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On the Gevrey index for some hypoelliptic opertors

By

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1. Introduction.

Let P be a partial differential operator with analytic coefficients in Ω which is an open set in $\boldsymbol{R}^d$. For $x\in\Omega$, we say P be $\gamma^{(s)}$-hypoelliptic in a neighborhood of x if there exists a neighborhood ω of x such that the following implication holds; for any open subset $\omega'\subset\omega$, any $u\in\mathcal{D}'(\omega)$

$$Pu\in\gamma^{(s)}(\omega')\Rightarrow u\in\gamma^{(s)}(\omega')\,.$$

Here $\gamma^{(s)}(\omega')$ means the Gevrey class of order s in ω'

$$f(x)\in\gamma^{(s)}(\omega') \quad \text{iff} \quad f(x)\in C^\infty(\omega') \quad \text{and satisfies}$$
$$^{\forall}K\Subset\omega',\ ^{\exists}C_0,\ ^{\exists}C_1,\ ^{\forall}\alpha\in\boldsymbol{N},\ \sup_K|D_x^\alpha f(x)|\leqslant C_0C_1^{|\alpha|}(\alpha!)^s\,.$$

Our concern is to determine the smallest numbers for P to be $\gamma^{(s)}$-hypoelliptic in a neighborhood of x. Let

$\Gamma_x(P)=\{s\in\boldsymbol{R};\ P$ is $\gamma^{(s)}$-hypoelliptic in a n.b.d. of $x\}$ and $\gamma_x(P)=\inf_{s\in\Gamma_x(P)} s$ which we call the Gevrey index for P. Here we define $\gamma_x(P)=\infty$ if $\Gamma_x(P)=\phi$. We think that in general, $\Gamma_x(P)$ is a closed connected set.

For the operator with constant coefficients and the semi-elliptic operator with variable coefficients, $\gamma_x(P)$ has been determined by L. Hörmander [3], and T. Ōkaji [7]. In this paper, we consider the following simple operator;

$$P=\sum_{j=1}^{n-1}D_{x_j}^2+ix_0^kD_{x_n}\,,\quad D_{x_j}=(1/i)\frac{\partial}{\partial x_j}\qquad (j=0,1,\cdots,n)\,,$$

where k is a positive integer.

From the result of [7], it follows that $\gamma_0(P)\geqslant 2$ since $\gamma_x(P)\geqslant 2$ for $x_0\neq 0$. On the other hand, since P is viewed as an operator of Hörmander type; $\sum_j X_j^2+X_0$, from the result of M. Derrdji and Ci. Zuily [1], it follows that $\gamma_0(P)\leqslant 2(k+2)$. But we have more sharp result for P as follows;

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Theorem 1. $\gamma_0(P) \leqslant k+2$.

Theorem 2. *When* $n=1$, $\gamma_0(P) \leqslant 2$.

Theorem 3. *When* $k=1$, $\gamma_0(P) \leqslant 2$.

Theorem 4. *When* $n \geqslant 2$ *and either* $k=3$ *or even number*, $\gamma_0(P) \geqslant k+2$.

Corollary. $\gamma_0(P) = \begin{cases} 2 & (n=1 \text{ or } k=1) \\ k+2 & (n\geqslant 2 \textit{ and either } k=3 \textit{ or even}) \end{cases}$

We conjecture that if $n \geqslant 2$ and $k \geqslant 2$, $\gamma_0(P)=k+2$, but possibly for a technically reason, we must make a restriction on k.

The plan of this paper is as follows. In §2, we shall show theorem 1 with aids of the result of M. Durand [2]. In §3, we shall prove theorem 2 by the method of Morrey and Nirenberg. In §4, we shall construct a parametrix of P, when $k=1$, which implies theorem 3. In the last section, we shall prove theorem 4 by using the Hörmander-Métivier's inequality.

2. Proof of theorem 1.

In 1978, M. Durand has proved that if the partial differential operator Q of order 2 with coefficients belonging to $\gamma^{(s)}(\Omega)$ satisfies the following estimate; for $t>0$, any $K \Subset \Omega$, and any $g \in C^\infty(\Omega)$, there exist constants $C_{K,t}$, $C_{K,t}(g)$ such that for any $u \in C_0^\infty(K)$,

$$(2.1) \qquad \begin{cases} \|u\|_t \leq C_{K,t}\{\|Qu\|_{t-a}+\|u\|_0\} \\ \|[Q, D_{x_j}]u\|_t \leq C_{K,t}\{\|Qu\|_{t+1-b}+\|u\|_0\}, \quad j=0, \cdots, n, \quad \text{and} \\ \|[Q, g]u\|_t \leq C_{K,t}(g)\{\|Qu\|_{t-b}+\|u\|_0\}, \end{cases}$$

where $C_{K,t}$, $C_{K,t}(g)$ are the constants depending uniformly on t when t is bounded, and

$$\|u\|_t^2 = \int (1+|\xi|)^{2t}|\hat{u}(\xi)|^2 d\xi ,$$

then for any $s \geq \max(1/b, 2/a)$, Q is $\gamma^{(s)}$-hypoelliptic in Ω.

For our operator P, from the result of [8] (c.f. th.18) it follows that

$$\sum_{j=0}^{n-1} \|D_{x_j}^2 u\|_t + \|\langle D_x\rangle^{2/(k+2)} u\|_t \leq C_{K,t}\{\|Pu\|_t+\|u\|_t\} ,$$

which implies (2.1) with $a=2/(k+2)$, $b=1/(k+2)$ since for any $\varepsilon>0$,

$$\|u\|_t \leq \varepsilon \|\langle D_x\rangle^\sigma u\|_t + C_{\varepsilon,\sigma,t}\|u\|_0 \qquad \text{if} \quad t, \sigma>0 .$$

Therefore, applying the result of M. Durand to P, we conclude that for any $s \geq k+2$, P is $\gamma^{(s)}$-hypoelliptic in a neighborhood of the origin. This proves theorem 1.

3. Proof of theorem 2.

In this section, we consider the case $n=1$, and denote D_{x_j} by D_j. We observe that from the result of [4] or [8], it follows that for $u\in C_0^\infty(\Omega)$, (Ω is sufficiently small neighborhood of the origin),

$$\sum_{j=0}^{2}||D_0^j u||+||x_0^k\langle D_1\rangle u||\leq C||Pu|| , \tag{3.1}$$

where $||\cdot||=||\cdot||_0$ and $\sigma(\langle D_1\rangle)=(1+|\xi|^2)^{1/2}$.

So, in order to prove theorem 2, it suffices to show

Proposition 3.1. *Let* $P=D_0^2+ix_0^k b(x)D_1$ *satisfy* (3.1), *where* b *is a real-valued function belonging to* $\gamma^{(s)}(\Omega)$ *and* $b(0)\neq 0$. *Then for any* $s\geq 2$, P *is* $\gamma^{(s)}$*-hypoelliptic in a neighborhood of the origin.*

We note that for $x_0\neq 0$, P is a semi-elliptic operator, so that if $u\in\mathcal{D}'(\Omega)$ and $Pu\in\gamma^{(s)}(\omega)$, $\Omega\supset\omega$, $s\geq 2$, then $u\in\gamma^{(s)}(\omega\setminus\{x_0=0\})$. Hence it suffices to show that if $u\in\mathcal{D}'(\Omega)$ and $Pu\in\gamma^{(s)}(\omega^0)$, $s\geq 2$, then $u\in\gamma^{(s)}(\omega^0)$, where $\omega^0=(-r,r)\times(-r,r)$, ($r$ is a small positive number.)

We prove this proposition by the method of Morrey and Nirenberg. We begin with some preparations. Let

$$\omega^d=(-r+d,\ r-d)\,,\quad \text{and}$$

denote by $\chi_{d,\varepsilon}(t)$ the characteristic function of

$$(-r+d+\varepsilon,\ r-d-\varepsilon)\,.$$

We define $\tilde{\phi}_{d,2\varepsilon}(t)=\varepsilon^{-1}\tilde{\phi}(t/\varepsilon)*\chi_{d,\varepsilon}(t)$, where $\tilde{\phi}(t)\in C_0^\infty((-r,r))$, $\tilde{\phi}(0)=1$, and $|\tilde{\phi}|\leq 1$ Let $\phi_{d,2\varepsilon}(x)=\prod_{i=0}^{1}\tilde{\phi}_{d,2\varepsilon}(x_i)$. Then $\phi_{d,2\varepsilon}$ has the following properties;

$$\phi_{d,2\varepsilon}=1 \quad\text{on}\quad \omega^{i+2\varepsilon},\ =0\quad\text{on}\quad \mathcal{C}\omega^d\,,\quad |\phi_{d,2\varepsilon}|\leq 1\,,\quad\text{and}$$

$$\sup|D_x^\alpha\phi_{d,\varepsilon}(x)|\leq C_\alpha\varepsilon^{-|\alpha|}\,, \tag{3.2}$$

where C_α is a constant independent of ε, d. We consider the norm;

$$N_\varepsilon(\langle D_1\rangle^j v)=\inf_{\Phi\in X_\varepsilon}||\langle D_1\rangle^j\Phi v||_{L^2((-r+\varepsilon,r-\varepsilon)\times R)}\,,\qquad j=0,\ 1/2,\ 1,\ 3/2\,,$$

where $X_\varepsilon=\{\Phi\in C_0^\infty(\boldsymbol{R}^2);\ \Phi=1\ \text{on}\ \omega^\varepsilon\}$. We note that $N_\varepsilon(v)=||v||_{L^2(\omega^\varepsilon)}$. Then we observe that for $u\in\mathcal{S}(\boldsymbol{R})$,

$$||[\langle D_1\rangle^{1/2},\ \tilde{\phi}_{d,\varepsilon}(x_1)]u||\leq C\varepsilon^{-1/2}||u||\,, \tag{3.3}$$

$$||\langle D_1\rangle[\langle D_1\rangle^{1/2},\ \tilde{\phi}_{d,\varepsilon}(x_1)]u||\leq C\{\varepsilon^{-1}||\langle D_1\rangle^{1/2}u||+\varepsilon^{-3/2}||u||\}\,, \tag{3.4}$$

$$||\langle D_1\rangle^{1/2}\tilde{\phi}_{d,\varepsilon}(x_1)u||\leq C\{||\langle D_1\rangle^{1/2}u||+\varepsilon^{-1/2}||u||\}\,,\quad\text{and} \tag{3.5}$$

$$||\langle D_1\rangle^{3/2}[D_1,\ \tilde{\phi}_{d,\varepsilon}(x_1)]u||\leq C\{\varepsilon^{-1}||\langle D_1\rangle^{1/2}u||+\varepsilon^{-3/2}||u||\}\,, \tag{3.6}$$

where C is a constant independent of d, ε, u. (c.f. [2])

Let

$$E_{d,\varepsilon,1}(v)=\sum_{p=0}^{2}N_{d+\varepsilon}(D_0^p\langle D_1\rangle^{1/2}v)+N_{d+\varepsilon}(x_0^k\langle D_1\rangle^{1/2}D_1v)\,,$$

$$E_{d,\varepsilon,0}(v)=\sum_{p=0}^{2}N_{d+\varepsilon}(D_0^p v)+N_{d+\varepsilon}(x_0^k\langle D_1\rangle v)\,,$$

$$R_{d,1}(v)=N_d(\langle D_1\rangle^{1/2}x_0^l D_0v)+\varepsilon^{-1/2}N_d(x_0^kD_0v)$$
$$+\varepsilon^{-1}N_d\langle D_1\rangle^{1/2}x_0v)+\varepsilon^{-3/2}N_d(x_0^kv)\,,\quad\text{and}$$

$$R_{d,0}(v)=N_d(x_0^kD_0v)+\varepsilon^{-1}N_d(x_0^kv)\,.$$

Then we obtain

Lemma 3.2. *Under the hypothesis of proposition* 3.1, *there are the constants* C_j $(j=1,2)$ *independent of* d, ε *such that for any* $v\in C^\infty(\Omega)$,

$$\varepsilon^2E_{d,\varepsilon,0}(v)\le C_1\{\varepsilon^2N_d(Pv)+\varepsilon R_{d,0}(v)\}\,,\quad\textit{and}\tag{3.7}$$

$$\varepsilon^{5/2}E_{d,\varepsilon,1}(v)\le C_2\{\varepsilon^{5/2}N_d(\langle D_1\rangle^{1/2}Pv)+\varepsilon^2N_d(Pv)$$
$$+\varepsilon^{3/2}R_{d,1}(v)+\varepsilon^{5/2}\sum_{p=0}^{2}N_d(D_0^pv)\}\,.\tag{3.8}$$

Proof. Let $u=\phi_{d,\varepsilon}(x)v$ in (3.1). We observe that

$$N_{d+\varepsilon}(x_0^k\langle D_1\rangle u)\le\|x_0^k\langle D_1\rangle\phi_{d,\varepsilon}u\|\,,\quad N_{d+\varepsilon}(D_0^pu)\le\|\phi_{d,\varepsilon}D_0^pu\|\,,$$
$$D_0^p\phi_{d,\varepsilon}=\phi_{d,\varepsilon}D_0^p+[D_0^p,\phi_{d,\varepsilon}]\,,\quad\text{and}$$
$$P\phi_{d,\varepsilon}=\phi_{d,\varepsilon}P+[D_0^2,\phi_{d,\varepsilon}]+ix_0^kb(x)[D_1,\phi_{d,\varepsilon}]\,.$$

Since $[D_0^p,\tilde\phi_{d,\varepsilon}(x_0)]$ vanishes on a neighborhood of the origin, (3.7) follows from (3.2). Let d_0 be a fixed number and $\psi(x)=\tilde\psi(x_0)\tilde\psi(x_1)$ such that

$$\psi\in C_0^\infty(\Omega)\,,\quad=1\qquad\text{on}\quad\omega^{d_0}\,.$$

For $d>d_0$, let $u=\psi\langle D_1\rangle^d\phi_{d,\varepsilon}v$ in (3.1). We observe that

$$N_{d+\varepsilon}(\langle D_1\rangle^{1/2}x_0^kD_1u)\le\|x_0^k\langle D_1\rangle^{3/2}\phi_{d,\varepsilon}u\|$$
$$\le\|P\psi\langle D_1\rangle^{1/2}\varphi_{d,\varepsilon}u\|+\|x_0^k\langle D_1\rangle[\langle D_1\rangle^{1/2},\psi]\phi_{d,\varepsilon}u\|\,,$$
$$N_{d+\varepsilon}(D_0^p\langle D_1\rangle^{1/2}u)\le\|\langle D_1\rangle^{1/2}\phi_{d,\varepsilon}D_0^pu\|$$
$$\le\|D_0^p[\langle D_1\rangle^{1/2},\psi]\phi_{d,\varepsilon}u\|+\|P\psi\langle D_1\rangle^{1/2}\phi_{d,\varepsilon}u\|+\|\langle D_1\rangle^{1/2}[\phi_{d,\varepsilon},D_0^p]u\|\,,$$
$$D_0^p\langle D_1\rangle^{1/2}\phi_{d,\varepsilon}=[D_0^p,\tilde\phi_{c,\varepsilon}(x_0)]\cdot[\langle D_1\rangle^{1/2},\tilde\phi_{d\,\varepsilon}(x_1)]\Phi+\phi_{d,\varepsilon}D_0^p\langle D_1\rangle^{1/2}\Phi$$
$$\text{for}\quad\Phi\in X_1\,,\quad\text{and}$$
$$P\psi\langle D_1\rangle^{1/2}\phi_{d,\varepsilon}=[P,\psi]\langle D_1\rangle^{1/2}\phi_{d,\varepsilon}+\psi[P,\langle D_1\rangle^{1/2}]\phi_{d,\varepsilon}+\psi\langle D_1\rangle^{1/2}P\phi_{d,\varepsilon}\,.$$

Then, similarly as before, from (3.3) (3.5), it follows that (3.8) holds.

Lemma 3.3. *We suppose the same hypothesis as proposition* 3.1. *Moreover suppose that for* $s\ge2$,

$$u \in \mathcal{D}'(\Omega), \quad Pu = f \in \gamma^{(s)}(\overline{\omega^0}).$$

Then theres exists a constant M independent of ε, j *such that for* $q \leq j$,

$$\varepsilon^{s(j+2)} \sum_{0 \leq p \leq 2} N_{(1+j)\varepsilon}(D_0^p D_1^q u) \leq M^{q+2}.$$

Proof. By C^∞-hypoellipticity of P, we may assume that $u \in C^\infty(\omega^0)$. So it suffices to show the estimate. We must prepare some consideration.

Let $K = \overline{\omega^0}$. Then by hypothesis, there is the constants A and B such that for any $\alpha \in \boldsymbol{N}^2$,

$$\sup_K |D_x^\alpha f| \leq A^{|\alpha|+1} \alpha!^s, \quad \text{and}$$

$$\sup_K |D_x^\alpha b| \leq B^{|\alpha|+1} \alpha!^s.$$

Let $j\varepsilon = d$. Then, we observe that for $|\alpha| = j$,

(3.9) $$\varepsilon^{sj} N_d(D_x^\alpha f) \leq A^{j+1},$$

(3.10) $$\varepsilon^{sj} \sup_{\omega^d} |D_x^\alpha b| \leq B^{j+1} \qquad \text{since} \quad \varepsilon j \leq 1,$$

(3.11) $$[P, D_1^q] = i x_0^k \sum_{0<r\leq q} \binom{q}{r} D_1^r b D_1^{q-r+1}, \quad \text{and}$$

(3.12) $$\binom{q}{r} r!^s j^{-jr} \leq 1 \qquad \text{for} \quad 0 < r \leq q.$$

Moreover, since we may assume that $\operatorname{supp} \phi_{d,\varepsilon} \subset \omega^{d+\varepsilon-\varepsilon^s/3}$, as the proof of lemma 2.4 in [2], we see that

(3.13) $$N_{d+\varepsilon}(\langle D_1 \rangle^{1/2} u) \leq C_3 \{\varepsilon^{s/2} N_{d+\varepsilon-\varepsilon^s/3}(\langle D_1 \rangle u) + \varepsilon^{-s/2} N_{d+\varepsilon-\varepsilon^s/3}(u)\}$$

where C_3 is a constant independent of d, ε, and u.

Then from (3.9), (3.13), it follows that

(3.14) $$\varepsilon^{s(j+1/2)} N_{j\varepsilon}(\langle D_1 \rangle^{1/2} D_1^j f) \leq C_3 (A+1) A^{j+1}.$$

From (3.10), (3.11), (3.12), it follows that

(3.15) $$\varepsilon^{s(q+1)} N_{q\varepsilon}([P, D_1^q] u) \leq \sum_{r=1}^q B^{r+1} \varepsilon^{s(q-r+1)} N_{q\varepsilon}(x_0^k D_1^{q-r+1} u), \quad \text{and}$$

(3.16) $$\varepsilon^{s(q+3/2)} N_{q\varepsilon+\varepsilon/2}(\langle D_1 \rangle^{1/2} [P, D_1^q] u)$$
$$\leq \sum_{r=1}^q C_3 \{B^{r+1} \varepsilon^{s(q-r+2)} N_{q\varepsilon+\varepsilon/2-\varepsilon^s/3}(x_0^k D_1^{q-r+2} u)$$
$$+ B^{r+1}(B+1) \varepsilon^{s(q-r+1)} N_{q\varepsilon+\varepsilon/2-\varepsilon^s/3}(x_0^k D_1^{q-r+1} u)\}.$$

Here we used the fact that $\|\langle D_1 \rangle u\| \sim \|u\| + \|D_1 u\|$.

These inequalities (3.15) and (3.16), and lemma 3.2 imply the desired estimate. In fact, we show this by induction on j. We assume that

$$(3.17)_j \qquad \begin{cases} \varepsilon_0^{s(q+1)} N_{(j+1)\varepsilon_0}(D_0^p D_1^q u) \leq M^{q+2}, & p=0,1,2, \\ \varepsilon_0^{s(q+1)} N_{(j+1)\varepsilon_0}(x_0^k \langle D_1 \rangle D_1^q u) \leq M^{q+2} & \text{for} \quad q \leq j, \end{cases}$$

and

$$(3.18)_{j-1} \qquad \begin{cases} \varepsilon_0^{s(q+3/2)} N_{(j+3/2)\varepsilon_0}(D_0^p \langle D_1 \rangle^{1/2} D_1^q u) \leq M^{q+3/2}, \\ \varepsilon_0^{s(q+3/2)} N_{(j+3/2)\varepsilon_0}(x_0^k \langle D_1 \rangle^{1/2} D_1^{q+1} u) \leq M^{q+3/2} \end{cases}$$

$$\text{for} \quad p=0,1,2, \quad \text{and} \quad q \leq j-1.$$

In (3.8), let $d=(2/3)\varepsilon_0(j+1)$, $\varepsilon=\varepsilon_0/2$, and

$$v = \varepsilon_0^{s(q+3/2)-2} D_1^q u .$$

If $s \geq 2$, we see that

$$s'+3/2 \geq s(q+1), \quad s'+1 \geq s(q+1),$$
$$s'+1/2 \geq s(q+1/2), \quad s' \geq sq,$$

where $s'=s(q+3/2)-2$. So, by (3.9), (3.14), (3.15), (3.16), we see that $(3.17)_j$ and $(3.18)_{j-1}$ imply $(3.18)_j$, if

$$M \geq \max\{2B, (32B(B+1)C_2C_3)^2, (C_3(A+1))^2, A^2\} .$$

For, from Schwartz inequality an $(3.17)_j$, it follows that since $t^{2k-1} \leq t^k$ if $k \geq 1$, and t is small positive,

$$\varepsilon_0^{s(q+1)} N_{(j+1)\varepsilon_0}(x_0^k \langle D_1 \rangle^{1/2} D_0 D_1^q u) \leq 2M^{q+2},$$

and it is seen that

$$\varepsilon_0^{s(q+3/2)} N_{(j+1)\varepsilon_0}(\langle D_1 \rangle^{1/2} P D_1^q u) \leq 2(B+1) B^2 C_3 M^{q+2} \sum_{r \geq 0} (B/M)^r$$
$$\leq 4(B+1) B^2 C_3 M^{q+2} \leq (1/8) M^{q+5/2},$$

and

$$\varepsilon_0^{s(q+1)} N_{(j+1)\varepsilon_0}(P D_1^q u) \leq 4B^2 M^{q+2} \leq (1/8) M^{q+5/2} .$$

The similar argument as above show that $(3.18)_j$ and $(3.17)_j$ imply $(3.17)_{j+1}$ if

$$M \geq \max\{2B, A^2, (12C_1)^2\} .$$

Therefore, by induction, we arrive at the conclusion of lemma 3.3 with $\varepsilon=\varepsilon_0$.

Lemma 3.4. *Under the same hypothesis of lemma* 3.3, *there exists a constant M independent of j and ε such that for any p, q satisfying $p+q \leq j$,*

$$\varepsilon^{s(p+q)} N_{j\varepsilon}(D_0^p D_1^q u) \leq M^{p+q+1} . \tag{3.19}$$

Proof. Instead of (3.7), (3.8), we start from the following inequality;

$$\sum_{p=0}^{2} \varepsilon^p N_{d+\varepsilon}(D_0^p u) \leq C\{\varepsilon^2 N_d(Pu) + \varepsilon N_d(D_0 u) + N_d(u)\} .$$

Then, by the similar argument in the previous lemma, from the result of lemma 3.3 it follows that (3.19) holds.

Proof of proposition 3.1. Let $\varepsilon j=d$. Then, from (3.19), it follows that for $p+q=j$,

$$||D_0^p D_1^q u||_{L^2(\omega^d)} \leqq N_d(D_0^p D_1^q u) \leq j^{sj} d^{-sj} M^{p+q+1} \qquad (s \geq 2),$$

which implies that u is in $\gamma^{(s)}(\omega^d)$. Combining this with the remark in the below of proposition 3.1, we obtain proposition 3.1.

4. Proof of theorem 3.

In this section, we consider the case $k=1$, and shall prove that for $s \geq 2$, P, given in the section 1, is $\gamma^{(s)}$-hypoelliptic in a neighborhood of the origin. It suffices to show that if $u \in \mathcal{E}'(\Omega)$, and $Pu \in \gamma^{(s)}(\omega)$, $s \geq 2$, then $u \in \gamma^{(s)}(\omega)$, where Ω and ω are the neighborhoods of the origin such that $\Omega \supset \omega$.

Let $\tilde{x}=(x_0, \cdots, x_{n-1})$, $x'=(x_1, \cdots, x_n)$, and $x''=(x_1, \cdots, x_{n-1})$, and denote their dual variables by $\tilde{\xi}$, ξ', ξ'', respectively. Define $\gamma_{\tilde{x}}^{(s)}(\Omega)$ by

$\{u \in \mathcal{D}'(\Omega)$; for any open sets $V \subset R^n$, and $W \subset R$ such that $V \times W \subset \Omega$, and any $\psi \in C_0^\infty(W)$, the distribution u_ψ on V given by

$$\langle u_\psi, \phi \rangle = \langle u, \phi(\tilde{x})\psi(x_n) \rangle$$

belongs to $\gamma^{(s)}(V)\}$,

and also define $\gamma_{x_n}^{(s)}(\Omega)$, similarly. Then, since there exists a constant C such that for any α, β, and $|\xi| \geq |\xi_n|^{1/2}$, $x \in \Omega$

$$|P_{(\beta)}^{(\alpha)}(x, \xi)|/|p(x, \xi)| \leq C^{|\alpha+\beta|+1} \alpha! \, (\beta!)^2 (1+|\xi|)^{-|\alpha|},$$

from the result of [5], it follows that

$$u \in \gamma_{\tilde{x}}^{(s/2)}(\Omega) \quad \text{if} \quad \gamma_{x_n}^{(s)}(\Omega) \in u .$$

So, in order to prove theorem 3, it suffices to show

Proposition 4.1. *Let* $u \in \mathcal{E}'(\Omega)$, *and* $Pu=f \in \gamma^{(s)}(\omega)$ $(\Omega \supset \omega)$. *If* $s \geq 2$, *then for any neighborhood* $\omega_0 \Subset \omega$ *of the origin, there are the constants C and L such that for any* $\phi \in C_0^\infty(\omega_0)$,

$$|\langle D_{x_n}^\alpha u, \phi \rangle| \leq C^{\alpha+1} (\alpha!)^s |\phi|_L , \tag{4.1}$$

where $|\phi|_L = \sup_{|\beta| \leq L} \left(\int |D_{\tilde{x}}^\beta \phi|^2 dx \right)^{1/2}$.

We begin with some preparations. By Fourier transformation with respect to x', or x, P is transformed into

$$P_1 = D_{x_1}^2 + |\xi''|^2 + ix_0\xi_n\,, \quad \text{or}$$

$$P_2 = i\xi_n D_{\xi_0} + |\tilde{\xi}|^2\,.$$

Let $A(z)$ and $B_\xi(z)$ are the linearly independent solution of the equation

$$(-(d/dz)^2 + z)V = 0$$

such that

$$A(z) \sim (1/2\sqrt{\pi}) \cdot z^{-1/4} e^{-2z^{3/2}/3}$$

as $z \to \infty$ in $\{|\arg z| < \pi\}$, and for $\sigma = \exp[2\pi i/(k+2)]$,

$$B_\xi(z) = \begin{cases} A(\sigma z) & \text{if } \xi_n > 0 \\ A(\sigma^{-1} z) & \text{if } \xi_n < 0\,. \end{cases}$$

We denote by W_ξ the wronskian of $\{A(z), B_\xi(z)\}$, and define the operator E on $\mathcal{S}(\boldsymbol{R}^{n+1})$ by

$$E\hat{f}(x, \xi') = \int_{-\infty}^{\infty} E(x_0, y_0, \xi')\hat{f}(y_0, \xi')dy_0\,, \quad (\hat{f}(t, \xi') = \int e^{-ix'\xi} f(t, x')dx')\,,$$

$$E(x_0, y_0, \xi') = \begin{cases} (i\xi_n)^{-1/3} W_\xi^{-1} A(z(x_0, \xi'))B_\xi(z(y_0, \xi')) & \text{if } x_0 > y_0\,, \\ E(y_0, x_0, \xi') & \text{if } x_0 < y_0\,, \end{cases}$$

where $z(t, \xi') = t(i\xi_n)^{1/3} + (i\xi_n)^{-2/3}|\xi''|^2$. Here we take a branch such that $z^{1/3}$ is real if $z > 0$.

Let us define another operator H on $\mathcal{S}(\boldsymbol{R}^{n+1})$ by

$$H\hat{f}(\xi) = \int_{-\infty}^{+\infty} H(\xi_0, \eta_0, \xi')\hat{f}(\eta_0, \xi')d\eta_0\,, \quad \hat{f}(\xi) = \int e^{-ix\xi} f(x)dx\,,$$

$$H(\xi_0, \eta_0, \xi') = \begin{cases} \xi_n^{-1} \exp(-\xi_n^{-1}[(\xi_0^3 - \eta_0^3)3^{-1} + |\xi''|^2(\xi_0 - \eta_0)]) & \\ \qquad\qquad \text{if } \xi_n > 0,\ \xi_0 > \eta_0\,, \\ 0 \qquad\qquad \text{if } \xi_n > 0,\ \xi_0 < \eta_0\,, \\ 0 \qquad\qquad \text{if } \xi_n < 0,\ \xi_0 > \eta_0\,, \\ \xi_n^{-1} \exp(-\xi_n^{-1}[(\xi_0^3 - \eta_0^3)3^{-1} + |\xi''|^2(\xi_0 - \eta_0)]) & \\ \qquad\qquad \text{if } \xi_n < 0,\ \xi_0 < \eta_0\,. \end{cases}$$

Now, we consider the equation

$$Pu = f\,, \tag{4.2}$$

where $f \in \mathcal{S}(\boldsymbol{R}^{n+1})$ such that $\hat{f}(t, \xi_n) = 0$ if $|\xi_n| \leq 1$. We observe that if $Pu = 0$, $u \in \mathcal{S}(\boldsymbol{R}^{n+1})$, then $u = 0$, since $(P_2\hat{u}, \hat{u}) = 0$ leads to $0 = \int |\tilde{\xi}|^2 |\hat{u}|^2 d\xi$. So, the solution u of (4.2) is uniquely given by

$$u(x) = \int e^{ix'\xi'}(E\hat{f})(x, \xi')(2\pi)^{-n} d\xi'\,, \quad \text{or} \tag{4.3}$$

(4.4) $$u(x)=\int e^{ix\xi}(H\hat{f})(\xi)(2\pi)^{-n-1}d\xi$$

We need some properties about E and H.

Lemma 4.2. *If* $|x_0-y_0|\geq\delta>0$, *then there exists a constant* $\varepsilon>0$ *such that*

(4.5) $$|E(x_0,y_0,\xi')|\leq Cte|\xi_n^{-1/3}|\exp(-\varepsilon|\xi_n^{1/2}|)$$

as $|\xi_n|\to\infty$.

Proof. We consider the case $x_0>y_0$. In the contrary case the similar argument hold, so we omit this. When both $z(x_0,\xi)$ and $z(y_0,\xi)$ tend to infinity, we have

$$\begin{aligned}E(x_0,y_0,\xi')&\sim Cte\,\xi_n^{-1/3}\{z(x_0,\xi')z(y_0,\xi')\}^{-1/4}\\&\quad\times\exp\{-(2/3)(z(x_0,\xi')^{3/2}-z(y_0,\xi')^{3/2})\}\\&\sim Cte\,\xi_n^{-1/3}\{\quad\quad\text{〃}\quad\quad\}^{-1/4}\\&\quad\times\exp\left(-\int_{y_0}^{x_0}\sqrt{i\xi_nt+|\xi''|^2dt}\right)\end{aligned}$$

where $\mathrm{Re}\sqrt{\xi_nt+|\xi''|^2}\geq0$. When $z(x_0,y_0,\xi')$ tends to infinity but $z(y_0,\xi')$ is bounded, we observe that

$$E(x_0,y_0,\xi')\sim Cte\,\xi_n^{-1/3}z(x_0,\xi')^{-1/4}\exp(-(2/3)z(x_0,\xi')^{3/2}),$$

and in this case, if $|x_0-y_0|\geq\delta>0$ and $|\xi_n|\to\infty$, then $|x_0|\geq\delta/2>0$. So, these consideration lead to (4.5).

Lemma 4.3. *Let* $\xi_m=t+i\rho\sqrt{|\xi_n|}$, $1\leq m\leq n-1$. *Then for* $|t|\leq|\xi_n|$, *we have, uniformly in* t,

$$\int_{-\infty}^{+\infty}|H(\xi_0,\eta_0,\xi')|d\eta_0,\quad \textit{and}\quad \int_{-\infty}^{+\infty}|H(\xi_0,\eta_0,\xi')|d\xi_0$$
$$\leq C|\xi_n^{-4/3}|\exp(C\rho^3|\xi_n^{1/2}|),$$

where C is a constant independent of ρ.

Proof. We observe that there exists constant $C_0>0$ such that

$$|H|\leq\exp(-C_0|\xi_n^{-}|^1|\xi_0-\eta_0|^3+\delta^2|\xi_0-\eta_0|).$$

Let $|\xi_n^{-1/3}|(\xi_0-\eta_0)=u$. Then, when $\xi_n>0$,

$$I=\int_{-\infty}^{+\infty}|H(\xi_0,\eta_0,\xi')|d\xi_0\leq\int_0^{+\infty}\exp(-(C_0/3)u^3+\delta^2\xi_n^{1/3}u)\xi_n^{-2/3}du$$

Let $u=v\delta\xi_n^{1/6}C_0^{-1/2}$. Then,

$$\begin{aligned}I&\leq\delta C_0^{-1/2}\xi_n^{-1/2}\int_0^{+\infty}\exp(C_0^{-1/2}\delta^2\xi_n^{1/2}(-(v^3/3)+v))dv\\&\sim\delta C_0^{-1/2}\xi_n^{-1/2}\sqrt{\pi}\,\exp((2/3)\delta^3C_0^{-1/2}\xi_n^{1/2})\sqrt{1/(C_0^{-1/2}\delta^3\xi_n^{1/2})}\end{aligned}$$

as $\xi_n \to \infty$. When $\xi_n < 0$, the similar argument holds. So, we obtain the half of lemma. As for the other half of this lemma, we can prove similarly as above. Q.E.D.

Lemma 4.4. *For* $|t| \leq \rho\sqrt{|\xi_n|}$, *we have, uniformly in t,*

$$|\xi_0 - \eta_0|^p |D^q_{\xi_n}| \{H(\xi_0, \eta_0, \xi'' + it\theta_m, \xi_n)|_{\xi_m = \xi_n}\}| \leq C_p^{q+1} q! |\xi_n|^{-q-1} \exp(-|\xi_0 - \eta_0| |\xi_n|/4) \quad \textit{for any } q, \tag{4.6}$$

where $\theta_{mj} = (d_j)_{1 \leq j \leq n-1}$ *with* $d_j = \delta_{m,j}$, $(1 \leq m \leq n-1$, $\delta_{i,j}$ *is a Kronecker's* $\delta)$

Proof. Writing $H(\xi_0, \eta_0, \xi'' + it\theta_m, \xi_n)|_{\xi_m = \xi_n} = \xi_n^{-1} \exp(-h(\xi_0, \eta_0, t, \xi_n))$, we observe that

$$-\mathrm{Re}\, h(\xi_0, \eta_0, t, \xi_0) \leq -3^{-1} |\xi_0^3 - \eta_0^3| |\xi_n|^{-1} - 1/2 |\xi_n| |\xi_0 - \eta_0|,$$

$$|D^q_{\xi_n} h(\xi_0, \eta_0, t, \xi_n)| \leq C^{q+1} q! \{3^{-1} |\xi_0^3 - \eta_0^3| |\xi_n|^{-1} + (1/2) |\xi_0 - \eta_0| |\xi_n|\} |\xi_n|^{-q}.$$

So, since $x^j e^{-x} \leq j!$, for $x \geq 0$, (4.6) follows from the well known fact that

$$D_x^\alpha \{e^{-f(x)}\} = \alpha! \sum \frac{i!}{i_1! \cdots i_k!} \frac{(-1)^i e^{-f}}{i!} \prod_{j=1}^{k} (f_{(\gamma^{(j)})}/\gamma^{(j)}!)^{i_j}$$

where $\sum_{j=1}^{k} i_j \gamma^{(j)} = \alpha$, $\sum i_j = i$, $\gamma^{(j)} \neq 0$, and $f_{(\gamma)} = D_x^\gamma f$.

Lemma 4.5. *For a measurable function* $K(t, s)$ *on* $\boldsymbol{R}^{2d}$, *suppose that* $\int_{\boldsymbol{R}^d} |K(t, s)| dt$, *and* $\int_{\boldsymbol{R}^d} |K(t, s)| ds \leq M$. *Then the integral operator K given by*

$$Ku = \int_{\boldsymbol{R}^d} K(t, s) u(s) ds$$

is a bounded operator on $L^2(\boldsymbol{R}^d)$ *with norm* $\leq M$.

This lemma is well-known. So we omit its proof.

Proof of proposition 4.1. Let $\chi \in \gamma^{(2)}(\boldsymbol{R})$ such that

$$0 \leq \chi \leq 1, \quad \chi(t) = 1 \quad \text{if} \quad |t| \geq 2, \quad \text{and} \quad = 0 \quad \text{if} \quad |t| \leq 1.$$

Then we see that for $\phi \in C_0^\infty(\omega_0)$,

$$(-1)^q \langle D^q_{x_n} u, \phi \rangle = \langle \chi(D_n) u, D^b_{x_n} \phi \rangle + \langle (1 - \chi(D_{x_n})) u, D^q_{x_n} \phi \rangle = I + I'.$$

Since $1 - \chi(D_{x_n})$ is a regularizing operator with respect to x_n, we have $(1 - \chi(D_{y_n})) u \in \gamma^{(2)}_{x_n}(\omega_0)$. So it suffices to show that there exist constants C_0, C_1, and L such that for any $\phi \in C_0^\infty(\omega_0)$, $|I| \leq C_0 C_1^q q!^s |\phi|_L$. $(s \geq 0)$.

Let $\psi \in \mathcal{S}$ such that ${}^t P \psi = g$, where $g = \chi(D_{x_n}) D^q_{x_n} \phi$. Then we observe that

$I=\langle u, {}^tP\psi\rangle=\langle f, \psi\rangle$, and

$$\psi(x) = \int e^{ix'\xi'}(Eg)(x', \xi')(2\pi)^{-n}d\xi' .$$

Let ω' is a neighborhood of the origin such that

$$\omega_0 \Subset \omega' \Subset \omega ,$$

and $h\in C_0^\infty(\omega')$ such that $h=1$ on ω_0. Then, I is written by

$$I = \langle hf, \psi\rangle+\langle(1-h)f, \psi\rangle = J+J' .$$

In view of $hf\in\gamma^{(s)}(\omega')$, we see that

$$J = \int e^{i\langle x'-y', \xi'\rangle}E(x_0, y_0, \xi')(1+|\xi'|^2)^{-N}\chi(\xi_n) \times D_{x_n}^q(1-\Delta_{x'})^N(hf)(x)\phi(y)dxdyd\xi'(2\pi)^{-n} .$$

Let $N=q+[n/2]+1$. Then this representation implies that

$$|J|\leq C_0C_1^q(q!)^s|\phi|_0 .$$

Next, we consider J'. We observe that, since $f\in\mathcal{E}'(\Omega)$, there exists a constant L such that $f=\sum_{|\beta|\leqq L} D_x^\beta f_\beta(x)$, where $f_\beta(x)\in L^2(\boldsymbol{R}^{n+1})$, with support contained in some compact set, from which it follows that

$$J' = \sum_{|\beta|\leq L}(-1)^\beta\langle f_\beta(x), D_x^\beta\{(1-h)\psi(x)\}\rangle = \sum_{|\beta|\leq L} J_\beta .$$

We note that $\operatorname{supp}h\cap\operatorname{supp}\phi=\phi$, so that we can take a partition of unity satisfying that

$$\sum \chi_j(x) = 1,\ \chi_j\in C_0^\infty(\boldsymbol{R}^{n+1}) ,$$

$\operatorname{supp}\chi_j$ is sufficiently small such that if $x\in\omega_0\cap\operatorname{supp}\chi_j$ and $y\in\operatorname{supp}(1-h)\cap\operatorname{supp}\chi_k$, then the one of the following cases holds:

(4.7–1) $\quad |x_0-y_0|\geq\delta>0$,

(4.7–2) $\quad |x_n-y_n|\geq\delta>0$

(4.7–3) $\quad$ for some $m, 1\leq m\leq n-1$, $|x_m-y_m|\geqq\delta>0$, and $|x_n-y_n|\leq\delta/2$.

In view of $\ker{}^tP=0$ in $\mathcal{S}$, we have the following representations;

$$J = \sum_{j,k} J_{j,k} ,$$

where for $\phi_k=\chi_k\phi$ and $f_j=\chi_jf(1-h)$,

$$J_{j,k} = \int D_x^\beta\{e^{i\langle x'-y', \xi'\rangle}\chi(\xi_n)\xi_n^qE(x_0, y_0, -\xi')\}\phi_k(y)f_j(x)dy'dy_0d\xi'dx(2\pi)^{-n} ,$$

or $$=\int e^{i\langle x-y,\xi\rangle}\tilde{\xi}^{\tilde{\beta}}\xi_n^{\beta_n+q}H(\xi_0,\eta_0,-\xi')e^{i\langle y_0,\xi_0-\eta_0\rangle}\phi_k(y)f_i(x)dyd\eta_0 d\xi dx(2\pi)^{-n-1},$$

where the integration is carried in its order, from left to right. We note that, by the construction of $\{\chi_j\}$, on the support of the integrand of $J_{j,k}$, one of (4.7) holds.

So, we investigate $J_{j,k}$, case by case. In the case (4.7–1), we take the first representation for $J_{j,k}$. Then, from lemma 4.2, it follows that there exists a constant C_β depending only on β, such that

$$\begin{aligned}|J_{j,k}|\le &\int|\xi|^{\beta}|\xi_n|^q\langle\xi''\rangle^{-2|\beta''|-2n}\exp(-\varepsilon|\xi_n|^{1/2})\\ &\times((1-\Delta_{x''})^{|\beta''|+n}\phi_k)(y)f_j(x)dyd\xi'dx(2\pi)^{-n}\\ \le & C^{q+1}(q!)^2|\phi_k|_{|\beta''|+n}|f_j|_0,\end{aligned}$$

where $\langle\xi''\rangle=(1+|\xi''|^2)^{1/2}$,

In the case (4.7–2), we take the second representation for $J_{j,k}$ since it is difficul to estimate $D^\alpha E$. We note that $\xi_0^{\beta}=\sum_{\gamma\le\beta_0}\binom{\beta_0}{\gamma}(\xi_0-\eta_0)^\gamma\eta_0^{\beta_0-\gamma}$. So, the integration by parts with respect to ξ_n implies that

$$\begin{aligned}J_{j,k}=\sum_{\gamma\le\beta_0}\binom{\beta_0}{\gamma}\int &(1/(x_n-y_n)^N)e^{i\langle x-y,\xi\rangle}\langle\tilde{\xi}\rangle^{-2n}D_{\xi_n}^N\{(\xi_0-\eta_0)^\gamma h_{\xi'}(\xi_0,\eta_0,y,q)\}\\ &\times(1-\Delta_{\tilde{y}})^n D_{\beta''}^{y''}(D_{y_0})^{\beta_0-\gamma}\phi_k(y)f_j(x)dyd\eta_0 d\xi dx(2\pi)^{-n-1},\end{aligned}$$

where $h_{\xi'}(\xi_0,\eta_0,y,q)=\xi_n^{\beta_n+q}\chi(\xi_n)H(\xi_0,\eta_0,-\xi')e^{i\langle y_0,\xi_0-\eta_0\rangle}$.

From lemma 4.4, it follows that for $N=\beta_n+q+[n/2]+1$, on the support of H,

$$\begin{aligned}&D_{\xi_n}^N\{(\xi_0-\eta_0)^\gamma h_\xi(\xi_0,\eta_0,y,q)\}\\ &=\sum_{k+i+j=N}\frac{N!}{i!j!k!}D_{\xi_n}^i(\xi_n^{\beta_n+q})D_{\xi_n}^j\chi(\xi_n)D_{\xi_n}^k H(\xi_0,\eta_0,-\xi')|\xi_0-\eta_0|^\gamma\\ &\le C_\beta^{q+1}q!^2\langle\xi_n\rangle^{-2}\exp(-(1/4)|\xi_0-\eta_0|\,|\xi_n|).\end{aligned}$$

So, by Parseval equality and lemma 4.5, we have

$$\begin{aligned}|J_{j,k}|\le\sum_{\gamma=0}^{\beta_0}\delta^{-N}C_\beta^{q+1}q!^2\{&\int\langle\xi\rangle^{-n}\langle\xi_n\rangle^{-2}d\xi\}^{1/2}\int_0^{+\infty}e^{-t/4}dt\\ &\times|\phi_k|_{|\tilde{\beta}|+[n/2]+1}|f_j|_0.\end{aligned}$$

Lastly, we consider the case (4.7–3). We also take the second representation for $J_{j,k}$. Let $\Phi(\xi_m,\xi_n)$ be a function such that $=1$ if $|\xi_m|\ge|\xi_n|$ and $=0$ if $|\xi_m|<|\xi_n|$. Then,

$$J_{j,k}=\int e^{i\langle x-y,\xi\rangle}h_{\xi'}(\xi_0,\eta_0,y,q)\Phi(\xi_m,\xi_n)\Phi_k(y)f_j(x)dyd\eta_0 d\xi dx(2\pi)^{-n-1}$$

$$+\int e^{i\langle x-y,\xi\rangle}h_{\xi'}(1-\phi)\phi_k(y)f_j(x)dyd\eta_0d\xi dx(2\pi)^{-n-1}$$
$$=K+K'.$$

As for K, the integration by parts with respect to ξ_m leads to

$$K=\int(x_m-y_m)^{-N}e^{i\langle x-y,\xi\rangle}\Phi D^N_{\xi_m}h_{\xi'}(\xi_0,\eta_0,y,q)\phi_k(y)f_j(x)$$
$$\times dyd\eta_0d\xi dx(2\pi)^{-n-1}$$
$$-\sum_{k=0}^{N-1}\int(x_m-y_m)^{-1-k}e^{i\langle x-y,\xi\rangle}\widetilde{D^k_{\xi_m}h_{\xi'}}\phi_k(y)f_j(x)dyd\eta_0$$
$$\times d\xi_0\cdots d\xi_{m-1}d\xi_{m+1}\cdots d\xi_n dx(2\pi)^{-n-1}$$
$$=K_1+K_2,\quad\text{where}\quad \widetilde{D^k_{\xi_m}h_{\xi'}}=[D^k_{\xi_n}h_{\xi'}]^{\xi_m=|\xi_n|}_{\xi_m=-|\xi_n|}.$$

Since on the support of Φ, $|\xi_m|^{-N}\leq|\xi_n|^{-N}$ and for some $\tilde{C}_\beta$, $\varepsilon>0$,

$$|D^N_\xi h_{\xi'}|\leq\tilde{C}^{N+1}_\beta N!|\xi_m|^{-N}\exp(-\varepsilon|\xi_0-\eta_0|^3|\xi_n|^{-1})$$

, in the same way as case (4.7–2), we obtain

$$|K_1|\leq C^{q+2}q!|\phi_k|_{|\tilde{\beta}|+2n}|f_j|_0.$$

On the other hand, since if $\xi_m=\pm|\xi_n|$,

$$|\partial/\partial\xi_n\{i\langle x-y,\xi\rangle\}|\geq\delta/2>0,$$

by the integration by parts with respect to ξ_n, we have

$$|K_2|\leq C^{q+1}_\beta q!|\phi_k|_{|\tilde{\beta}|+2n}|f_j|_0.$$

Now, we consider K'. To treat this, we use a complex integral with respect to ξ_m. We may assume that $x_m-y_m>\delta$. In the contrary case, the similar argument holds. Then,

$$K'=\int_\Gamma e^{i\langle x-y,\xi\rangle}h_{\xi'}(\xi_0,\eta_0,y,q)\phi_k(y)f_j(x)d\mu$$
$$+\int_{\Gamma'}\{\qquad ''\qquad\}d\mu$$
$$+\int_{\Gamma''}\{\qquad ''\qquad\}d\mu$$
$$=L+L'+L'',$$

where $d\mu=dyd\eta_0d\xi_m d\xi_0\cdots d\xi_{m-1}d\xi_{m+1}\cdots d\xi_n dx(2\pi)^{-n-1}$, and the paths are taken as follows; for $\rho>0$,

$$\Gamma=\{\xi_m=-|\xi_n|+it;\ 0\leq t\leq\rho\sqrt{|\xi_n|}\}$$
$$\Gamma'=\{\xi_m=t+i\rho\sqrt{|\xi_n|};\ -|\xi_n|\leq t\leq|\xi_n|\}$$
$$\Gamma''=\{\xi_m=|\xi_n|+it;\ \rho\sqrt{|\xi_n|}\geq t\geq 0\}.$$

Since if $\xi_m \in \Gamma$ or Γ'',

$$|\partial/\partial\xi_n i\langle x-y,\xi\rangle| \geq \delta/2 > 0\,,$$

by lemma 4.4, the integration by parts with respect to ξ_n leads to

$$|L|+|L''| \leq C_\beta^{q+1} q!\,|\phi_k|_{|\tilde{\beta}|+2n}|f_j|_0\,.$$

On the other hand, we observe that for $\xi_m \in \Gamma'$,

$$i\langle x_m - y_m, \xi_m\rangle = i\langle x_m - y_m, t\rangle - (x_m - y_m)\rho\sqrt{|\xi_n|}\,.$$

So, from lemma 4.3, it follows that if ρ is sufficiently small, there are the constant ε and $C>0$ such that

$$|L'| \leq C\{\int (e^{-\varepsilon|\xi_n|^{1/2}}\langle\tilde{\xi}\rangle^{-2n}\chi(\xi_n)|\xi_n|^{q+\beta_n-3/4})^2 d\xi\}^{1/2}|\phi_k|_{|\tilde{\beta}|+2n}|f_j|_0,$$

which implies that for some C_β ,

$$|L'| \leq C_\beta^{q+1} q!^2 |\phi_k|_{|\tilde{\beta}|+2n}|f_j|_0\,.$$

Summing up the above argument, we obtain (4.1) Q.E.D.

5. Proof of theorem 4.

We use the Hörmander-Métivier inequality as follows;

Lemma 5.1. *If P is $\gamma^{(s)}$-hypoelliptic in a neighborhood of the origin, then for any sufficiently small, neighborhood $\omega' \Subset \omega$ of the origin, there exist constants C and L such that for any $u \in C^\infty(\omega)$, and any $l \in N$.*

$$(5.1)\qquad \{\sum_{|\alpha|\leq l} \|D_x^\alpha u\|^2_{L^2(\omega')}\}^{1/2} \leq CL^l\{\sum_{|\alpha|\leq l} l^{s(l-|\alpha|)}\|D_x^\alpha Pu\|_{L^2(\omega)} + (l!)^s\|u\|_{L^2(\omega)}\}\,.$$

Proof. P has a right inverse in $L^2(\Omega)$ if Ω is small. So, (5.1) follows from [6] (c.f. p. 112 in [3] and [7]).

Proposition 5.2. *If there exists a non-zero complex number λ such that the equation $\{\hat{P}(t,\xi',D_t)|_{\xi=(\lambda,0,\cdots,0,-1)}\}\, v(t)=0$:*

$$(5.2)\qquad \{D_t^2 - it^k\}\, v(t) = -\lambda^2 v(t)$$

has a non-trivial solution in $L^2(\boldsymbol{R})$, then for $1\leq s<k+2$, P is not $\gamma^{(s)}$-hypoelliptic in any neighborhood of the origin.

Proof. Let $v(t)$ be a non-trivial solution in $L^2(\boldsymbol{R})$ of (5.2.) Then, we note that $v \in C^\infty(\boldsymbol{R})$. For a large parameter ρ, let $V(x,\rho) = v(\rho x_0)\exp(i\lambda x_1\rho - i\rho^{k+2}x_n)$. Then, this function satisfies the equation

$$PV=(\sum_{j=0}^{n-1} D_{x_j}^2+ix_0^k D_{x_n})V(x,\rho)=0\,.$$

Moreover, we observe that for any $N\in \boldsymbol{N}$,

$$|D_{x_n}^N V(\bar{x},\rho)|\geq c\rho^{(k+2)N}\,, \quad \text{and} \tag{5.3}$$

$$\int_\omega |V(x,\rho)|^2 dx \leq c'\rho^{-1}e^{\delta\rho}\,, \tag{5.4}$$

where c, c', δ are the positive constants independent of ρ, N, and $\bar{x}=\rho^{-1}\bar{y}$ such that $v(\bar{y})\neq 0$.

Let $u=V(x,\rho)$ in (5.1). Then, from (5.3), (5.4) it follows that

$$c\rho^{(k+2)(l-[n/2]-1)}\leq CL^l(l!)^s c'\rho^{-1}e^{\delta\rho}\,,$$

which implies that

$$1\leq c^{-1}c'C\exp(l\log L+ls\log l+\delta\rho-(k+2)l\log\rho+[n/2](k+s)\log\rho)\,.$$

In the last inequality, let $\rho=l$ and $l\to+\infty$. Then, if $s<k+2$, the right hand side of it tends to 0, which leads to a contradiction. Q.E.D.

Proof of theorem 4. We claim that the condition of proposition 5.2 is satisfied if $k=3$ or even number when $n\geq 2$. First, we consider the case $k=3$, $n\geq 2$. Let

$$S=\{z\in\boldsymbol{C};\ |\arg z|<\pi/5\}\,, \quad \text{and} \quad S'=\{z\in\boldsymbol{C};\ 3\pi/5<\arg z<\pi\}\,.$$

From the result of Y. Sibuya [9]. Chap. 6, it follows that there exist the non-zero complex number μ and the non-trivial solution $V(z)$ of equation

$$(D_z^2+z^3)V=\mu V$$

such that $V(z)$ is exponentially decreasing as $z\to\infty$ in S and S'. So, it is easily seen that $V(te^{-\pi i/10})$ satisfies the condition of proposition 5.2 with $-\lambda^2=\mu e^{\pi i/5}$.

Secondly, we consider the case that k is even and $n\geq 2$. Let $S_0=\{z\in\boldsymbol{C};\ |\arg z|<\pi/(k+2)$ and $S_1=\{z\in\boldsymbol{C};\ |\arg z-\pi|<\pi/(k+2)\}$. Then, for each μ, there exists a fundamental system of solution $\{V_j^\pm\}$ of the equation

$$(D_z^2+z^k)V=\mu V$$

such that V_j^+ is exponentially increasing and V_j^- is exponentially decreasing as $z\to\infty$ in S_j, $j=0, 1$. On the other hand, it is well known that the operator

$$-(\partial/\partial y)^k+y^k \quad \text{on} \quad L^2(\boldsymbol{R})$$

has infinitely many eigenvalues if k is even. Let μ be a non-zero eigenvalue and V be its eigenfunction belonging to $L^2(\boldsymbol{R})$. We note that V is an entire function. Since the real line $\boldsymbol{R}$ is properly contained in S_j ($j=0, 1$), $\{V, V_j^-\}$ is linearly dependent, for $j=0, 1$, which implies that $V(te^{-\pi i/(2k+4)})$ satisfies the condition of proposition 5.2 with $-\lambda^2=\mu e^{\pi i/(k+2)}$.

Therefore, we can use proposition 5.2 and obtain theorem 4.

Remark. For $\lambda \in \boldsymbol{R}$, the condition of proposition 5.2 is never satisfied. This is connected to C^{∞}-hypoellipticity of P. Of course, for the case $n=1$ (i.e. $\lambda=0$) and $k=1$, the condition of proposition is not satisfied.

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