integral coefficients and satisfy (R0)—(R2).

Then by Lemma 4.1, we can find representations π_1, π_2 of $SL(2, K)$ such that $f_2(T)=c_{\pi_1}(T)$, $f_1(T)=c_{\pi_2}(T)$. For these representations π_1, π_2, we see $\pi_1\not\sim\pi_2$ and $\wedge^2\pi_1\sim\wedge^2\pi_2$.

Thus to find a counterexample for Problem II (iii) is reduced to looking for a non-zero Laurent polynomial $p(T)$ satisfying (*1) and (*2). Moreovre such a $p(T)$ gives a counterexample for Problem $\mathrm{I}_1(O(n, K))$ if both fo $f_1(T)$, $f_2(T)$ in (*4.2) satisfy (M2′). In this case we have $n=f_1(1)=f_2(1)=q(1)/2$.

The following examples demonstrate the existence of $p(T)\in\boldsymbol{Z}[T, T^{-1}]$ satisfying (*1) and (*2). The detailed discussion for those examples will be given in §4.6.

Example 1. For $k\geqq 1$, $p(T)=(T-1)^{2k}T^{-k}$ satisfies (*1) and (*2). Moreover if k is even, then both of $f_1(T)$ and $f_2(T)$ satisfy (M2′).

Example 2. For $k\geqq 0$, $p(T)=(T^2-1)^2(T-1)^{12+2k}T^{-8-k}$ satisfies (*1) and (*2). Moreover if k is even, then both of $f_1(T)$ and $f_2(T)$ satisfy (M2′).

Note $f_1(1)=f_2(1)=2^{2k-1}$ in Example 1, and $f_1(1)=f_2(1)=2^{13+2k}$ in Example 2. Thus we see from these examples the following

Theorem 4. (i) *For $k\geqq 1$, there exists a pair π_1, π_2 of rational representations of $SL(2, K)$ of dimension 2^{2k-1} such that $\pi_1 \not\sim \pi_2$ and $\wedge^2\pi_1 \sim \wedge^2\pi_2$.*

(ii) *For $r\geqq 1$ except for $r=2, 4$, Problem $\mathrm{I}_1(O(2^{2r+1}, K))$ is negative. Namely, for such an r, there exist nilpotent elements $X, X'\in\mathfrak{o}(2^{2r+1}, K)$ such that $X\not\sim X'$ under $O(2^{2r+1}, K)$ but $\mathrm{ad}(X)\sim\mathrm{ad}(X')$ under $GL(\mathfrak{o}(2^{2r+1}, K))$.*

4.3. In the above subsection, we saw that for a counterexample of Problem II (iii) (resp. of Problem $\mathrm{I}_1(O(n, K))$), there corresponds a Laurent polynomial satisfying (*1) and (*2) (resp. and besides, the condition (M2′) on $f_1(T)$ and $f_2(T)$). In the following, we translate these correspondences from Laurent polynomials to usual polynomials, which are easier to treat in some cases.

Let us begin with definitions.

Definition 4.1. Let s be a positive integer and $v\in\frac{1}{2}\boldsymbol{Z}$. A Laurent polynomial $f(T)=\sum_{j\in\boldsymbol{Z}} a_jT^j$ is called *symmetric* and *unimodal* in step s with the peak at v if it satisfies

(1) (symmetric) $\quad a_j=a_{2v-j} \quad$ for all $j\in\boldsymbol{Z}$,

(2) (unimodal) $\quad a_j\geqq a_{j+s} \quad$ for all $j\geqq v$.

It should be noted that from (2) and $a_j=0$ for sufficiently large j, one sees $a_j\geqq 0$ for all j.

For a polynomial $f(T)$, we say simply symmetric and unimodal, when we take the step $s=1$ and the peak $v=\frac{1}{2}\deg f(T)$. (C.f. Kac [5, Ex. 10.12], Macdonald [8, p. 28], and White [13].)

Definition 4.2. (i) A polynomial $P(T)\in\boldsymbol{Z}[T]$ is called *distinguished* if it satisfies the following (D0)—(D4):

(D0) $\deg P(T)$ is even.

(D1) $P(T)$ divides $P(T^2)$.

(D2) Both $F_1(T)$ and $F_2(T)$ have integral coefficients and are symmetric and unimodal in step 2 with the peak at $\frac{1}{2}\deg P(T)$, where

$$Q(T)=P(T^2)/P(T), \tag{4.1}$$

$$\begin{cases} F_1(T)=(Q(T)+P(T))/2, \\ F_2(T)=(Q(T)-P(T))/2. \end{cases} \tag{4.2}$$

(D3) $P(1)=0$.

(D4) $P(T)$ is monic, i.e., the coefficient of the highest degree is 1.

(ii) A distinguished polynomial $P(T)$ is called *strongly distinguished* if both of

$F_1(T)$, $F_2(T)$ in (4.2) satisfy the following condition (M2''):

(M2'') the coefficient of T^j is even, for any j with $j+1\equiv\frac{1}{2}\deg P(T) \bmod 2$.

Remark 5. (1) In (D2), the part of condition that $F_1(T)$ and $F_2(T)$ have integral coefficients is in fact redundant, and is deduced from (D1) and (D4) (see §5, Remark 9).

(2) From the definition (4.2), we see

$$\begin{cases} P(T) = F_1(T)-F_2(T)\,, \\ Q(T) = F_1(T)+F_2(T)\,. \end{cases} \tag{4.3}$$

Then it follows from the condition (D2) that $P(T)$ is symmetric and that $Q(T)$ is symmetric and unimodal in step 2.

4.4. Let $p(T)=\sum_{j=-L}^{L} b_jT^j$ with $b_L\neq 0$ be a Laurent polynomial satisfying (*1) and (*2). We define a polynomial

$$P(T) = T^Lp(T)\,. \tag{4.4}$$

Then $\deg P(T)=2L$, and $P(T)$ satisfies (D1) and (D2) according as $p(T)$ satisfies (*1) and (*2), respectively. Indeed, we see the correspondence between Laurent polynomials $q(T)$, $f_1(T)$, $f_2(T)$ in (*4.1)–(*4.2) and polynomials $Q(T)$, $F_1(T)$, $F_2(T)$ in (4.1)–(4.2) as

$$Q(T) = T^Lq(T)\,,\quad F_1(T) = T^Lf_1(T)\,,\quad F_2(T) = T^Lf_2(T)\,. \tag{4.5}$$

We now recall Proposition 2.1, which states (i) $p(1)=0$ and (ii) $b_L=\pm 1$. Normalizing as $b_L=1$ (multiply by -1 if necessary), we see that $P(T)$ satisfies (D3) and (D4). Note that $p_{\pi_1,\pi_2}(T)=-p_{\pi_2,\pi_1}(T)$, so that the normalization $b_L=1$ means only a permutation of π_1, π_2 associated with $p(T)$. Moreover, the condition (M2') on $f_1(T)$, $f_2(T)$ corresponds to (M2'') on $F_1(T)$, $F_2(T)$. Thus for $p(T)$ satisfying (*1) and (*2), the polynomial $P(T)=T^Lp(T)$ is distinguished. And if $f_1(T)$ and $f_2(T)$ satisfy (M2') in addition, then $P(T)$ is strongly distinguished.

Conversely, for a distinguished polynomial $P(T)$ of degree $2L$, we put $p(T)=T^{-L}P(T)$. Then clearly $p(T)$ satisfies (*1) and (*2). Thus from the argument in §4.2, we see

Theorem 5. (i) *There is a one-to-one correspondence between the set of distinguished polynomials and the set of pairs π_1, π_2 (up to equivalence) of rational representations of $SL(2, K)$ such that $\pi_1 \not\sim \pi_2$ and $\wedge^2\pi_1\sim\wedge^2\pi_2$.*

(ii) *There is a one-to-one correspondence between the set of strongly distinguished polynomials $P(T)$ with $n=Q(1)/2$ and the set of pairs X, X' (up to conjugation under $O(n, K)$) of nilpotent elements in $\mathfrak{o}(n, K)$ such that $X\not\sim X'$ under $O(n, K)$ and that* $\mathrm{ad}(X)\sim\mathrm{ad}(X')$ *under $GL(\mathfrak{o}(n, K))$. Here $Q(T)$ is defined as $Q(T)=P(T^2)/P(T)$.*

4.5. We observe here a fundamental property of distinguished polynomials.

For further properties, we study in the next section.

Let $P(T)$ be a polynomial satisfying (D1) of degree $2L$. Define $F_1(T)$, $F_2(T)$ by (4.2), and $f_1(T)$, $f_2(T)$ by (4.5). Then for $P(T)$, we define two 2×2 matrices as follows.

$$U_{P(T)}=\begin{pmatrix} F_1(T) & F_2(T) \\ F_2(T) & F_1(T) \end{pmatrix}, \tag{4.6}$$

$$u_{P(T)}=\begin{pmatrix} f_1(T) & f_2(T) \\ f_2(T) & f_1(T) \end{pmatrix}. \tag{4.7}$$

An easy calculation shows the following.

Lemma 4.2. *Let $P(T)$ and $P'(T)$ be two polynomials satisfying the condition $(D1)$. Then we have*

$$U_{P(T)P'(T)}=U_{P(T)}U_{P'(T)}, \tag{4.8}$$

$$u_{P(T)P'(T)}=u_{P(T)}u_{P'(T)}. \tag{4.9}$$

From this lemma, we see the property that the set of distinguished polynomials is stable under multiplication as follows.

Proposition 4.3. *Let $P(T)$ and $P'(T)$ be distinguished polynomials. Then the product $P(T)P'(T)$ is also distinguished.*

Proof. Let, π_1 π_2 and π_1', π_2' be the pairs of representations associated with $P(T)$ and $P'(T)$, respectively. Then from (4.9), we see the pair $(\pi_1\otimes\pi_1')\oplus(\pi_2\otimes\pi_2')$, $(\pi_1\otimes\pi_2')\oplus(\pi_2\otimes\pi_1')$ is associated with $P(T)P'(T)$. Q.E.D.

4.6. Here we verify the assertions in Examples 1 and 2 in §4.2. First, it is clear that the polynomial $(T-1)^2$ is distinguished, so that $(T-1)^{2k}$ is distinguished for $k\geqq 1$ by Proposition 4.3. We see also $P(T)=(T^2-1)^2(T-1)^{12}$ is distinguished. In fact, it is clear that $P(T)$ satisfies the conditions in Definition 4.2 (i) except (D2). Let $F_1(T)$, $F_2(T)$ be in (4.2). Then (D2) is seen from the following explicit calculation of $F_1(T)$ and $F_2(T)$:

$$\begin{aligned} F_1(T) &= (1+66T^2+496T^4+990T^6+990T^8+990T^{10}+496T^{12}+66T^{31}+T^{16})+ \\ &\quad +(24T^3+440T^5+1584T^7+1584T^9+440T^{11}+24T^{13}), \\ F_2(T) &= (2T^2+132T^4+990T^6+1848T^8+990T^{10}+132T^{12}+2T^{14})+(12T+ \\ &\quad +220T^3+804T^5+1012T^7+1012T^9+804T^{11}+220T^{13}+12T^{15}). \end{aligned}$$

Now, by Proposition 4.3, $(T^2-1)^2(T-1)^{12+2k}$ is distinguished for $k\geqq 0$.

Next, for the condition (M2″), we note the following congruences in case k is even.

$$(T\pm 1)^{2k}\equiv(T^4+2T^2+1)^{k/2} \mod 4,$$

$$(T^2\pm 1)^2(T\pm 1)^{12+2k}\equiv(T^4+2T^2+1)^{4+k/2} \mod 4.$$

This shows that the coefficients of odd degrees of $P(T)$ and $Q(T)$ in Examples 1 &2 are all divided by 4. So that the coefficients of odd degrees of $F_1(T)$ and $F_2(T)$ are all divided by 2 in this case. Thus for even k, $(T-1)^{2k}$ and $(T^2-1)^2(T-1)^{12+2k}$ are strongly distinguished.

§5. Properties of distinguished polynomials.

We hope to determine explicitly all the distinguished polynomials, which we have not yet succeeded in. In this section, we observe some properties of distinguished and strongly distinguished polynomials.

5.1. It is convenient to determine the polynomials $P(T)$ of integral coefficients which satisfy the condition (D1). For this purpose we recall some elementary properties of cyclotomic polynomials.

The m-th cyclotomic polynomial $\Phi_m(T)$ is by definition

$$\Phi_m(T) = \prod_{\zeta}(T-\zeta),$$

where ζ runs over the primitive m-th roots of unity. Then the following are known. (See e.g. Lang [7, Ch. VIII, §3].)

(CP 1) $\Phi_m(T)$ has integral coefficients and irreducible over $\boldsymbol{Z}$, and its degree is $\varphi(m)$, where $\varphi(m)=\sharp(\boldsymbol{Z}/m\boldsymbol{Z})^{\times}$ the Euler function.

(CP 2) $$T^m-1 = \prod_{d|m} \Phi_d(T),$$

and conversely

(CP 3) $$\Phi_m(T) = \prod_{d|m} (T^d-1)^{\mu(m/d)},$$

where $\mu(\cdot)$ is the Möbius function: $\mu(a)=(-1)^r$ if a is square-free and the product of r primes, and $\mu(a)=0$ otherwise.

Moreover from the definition of the cyclotomic polynomial, we see easily

Lemma 5.1. *Let m and m' be mutually relative prime integers. Then*

$$\Phi_m(T^{m'}) = \prod_{d|m'} \Phi_{dm}(T). \tag{5.1}$$

Especially, for a prime number p not dividing m, and $r\geqq 0$,

$$\Phi_m(T^{p^r}) = \prod_{0\leqq t\leqq r} \Phi_{p^t m}(T), \tag{5.2}$$

and

$$\Phi_{p^{r+1}m}(T) = \Phi_m(T^{p^{r+1}})/\Phi_m(T^{p^r}). \tag{5.3}$$

5.2. Let us now consider a polynomial $P(T)$ with integral coefficients satisfying, for fixed $h\in\boldsymbol{Z}_+$, the condition (**h) $$P(T) \ \text{divides} \ P(T^h).$$

Then we have

Lemma 5.2. *Let $P(T)$ be a monic polynomial with integral coefficients. Assume $P(0) \neq 0$ and $P(T)$ satisfies (**h). Then $P(T)$ is written as a product of cyclotomic polynomials. And also $P(T)$ is written in the form*

$$P(T) = \prod_{s \geqq 1} (T^s - 1)^{e_s} \quad \text{with} \quad e_s \in \boldsymbol{Z}. \tag{5.4}$$

Proof. For the first assertion, it is sufficient to prove that every root of $P(T)$ is a root of unity. Let ζ be a root of $P(T)$. Then by (**h), ζ^h is also a root of $P(T)$. So that $\zeta, \zeta^h, \zeta^{h^2}, \zeta^{h^3}, \cdots$ are also roots of $P(T)$. Since the number of roots is finite, we see $\zeta^{h^r} = \zeta^{h^{r'}}$ for some $r \neq r'$. Then from our assumption that 0 is not a root of $P(T)$, we conclude that ζ is a root of unity.

The second assertion follows from the first one and (CP 3). Q.E.D.

Remark 6. In the expression (5.4), the exponents $e_s \in \boldsymbol{Z}$ are uniquely determined by $P(T)$. In fact, from the first assertion of Lemma 5.2, $P(T)$ is written as

$$P(T) = \prod_{m \geqq 1} \Phi_m(T)^{f_m} \quad \text{with} \quad f_m \in \boldsymbol{Z}_+. \tag{5.4'}$$

Here f_m is uniquely determined, because $\Phi_m(T)$ is irreducible over $\boldsymbol{Z}$. Apply (CP 2) to (5.4), and compare the exponent of $\Phi_m(T)$ in (5.4′) and (5.4). Then we see $f_m = \sum_{m|s} e_s$. From this, e_s's are uniquely determined, because the matrix expressing such a system of linear equations is of an upper triangular form with diagonal components 1.

Explicitly, e_s's are written by f_m's as

$$e_s = \sum_{s|m} \mu(m/s) f_m.$$

Later we need only results for $h=2$ and 3, so that for brevity we restrict ourselves to the case where $h=p$ is a prime number. Here we determine more precisely the polynomials satisfying (**p). The following result may be well-known.

Proposition 5.3. *Let p be a prime number, and $P(T)$ a monic polynomial with integral coefficients. We assume $P(0) \neq 0$. Then $P(T)$ satisfies (**p) if and only if $P(T)$ is of the form*

$$P(T) = \prod_{m, i} \Phi_m(T^{p^i})^{f_{m,i}} \quad \text{with} \quad f_{m,i} \in \boldsymbol{Z}_+, \tag{5.5}$$

where m runs over positive integers not divided by p, and $i \in \boldsymbol{Z}_+$.

Proof. It is clear from (5.3) that a polynomial $P(T)$ of the form (5.5) satisfies (**p). Conversely, let $P(T)$ satisfy (**p). We prove by induction on $\deg P(T)$ that $P(T)$ is of the form (5.5).

As is seen in Lemma 5.2, every root of $P(T)$ is a root of unity. Let ζ be a root of $P(T)$ with maximum multiplicative order, and express the order of ζ as $p^r m$ with m not divided by p and $r \geqq 0$. From (**p), we see that all of $\zeta, \zeta^p, \zeta^{p^2}, \cdots, \zeta^{p^r}$ are roots of $P(T)$. We note that ζ^{p^t} is a primitive p^{r-t}-th root of unity for $0 \leqq t \leqq r$. Then by the irreducibility of the cyclotomic polynomials, $P(T)$ is divided by $P_0(T) = \prod_{0 \leqq t \leqq r} \Phi_{p^t m}(T)$. Note that $P_0(T) = \Phi_m(T^{p^r})$ by (5.2), which is of the form (5.5). Let $P_1(T)$ be the quotient $P(T)/P_0(T)$. Assume that we have proved that $P_1(T)$ satisfies (**p). Then by the induction assumption, $P_1(T)$ is of the form (5.5), therefore so is $P(T) = P_0(T)P_1(T)$. This concludes the proof.

By (**p), there exists a polynomial $S(T)$ such that $P(T^p) = P(T)S(T)$. So we have $P_0(T^p)P_1(T^p) = P_0(T)P_1(T)S(T)$. On the other hand, we have $P_0(T^p) = P_0(T) \times \Phi_{p^{r+1}m}(T)$ by (5.3). Then from these two equalities, we obtain $P_1(T^p)\Phi_{p^{r+1}m}(T) = P_1(T)S(T)$. Note that $\Phi_{p^{r+1}m}(T)$ is irreducible, and that it does not divide $P_1(T)$, because we have taken ζ as a root of $P(T)$ with maximum order $p^r m$. Thus $\Phi_{p^{r+1}m}(T)$ divides $S(T)$, so that $P_1(T)$ divides $P_1(T^p)$, that is, $P_1(T)$ satisfies (**p), as desired. Q.E.D.

We note that in the expression (5.5), the exponents $f_{m,i}$ are uniquely determined by $P(T)$. This can be shown by a similar argument as in Remark 6.

5.3. As a corollary to Proposition 5.3 for $p=2$, we have

Proposition 5.4. *If a polynomial $P(T)$ is distinguished, then it is of the form*

$$P(T) = \prod_{m,i} \Phi_m(T^{2^i})^{f_{m,i}} \qquad \text{with} \quad f_{m,i} \in \boldsymbol{Z}_+ , \tag{5.6}$$

where m runs over odd integers, and $i \in \boldsymbol{Z}_+$.

It should be noted that a polynomial of the form (5.6) is not necessarily distinguished. (C.f. Proposition 5.6 below.)

Let us introduce some words.

Definition 5.1. (i) A monic polynomial $P(T)$ with integral coefficients is said to be *admissible* if it satisfies (**p) for $p=2$ and $P(0) \neq 0$. (By Proposition 5.3 for $p=2$, $P(T)$ is admissible if and only if it can be expressed as (5.6).)

(ii) An admissible polynomial is said to be *pure if* $f_{m,i}=0$ for $i>0$ in the expression (5.6).

(iii) An admissible polynomial is said to be *properly impure* if $f_{m,0}=0$ for all odd m.

(iv) For an admissible polynomial $P(T)$, there exists a unique decomposition $P(T) = P'(T)P''(T)$ such that $P'(T)$ is pure and $P''(T)$ is properly impure. We call $P'(T)$ or $P''(T)$ the *pure part* or the *impure part* of $P(T)$, respectively. We have explicitly, for $P(T)$ in the form (5.6),

$$P'(T) = \prod_m \Phi_m(T)^{f_{m,0}} , \tag{5.7}$$

(5.8) $$P''(T) = \prod_{m,i>0} \Phi_m(T^{2^i})^{f_{m,i}}.$$

Remark 7. By Lemma 5.1, an admissible polynomial $P(T)$ has also an expression as (5.4). Then $P(T)$ is pure if and only if $e_s=0$ for any even s in such an expression. And $P(T)$ is properly impure if and only if $e_s=0$ for any odd s.

Remark 8. Let $P(T)$ be adimssible, and express it in the form (5.4). Then $P(T)$ is symmetric if and only if $\sum_{s\geq 1} e_s$ is even. From this, we see that $Q(1)=2^{\Sigma e_s}$ is a power of 4, when $P(T)$ is distinguished. Here $Q(T)=P(T^2)/P(T)$. In this case $F_1(1)=F_2(1)$ is odd times power of 2, where $F_1(T)$ and $F_2(T)$ are given in (4.2). Thus we again obtain Thoerem 1 (iii) via Theorem 5.

Remark 9. Here we verify the assertion of Remark 5 (1) in §4. More precisely, we claim that both the polynomials $F_1(T)$ and $F_2(T)$ have integral coefficients if $P(T)$ is admissible. Here $F_1(T)$, $F_2(T)$ are in (4.2).

By (4.8) in Lemma 4.2, it suffices to verify the claim for $P(T)=\Phi_m(T^{2^i})$ for odd m. Moreover, by (5.3) we have $Q(T)=\Phi_{2m}(T^{2^i})$ for $P(T)=\Phi_m(T^{2^i})$, where $Q(T)=P(T^2)/P(T)$. Then we have only to verify the claim for $\Phi_m(T)$ for m odd. For $m=1$,

(5.9) $$F_1(T) = T \quad\text{and}\quad F_2(T) = 1.$$

For $m>1$, noting that $Q(T)=P(-T)$, we have

(5.10) $$\begin{cases} F_1(T) = \text{the sum of terms in } Q(T) \text{ of even degrees,} \\ F_2(T) = \text{the sum of terms in } Q(T) \text{ of odd degrees.} \end{cases}$$

Hence the claim.

Remark 10. Let $P(T)$ be admissible, pure, and symmetric. Then $Q(T)=P(-T)$, so that we have (5.10) as well. Here $Q(T)$, $F_1(T)$, and $F_2(T)$ are in (D2). Thus for a pure polynomial, the condition (D2) is easier to test than general case.

Remark 11. Let $P(T)$ be admissible. Then from (4.8), (5.9), and (5.10), we see that

(5.11) $$F_1(T) \quad \text{is monic,} \quad \text{and}$$

(5.12) $$\deg F_1(T) > \deg F_2(T).$$

It follows from (5.11) and (M2″) that the degree of a strongly distinguished polynomial is divided by 4.

5.4. We give here a result that some types of admissible polynomials can never be distinguished.

Lemma 5.5. *Let $m\in \boldsymbol{Z}_+$ be odd. Then*
(i) $\Phi_m(1)=0$ *and* $\Phi_m(-1)=-2$ *if* $m=1$.

(ii) $\Phi_m(1)=p$ *and* $\Phi_m(-1)=1$ *if* m *is a power of prime* p.
(iii) $\Phi_m(1)=1$ *and* $\Phi_m(-1)=1$ *if* m *is divided by two distinct primes.*

Corollary. *Let* $m\in \boldsymbol{Z}_+$ *be odd, and* $P(T)=\Phi_m(T^{2^i})$ *with* $i\geqq 0$. *Define* $F_1(T)$ *and* $F_2(T)$ *by* (4.2). *Then*
(i) $F_1(1)=F_2(1)=1$ *if* $m=1$.
(ii) $F_1(1)=(p+1)/2$ *and* $F_2(1)=(1-p)/2$ *if* m *is a power of prime* p.
(iii) $F_1(1)=1$ *and* $F_2(1)=0$ *if* m *is divided by two distinct primes.*

Proof of Lemma 5.5. Since $\Phi_1(T)=T-1$, (i) is clear. Taking $m=1$ in (5.3), we obtain for $r\geqq 0$,

$$\Phi_{p^{r+1}}(T) = (T^{p^{r+1}}-1)/(T^{p^r}-1)\,.$$

Let $T\to\pm 1$ in this equality. Then we see (ii).

For (iii), we take an odd prime p such that $m=p^{r+1}m_0$ with $m_0>1$ not divided by p and $r\geqq 0$. Then again by (5.3),

$$\Phi_m(T) = \Phi_{m_0}(T^{p^{r+1}})/\Phi_{m_0}(T^{p^r})\,.$$

Put $T=\pm 1$ in this equality. Then we get $\Phi_m(\pm 1)=1$. In fact, since m_0 is odd and not equal to 1, both of the numerator and the denominator in the right-hand side have the common values $\Phi_{m_0}(\pm 1)\neq 0$ at $T=\pm 1$. Q.E.D.

Proposition 5.6. *Let* $P(T)$ *be an admissible polynomial. Assume that* $P(T)$ *is not pure, and that* $f_{1,i}=0$ *for* $i>0$ *in the expression* (5.6). *Then* $P(T)$ *cannot be distinguished.*

Proof. Let $F_1(T)$ and $F_2(T)$ be the polynomials defined by (4.2) for an admissible polynomial $P(T)$. Let us consider a condition (※) on $P(T)$:

$$(※)\qquad F_1(1)>0 \quad\text{and}\quad F_2(1)\leqq 0\,.$$

Then from the formula (4.8), we see that the product of admissible polynomials satisfying (※) also satisfies (※). We know from (ii) and (iii) of Corollary to Lemma 5.5 that $\Phi_m(T^{2^i})$ satisffies (※) for odd $m>1$.

Now let $P(T)$ be an admissible polynomial satisfying the assumptions in Proposition 5.6, and let $P'(T)$ and $P''(T)$ be the pure and the impure part of $P(T)$ respectively. Then $P''(T)\neq 1$, from the assumption that $P(T)$ is not pure. And $P''(T)$ satisfies the condition (※), from the assumption that $f_{1,i}=0$ for $i>0$. Let $F_1'(T)$, $F_2'(T)$ or $F_1''(T)$, $F_2''(T)$ be the polynomials defined as in (4.1) corresponding to $P'(T)$ or $P''(T)$, respectively. Then from (4.8),

$$F_1(T) = F_1'(T)F_1''(T)+F_2'(T)F_2''(T)\,.$$

We may assume that $P(T)$ is symmetric, so that $P'(T)$ is symemtric. Since $P'(T)$ is symmetric and pure, we see from Remark 10 that $F_1'(T)$ consists of terms of even degrees and $F_2'(T)$ of odd degrees. On the other hand, since $P''(T)$ is properly impure, both $F_1''(T)$ and $F_2''(T)$ consist of terms of even degrees. Therefore

$F_2'(T)F_2''(T)$ is the part of odd degrees of $F_1(T)$.

Now assume that $P(T)$ is distinguished. Then $P(1)=0$, so that $P'(1)=0$. Therefore we have $F_1'(1)=F_2'(1)>0$. On the other hand, $F_2''(1)\leqq 0$, because $P''(T)$ satisfies (※). Thus we see $F_2'(1)F_2''(1)\leqq 0$. However, by the condition (D2), all the coefficients of $F_1(T)$, hence of $F_2'(T)F_2''(T)$, are non-negative. This implies $F_2'(T)F_2''(T)=0$, which contradicts our assumption that $P'(T)\neq 1$ and $P''(T)\neq 1$. Thus $P(T)$ cannot be distinguished. Q.E.D.

5.5. As wehave mentioned in Remark 10, a pure polynomial is easier to treat in some sense. However, we cannot obtain counterexamples for Problem $\mathrm{I}_1(O(2^{2r+1}, K))$ with r even, if we consider pure polynomials only. In fact, we have the following.

Proposition 5.7. *Let $P(T)$ be a strongly distinguished polynomial. If $P(T)$ is pure, then $Q(1)$ is a power of* 16. *Here $Q(T)=P(T^2)/P(T)$.*

For the proof, we prepare a lemma.

Lemma 5.8. *Let e be an even integer, and j a non-negative odd integer. Then the binomial coefficient $\binom{e}{j}$ is even.*

Proof. Clear from the congruence equality

$$(T-1)^e \equiv (T^2-1)^{e/2} \qquad \mathrm{mod}\ 2\,.$$

Q.E.D.

Proof of Proposition 5.7. Let us write $P(T)$ in the form (5.4). By the assumption that $P(T)$ is pure, the exponent e_s is zero for any even s. Clearly we have $Q(T)=\prod_s(T^s+1)^{e_s}$ and $Q(1)=2^{\Sigma e_s}$.

We claim that every e_s is even. Let us prove this by contradiction. Suppose there exists an s such that e_s is odd, and put $h=\min\{s;\, e_s \text{ is odd}\}$. Note then h is odd. We expand $Q(T)$ by the binomial theorem, and get

$$Q(T)=\sum_{j_s\geqq 0}\binom{e_1}{j_1}\binom{e_3}{j_3}\cdots\binom{e_M}{j_M}T^{j_1+3j_3+\cdots+Mj_M}\,.$$

Here $M=\max\{s;\, e_s\neq 0\}$, and the right-hand side should be understood to be a formal power series in $\boldsymbol{Z}[[T]]$, because e_s may possibly be negative. But it is in fact a finite series. Now look at the coefficient of T^h in this expansion, which reads as follows:

$$\binom{e_h}{1}+\sum_{j_1,\cdots,j_{h-2}}\binom{e_1}{j_1}\binom{e_3}{j_3}\cdots\binom{e_{h-2}}{j_{h-2}},$$

where the sum in the second term is taken over $j_1, j_3, \cdots, j_{h-2}$ with $j_1+3j_3+\cdots+(h-2)j_{h-2}=h$. Then in the second term, not all j_i's are even, because h is odd. Therefore by Lemma 5.8, the second term is even; and by definition $\binom{e_h}{1}=e_h$

is odd. So the coefficient of T^h in $Q(T)$ is odd. On the other hand, we know that $Q(T)$ is monic. Thus there appear odd coefficients in $Q(T)$, both in the terms of odd degrees (term of T^h) and of even degrees (top term). This contradicts the condition (M2″) in §4.3, because as seen in Remark 10, $F_1(T)$ and $F_2(T)$ defined in (D2) are given by (5.10) for the pure polynomial $P(T)$. We have thus proved our claim that every e_s is even.

For Proposition 5.7, it suffices to prove that $\sum_s e_s$ is divided by 4. We have seen in Remark 11 that $\deg P(T)$ is divided by 4, if $P(T)$ is strongly distinguished. Therefore

$$\deg P(T) = \sum_{s:\,\mathrm{odd}} se_s \equiv 0 \qquad \mod 4\,.$$

On the other hand, we have seen above that every e_s is even, so that $(s-1)e_s \equiv 0 \mod 4$ for s odd. Then we obtain

$$\sum e_s \equiv \sum (e_s + (s-1)e_s) \equiv \sum se_s \equiv 0 \qquad \mod 4\,.$$

This completes our proof. Q.E.D.

5.6. Here we give some interesting examples of distinguished polynolials.

Example 3. For $l \geqq 1$, $P(T) = \prod_{i=1}^{l} (T^{2i-1}-1)^2$ is distinguished. We prove this fact in the next section.

Example 4. $P(T) = (T-1)^{88}\Phi_9(T) = (T-1)^{88}(T^9-1)(T^3-1)^{-1}$ is distinguished. This gives an example that there actually appears a negative exponent e_s in the expression (5.4).

Example 5. (Conjecture) $P(T) = (T^l-1)(T-1)^{l^2}$ will be distinguished for odd $l \geqq 1$. This is conjectured from the central limit theorem and a property of the function e^{-x^2} corresponding to Gaussian distribution. For $l \leqq 11$, this is ascertained by direct calculation.

The author computed some other examples, but cannot find until now the general rule to exhaust the distinguished polynomials. It seems a rather complicated problem.

§6. On the alternating tensor product of degree 2.

6.1. In this section, we prove that the polynomials

$$P(T) = (T-1)^2(T^3-1)^2 \cdots (T^{2l-1}-1)^2$$

are distinguished. This is in fact related to some more general result, which states as follows.

Theorem 6. *Let Γ be a group, and π a finite dimensional representation of Γ. Put*

(6.1) $$\pi_1 = \bigoplus_{i\,:\,\mathrm{even}} \wedge^i \pi\,, \quad \textit{and} \quad \pi_2 = \bigoplus_{i\,:\,\mathrm{odd}} \wedge^i \pi\,.$$

Assume that π_1 *and* π_2 *are completely reducible. Then* $\wedge^2\pi_1$ *and* $\wedge^2\pi_2$ *are equivalent to each other.*

Proof. Let θ_1 and θ_2 be the traces of π_1 and π_2, respectively. Since π_1 and π_2 are assumed to be completely reducible, we can argue using their traces. Therefore we see from (2.3) that $\wedge^2\pi_1 \sim \wedge^2\pi_2$ if and only if the equality

$$(\theta_1(g)-\theta_2(g))(\theta_1(g)+\theta_2(g)) = \theta_1(g^2)-\theta_2(g^2)$$

holds for all $g\in\Gamma$. Now we put $D(T;g)=\det(1+T\pi(g))$. Then

$$D(T;g) = \sum_{i\geqq 0} T^i \operatorname{Tr}(\wedge^i \pi(g))\,.$$

Therefore we have

$$D(1;g) = \theta_1(g)+\theta_2(g)\,, \quad \text{and} \quad D(-1;g) = \theta_1(g)-\theta_2(g)\,.$$

Thus

$$\begin{aligned}(\theta_1(g)-\theta_2(g))(\theta_1(g)+\theta_2(g)) &= D(-1;g)D(1;g)\\ &= \det(1-\pi(g))\det(1+\pi(g)) = \det(1-\pi(g^2))\\ &= D(-1;g^2) = \theta_1(g^2)-\theta_2(g^2)\,.\end{aligned}$$

This completes the proof. Q.E.D.

Corollary 1. *Let* Γ, π, π_1, π_2 *be as above. Assume that there exists a* $g_0\in\Gamma$ *such that* $\pi(g_0)$ *does not have* 1 *as its eigenvalue. Then the pair* π_1, π_2 *gives a counter-example for Problem* III$(\Gamma;(1^2))$, *i.e.*, $\pi_1 \not\sim \pi_2$ *and* $\wedge^2\pi_1 \sim \wedge^2\pi_2$.

Proof. We have only to see $\pi_1 \not\sim \pi_2$. This follows from the condition that the function $\operatorname{Tr}\pi_1(g)-\operatorname{Tr}\pi_2(g)=\det(1-\pi(g))$ does not identically vanish on Γ. Q.E.D.

Corollary 2. *Let* $\mathfrak{g}$ *be a semi-simple Lie algebra over* K, *and* π *be a finite dimensional representation of* $\mathfrak{g}$. *Assume that* 0 *is not a weight of* π. *Put* π_1 *and* π_2 *as* (6.1). *Then* $\pi_1 \not\sim \pi_2$ *and* $\wedge^2\pi_1 \sim \wedge^2\pi_2$.

Remark 12. For the groups $\Gamma=SL(l+1,K)$, $Sp(2l,K)$, $SO(2l,K)$ (simple algebraic groups of classical type A_l, C_l, D_l), we can take, as the representation π in Corollary 1, the identical representation of Γ. Also we can take the identical representation for $\Gamma=O(2l+1,K)$ as π. But for $\Gamma=SO(2l+1,K)$, every rational representation π satisfies $\det(1-\pi(g))=0$ for all $g\in\Gamma$. Therefore to apply Corollary 1 to the connected groups of type B_l, we must go to $Spin(2l+1,K)$, the double covering of $SO(2l+1,K)$.

6.2. As an application of Theorem 6, we verify that the polynomial $P(T)=\prod_{i=1}^{l}(T^{2i-1}-1)^2$ is distinguished.

In Theorem 6, we take $\Gamma=SL(2,K)$ and π the $2l$-dimensional irreducible representation. Then for $g=h(t)=\mathrm{diag}(t, t^{-1})$ with $t\in K^\times$, the eigenvalues of $\pi(g)$ are $t^{2l-1}, t^{2l-3}, \cdots, t^{-2l+1}$. Therefore for $g=h(t)$,

$$\mathrm{Tr}\,\pi_1(g)-\mathrm{Tr}\,\pi_2(g) = \det(1-\pi(g)) = \prod_{i=-l+1}^{l}(1-t^{2i-1}),$$

where π_1 and π_2 are as in (6.1). Thus we obtain the Laurent polynomail $p(T)$ associated with the pair π_1, π_2 as $p(T)=\prod_{i=-l+1}^{l}(1-T^{2i-1})$, so that we have $P(T)$ defined by (4.5) as $P(T)=\prod_{i=1}^{l}(T^{2i-1}-1)^2$. This shows that $P(T)$ is distinguished (see Theorem 5 (i)).

6.3. We give here another calculative proof for the fact above that $P(T)=\prod_{i=1}^{l}(T^{2i-1}-1)^2$ is distinguished. It suffices to see that $P(T)$ satisfies (D2), because the other conditions in Definition 4.2 (i) are obvious for $P(T)$.

We recall first a q-analogue of the binomial theorem. Let x and q be indeterminates. Then we have the following identity.

$$\prod_{i=l}^{l}(1+q^{2i-1}x)(1+q^{2i-1}x^{-1}) = \sum_{j=-l}^{l}\begin{bmatrix}2l\\ l+j\end{bmatrix}; q^2\Bigg]q^{j^2}x^j. \tag{6.2}$$

Here $\begin{bmatrix}n\\ r\end{bmatrix}; q\Big]$ is the q-binomial coefficient or the Gaussian polynomial defined as

$$\left[\begin{matrix}n\\ r\end{matrix}; q\right] = \frac{[n]!}{[r]!\,[n-r]!} \qquad \text{with} \quad [a]! = (q^a-1)(q^{a-1}-1)\cdots(q-1).$$

The proof of (6.2) is as follows. Denote by $f(x)$ the left-hand side of (6.2), and write $f(x)=\sum_{j=-l}^{l} a_j x^j$. Then there holds an equality

$$(1+q^{2l-1}x^{-1})qx\cdot f(q^2x) = (1+q^{2l+1}x)\cdot f(x).$$

Comparing the coefficients of x^j in both sides, we obtain

$$(q^{2l+2j}-1)a_j = (q^{2l+1}-q^{2j-1})a_{j-1}.$$

From this and $a_l=q^{l^2}$, we can easily deduce (6.2).

Now putting $q=T$ and $x=1$ in (6.2), we have

$$Q(T) = \sum_{j=-l}^{l}\left[\begin{matrix}2l\\ l+j\end{matrix}; T^2\right]T^{j^2}, \tag{6.3}$$

where $Q(T)=P(T^2)/P(T)$. As $P(T)$ is pure, we see from Remark 10,

$$\begin{cases} F_1(T) = \displaystyle\sum_{j\,:\,\text{even}}\left[\begin{matrix}2l\\ l+j\end{matrix}; T^2\right]T^{j^2}, \\ F_2(T) = \displaystyle\sum_{j\,:\,\text{odd}}\left[\begin{matrix}2l\\ l+j\end{matrix}; T^2\right]T^{j^2}. \end{cases} \tag{6.4}$$

Here $F_1(T)$ and $F_2(T)$ are defined by (4.2). On the other hand, we know that the Gaussian polynomial $\begin{bmatrix} n \\ r \end{bmatrix}; T\Big]$ is symmetric and unimodal. This fact can be proved in various ways (see e.g. Kac [5, Ex. 10.13], Macdonald [8, Ch. I.8, Ex. 4], White [13]). Therefore $\left[\begin{matrix} 2l \\ l+j \end{matrix}; T^2\right]$ symmetric and unimodal in step 2. Furthermore, since $\deg\left[\begin{matrix} n \\ r \end{matrix}; T\right]=r(n-r)$, we have $\frac{1}{2}(\deg\left[\begin{matrix} 2l \\ l+j \end{matrix}; T^2\right]T^{j^2})=l^2$. Thus in the summations in (6.4), each summand $\left[\begin{matrix} 2l \\ l+j \end{matrix}; T^2\right]T^{j^2}$ is symmetric and unimodal in step 2 with the peak at l^2 independent of j, so that $F_1(T)$ and $F_2(T)$ are also symmetric and unimodal in step 2 with their peaks at l^2. This shows that $P(T)$ satisfies (D2), and completes our proof.

§7. On Problem III (Γ; (2, 1)) for $\Gamma=SL(2, K)$.

In §3, we see from Theorem 2 and Remark 4 (2) that for a connected algebraic group Γ, Problem III(Γ; (2, 1)) is affirmative if the dimensions of the representations are not power of 3. In this section, we investigate Problem III(Γ; (2, 1)) for $\Gamma=SL(2, K)$ along the same line as in §§2, 4, and 5. Then we can give more precise result, which does ont require the condition (♯) on the dimensions of representations, as follows.

Theorem 7. *Let π_1 and π_2 be finite dimensional rational representations of $SL(2, K)$. Then $S_{(2,1)}\pi_1 \sim S_{(2,1)}\pi_2$ implies $\pi_1 \sim \pi_2$.*

For the proof of this theorem, we recall some elementary facts on cyclotomic fields.

Let $\zeta_m \in \boldsymbol{C}$ denote a primitive m-th root of unity. We fix a primitive cubic root of unity denoted by ω.

First of all, we know the irreducibility of $\Phi_m(T)$ over $\boldsymbol{Q}$, so that $[\boldsymbol{Q}(\zeta_m): \boldsymbol{Q}]=\varphi(m)$. From this we see $\boldsymbol{Q}(\zeta_m) \cap \boldsymbol{Q}(\zeta_{m'})=\boldsymbol{Q}$ if m and m' are relatively prime to each other. Especially, $\Phi_m(T)$ is irreducible over $\boldsymbol{Z}[\omega]$ if m is not divided by 3.

On the contrary, if n is divided by 3, we have $\boldsymbol{Q}(\zeta_n) \supset \boldsymbol{Q}(\omega)$ and $[\boldsymbol{Q}(\zeta_n): \boldsymbol{Q}(\omega)]=\varphi(n)/2$. Therefore $\Phi_n(T)$ is factorized into two polynomials, both irreducible over $\boldsymbol{Z}[\omega]$ of degree $\varphi(n)/2$. We see this factorization explicitly in the following equality (7.2).

Lemma 7.1. *Let m be a positive integer not divided by* 3. *Then we have*

$$\Phi_m(T^3) = \Phi_m(T)\Phi_m(\omega T)\Phi_m(\omega^2 T)\,, \tag{7.1}$$

and for $r \geqq 0$,

$$\Phi_{3^{r+1}m}(T) = \Phi_m(\omega T^{3^r})\Phi_m(\omega^2 T^{3^r})\,. \tag{7.2}$$

Proof. The equality (7.1) is easily deduced from (CP 3) in §5.1. In fact,

$$\begin{aligned}\Phi_m(T^3) &= \prod_{d|m} (T^{3d}-1)^{\mu(m/d)} \\ &= \prod_{d|m} ((T^d-1)(\omega T^d-1)(\omega^2 T^d-1)^{\mu(m/d)} \\ &= \prod_{d|m} ((T^d-1)(\omega^d T^d-1)(\omega^{2d} T^d-1))^{\mu(m/d)} \\ &= \Phi_m(T)\Phi_m(\omega T)\Phi_m(\omega^2 T)\,.\end{aligned}$$

For (7.2), replace T by T^{3^r} in (7.1). Then we have

$$\Phi_m(T^{3^{r+1}}) = \Phi_m(T^{3^r})\Phi_m(\omega T^{3^r})\Phi_m(\omega^2 T^{3^r})\,.$$

Recalling (5.3) with $p=3$, we see (7.2) clearly from this. Q.E.D.

We see from the remark just before the lemma that $\Phi_m(\omega T^{3^r})$ and $\Phi_m(\omega^2 T^{3^r})$ are irreducible over $\boldsymbol{Z}[\omega]$, for m not divided by 3.

For the proof of Theorem 7, we prepare the following

Lemma 7.2. *Put* $\phi_m(T)\in\boldsymbol{Z}[\omega][T^{1/2}, T^{-1/2}]$ *as*

$$\phi_m(T) = \Phi_m(\omega T)T^{-\varphi(m)/2}\,. \tag{7.3}$$

Then we have

$$\phi_m(T^{-1}) = \eta_m\bar{\phi}_m(T)\,. \tag{7.4}$$

Here η_m *is a unit of* $\boldsymbol{Z}[\omega]$ *(i.e.,* $\eta_m=\pm 1, \pm\omega, \pm\omega^2$*), and* — *(bar) means the complex conjugate.*

Proof. From (CP 3), we see

$$\Phi_m(T^{-1}) = \prod_{d|m} (T^{-d}-1)^{\mu(m/d)} = \pm\Phi_m(T)T^{-\varphi(m)}\,.$$

Therefore

$$\Phi_m(\omega T^{-1}) =\Phi_m((\bar{\omega}T)^{-1}) = \pm\Phi_m(\bar{\omega}T)\cdot(\bar{\omega}T)^{-\varphi(m)} = \eta_m\bar{\Phi}_m(\omega T)T^{-\varphi(m)}\,,$$

where $\eta_m=\pm\omega^{\varphi(m)}$. Then (7.4) is clear from this. Q.E.D.

Proof of Theorem 7. We put $p(T)=c_{\pi_1}(T)-c_{\pi_2}(T)$ (for $c_\pi(T)$, see §4.1), and

$$\begin{cases} s_1(T) = c_{\pi_1}(T)-\omega c_{\pi_2}(T)\,, \\ s_2(T) = c_{\pi_1}(T)-\omega^2 c_{\pi_2}(T)\,, \end{cases} \tag{7.5}$$

$$s(T) = s_1(T)s_2(T) = c_{\pi_1}(T)^2+c_{\pi_1}(T)c_{\pi_2}(T)+c_{\pi_2}(T)^2\,. \tag{7.6}$$

Then by Remark 4 (2), the assumption $S_{(2,1)}\pi_1\sim S_{(2,1)}\pi_2$ amounts to

$$p(T)s(T) = p(T^3)\,. \tag{**}$$

Put $T=1$ in (**), and note that $c_{\pi_1}(1)$ and $c_{\pi_2}(1)$ are positive. Then we see

$p(1)=0$, whence $c_{\pi_1}(1)=c_{\pi_2}(1)$.

For Theorem 7, it suffices to show that (**) implies $p(T)=0$. We prove this by contradiction. Assume that $p(T)\neq 0$, and write it as $p(T)=\sum_{j=-L}^{L} b_j T^j$ with $b_L\neq 0$. Moreover, we define the polynomials $P(T)$, $S_1(T)$, $S_2(T)$, and $S(T)$ as

$$(7.7)\qquad \begin{cases} P(T) = T^L p(T)\,, \\ S_1(T) = T^L s_1(T) \quad\text{and}\quad S_2(T) = T^L s_2(T)\,, \\ S(T) = S_1(T)S_2(T) = T^{2L}s(T)\,. \end{cases}$$

Then from (**), we have

$$(***)\qquad P(T)S(T) = P(T^3)\,.$$

Now by Proposition 5.3, $P(T)$ is written, up to a constant factor, as a product of the polynomials $\Phi_m(T^{3^i})$, where m is not divided by 3 and $i\geqq 0$. Therefore by (5.3), we see that $S(T)$ is written as a product of the polynomials $\Phi_{3^{i+1}m}(T)$ with m not divided by 3 and $i\geqq 0$. We note that $\boldsymbol{Z}[\omega]$ is a U.F.D. (=unique factorization domain). So that the polynomial ring $\boldsymbol{Z}[\omega][T]$ is also a U.F.D.. Then by Lemma 7.1, we see that up to a unit factor, $S_1(T)$ is written as a product of the irreducible polynomials $\Phi_m(\omega T^{3^i})$ and $\Phi_m(\omega^2 T^{3^i})$ with m not divided by 3 and $i\geqq 0$. Thus up to a unit factor, $s_1(T)$ is a product of Laurent polynomials $\phi_m(T^{3^i})$ and $\bar{\phi}_m(T^{3^i})$ with m not divided by 3 and $i\geqq 0$. From this and Lemma 7.2, we see $s_1(T^{-1})=\eta\cdot\bar{s}_1(T)$ with some unit η. On the other hand, since $c_{\pi_1}(T)$ and $c_{\pi_2}(T)$ are invariant under $T\mapsto T^{-1}$, we have $s_1(T)=s_1(T^{-1})$. Therefore we obtain $s_1(T)=\eta\cdot\bar{s}_1(T)$, whence

$$(1-\eta)\cdot c_{\pi_1}(T) = \omega(1-\eta\omega)\cdot c_{\pi_2}(T)\,.$$

Put $T=1$ in this equality, and recall $c_{\pi_1}(1)=c_{\pi_2}(1)>0$.Then we see $1-\eta=\omega(1-\eta\omega)$. If $\eta=1$, then we get $1-\omega=0$, a contradiction. And if $\eta\neq 1$, then we obtain $c_{\pi_1}(T)=c_{\pi_2}(T)$, which contradicts our assumption $p(T)\neq 0$. Hence the theorem.

Q.E.D.

Department of Mathematics
Kyoto University

References

[1] N. Bourbaki, Groupes et algèbres de Lie, Chapitres 7 et 8, Harmann, Paris, 1975.
[2] T. Hirai, Structure of unipotent orbits and Fourier transform of unipotent orbital integrals for semisimple Lie groups, in "Lectures on Harmonic Analysis on Lie Groups and Related Topics", Lectures in Mathematics, Kyoto Univ. 14 (1982), pp. 75–138.
[3] T. Hirai, On semisimple Lie algebras over algebraically closed fields, Proc. Japan Acad. **60**, Ser A (1984), 225–228.
[4] N. Jacobson, Lie algebras, Intersc. Tracts 10, J. Wiley, New York, 1962.
[5] V. G. Kac, Infinite dimensional Lie algebras, Progress in Math. 44, Birkhäuser, Boston-Basel-Stuttgart, 1983.

[6] B. Kostant, On Macdonald's η-function formula, the Laplacian and generalized exponents, Advances in Math., **20** (1976), 179–212.
[7] S. Lang, Algebra, Addison-Wesley, 1965.
[8] I. G. Macdonald, Symmetric functions and Hall polynomials, Oxford Univ. Press, 1979.
[9] J.-P. Serre, Représentations linéaires des groupes finis, Hermann, Paris, 1971.
[10] T. A. Springer and R. Steinberg, Conjugacy classes, in Lecture note in Math. vol. 131 (1970), Springer-Verlag, pp. 167–266.
[11] J. H. M. Wedderburn, Lectures on matrices, Amer. Math. Soc., 1934.
[12] H. Weyl, The classical groups, Princeton Univ. Press, 1946.
[13] D.E. White, Monotonicity and unimodality of the pattern inventory, Advances in Math., **38** (1980), 101–108.