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Semi-ampleness of the numerically effective part of Zariski decomposition

By

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§0. Introduction.

The minimal model conjecture is the central problem in the classification theory of algebraic varieties. Though big progress was made in this conjecture by the effort of Kawamata [Ka3], many difficulities yet remain to be overcome.

Let $f: X \to Y$ be the contraction with respect to an extremal ray which is isomorphic in codimension one. The existence of a flip of $f: X \to Y$ is one of key steps to prove the minimal model conjecture (for the definition of a flip, see §4). As will be showed in §4, it is equivalent to saying that the pluricanonical ring $R = \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nd(K_X + f^*(rA))))$ is finitely generated, where A is an ample divisor on Y, r is a sufficiently large integer and d is an integer such that dK_X is a Cartier divisor. The main aim of this article is to show that the existence of suitable decomposition of $K_X + f^*(rA)$ is enough for R to be finitely generated. Indeed we have the following theorem which is, in some sense, a generalization of the theorem in [Ka3, Theorem 6.1] and (A.5) in [F4].

Theorem 0. *Let (X, Δ) be a normal complete variety with only log-terminal singularities and D a nef and good $\boldsymbol{Q}$-Cartier divisor on X. Assume that $(K_X + \Delta) + D$ admits a Zariski decomposition with rational coefficients $(K_X + \Delta) + D = P + N$, where P is the nef part of this decomposition. If there are two positive rational numbers a, b such that $\nu(X, D) \geqq \nu(X, D + aP)$ and $\kappa(X, D + bP) \geqq 0$, then P is semi-maple.*
(See §1, for the definitions of the terminologies used in the theorem.)

Applying the theorem to the case where D is the pullback of rA by some birational morphism, we see that if $K_X + f^*(rA)$ admits a Zariski decomposition in the sense of §4 by changing X, then there exists a flip of $f: X \to Y$. It seems to the author that the theory of generalized Zariski decomposition must play a key role to establish the minimal model conjecture.

§1 is preliminaries to prepare our notation and to recall some of known results. §2 is devoted to prove the Theorem 0. Our proof of the theorem is just

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the same as Kawamata [Ka3] except for the final step. In §3, as application of our theorem, we prove that log-canonical ring with $\kappa \leqq 2$ is finitely generated. In §4, we discuss the reltaionship among several conjectures which are relevant to our theorem.

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§1. Preliminaries.

(1.1) Notation.

Throughout this paper, all the varieties are defined over an algebraically closed field k of characteristic zero.

(1.1.1) Let X be a normal complete variety of dimension n. We set

$$Z_{n-1}(X) = \{\text{Weil divisors on } X\}, \quad \text{and}$$
$$\mathrm{Div}(X) = \{\text{Cartier divisors on } X\}.$$

Then there is a natural injective homomorphism

$$\mathrm{Div}(X) \to Z_{n-1}(X).$$

A $\boldsymbol{Q}$-divisor (or $\boldsymbol{Q}$-Cartier divisor) on X is an element of $Z_{n-1}(X)\otimes\boldsymbol{Q}$ (or $\mathrm{Div}(X)\otimes\boldsymbol{Q}$, resp.). A normal variety is said to be $\boldsymbol{Q}$-factorial if every Weil divisor is a $\boldsymbol{Q}$-Cartier divisor. For any surjective morphism $f: X \to Y$ of normal complete varieties, we have a homomorphism

$$f^* : \mathrm{Div}(Y)\otimes\boldsymbol{Q} \to \mathrm{Div}(X)\otimes\boldsymbol{Q}.$$

Furthermore, if f is birational, we can define a homomorphism

$$f_* : Z_{n-1}(X)\otimes\boldsymbol{Q} \to Z_{n-1}(Y)\otimes\boldsymbol{Q}.$$

Clearly $f_* f^* = 1_{\mathrm{Div}(X)\otimes\boldsymbol{Q}}$.

A Q-Cartier divisor D on X is called nef (or numerically effective) if $(D\cdot C)\geqq 0$ for any integral curve C on X. Let D be a Q-Cartier divisor on X and d a positive integer such that dD is a Cartier divisor. Then we define

$$\kappa(X, D) = \kappa(X, dD) = \begin{cases} -\infty & \text{if } h^0(X, \mathcal{O}_X(mdD)) = 0 \text{ for any } m \geqq 1, \\ \max\{\dim \Phi_{|mdD|}(X) \mid m \geqq 1\}, & \text{otherwise,} \end{cases}$$

which is called D-dimension. Let $D_1, \cdots, D_e$ be $\boldsymbol{Q}$-Cartier divisors on X. We define $D_1 \cdots D_e \equiv 0$ if for any e-cycle W on X, $(D_1 \cdots D_e \cdot W) = 0$. If X is projective and $D_1, \cdots, D_e$ are nef, then $(D_1 \cdots D_e \cdot A^{n-e}) = 0$ for some ample divisor A implies $D_1 \cdots D_e \equiv 0$. The numerical Kodaira dimension of a nef $\boldsymbol{Q}$-Cartier divisor D on X is defined to be the integer

$$\nu(X, D) = \max\{e \mid D^e \not\equiv 0\}.$$

It is well known that $\kappa(X, D) \leqq \nu(X, D)$ (cf. [Ka3, Proposition 2.2]). If the equality holds in the above inquality, then D is said to be good. D is called big if $\nu(X, D)=n$. Due to [Ka1, Lemma 3], D is big if and only if $\kappa(X, D)=n$.

(1.1.2) Let U be a smooth part of X and $i: U \to X$ the inclusion map. Then we define $\omega_X = i_*(\Omega_X^n)$. It is easy to see that there is a Weil divisor K_X such that $\mathcal{O}_X(K_X)=\omega_X$. K_X is called a canonical divisor of X. X is said to be $\boldsymbol{Q}$-Gorenstein if K_X is a $\boldsymbol{Q}$-Cartier divisor. A $\boldsymbol{Q}$-Gorenstein variety X has only canonical singularities (or terminal singularities) if there is a desingularization $\mu: Y \to X$ such that $K_Y = \mu^*(K_X) + \Sigma_i a_i E_i$ for some non-negative (or positive, resp.) rational numbers a_i, where $\Sigma_i E_i$ is the exceptional locus of μ.

(1.1.3) According to Kawamata, the rouding-up $\ulcorner r \urcorner$ of a real number r is the integer $\min\{t \in \boldsymbol{Z} \mid r \leqq t\}$. Let $D = \Sigma_i a_i D_i$ be a $\boldsymbol{Q}$-divisor, where $a_i \in \boldsymbol{Q}$ and D_i's are mutually distinct prime divisors. Then we define

$$\begin{aligned} \ulcorner D \urcorner &= \Sigma_i \ulcorner a_i \urcorner D_i \,, \\ [D] &= \Sigma_i -\ulcorner -a_i \urcorner D_i \,, \\ \{D\} &= D - [D] \,. \end{aligned}$$

(1.1.4) Let Δ be an effective $\boldsymbol{Q}$-divisor on X such that $[\Delta]=0$. The pair (X, Δ) is said to have only log-terminal singularities if $K_X + \Delta$ is a $\boldsymbol{Q}$-Cartier divisor and there is a desingularization $\mu: Y \to X$ such that $E = ((\text{the exceptional locus of } \mu) \cup \mu^{-1}(\operatorname{Supp} \Delta))_{\mathrm{red}}$ is a divisor with only normal crossings and that $K_Y = \mu^*(K_X + \Delta) + \Sigma_i a_i E_i$ for some $a_i \in \boldsymbol{Q}$ with $a_i > -1$, where $E = \Sigma_i E_i$. Let $\mu': Y' \to X$ be an arbitrary desingularization such that $E' = ((\text{the exceptional locus of } \mu') \cup \mu'^{-1}(\operatorname{Supp} \Delta))_{\mathrm{red}}$ is a divisor with only normal crossings. Then the above condition with respect to E' is satisfied (cf. [Ka2, Lemma 1.5]).

(1.1.5) A $\boldsymbol{Q}$-Cartier divisor D on X is said to have a Zariski decomposition with rational coefficients if D has a decomposition $D = P + N$ such that

(i) P is a nef $\boldsymbol{Q}$-Cartier divisor on X and N is an effective $\boldsymbol{Q}$-Cartier divisor on X, and

(ii) there is a positive integer d such that dP and dD are Cartier divisors, and that $h^0(X, \mathcal{O}_X(mdP)) = h^0(X, \mathcal{O}_X(mdD))$ for any integer $m \geqq 0$.

P is called the nef part (or numerically effective part) of this decomposition and N is said the negative part.

Proposition (1.1.6) (cf. [F4, (A.3)]) *The notation being as above, we assume that $\kappa(X, D) = \dim X$. Then a Zariski decomposition with rational coefficients of D is uniquely determined if it exists.*

Proof. It is sufficient to show that for any effective $\boldsymbol{Q}$-Cartier divisor E on X such that $D - E$ is nef, $E - N$ is effective. We may assume that D, P, N and E are Cartier divisors, considering mD, mP, mN and mE for some positive integer m. We take a resolution of singularities $\mu: Y \to X$ such that there are effective divisors H, N' and E' on Y such that $\mu^* N = H + N'$, $\mu^* E = H + E'$ and $\operatorname{Supp} N' \cap \operatorname{Supp} E' = \phi$. We set $L = \mu^* D - H$. From the facts that $\mu^*(D - E)$ and $\mu^*(D - N)$

are nef and $L-\mu^*P=N'$, L is nef and big. Therefore due to [F1. Theorem (6.12)], there is an effective divisor Γ on Y such that $Bs|tL-\Gamma|=\phi$ for any interger $t\gg 0$. On the other hand, since $L=\mu^*P-N'$, we have

$$h^0(Y, \mathcal{O}_Y(tL-tN'))\leqq h^0(Y, \mathcal{O}_Y(tL))\leqq h^0(Y, \mathcal{O}_Y(t\mu^*D)), \quad \text{and}$$
$$h^0(Y, \mathcal{O}_Y(tL-tN')) = h^0(Y, \mathcal{O}_Y(t\mu^*(P)).$$

Hence from the definition of the Zariski decomposition,

$$H^0(Y, \mathcal{O}_Y(tL-tN'))\simeq H^0(Y, \mathcal{O}_Y(tL)).$$

Therefore tN' is a sum of fixed components of $|tL|$. Combining these facts, we have $tN'\leqq\Gamma$ for all $t\gg 0$. Hence $N'=0$. Thus we get $E-N$ is effective. q.e.d.

(1.2) Known results.

Let X be a complete irreducible variety and D a Cartier divisor on X. We define

$$SBs(D) = \bigcap_{n>0} Bs|nD|,$$

which is called the stable base locus of D. Fujita [F1] proved the following two results.

(1.2.1) [F1, Proposition (1.18)] There exists a positive integer m such that $Bs|mD|=SBs(D)$.

(1.2.2) [F1, Theorem (1.20)] Let $f: X\to Y$ be a surjective morphism of an irreducible variety X onto a normal variety Y. Then $SBs(f^*D)=f^{-1}(SBs(D))$ for any Cartier divisor D on Y.

A Cartier divisor D on a normal variety X is said to be semi-ample if $SBs(D)=\phi$. We say that a $\boldsymbol{Q}$-Cartier divisor D is semi-ample if there is a positive integer d suhc that dD is a semi-ample Cartier divisor. Kawamata [Ka3] genaralized slightly a result in Kollár [Ko2].

(1.2.3) [Ka3, Theorem (3.2)] Let X be a non-singular projective variety, L a semi-ample $\boldsymbol{Q}$-Cartier divisor on X such that L_{red} has only normal crossings, and D be an effective divisor on X. Assume that there is an effective divisor D' such that $D+D'\in|mL|$ and mL is a Cartier divisor for some integer m. Then homomorphisms induced by multiplying D

$$\phi_D^i : H^i(X, \mathcal{O}_X(\ulcorner L\urcorner+K_X)) \to H^i(X, \mathcal{O}_X(\ulcorner L\urcorner+D+K_X))$$

are injective for all $i\geqq 0$.

Next we introduce the notion of a normal crossing variety. A reduced equidimenisonal projective scheme X is called a normal crossing variety if the completion of a local ring at each point of X is isomorphic to

$$k[[x_0, \cdots, x_d]]/(x_0 \cdots x_r).$$

Since a normal crossing variety is a locally complete intersection, X has the invertible dualizing sheaf, which is denoted by $\mathcal{O}_X(K_X)$. Let X_0 be a normalization

of X. We set

$$X_n = X_0 \underset{X}{\times} \cdots \underset{X}{\times} X_0 \quad (n+1 \text{ factors}), \quad \text{and}$$

$$\varepsilon_n : X_n \to X \quad (\text{natural projection}),$$

for any integer $n \geqq 0$. It is easy to see that X_n is non-singular for all n and a correspondance $\Delta_n \mapsto X_n$ gives a simplicial scheme X. in the sence of Deligne [D]. The union of the images of lower dimensional irreducible components of $X_{n'}$ for $n' > n$ forms a divisor on X_n with only normal crossings which is denoted by B_n. A Cartier divisor D on X is said to be permissible if the pullback of D by ε_n induces a Cartier divisor on X_n for all n, i.e., locally no strata of X are contained in Supp (D). An element of {permissible Cartier divisors on X} $\otimes \boldsymbol{Q}$ is called a permissible $\boldsymbol{Q}$-Cartier divisor. We say that a permissible Cartier divisor on X has only normal crossings if $\varepsilon_n^*(D) \cup B_n$ has only normal crossings on X_n for all n. Kawamata [Ka3, Theorem 5.1] proved the following theorem which is a generalization of non-vanishing theorem of Shokurov [S].

(1.2.4) Let X be a normal corssing variety, Z a projective variety, $f: X \to Z$ a morphism, H a permissible Cartier divisor on X, A a permissible $\boldsymbol{Q}$-Cartier divisor on X, and let q be a positive integer. Assume that the following conditions are satisfied.

(i) f induces a surjective morphism from each irreducible component of X_n onto Z for all n.

(ii) Supp(A) has only normal crossings and $\ulcorner A \urcorner$ is effective.

(iii) There is a nef Cartier divisor H_0 on Z such that $\mathcal{O}_X(qH) = f^*(\mathcal{O}_Z(H_0))$.

(iv) There is an ample divisor L_0 on Z such that $\mathcal{O}_X(q(H+A-K_X)) = f^*(\mathcal{O}_Z(L_0))$, where qA is a permissible Cartier divisor.

Then there is a positive integer p such that

$$H^0(X, \mathcal{O}_X(ptH + \ulcorner A \urcorner)) \neq 0$$

for all sufficiently large integer t.

Finally we recall Hironaka's result on resolution of singularities. Let X be a non-singular variety and B a non-singular subvarity of X. We say that an effective divisor E on X has only normal crossings with B if for any point $x \in B$, there exists a regular system of parameters of $\mathcal{O}_{X,x}$, say $(z_1, \cdots, z_n)$, such that the ideal in $\mathcal{O}_{X,x}$ of each irreducible component of E containing x is generated by one of z_i's, and that the ideal of B in $\mathcal{O}_{X,x}$ is generated by some of z_i's. A resolution datum R on X is an object of the following form;

$$E = (E_1, \cdots, E_\alpha : D),$$

where $\Sigma_i E_i$ is an effective divisor on X with only normal crossings and D is an effective divisor on X. A resolution datum R on X is said to be resolved if D_{red} has only normal crossings. Let C be a non-singular subvariety of X and $f: X' \to X$ a blowing-up with center C. We say that f is permissible with respect to a resolu-

tion datum $R=(E_1, \cdots, E_\alpha : D)$ if $\Sigma_i E_i$ has only normal crossings with C. Then we set

$$f^*R = (E'_1, \cdots, E'_\alpha, E'_{\alpha+1} : f^*D),$$

where E'_i is the strict transform of E_i by f for $1 \leqq i \leqq \alpha$ and $E'_{\alpha+1} = f^{-1}(D)$. Clearly f^*R is also a resolution datum on X'. By the fundamental theorem $I_2^{N,n}$ [H1, p170], we have the following theorem.

(1.2.5) Let R be an arbitrary resolution datum on X. Then there is a sequence of birational morphisms;

$$X_r \xrightarrow{f_{r-1}} X_{r-1} \xrightarrow{f_{r-2}} X_{r-2} \to \cdots \to X_1 \xrightarrow{f_0} X_0 = X$$

such that

(i) $X_{i+1} \xrightarrow{f_i} X_i$ $(0 \leqq i < r)$ is a blowing-up with smooth center C_i,
(ii) $R_0 = R$,
(iii) f_i is permissible with respect to R_i $(0 \leqq i < r)$,
(iv) $R_{i+1} = f_i^* R_i$ $(0 \leqq i < r)$, and that
(v) R_r is resolved.

Furthermore, we need Hironaka's flattening theorem.

(1.2.6) [H2. Flattening theorem] Let $f: V \to S$ be a proper surjective morphism of algebraic varieties. Then there exists a birational morphism $\pi: S' \to S$, which is obtained by a sequence of blowing-ups, such that $f': V \underset{S}{\times} S'/\text{torsion} \to S'$ is a flat morphism.

$$\begin{array}{ccc} V \underset{S}{\times} S'/\text{torsion} & \xrightarrow{\pi'} & V \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{\pi} & S \end{array}$$

We say that $f': V \underset{S}{\times} S/\text{torsion} \to S'$ is a flat model of $f: V \to S$.

§2. Proof of Theorem 0.

This theorem is proved by the same method as in Kawamata [Ka3].

Let us consider the following diagram;

$$\begin{array}{ccc} V' & \xrightarrow{\pi'} & V \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow{\pi'} & T \end{array}$$

(i) V, T and T' are non-singular projective varieties, and V' is a projective variety, and

(ii) π and π' are birational morphisms, f is a surjective morphism with con-

nected fibres and f' is a flat morphism. Then we have,

Lemma (2.1) *The notation being as above, let E be an effective $\boldsymbol{Q}$-divisor on V such that the support of E is not mapped onto T and E is f-nef, i.e., for any curve Γ on V such that $f(\Gamma)$ is a point, $(E\cdot\Gamma)\geqq 0$. Then there is an effective $\boldsymbol{Q}$-divisor F on T' such that $f'^*F=\pi'^*E$.*

Proof. We set $d=\dim V$ and $e=\dim T$. We may assume $d-e\geqq 1$. Let $\lambda\colon W\to V'$ be a desingularization of V' and h the composition of f' and λ. Let F be a maximal effective $\boldsymbol{Q}$-divisor on T' such that $X=\pi'^*E-f'^*F$ is effective. It is obvious that X is f'-nef. Suppose that $X\neq 0$. Then there is a prime divisor D on W such that $h(D)\subset f'(\operatorname{Supp}X)$, $h(D)$ is a divisor on T' and that D is not contained in λ^*X. Let A, B be very ample divisors on T', W respectively and we take general members

$$A_1, \cdots, A_{e-1}\in |A|, \quad B_1, \cdots, B_{d-e-1}\in |B|.$$

We set

$$C = A_1\cap\cdots\cap A_{e-1}, \quad S = h^{-1}(C)\cap B_1\cap\cdots\cap B_{d-e-1}.$$

Then the following conditions are satisfied.

(1) $h'=h|_S\colon S\to C$ is a surjective morphism of a non-singular surface onto a non-singilar curve with connected fibres.

(2) $\lambda^*X|_S$ and $D|_S$ are non-zero effective divisors.

Hence we have a curve Γ on S such that $h'(\Gamma)$ is a point and $(\lambda^*X|_S\cdot\Gamma)<0$. This is a contradiction. q.e.d.

Now we start the proof of the theorem.

Step 1. First we claim that there is a diagram;

$$\begin{tikzcd} & Y \arrow[dl, "\mu"'] \arrow[dr, "f"] & \\ X & & Z \end{tikzcd}$$

such that

(2.2) Y and Z are non-singular projective varieties,

(2.3) μ is a birational morphism and f is a surjective morphism with connected bfires,

(2.4) there is a nef and big $\boldsymbol{Q}$-Cartier divisor D_0 on Z such that $f^*D_0=\mu^*D$, and that

(2.5) there is a nef $\boldsymbol{Q}$-Cartier divisor P_0 on Z such that $f^*P_0=\mu^*P$.

We set $\kappa=\nu(X, D)=\kappa(X, D)$ and $n=\dim X$. Fix a positive integer s such that the dimension of the image of

$$\Phi_{|sD|}\colon X\cdots\to \operatorname{Proj}\bigoplus_{n=0}^{\infty}\operatorname{Sym}^n(H^0(X, \mathcal{O}_X(dsD)))$$

is equal to κ. By resolution of singularities and the Stein factorization, we obtain the following diagram;

$$\begin{tikzcd} Y \arrow[r, "\mu"] \arrow[d, "f"'] & X \arrow[d, dotted] \\ Z \arrow[r, "\pi"] & Z_0 = \Phi_{|sD|}(X) \end{tikzcd}$$

such that

(a) Y and Z are non-singular projective varieties,

(b) μ is a birational morphism, f is a surjective morphism with connected fibres and π is a generically finite morphism, and that

(c) there is a nef and big Cartier divisor M_0 on Z such that M_0 is base-point free and $\mu^*(sD)=f^*M_0+F$, where F is the fixed part of $|\mu^*(sD)|$.

Let H be an ample divisor on Y. we caluculate

$$\begin{aligned} 0 &= (\mu^*(sD)^{\kappa+1}\cdot H^{n-\kappa-1}) = ((M+F)^{\kappa+1}\cdot H^{n-\kappa-1}) \\ &= \sum_{i=1}^{\kappa} s^{\kappa-i}((\mu^*D)^{\kappa-i}\cdot M^i\cdot H^{n-\kappa-i}\cdot F)+(M^{\kappa+1}\cdot H^{n-\kappa-1}), \end{aligned}$$

where $M=f^*M_0$.

Since M_0 is base-point free and D is a nef, we have

$$((\mu^*D)^{\kappa-i}\cdot M^i\cdot H^{n-\kappa-i}\cdot F)=0 \quad \text{for} \quad 0\leqq i\leqq\kappa \quad \text{and} \quad M^{\kappa+1}\equiv 0.$$

In particular, $(M^\kappa\cdot F\cdot H^{n-\kappa-1})=0$. Hence no irreducible components of F are mapped onto Z. Moreover, it is obvious that F is f-nef. By lemma (2.1), taking a flat model of $f\colon Y\to Z$ and resolution of singularities, if necessary, we may assume that there is an effective $\boldsymbol{Q}$-divisor F_0 on Z such that $F=f^*F_0$. Hence we have a nef and big $\boldsymbol{Q}$-divisor D_0 on Z such that $f^*D_0=\mu^*D$. On the other hand, since D and P are nef, $(D+aP)^{\kappa+1}\equiv 0$ implies $P\cdot D^\kappa\equiv 0$, considering the expansion

$$(D+aP)^{\kappa+1}\equiv \sum_{i=0}^{\kappa} aD^i\cdot(D+aP)^{\kappa-i}\cdot P+D^{\kappa+1}.$$

By the same argument as above, $P\cdot D^\kappa\equiv 0$ implies $\mu^*(P)\cdot M^\kappa\equiv 0$. Hence we have

$$\mu^*(D+P)\cdot M^\kappa\equiv(1/s)(M^{\kappa+1}+F\cdot M^\kappa)+b\mu^*(P)\cdot M^\kappa\equiv 0.$$

Therefore our claim is clear by lemma (2.1).

Step 2. $\kappa(X, P)\geqq 0$.

Fix a positive integer d such that dP, dP_0 and dN are Cartier divisors. Thanks to [Ka1, Lemma 3 and 4], there exists a $\boldsymbol{Q}$-divisor D_1 on Z such that $D_0-\delta D_1$ is ample for all rational number δ with $0<\delta<1$. Changing a model of Y, if necessary, we may assume (2.6), (2.7), (2.8) and (2.9).

(2.6) There is a divisor $E=\Sigma_{i\in I}E_i$ on Y with only normal crossings, where $I=\{1, \cdots, \alpha\}$.

(2.7) $(\mu^*N)_{\mathrm{red}}\leqq E$.

(2.8) $(T \cup \mu^{-1}(\mathrm{Supp}\,\Delta))_{\mathrm{red}}$ is a divisor whose support is contained in E, where T is the exceptional locus of μ.

(2.9) $f^*D_1 = \Sigma_{i\in I} b_i E_i$ with b_i non-negative rational numbers.

By our assumotion, if we set

$$K_Y - \mu^*(K_X+\Delta) + \mu^*N = \Sigma_{i\in I} a_i E_i$$

with $a_i \in \boldsymbol{Q}$, then $a_i > -1$ for $i \in I$. We can check easily that

(2.10) (i) $K_Y + \mu^*D = \mu^*P + \Sigma_{i\in I} a_i E_i$,
(ii) $\mu_*(\ulcorner K_Y - \mu^*(K_X+\Delta)\urcorner) = \ulcorner -\Delta \urcorner = 0$, and
(iii) $\mu_*(\ulcorner \mu^*N \urcorner) = \ulcorner N \urcorner$.

We define

$$A_1 = \Sigma_{i\in I}(a_i - \delta b_i)E_i\,.$$

Then $\ulcorner A_1 \urcorner$ is effective if δ is small enough, and

$$\mu^*(dP) + A_1 - K_Y = f^*((d-1)P_0 + (D_0 - \delta D_1))$$

Hence by (1.2.4), there is a positive integer p_1 such that

$$H^0(Y, \mathcal{O}_Y(tp_1\mu^*(dP) + \ulcorner A_1 \urcorner)) \neq 0$$

for $t \gg 0$. On the other hand, by (2.10),

$$\begin{aligned}\mu_*(\ulcorner A_1 \urcorner) &\leqq \mu_*(\ulcorner K_X - \mu^*(K_X+\Delta) + \mu^*N \urcorner)\\ &\leqq \mu_*(\ulcorner K_Y - \mu^*(K_X+\Delta)\urcorner + \ulcorner \mu^*N \urcorner)\\ &= \ulcorner N \urcorner\,.\end{aligned}$$

Hence we have

$$\ulcorner A_1 \urcorner \leqq t(T + p_1\mu^*(dN))$$

for $t \gg 0$. Therefore, we obtain the following inequalities

$$\begin{aligned}h^0(Y, \mathcal{O}_Y(tp_1\mu^*(dP))) &\leqq h^0(Y, \mathcal{O}_{Y1}(tp_1\mu^*(dP) + \ulcorner A_1 \urcorner))\\ &\leqq h^0(Y, \mathcal{O}_Y(\mu^*(tp_1 d(P+N)) + tT))\,.\end{aligned}$$

By the definition of Zariski decomposition with rational coefficients and the fact that T is the exceptional locus of μ,

$$h^0(Y, \mathcal{O}_Y(tp_1\mu^*(dP)) = h^0(Y, \mathcal{O}_Y(\mu^*(tp_1 d(P+N)) + tT))\,.$$

Thus we get $\kappa(K, P) \geqq 0$.

Step 3. P is semi-ample.

By (1.1), there is a positive integer m_0 such that $SBs(dP) = Bs\,|m_0 dP|$. We suppose that $SBs(dP)$ is not empty. Taking a sequence of blowing-ups, if necessary, it may be assumed that

(2.11) $\mu^*(m_0 dP) = L + \Sigma_{i\in I} r_i E_i$ with non-negative integer r_i, where L is base-point free and $E_i \subset SBs(\mu^*(dP)) = \mu^{-1}(SBs(dP))$ if $r_i > 0$.

Let $f'\colon Y'\to Z'$ be a flat model of $f\colon Y\to Z$ and $g\colon \tilde{Y}\to Y'$ a nomalization of Y'. We set π, π' and $\tilde{f}$ as following diagram.

$$\begin{tikzcd} \tilde{Y} \arrow[r, "g"] \arrow[d, "\tilde{f}"'] & Y' \arrow[r, "\pi'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ Z' \arrow[r, equal] & Z' \arrow[r, "\pi"'] & Z \end{tikzcd}$$

Let $|L_0|$ be the movable part of $|\pi^*(m_0dP_0)|$. Then from equidimentionality of $\tilde{f}$ and $\tilde{f}^*|\pi^*(m_0dP_0)|=|g^*\pi'^*\mu^*(m_0dP)|$, $\tilde{f}^*|L_0|=|g^*\pi'^*L|$. Hence we may assume that

(2.12) there is a nef Cartier divisor L_0 on Z such that $f^*L_0=L$.

We set

$$c=\min\left\{\frac{a_i+1-\delta b_i}{r_i}\,\middle|\, i\in I\right\}, \quad \text{and}$$
$$I_0=\{i\in I\,|\,cr_i=a_i+1-\delta b_i\}.$$

Then if $0<\delta\ll 1$, c is positive. Let $\{G_\beta\}$ be the collection of subvarieties of Z such that

$$\{G_\beta\}=\{\text{one of irreducible components of } f(E_{i_1}\cdots E_{i_t})\,|\,i_1,\cdots,i_t\in I_0\}.$$

We take a minimal element G_1 with respect to the inclusion relation among G_β's. We choose a menber

$$D_2\in|q(D_0-D_1)|$$

for some positive integer q such that $G_1\subset D_2$ and $G_\beta\not\subset D_2$ for $\beta\neq 1$. Let R be a resolution datum $(E_1,\cdots,E_\alpha\colon E_1+\cdots+E_\alpha+f^*D_2)$ and $\nu\colon Y'\to Y$ a blowing-up with smooth center C which is permissible with respect to R. Then we set

$$\nu^*R=(E'_1,\cdots,E'_\alpha,E'_{\alpha+1}\colon \nu^*(E_1+\cdots+E_\alpha+f^*D_2)), \quad \text{and}$$
$$I'=\{1,2,\cdots,\alpha,\alpha+1\}.$$

Let a'_i, b'_i and r'_i be rational numbers determined by the same manner as before. Then we have

(1) $c=\min\left\{\dfrac{a'_i+1-\delta b'_i}{r'_i}\,\middle|\, i\in I'\right\}$, and

(2) if we set $I'_0=\{i\in I'\,|\,cr'_i=a'_i+1-\delta b'_i\}$ and $J=\{i\in I\,|\,C\subset E_i\}$, then

$$I'_0=\begin{cases} I_0\cup\{\alpha+1\} & \text{if } J\subset I_0 \text{ and } \#J=\operatorname{Codim}_Y C. \\ I_0 & \text{otherwise.}\end{cases}$$

This is an easy consequence of the following inequality.

(2.13) Let $\sigma_1,\cdots,\sigma_l,\tau_1\cdots,\tau_l$ be non-negative real numbers such that $(\sigma_i,\tau_i)\neq(0,0)$ for $1\leqq i\leqq l$. Then we get the inequality

$$\frac{\sigma_1+\cdots+\sigma_l}{\tau_1+\cdots+\tau_l} \geqq \min\left\{\frac{\sigma_i}{\tau_i} \mid 1 \leqq i \leqq l\right\},$$

where equality holds if and only if $\sigma_1/\tau_1=\cdots=\sigma_l/\tau_l$. Hence by (1.2.5), we may assume that

(2.14) $f^*D_2=\Sigma_{i\in I} s_i E_i$ with non-negative intergers s_i.

We set

$$c' = \min\left\{\frac{a_i+1-\delta b_i}{r_i+\delta' s_i} \mid i\in I\right\}, \quad \text{and}$$
$$I_1 = \{i\in I \mid a_i+1-\delta b_i = c'(r_i+\delta' s_i)\}.$$

Clearly, if δ' is small enough, then $c'<c$, $I_1\subseteq I_0$ and $s_i>0$, $r_i>0$ for all $i\in I_1$. Therefore, by our construction we obtain the following.

(2.15) The E_i with $i\in I_1$ and their intersections are mapped surjectively onto G_1 by f.

We define

$$A = \textstyle\sum_{i\in I-I_1}(-c'(r_i+\delta' s_i)+a_i-\delta b_i)E_i, \quad \text{and}$$
$$B = \textstyle\sum_{i\in I_1} E_i.$$

Since $f_*(\ulcorner A\urcorner)\leqq\ulcorner N\urcorner$, by the same argument as in Step 2, we have

$$H^0(Y, \mathcal{O}_Y(n\mu^*(dP))) = H^0(Y, \mathcal{O}_Y(n\mu^*(dP)+\ulcorner A\urcorner))$$

if n is sufficiently large. Hence B is a sum of fixed components of $|n\mu^*(dP)+\ulcorner A\urcorner|$ because B is a sum of fixed components of $|n\mu^*(dP)|$ by (2.11). Let us consider a $\boldsymbol{Q}$-divisor on Y

$$W = n\mu^*(dP)+A-B-K_Y.$$

By (2.4), (2.5), (2.9), (2.10), (2.11), (2.12) and (2.14), we caluculate

$$\begin{aligned} W &= (nd-c'm_0d-1)\mu^*(P)+c'L+(1-c'\delta' q)f^*(D_0-\delta D_1) \\ &= f^*((nd-c'm_0d-1)P_0+c'L_0+(1-c'\delta' q)(D_0-\delta D_1)). \end{aligned}$$

Therefore, if $n\gg 0$ and $\delta'\ll 1$, we have

$$\begin{aligned} m_0dW-B &\geqq (nd-c'm_0d-1)(\mu^*(m_0dP)-B)+m_0dc'L \\ &\quad +m_0d(1-c'\delta' q)(D_0-\delta D_1) \\ &\geqq 0. \end{aligned}$$

Hence applying (1.2.3), if $n\gg 0$ and $\delta'\ll 1$,

$$H^1(Y, \mathcal{O}_Y(n\mu^*(dP)+\ulcorner A\urcorner-B)) \to H^1(Y, \mathcal{O}_Y(n\mu^*(dP)+\ulcorner A\urcorner))$$

is injective. Considering the cohomology sequence derived from the short exact sequence

$$0 \to \mathcal{O}_Y(n\mu^*(dP)+\ulcorner A\urcorner-B) \to \mathcal{O}_Y(n\mu^*(dP)+\ulcorner A\urcorner) \to \mathcal{O}_B(n\mu^*(dP)+\ulcorner A\urcorner) \to 0$$

we get

$$H^0(B, \mathcal{O}_B(n\mu^*(dP)+\ulcorner A\urcorner)) = 0 .$$

On the other hand, by (2.14), applying (1.2.4) to

$$\begin{aligned} W|_B &= (n\mu^*(dP)+A)|_B - K_B \\ &= (f|_B)^*(((nd-c'm_0d-1)P_0+c'L_0+(1-c'\delta'q)(D_0-\delta D_1))|_{G_1}) , \end{aligned}$$

there exists a positive integer p such that

$$H^0(B, \mathcal{O}_B(tp\mu^*)(dP)+\ulcorner A\urcorner)) \neq 0$$

for $t \gg 0$. This is a contradiction. q.e.d.

Remark (2.16) If the singularities of X are worse than log-terminal, then this theorem is not necessarily true. Indeed Wilson [W] gave a normal projective Gorentein variety of general type with nef dualizing sheaf whose canonical ring is not finitely generated.

§3 Log-canonical ring with $\kappa \leqq 2$.

In this section, we prove the following theorem, which is a generalization of Fujita's result [F4] that the canonical rings of elliptic threefolds are finitely generated.

Theorem (3.1) *Let (X, Δ) be a normal complete variety with only log-terminal singularities. Assume that $0 \leqq \kappa(X, K_X+\Delta) \leqq 2$. Then there exists a birational morphism $f: Y \to X$ from non-singular complete variety Y to X such that $f^*(K_X+\Delta)$ has a Zariski decomposition with rational coefficients $f^*(K_X+\Delta)=P+N$ and that the nef part P is semi-ample. In particular, the log-canonical ring $\bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_Y([n(K_X+\Delta)]))$ is finitely generated.*

First, we prove the following lemma.

Lemma (3.2) *Let X be a normal complete variety and D a $\boldsymbol{Q}$-Cartier divisor on X with $0 \leqq \kappa(X, D) \leqq 2$. Then there exists a birational morphism $f: Y \to X$ from a non-singular complete variety Y to X such that $f^*(D)$ has a Zariski decomposition with rational coefficients $f^*(D)=P+N$ and that the nef part P is nef and good.*

Proof. Clearly we may assume that D is effective. For sufficiently large integer m, we can construct the following commutative diagram.

$$\begin{array}{ccc} Y & \xrightarrow{h} & W \\ f\downarrow & & \downarrow g \\ X & \overset{\phi_{|mD|}}{\dashrightarrow} & W_0 = \operatorname{Im} \phi_{|mD|} \end{array}$$

(i) Y and W are non-singular complete varieties,

(ii) f and g are birational morphisms and

(iii) h is a fibre space.

Then, by the argument in [I, proof of theorem 1 and 2], we have an effective $\boldsymbol{Q}$-divisor M on W such that $h^*M \leqq f^*D$ and that $H^0(Y, \mathcal{O}_Y(h^*([mM]))) \simeq H^0(Y, \mathcal{O}_Y([mf^*(D)]))$ for all $m \geq 0$. Since $\dim W \leqq 2$, there exists a Zariski decomposition with rational coefficients $M = P_0 + N_0$ such that the nef part P_0 is nef and big. Set $P = h^*(P_0)$ and $N = f^*(D) - P$. Then $f^*(D) = P + N$ is a desired Zariski decomposition. q.e.d.

Proof of Theorem (3.1). By lemma (3.2), there exists a birational morphism f: $Y \to X$ such that

(i) Y is a non-singular complete variety,

(ii) there exists a normal crossing divisor $F = \Sigma_i F_i$ on Y,

(iii) $K_Y = f^*(K_X + \Delta) + \Sigma_i a_i F_i$ for some $a_i \in \boldsymbol{Q}$ with $a_i > -1$ and that

(vi) $f^*(K_X + \Delta)$ has a Zariski decomposition $f^*(K_X + \Delta) = P + N$ such that the nef part is nef and good.

Set $\Delta' = \sum_{a_i \leqq 0} -a_i F_i$ and $N' = N + \sum_{a_i > 0} a_i F_i$. Then (Y, Δ') has only log-terminal singularities and $K_Y + \Delta' = P + N'$ is a Zariski decomposition since the support of $\sum_{a_i > 0} a_i F_i$ is exceptional by (2.10). Hence P is semi-ample by Theorem 0. q.e.d.

§4 Some remarks on the related topics.

Let X be a non-singular complete variety and D an effective element of $\mathrm{Div}(X) \otimes \boldsymbol{R}$. We say that D has a Zariski decomposition if there is an effective element N of $\mathrm{Div}(X) \otimes \boldsymbol{R}$ such that $P = D - N$ is nef and for some positive integer d, $h_0(X, \mathcal{O}_X([ndP])) = h^0(X, \mathcal{O}_X([ndD]))$ for all $n \geqq 0$. If the nef part P is nef and good, then this Zariski decomposition is called a good Zariski decomposition. (The goodness of nef divisor with real coefficients is defined by similar manner as in §1). Recently S.D. Cutkosky [C] gave an example of a big divisor whose Zariski decomposition has no rational coefficients. Hence the fundamental conjecture of Zariski decomposition is the following form.

(ZD)$_n$: For any effective element D of $\mathrm{Div}(X) \otimes \boldsymbol{R}$ on an n-dimensional non-singular complete variety X, there is a birational morphism of non-singular compelete varieties f: $Y \to X$ such that $f^*(D)$ has a Zariski deciomposition.

Recently we prove the following theorem. First, Kawamata [Ka4] proved this in the case where $K_X + \Delta$ is a big divisor. Details and proof of this theorem will be written in my master thesis [M].

Theorem (4.1) *Let X be a non-singular projective variety and Δ an effective $\boldsymbol{Q}$-divisor on X such that (X, Δ) has only log-terminal singularities. Assume that $K_X + \Delta$ has a good Zariski decomposition $K_X + \Delta = P + N$. Then P is a $\boldsymbol{Q}$-divisor.*

Let X be a normal algebraic variety and $f\colon Y\to X$ a desingularization of X. Then it is easy to check that $\mathcal{O}_X$-algebra $\bigoplus_{n=0}^{\infty} f_*(\omega_Y^{\otimes n})$ is independent of the choice of the resolution f. Thus we define

$$LR(X)=\bigoplus_{n=0}^{\infty} f_*(\omega_Y^{\otimes n}),$$

which is called the local canonical ring of X. M. Reid [R1] raised the following problem.

(L_n): LR(X) is finitely generated $\mathcal{O}_X$-algebra for any n-dimensional normal variety X. A global verions of this problem is

$(\mathrm{CG})_n$: For any n-dimensional projective variety X having only canonical singularities with maximal Kodaira dimension, the pluricanonical ring $\bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nK_X))$ is finitely generated.

The minimal model conjecture of dimension n is

$(\mathrm{MMC})_n$: Every n-dimensional algebraic variety X is birationally equivalence to a projective variety $\bar{X}$ having only terminal singularities which satisfies one of the following;

(i) $K_{\bar{X}}$ is a nef $\boldsymbol{Q}$-Cartier divisor,

(ii) there is a surjective morphism $\phi\colon \bar{X}\to Y$ with connected fibres such that $\dim Y<\dim X$ and $-K_{\bar{X}}$ is ϕ-ample.

We recall M. Reid and Y. Kawamata's program for $(\mathrm{MMC})_n$ in short (for detail, see [Ka2] or [R2]). Let X be an n-dimensional normal $\boldsymbol{Q}$-factorial projective variety with only terminal singularies. We set

$$\begin{aligned} N_1(X) &= \{\text{1-cycles on } X\}/\equiv\otimes\boldsymbol{R}, \\ \overline{NE}(X) &= \text{the closure of the cone in } N_1(X) \text{ generated} \\ &\qquad \text{by effective 1-cycles}, \\ \overline{NE}_D(X) &= \{z\in\overline{NE}(X)\mid (D\cdot z)\geqq 0\}, \end{aligned}$$

where $\equiv$ is the numerical equivalence and D is a $\boldsymbol{Q}$-Cartier divisor on X. We say a half line $\boldsymbol{R}_+z$ in $\overline{NE}(X)$ to be an extremal ray if $(K_X\cdot z)<0$ and for any $z_1, z_2\in\overline{NE}(X)$ such that $z_1+z_2\in\boldsymbol{R}_+z$, $z_1, z_2\in\boldsymbol{R}_+z$. By [Ko1, Theorem 1], we have the following theorem.

(4.2) *For any $\varepsilon>0$ and any ample divisor A on X, there are integral cureves $l_1, \cdots, l_r$ such that*

(i) $\overline{NE}(X)=\boldsymbol{R}_+[l_1]+\cdots+\boldsymbol{R}_+[l_r]+\overline{NE}_{K_X+\varepsilon A}(X)$, *and*

(ii) $\boldsymbol{R}_+[l_1]$ *is an extremal ray for* $i=1, \cdots, r$,

where $[l_i]$ $(i=1, \cdots, r)$ *denotes the class of* l_i *in* $N_1(X)$.

Furthermore, due to [Ka2, Theorem 2.6 or our theorem 0], we obtain the contrac-

tion theorem.

(4.3) *For every extremal ray R, we have the contraction $\phi: X \to Y$ such that*
(i) *ϕ is a morphism of X onto a normal projective variety Y with connected fibres,*
(ii) *$-K_X$ is ϕ-ample, and*
(iii) *for any curve C on X, $[C] \in R$ if and only if C is contracted by ϕ.*

We can classify the extremal rays into 3-types by the properties of contractions as follows.

(a) $\dim Y < \dim X$. Then ϕ gives $\boldsymbol{Q}$-fano fibring.

(b) ϕ is a birational morphism which is not isomorphic in codimension one. In this case, Y is also $\boldsymbol{Q}$-factorial variety with only terminal singularities.

(c) ϕ is a birational morphism which is isomorphic in codimension one. Then Y has only rational singularities, but is not $\boldsymbol{Q}$-Gorenstein.

In order to avoid the case of (c), we need some modification of $\phi: X \to Y$. For the existence of this modifiaction, there is the following conjecture.

$(\mathrm{MMC}_1)_n$: There exists a $\boldsymbol{Q}$-factorial normal projective variety X_+ with only terminal singularities such that

(1) there is a birational morphism $\phi_+: X_+ \to Y$ which is isomorphic in codimension one, and

(2) K_{X_+} is ϕ_+-ample.

$\phi_+: X_+ \to Y$ is called a fllip of $\phi: X \to Y$. Assuming $(\mathrm{MMC}_1)_n$, the conjecture of inductinon is as follows.

$(\mathrm{MMC}_2)_n$: After a finite number of contractions and flips, we can get a minimal model as in $(\mathrm{MMC})_n$.

A relationship of these conjectures is given by the following.

Proposition (4.4)

(a) *$(MMC)_n$ implies $(CG)_n$.*
(b) *$(ZD)_n$ implies $(CG)_n$.*
(c) *$(CG)_n$ implies $(LC)_n$.*
(d) *$(LC)_n$ implies $(MMC_1)_n$.*
(e) *$(MMC_1)_n$ and $(MMC_2)_n$ imply $(MMC)_n$.*

$$\begin{tikzcd}
 & (MMC)_n \arrow[dr, Rightarrow, "(a)"] & & \\
(MMC_1)_n \arrow[ur, Rightarrow, "(e)"] \arrow[rr, Rightarrow, "+(MMC_2)_n"] & & (CG)_n \arrow[dl, Rightarrow, "(c)"] & (ZD)_n \arrow[l, Rightarrow, "(b)"'] \\
 & (LC)_n \arrow[ul, Rightarrow, "(d)"] & &
\end{tikzcd}$$

Proof. By our Theorem 0 and Thhorem (4.1), (a) and (b) are obvious. (e) is the program of M. Reid and Y. Kawamata.

(c): Let X be a normal variety and $f: X \to X$ a resolution of singularities of X. Clearly we can reduce to the case where X is projective. Let A be an ample divisor on X and r a sufficiently large integer. Taking a general menber $D \in |2rA|$ such that f^*D is smooth on Y, we have a double covering $\pi: \tilde{Y} \to Y$ reamifying along f^*D. It is easy to see that Y is a smooth projective variety of general type and $K_{\tilde{Y}} = \pi^*(K_Y + f^*(rA))$. Hence we have

$$(4.4.1) \quad \pi_* \mathcal{O}_{\tilde{Y}}(nK_{\tilde{Y}}) = \mathcal{O}_Y(n(K_Y + f^*(rA))) \otimes \pi_* \mathcal{O}_{\tilde{Y}} .$$

Let G be the Galois group $\mathrm{Gal}(\tilde{Y}/Y)$ of the covering. Then from (4.4.1),

$$(\bigoplus_{n=0}^{\infty} H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(nK_{\tilde{Y}})))^G = \bigoplus_{n=0}^{\infty} H^0(Y, \mathcal{O}_Y(n(K_Y + f^*(rA)))) .$$

Therefore, by the assumption of $(CG)_n$, $\bigoplus_{n=0}^{\infty} H^0(Y, \mathcal{O}_Y(n(K_Y + f^*(rA))))$ is finitely generated. It is well known that

$$\mathrm{Proj}_X(\bigoplus_{n=0}^{\infty} f_*(\omega_Y^{\otimes n})) = \mathrm{Proj} \bigoplus_{n=0}^{\infty} (H^0(Y, \mathcal{O}_Y(n(K_Y + f^*(rA))))$$

(See [R1]). Hence $LR(X)$ is finitely generated.

(d): Let $f: X \to Y$ be the contraction with respect to an extremal ray which is isomorphic in codimension one. A flip of $f: X \to Y$ is nothing more than $\mathrm{Proj}_Y(LR(Y))$ (See [Ka2]). q.e.d.

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Reference

[C] S. D. Cutkoski, Zariski decomposition of divisors on algebraic varieties, Duke Math. J., Vol. 53, No. 1, (1986), 149–156.

[D] P. Deligne, Theorie de Hodge III, Publ. Math. IHES., **44** (1974), 5–78.

[F1] T. Fujita, Semi-positive line bundle, J. Fac. Sci. Univ. Tokyo Sec. IA, **30** (1984), 353–378.

[F2] T. Fujita, Cnaonical rings of algebraic varieties, Classification of Algebraic and Analytic Manifolds, K. Ueno, ed., Progress in Math. 39 (1983), Birkhäuser, Boston-Basel-Stuttgent, 65–70.

[F3] T. Fujita, Fractionally logarithmic canonical rings of surfaces, J. Fac. Sci. Univ. Tokyo Sec. IA, **30** (1984), 685–696.

[F4] T. Fujita, Zariski decomposition and canonical rings of elliptic threefolds, J. Math. Soc. Japan, Vol. 38, No. 1 (1986), 19–37.

[H1] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero I-II, Ann. Math., **79** (1964), 109–326.

[H2] H. Hironaka, Flattening theorem in complex-analytic geometry, Amer. J. Math., **97** (1975), 503–547.

[I] S. Iitaka, On D-dimensions of algebraic varieties, J. of Math. Soc. of Japan, **23** (1971), 356–373.

[Ka1] Y. Kawamata, A generalization of Kodaira-Ramanujam's vanishing theorem, Math. Ann., **261** (1982), 43–46.
[Ka2] Y. Kawamata, The cone of curves of algebraic varieties, Ann. Math., **119** (1984), 603–633.
[Ka3] Y. Kawamata, Pluricanonical systems on minimal algebraic varieties, Invent. Math., **79** (1985), 567–588.
[Ka4] Y. Kawamata, The Zariski decomposition of log-canonical divisors, preprint.
[Ko1] J. Kollár, The cone theorem, Ann. Math., **120** (1984), 1–5.
[Ko2] J. Kollár, Highter direct image of dualizing sheaves, Ann. Math., 123 (1986), 11–42.
[R1] M. Reid, Canonical 3-folds, Géométrie Algébrique, Anger 1979, A. Beauville ed., 1980, Sijithoff & Noordhoff. The Netherlands, 273–310.
[R2] M. Reid, Minimal model of canonical 3-folds, Algebraic Varirties and Analytic Varieties, S. Iitaka ed., Advanced Studies in Pure Math. 1, 1983, Kinokuniya, Tokyo and North-Holland, 131–180.
[M] A. Moriwaki, Zariski decomposition on higher dimensional algebraic variety (in Japanese), Master Thesis, Kyoto Univ., (1986).
[S] V.V. Shokurov, Non-vanishing theorem, preprint.
[V] E. Viehweg, Vanishing theorem, J. rein. angrew. Math., **375** (1982), 1–8.
[W] P. H. M. Wilson, On the canonical ring of algebraic varieties, Comp. Math., **43** (1981), 365–385.

Added in proof.; ① Theorem 0 can be generalized to the case where X is a Kähler manifold. Hence we have the canonical ring of X is finitely generated if $\kappa(X)\leqq 2$.

② Some of results in [M] will be found in A. Moriwaki, Relative Zariski decomposition on higher dimensional algebraic varieties, Proc. Japan Acad., Vol. 62, Ser. A, No. 3 (1986), 108–111.