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On singular solutions for some first order partial differential equations with degenerate principal symbols

Dedicated to Professor Sigeru Mizohata on his sixtieth birthday

By

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1. Introduction.

In this paper, we consider the following equation:

$$L(u)\equiv\sum_{i=1}^{n}(\lambda_i x_i+a_i)\partial u/\partial_i x=u\,, \tag{1.1}$$

where $\lambda_i\in\boldsymbol{C}$ $(i=1, 2, \cdots, n)$ and a_i $(i=1, 2, \cdots, n)$ are all holomorphic in a neighborhood of the origin of $\boldsymbol{C}^n$. We assume, moreover, that each a_i vanishes at the origin with an order larger than two.

In [4] and [5], we have concerned with the holomorphic solutions of the equation of the type (1.1), but of fairly particular one. Here, we shall wholly concern with the singular solutions.

To clarify the motivation of this study, we make a slight review to the case of ordinary differential equations:

$$P(u)\equiv\sum_{i=0}^{m}\{p_i x^i+q_i(x)\}u^{(i)}=0\,, \tag{1.2}$$

where p_i $(i=1, 2, \cdots, n)\in\boldsymbol{C}$ with $p_m=1$ and each $q_i(x)$ is a holomorphic function in a neighborhood of the origin of $\boldsymbol{C}$, vanishing at the origin with an order larger than $i+1$. Now, let P_0 be as follows:

$$P_0=\sum_{i=1}^{m}p_i x^i d^i/dx^i\,, \tag{1.3}$$

then, for a certain solution $u_0(x)$ of the equation:

$$P_0(v)=0\,, \tag{1.4}$$

we can always find a solution $u(x)$ of (1.2) having the form:

$$u(x)=u_0(x)H(x)\,, \tag{1.5}$$

where $H(x)$ $(\not\equiv 0)$ is holomorphic at the origin.

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We shall intend to adapt what is stated above to the equation (1.1), that is, we shall look for a solution $u(x_1, \cdots, x_n)$ having the form:

$$u(x_1, \cdots, x_n) = u_0(x_1, \cdots, x_n)H(x_1, \cdots, x_n)\,, \tag{1.6}$$

where $u_0(x_2, \cdots, x_n)$ is a solution of the equation:

$$L_0(v) \equiv \sum_{i=0}^{n} \lambda_i x_i \partial v/\partial x_i = v\,, \tag{1.7}$$

and $H(x_1, \cdots, x_n)$ is a holomorphic function at the origin.

In the case of ordinary differential equations, the number of the solutions of (1.4) which are linearly independent are finite, and this makes it simple to study the problem. However, it is infinite in the case of partial differential equations, and this will make it fairly hard to study the problem. Nevertheless, we can make a characterization of the solutions of (1.7), that is, we can see that every solution $v(x_1, \cdots, x_n)$ of (1.7) has the following property:

$$v(t^{\lambda_1}x_1, \cdots, t^{\lambda_n}x_n) = tv(x_1, \cdots, x_n) \qquad (t \neq 0)\,, \tag{1.8}$$

In fact, it is easy to see that a function $v(x_1, \cdots, x_n)$ satisfying (1.8) is a solution of (1.7). Conversely, let $v(x_1, \cdots, x_n)$ be a solution of (1.7), then substituting $(t^{\lambda_1}x_1, \cdots, t^{\lambda_n}x_n)$ for $(x_1, \cdots, x_n)$, we have

$$t dv(t^{\lambda_1}x_1, \cdots, t^{\lambda_n}x_n)/dt = v(t^{\lambda_1}x_1, \cdots, t^{\lambda_n}x_n)\,, \tag{1.9}$$

and this implies (1.8). It should be noted that (1.8) shows that the solution of (1.7) is a homogeneous function of degree one in $(x_1^{1/\lambda_1}, \cdots, x_n^{1/\lambda_n})$, if none of λ_i $(i=1, 2, \cdots, n)$ is zero.

Remark. When $\lambda_i \neq 0$ $(i=1, 2, \cdots, p)$ and $\lambda_k = 0$ $(k=p+1, \cdots, n)$, we can show that the solution of (1.7) is given by

$$v(x_i, \cdots, x_n) = v_0(x_1, \cdots, x_p)w(x_{p+1}, \cdots, x_n)\,, \tag{1.10}$$

where v_0 is a homogeneous function of degree one in $(x_1^{1/\lambda_1}, \cdots, x_p^{1/\lambda_p})$ and w an arbitrary function of $(x_{p+1}, \cdots, x_n)$.

Substituting $u(x_1, \cdots, x_n)$ given in (1.6) into (1.1), we have immediately

$$u_0 L(H) + \Big(\sum_{i=1}^{n} a_i \partial u_0/\partial x_i\Big)H = 0\,. \tag{1.11}$$

Hence, if it holds

$$L_a(u_0) \equiv \sum_{i=1}^{n} a_i \partial u_0/\partial x_i = bu_0\,, \tag{1.12}$$

with a holomorphic function b vanishing at the origin, it follows

$$L(H) + bH = 0\,. \tag{1.13}$$

As is shown in the next section, the equation (1.13) admits always a holomorphic solution, and this means that (1.12) gives a sufficient condition in order that (1.1) has a solution such as given by (1.5). I, myself, anticipate that (1.12) is also necessary for the equation (1.1) to have a solution such as given by (1.5), however, to my regret, I will show it only for the following three cases:

(I) $u_0(x_1, \cdots, x_n) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ $\quad (\sum_{i=1}^{n} \alpha_i \lambda_i = 1)$

(II) $u_0(x_1, \cdots, x_n) = (\sum_{i=1}^{n} c_i x_i^{1/\lambda_i r})^r$ $\quad (r \neq 0)$

(III) $u_0(x_1, \cdots, x_n) = (\sum_{i=1}^{n} c_i x_i^{q/\lambda_i})^p (\sum_{i=1} d_i x_i^{r/\lambda_i})^s$ $\quad (pq+rs = 1)$,

under the assumption

(1.14) $$\Lambda \not\ni 0,$$

where α_i, c_i, d_i ($i=1, 2, \cdots, n$), p, q, r and s are all in $\boldsymbol{C}$ and Λ denotes the convex hull of $\lambda_1, \cdots, \lambda_n$.

2. The case (I).

In this section, we consider the case (I). In virtue of (1.11), we have

(2.1) $$x_1 \cdots x_n L(H) + (\sum_{i=1}^{n} \alpha_i x_1 \cdots x_{i-1} a_i x_{i+1} \cdots x_n) H = 0.$$

Lemma 2.1. *Let* $\alpha_i \neq 0$, *then*

(2.2) $$a_i \equiv 0 \quad (\mathrm{mod}\ x_i),$$

is necessary for the equation (2.1) *to have a non-trivial solution.*

Proof. We may assume that $\alpha_1 \neq 0$. If (2.2) does not hold for $i=1$, it follows immediately that $H \equiv 0$ (mod x_1), that is, $H = x_1^{\nu} G$, where $\nu \in \boldsymbol{N}$ and G is a holomorphic function $\not\equiv 0$ (mod x_1). Thus, we obtain that $a_1 G \equiv 0$ (mod x_1), but this is a contradiction. Q.E.D.

Now, let $\alpha_i \neq 0$ ($i=1, 2, \cdots, p$) and $\alpha_k = 0$ ($k=p+1, \cdots, n$), then we have

(2.3) $$a_i = x_i b_i \quad (i = 1, 2, \cdots, p),$$

where b_i ($i=1, 2, \cdots, p$) are all holomorphic. Hence we have

(2.4) $$L(H) + (\sum_{i=1}^{n} \alpha_i b_i) H = 0.$$

We can show that the equation (2.4) has always a non-trivial solution by using the method of dominant series. In fact, taking the dominant series as follows:

(2.5) $$\sum_{i=1}^{n} \alpha_i b_i \ll M(x_1 + \cdots + x_n) / \{1 - \rho(x_1 + \cdots + x_n)\},$$

where M and ρ are positive constants chosen suitably, and letting ε be the distance between Λ and the origin, we see that the formal solution H of (2.4) is dominated as follows:

$$H(x_1, \cdots, x_n) \ll H_0(x_1 + \cdots + x_n)\,, \tag{2.6}$$

where $H_0(z)$ is a solution of the auxiliary equation:

$$\{\varepsilon - Mz/(1-\rho z)\}\, dH_0/dz = MH_0/(1-\rho z)\,, \tag{2.7}$$

satisfying $H_0(0) \geqq |H(0, \cdots, 0)|$.

Thus, we can obtain the

Theorem 2.1. *In the case of* (I), *the equation* (1.1) *has a solution of the type* (1.6), *if and only if* $a_k \equiv 0 \pmod{x_k}$ *for* k *such that* $\alpha_k \neq 0$.

Remark. Theorem 2.1 shows us the necessity of (1.12) for the existence of the solution of the type (1.6).

3. The case (II)-I.

In this section, we consider the case (II) under the assumption:

$$1/\lambda_i r \notin \boldsymbol{Z} \qquad \textit{for all } i\,. \tag{A}$$

We begin with the following lemma.

Lemma 3.1. *Let* $u_1, \cdots, u_p$ *and* v *be holomorphic functions in a neighborhood of the origin of* $\boldsymbol{C}^n$. *If it holds*

$$\sum_{i=1}^{p} x_{k_i}^{\alpha_i} u_i + v = 0 \qquad \{k_1, \cdots, k_p\} \subset \{1, \cdots, n\}\,, \tag{3.1}$$

for some α_i $(i=1, 2, \cdots, p) \notin \boldsymbol{Z}$, *then it follows that* $u_i = 0$ $(i=1, 2, \cdots, p)$ *and* $v=0$.

Proof. It is easy. Q.E.D.

From (1.11) we have

$$\Big(\sum_{i=1}^{n} c_i x_i^{1/\lambda_i r}\Big) L(H) + \Big(\sum_{i=1}^{n} c_i x_i^{1/\lambda_i r} a_i/\lambda_i x_i\Big) H = 0\,, \tag{3.2}$$

Hence by the assumption (A) and Lemma 3.1, we obtain

$$\lambda_i x_i L(H) + a_i H = 0\,, \tag{3.3}$$

for all i such that $c_i \neq 0$.

Lemma 3.2. *It must hold that*

$$a_i \equiv 0 \qquad (\mathrm{mod}\ x_i)\,, \tag{3.4}$$

for the equation (3.3) *to have a non-trivial solution.*

Proof. Let $i=1$. If (3.4) does not hold for $i=1$, it follows that $H=x_1^\nu G$ ($\nu\in\boldsymbol{N}$ and $G\not\equiv 0 \pmod{x_1}$). Thus, we obtain

$$\nu\lambda_1+1=0 \tag{3.5}$$

and

$$L(G)=G\,. \tag{3.6}$$

For the equation (3.6) to have a non-trivial holomorphic solution G it must hold

$$\sum_{i=1}^{n}\alpha_i\lambda_i=1\,, \tag{3.7}$$

for some $(\alpha_1,\cdots,\alpha_n)\in\bar{\boldsymbol{N}}^n$. However, this contradicts (1.14), since $\lambda_1<0$ by (3.5).
Q.E.D.

By (3.3) we can see that $a_i/\lambda_i x_i$ does not depend on i. This implies that $a_i=\lambda_i x_i b$ for some holomorphic function b.

Theorem 3.1. *To the case* (II), *under the assumption* (A), *the equation* (1.1) *admits a solution given by* (1.6), *if and only if it holds*

$$a_i=\lambda_i x_i b\quad (b\ \textit{is holomorphic})\,, \tag{3.8}$$

for all i *such that* $c_i\neq 0$.

Remark. Theorem 3.1 shows us the necessity of (1.12) for the existence of the solution given by (1.6).

4. The case (II)-II.

We should take the case where $1/\lambda_i r$ ($i=1, 2, \cdots, n$) belong to $\boldsymbol{Z}$. We start with the

Lemma 4.1. *Let* $1/\lambda_k r$ $(k=1, 2, \cdots, n)\in\boldsymbol{Z}$, *then* $\lambda_j\lambda_k r^2>0$ *for all* j *and* k.

Proof. Let $\lambda_k=\sigma_k+\mathrm{i}\tau_k$ and $r=\alpha+\mathrm{i}\beta$, then we have

$$\alpha\tau_k+\beta\sigma_k=0\qquad (k=1, 2, \cdots, n)\,. \tag{4.1}$$

If $\lambda_j\lambda_k r^2<0$ for some j and k, it follows

$$(\alpha\sigma_j-\beta\tau_j)(\alpha\sigma_k-\beta\tau_k)>0\,. \tag{4.2}$$

When $\beta=0$, we see that $\sigma_j\sigma_k<0$, and this contradicts (1.14). When $\beta\neq 0$, we see that $\lambda_k=(-\alpha/\beta+\mathrm{i})\tau_k$ for all k and that $\tau_j\tau_k<0$. This contradicts also (1.14)
Q.E.D.

In this section, we study the case (II) under the assumption:

$$1/\lambda_j r\in\boldsymbol{N}\qquad\text{for all}\quad j\,. \tag{B}$$

Let us put

$$1/\lambda_j r = m_j + 1 \qquad (m_j \in \boldsymbol{N}), \tag{4.3}$$

then we obtain from (1.11)

$$\Phi L(H) + L_a(\Phi) H = 0, \tag{4.4}$$

where $\Phi = \sum_{i=1}^{n} c_i x_i^{m_i+1}$. We divide our study into the following two cases:

(i) Φ is irreducible in $\boldsymbol{C}[x_1, \cdots, x_n]$

(ii) Φ is reducible in $\boldsymbol{C}[x_1, \cdots, x_n]$.

We consider first the case (i).

Lemma 4.2. *Suppose that Φ is irreducible and that f and g are holomorphic in a neighborhood of the origin of $\boldsymbol{C}^n$. If $fg \equiv 0$ (mod Φ), that is, if there exists a holomorphic function h such tha) $fg = h\Phi$, then it follows that $f \equiv 0$ (mod Φ) or $g \equiv 0$ (mod Φ).*

Proof. It is almost obvious. Q.E.D.

Now, let us show the following proposition.

Proposition 4.1. *Suppose that Φ is irreducible. The following condition*

$$L_a(\Phi) \equiv 0 \qquad (\text{mod } \Phi), \tag{4.5}$$

is necessary and sufficient in order that the equation (4.4) has a non-trivial solution.

Proof. It suffices to show the necessity. Let us assume that (4.5) does not hold, then we see by Lemma 4.2 that $H \equiv 0$ (mod Φ). Putting $H = \Phi^\nu G$ ($\nu \in \boldsymbol{N}$ and $G \not\equiv 0$ (mod Φ)) and noticing

$$rL(\Phi) = \Phi + L_a(\Phi), \tag{4.6}$$

we obtain

$$\Phi(rL(G) + \nu G) + (\nu + r) L_a(\Phi) G = 0. \tag{4.7}$$

Hence we obtain that $\nu + r = 0$ and therefore

$$L(G) = G. \tag{4.8}$$

Thus, it must hold

$$\sum_{i=1}^{n} \alpha_i \lambda_i = 1, \tag{4.9}$$

for some $(\alpha_i, \cdots, \alpha_n) \in \bar{\boldsymbol{N}}^n$.

On the other hand, we see that λ_i $(i = 1, 2, \cdots, n) < 0$, since $r < 0$. This is not compatible with (4.9). Q.E.D.

Next, we shall consider the case (ii). In this case, we may assume that $c_1c_2 \neq 0$ and $c_i=0$ $(i=3, \cdots, n)$. Let Φ be as follows:

$$\Phi = \prod_{i=1}^{p} \phi_i \qquad (p \geqq 2), \tag{4.10}$$

where ϕ_j and ϕ_k are mutually prime. As is easily seen, Φ has no diviser such as ϕ_k^s $(s \geqq 2)$. Let us assume that, for example,

$$L_a(\Phi) \not\equiv 0 \qquad (\mathrm{mod}\ \phi_1), \tag{4.11}$$

then we obtain that $H=\phi_1^{\nu}G$ ($\nu \in \boldsymbol{N}$ and $G \not\equiv 0$ (mod ϕ_1)). Substituting H into (4.5), we get

$$\Phi(rL(G)+\nu G)-\nu r\phi_1 L(\psi)G+(\nu+r)L_a(\Phi)G = 0, \tag{4.12}$$

where $\psi=\prod_{i=2}^{p}\phi_i$. Hence we obtain that $\nu+r=0$ and therefore that

$$\psi L(G) = (\nu L(\psi)+\psi)G. \tag{4.13}$$

Lemma 4.3. *If it holds*

$$L(\psi) \not\equiv 0 \qquad (\mathrm{mod}\ \psi), \tag{4.14}$$

then the equation (4.13) *has no non-trivial solution.*

Proof. Let us assume that

$$\nu L(\psi) = (\gamma+h), \tag{4.15}$$

where $\gamma \in \boldsymbol{C}$ and h is a holomorphic function vanishing at the origin. Then, we have

$$L(G) = (1+\gamma+h)G. \tag{4.16}$$

If we put

$$\phi_i = x_1^{\sigma_i}+\cdots+\alpha_i x_2^{\tau_i} \qquad (\alpha_i \neq 0;\ i = 2, \cdots, p), \tag{4.17}$$

then we obtain from (4.15)

$$\lambda_1\sigma = \lambda_2\tau = \gamma/\nu, \tag{4.18}$$

where $\sigma=\sum_{i=2}^{p}\sigma_i$ and $\tau=\sum_{i=2}^{p}\tau_i$. Moreover, from (4.16) we obtain

$$\sum_{i=1}^{n}\alpha_i\lambda_i = 1+\gamma, \tag{4.19}$$

for some $(\alpha_1, \cdots, \alpha_n) \in \bar{\boldsymbol{N}}^n$.

On the other hand, we see that $\lambda_1, \lambda_2<0$, since $r<0$ and from (4.18)

$$\gamma = \nu\lambda_1\sigma = -r\lambda_1\sigma = -\sigma_1/(m_1+1) > -1, \tag{4.20}$$

and this contradicts (4.19). Q.E.D.

If we assume $L(\psi)\not\equiv 0 \pmod{\phi_2}$, then putting $\psi=\phi_2\psi_0$, we have

$$\psi_0 L(G_0)=(\nu L(\psi_0)+\psi_0)G_0\,, \tag{4.21}$$

where G_0 is holomorphic. Applying Lemma 4.3 to (4.21) again and again, we obtain finally

$$L(g)=g\,, \tag{4.22}$$

for some holomorphic function g. However, the solution of (4.22) is no more than the trivial one, since $\lambda_i\ (i=1, 2, \cdots, n)<0$. Thus, we can have the same result as Proposition 4.1, even if Φ is reducible.

5. The case (II)-III.

In this section, we study the case (II) under the assumption:

$$-1/\lambda_i r\in \boldsymbol{N} \qquad \textit{for all} \quad i \tag{C}$$

Putting

$$\Phi=\sum_{i=1}^{n} c_i x_i^{-m_i}=\Big(\prod_{i=1}^{n} x_i^{-m_i}\Big)\Psi\,, \tag{5.1}$$

we have from (1.11)

$$\Phi L(H)+L_a(\Phi)H=0\,, \tag{5.2}$$

where $m_i=-1/\lambda_i r$. Moreover, we introduce a polynomial Q by

$$Q=\Big(\prod_{i=1}^{n} x_i^{m_i+1}\Big)L_a(\Phi)\,, \tag{5.3}$$

then we obtain

$$x_1\cdots x_n\Psi L(H)+QH=0\,. \tag{5.4}$$

Now, let us divide our study into the following two cases:

(i) Ψ is irreducible in $\boldsymbol{C}[x_1, \cdots, x_n]$
(ii) Ψ is reducible in $\boldsymbol{C}]x_1, \cdots, x_n]$.

Let us consider the case (i). We start with the

Lemma 5.1. *It must hold*

$$Q\equiv 0 \qquad (\mathrm{mod}\ \Psi)\,, \tag{5.5}$$

for (5.4) *to have a non-trivial solution.*

Proof. Let us assume the contrary. Putting $H=\Phi^{\nu}G$ ($\nu\in\boldsymbol{N}$ and $G\not\equiv 0$ (mod Ψ)), we get

(5.6) $$x_1 \cdots x_n(\Psi L(G)+\nu L(\Psi)G)+QG = 0\,.$$

Using (5.2), (5.3) and

(5.7) $$rL(\Phi) = \Phi + rL_a(\Phi)\,,$$

we obtain that $\nu+r=0$ and

(5.8) $$x_1 \cdots x_n(L(G)+(n-1)G)+AG = 0\,,$$

where $A=\sum_{i=1}^{n} x_1 \cdots x_{i-1}a_i x_{i+1} \cdots x_n/\lambda_i$. We note that $\lambda_i>0$ $(i=1, 2, \cdots, n)$, since $r<0$. Now, if $A\equiv 0$ (mod $x_1 \cdots x_n$), we get

(5.9) $$L(G)+(n+1)G = 0\,.$$

However, (5.9) has no more than the trivial solution, because $\lambda_i>0$ $(i=1, 2 \cdots, n)$. Hence, for example, $A\not\equiv 0$ (mod x_1). Then, putting $G=x_1^{\mu}G_0$ ($\mu\in \boldsymbol{N}$ and $G_0\not\equiv 0$ (mod x_1)), we have

(5.10) $$\lambda_1\mu+1 = 0\,,$$

and this contradicts that $\lambda_1>0$. Q.E.D.

In virtue of Lemma 5.1, we may put

(5.11) $$Q = c\Psi\,,$$

where c is a holomorphic function. Hence we obtain

(5.12) $$x_1 \cdots x_n L(H)+cH = 0\,.$$

Lemma 5.2. *If* (5.12) *has a non-trivial solution, then*

(5.13) $$c\equiv 0 \qquad (\text{mod}\ x_1 \cdots x_n)\,.$$

Proof. If, for example, $c\not\equiv 0$ (mod x_1), it follows that $a_1\not\equiv 0$ (mod x_1) and $H\equiv 0$ (mod x_1). Putting $H=x_1^{\rho}G$ ($\tau\in \boldsymbol{N}$ and $G\not\equiv 0$ (mod x_1)), we obtain

(5.14) $$\rho a_1\Phi + c_1 a_1 x_1^{-m_1}/\lambda_1 = x_1\omega\,,$$

where ω is meromorphic but holomorphic in x_1. Since $a_1\not\equiv 0$ (mod x_1), we have $\lambda_1\rho+1=0$ and

(5.15) $$\rho\left(\sum_{i=2}^{n} c_i x_i^{-m_i}\right)a_1 = x_1\omega\,.$$

However, this is a contradiction. Q.E.D.

Summing up the results obtained above, we can see that (1.12) is also necessary to the existence of the solution of the form (1.6), if Ψ is irreducible.

Next, we consider the case (ii). Let ψ_i $(i=1, 2, \cdots, s)$ be the irreducible factors of Ψ.

Lemma 5.3. *For any* $i \in \{1, \cdots, s\}$, ψ_i^p $(p \geqq 2)$ *is not the factor of* Ψ.

Proof. It is easy. Q.E.D.

Let us assume that, for example, ψ_1 is not a factor of $L_a(\Phi)$ (or Q), then Putting $H=\psi_1^\nu G$ ($\nu \in \boldsymbol{N}$ and $G \not\equiv 0 \pmod{\psi_1}$) and $\Psi=\psi\psi_1$ and using (5.7), we obtain that $\nu+r=0$, and consequently

$$x_1 \cdots x_n \{L\psi(G)+(n-1)\psi G+rL(\psi)\} G+\psi AG = 0 . \tag{5.16}$$

we note here that $\lambda_i>0$ $(i=1, 2, \cdots, n)$, since $r=-\nu$.

Lemma 5.4. *Let* $c_i \neq 0$, *then it must be* $a_i \equiv 0 \pmod{x_i}$ *in order that* (5.16) *has a non-trivial solution.*

Proof. Let us assume that $c_1 \neq 0$and $a_1 \not\equiv 0 \pmod{x_1}$, then putting $G=x_1^\mu G_0$ ($\mu \in \boldsymbol{N}$ and $G_0 \not\equiv 0 \pmod{x_1}$), we can see easily that $\lambda_1\mu+1=0$. But this contradicts that $\sigma+r=0$ and (C). Q.E.D.

For the simplicity, we assume that $c_i \neq 0$ for all i. By virtue of Lemma 5.4, we can set $a_i=\lambda_i x_i b_i$ for all i, and then we obtain from (5.16)

$$\psi\{L(G)+(n-1)G+BG\}+rL(\psi)G = 0 , \tag{5.17}$$

where $B=\sum_{i=1}^n b_i$

Lemma 5.5. *If* $L(\psi) \equiv 0 \pmod{\psi}$, *then* (5.17) *can not have a non-trivial solution.*

Proof. Let $rL(\psi)=(\gamma+h)\psi$, where γ is a constant and h a holomorphic function vanishing at the origin, then we obtain

$$L(G)+(\gamma_0+h+B)G = 0 , \tag{5.18}$$

where $\gamma_0=(n-1)+\gamma>0$. But (5.18) can not have a non-trivial solution, since $\lambda_i>0$ $(i=1, 2, \cdots, n)$. Q.E.D

In virtue of Lemma 5.5, we obtain, for example, that $L(\psi) \neq 0 \pmod{\psi_2}$. Then, putting $G=\psi_2^\mu G_0$ as usual and $\psi=\psi_2\psi_0$, we obtain $\mu+r=0$, and consequently

$$\psi_0\{L(G_0)+(n-1)G_0+BG_0\}+rL(\psi_0)G_0 = 0 . \tag{5.19}$$

Repeating the above arguments, we atatin finally

$$L(g)+(n-1)g+Bg = 0 \qquad (g \text{ is holomorphic}). \tag{5.20}$$

However, (5.20) has no non-trivial solution, since $\lambda_i>0$ for all i. Thus, we can have the following lemma.

Lemma 5.6. *It is necessary that*

$$L_a(\Phi) \equiv 0 \qquad (\mathrm{mod}\ \Psi) , \tag{5.21}$$

in order that the equation (5.1) *has a non-trivial solution.*

Summing up the results obtained above, we can show the following theorem.

Theorem 5.1. *In the case* (II), *under the assumption* (C), *the equation* (5.4) *admits a non-trivial solution if and only if* $L_a(\Phi)\equiv 0 \pmod{\Phi}$.

Remark. in the case where $r/\lambda_i \notin \boldsymbol{Z}$ $(i=1, \cdots, p)$ and $r/\lambda_k \in \boldsymbol{Z}$ $(k=p+1, \cdots, n)$, Theorem 5.1 is also valid.

Remark. Theorem 5.1 means that (1.12) is also necessary in order that the equation (1.1) has a solution of the form (1.6).

6. The case (III)-I.

In this section, let us consider the case (III). The methods used to the case (III) is almost same as those to the case (II), therefore we shall give only a sketch of our arguments. First, we study the problem under the assumption:

(D) $\qquad r/\lambda_i \notin \boldsymbol{Z} \qquad$ *for some* i.

For the simplicity, we assume that c_i, $d_i \neq 0$ for all i. We may assume that $i=1$ in (D) without loss of generality.

If $(r+q)/\lambda_1 \notin \boldsymbol{Z}$, we have immediately

$$\lambda_1 x_1 L(H) + a_1 H = 0 , \tag{6.1}$$

and this implies that

$$a_1 = \lambda_1 x_1 b \qquad (b \text{ is holomorphic}) . \tag{6.2}$$

This shows us that (1.12) is necessary for the existence of the solution such as in (1.6).

Let us consider the case where $(r+q)/\lambda_1$ $(=m) \in \boldsymbol{Z}$ and $q \neq r$, then using the following notations:

$$c(x') = \sum_{i=2}^{n} c_i x_i^{q/\lambda_i} , \quad c_a(x) = \sum_{i=2}^{n} c_i x_i^{q/\lambda_i} a_i/\lambda_i x_i$$

$$d(x') = \sum_{i=2}^{n} d_i x_i^{r/\lambda_i} , \quad d_a(x) = \sum_{i=2}^{n} d_i x_i^{r/\lambda_i} a_i/\lambda_i x_i ,$$

we obtain

$$c(x')L(H) + (src(x')a_i/\lambda_1 x_1 + pqc_a(x))H = 0 , \tag{6.3}$$

$$d(x')L(H) + (srd_a(x) + pqd(x')a_1/\lambda_1 x_1)H = 0 , \tag{6.4}$$

$$\begin{aligned}(c_1 d_1 x_1^m + c(x')d(x'))L(H) &+ c_1 d_1 x_1^m a_1 H/\lambda_1 x_1 \\ &+ (src(x')d_a(x) + pqd(x')c_a(x))H = 0 .\end{aligned} \tag{6.5}$$

Lemma 6.1. *If the equations* (6.3) (6.5) *have a non-trivial solution, then* $a_1 \equiv 0$ (mod x_1).

Proof. If $a_1 \not\equiv 0$ (mod x_1), then it follows that $H = x_1^\nu G$ ($\nu \in \boldsymbol{N}$ and $G \not\equiv 0$ (mod x_1)). Then, it follows

$$(6.6) \qquad \lambda_1 \nu + pq = 0\,, \quad \lambda_1 \nu + rs = 0\,,$$

and consequently

$$(6.7) \qquad (c_a(x)/c(x') = d_a(x)/d(x')(= \omega(x))\,.$$

We notice that $\omega(x)$ is holomorphic in x_1. Using (6.5) and (6.7), we obtain

$$(6.8) \qquad (c_1 d_1 x_1^m - c(x')d(x'))(a_1 - \lambda_1 x_1 \omega(x)) = 0\,.$$

This shows that $a_1 = \lambda_1 x_1 \omega(x)$, and consequently that $\omega(x)$ is holomorphic. This is a contradiction. Q.E.D.

Now, putting $a_1 = \lambda_1 x_1 b$ (b is holomorphic), we get also (6.7) from (6.3)$\sim$(6.5). This implies that (1.12) is necessary for the existence of the solution such as in (1.6).

Next, we consider the case where $(r+q)/\lambda_1 \in \boldsymbol{Z}$ and $q=r$. In this case, we obtain

$$(6.9) \qquad (c_1 d(x') + d_1 x(c'))L(H) + (d_1 src(x') + c_1 pqd(x'))a_1 H/\lambda_1 x_1$$
$$+ (d_1 pqc_a(x) + c_1 rsd_a(x))H = 0\,,$$

$$(6.10) \qquad (c_1 d_1 x_1^m + c(x')d(x'))L(H) + c_1 d_1 x_1^m a_1 H/\lambda_1 x_1$$
$$+ (pqd(x')c_a(x) + rsc(x')d_a(x))H = 0\,.$$

Of course, we may assume that $(q+r)/\lambda_i = 2q/\lambda_i \in \boldsymbol{Z}$ for other i's.

If $c_1 d(x') + d_1 c(x') = 0$, we have from (6.9)

$$(6.11) \qquad (p-s)\sum_{i=2}^{n} c_i x_i^{q/\lambda_i}(a_1/\lambda_1 x_1 - a_i/\lambda_i x_i) = 0\,.$$

When $p=s$, we have from (6.10)

$$(6.12) \qquad (c_1^2 x_i^m - c(x')^2)L(H) + (c_1^2 x_1^m a_1/\lambda_1 x_1 - c(x')c_a(x))H = 0\,.$$

We can show that $(c_1^2 x_1^m a_1/\lambda_1 x_1 - c(x')c_a(x))/(c_1^2 x_1^m - c(x')^2)$ must be holomorphic for the equation (6.12) to have a non-trivial solution by the techniques used in the sections 4 and 5.

When $p \neq s$, we see that $a_1 = \lambda_1 x_1/b$ (b is holomorphic) and have

$$(6.13) \qquad (c_1^2 x_1^m - c(x')^2)L(H) + (c_1^2 x_1^m b - c(x')c(x')c_a(x))H = 0\,,$$

and we can have the same result as the case when $p=s$.

Let us consider the case where $c_1d(x)+d_1c(x)\equiv 0$. We assume, for the simplicity, that $q/\lambda_1\in \boldsymbol{Z}$ for other i's.

Lemma 6.2. *If the equations* (6.9) *and* (6.10) *have a non-trivial solution, then* $a_1\equiv 0$ (mod x_1).

Proof. Let us assume that $a_1\not\equiv 0$ (mod x_1), then we obtain with some $\nu\in \boldsymbol{N}$,

$$(6.14)\qquad (\lambda_1\nu+qs)d_1c(x')+(\lambda_1\nu+pq)c_1d(x') = 0\,.$$

If $\lambda_1\nu+qs=0$, then $\lambda_1\nu+pq=0$. Then, we have $p=s$, and $2\lambda_1\nu+1=0$, since $pq+qs=1$. Thus, we obtain from (6.9) and (6.10)

$$(6.15)\qquad \begin{aligned}&(c_1d(x')+d_1c(x'))(2c_1d_1x_1^m a_1/\lambda_1x_1+c(x')d_a(x)+d(x')c_a(x))\\ &= (c_1d_1x_1^m+c(x')d(x'))(d_1c_a(x)+c_1d_a(x)+(d_1c(x')+c_1d(x'))a_1/\lambda_1x_1)\,.\end{aligned}$$

And (6.15) shows that $a_1\equiv 0$ (mod x_1).

If $\lambda_1\nu+qs\neq 0$, it follows from (6.14)

$$(6.16)\qquad c_1d(x') = -d_1(\lambda_1\nu+qs)c(x')/(\lambda_1\nu+pq)$$

$$(6.17)\qquad c_1d_a(x) = -d_1(\lambda_1+qs)c_a(x)/(\lambda_1\nu+pq)\,.$$

Substituting (6.16) and (6.17) into (6.9) and (6.10), we have

$$(6.18)\qquad c(x')L(H)-q(\nu a_1c(x')-x_1-(\lambda_1\nu+1)c_a(x))H = 0\,,$$

$$(6.19)\qquad \begin{aligned}&(c_1^2(\lambda_1\nu+pq)x_1^m-(\lambda_1\nu+qs)c(x)^2L(H)\\ &+c_1^2(\lambda_1\nu+pq)x_1^m a_1/\lambda_1x_1-(\lambda_1\nu+qs)c(x')c_a(x)H = 0\,.\end{aligned}$$

From (6.18) and (6.19), we can show that $a_1\equiv 0$ (mod x_1). Q.E.D.

By virtue of Lemma 6.2, we put $a_1=\lambda_1x_1b$, where b is a holomorphic function. Substituting it into (6.9), we have

$$(6.20)\qquad \begin{aligned}&(c_1d(x')+d_1c(x'))L(H)+q\{(d_1sc(x')\\ &+c_1pd(x'))b+d_1pc_a(x)+c_1sd_a(x)\}H = 0\,.\end{aligned}$$

We shall study the case where $q/\lambda_i\in \boldsymbol{N}$ $(i=2,\cdots,n)$ and $\Omega(x')=c_1d(x')+d_1c(x')$ is irreducible.

Now, let us assume that

$$(6.21)\qquad (d_1sc(x')+c_1pd(x'))b+d_1pc_a(x)+c_1sd_a(x)\not\equiv 0 \qquad (\text{mod } \Omega)\,,$$

putting $H=\Omega^\nu G$ ($\nu\in \boldsymbol{N}$ and $G\not\equiv 0$ (mod Ω)) and noticing

$$(6.22)\qquad L(\Omega) = q\Omega+q(c_1d_a(x)+d_1c_a(x))\,,$$

we obtain from (6.20)

(6.23) $$(d_1 sc(x')+c_1 pd(x'))b+d_1(-+p)c_a(x)+c_1(\nu+s)d_a(x) = \omega(x)\Omega(x'),$$

where $\omega(x)$ is a holomorphic function vanishing at the origin. Hence if (6.20) has a non-trivial solution, it must hold

(6.24) $$\sum_{i=1}^{n} \alpha_i\lambda_i+\nu q = 0,$$

for some $(\alpha_1, \cdots, \alpha_n)\in\bar{\boldsymbol{N}}^n$. However, this is impossible, since $q/\lambda_i>0$ for all i.

When $\Omega(x')$ is reducible or when $q/\lambda_i<0$ for all i, repeating the almost arguments as those of the sections 4 and 5, we can obtain the same result. Thus, we can attain the

Theorem 6.1. *In the case* (III), *under the assumption* (D), (1.12) *is necessary in order that the equation* (1.1) *has a solution such as in* (1.6).

7. The case (III)-II.

Here we consider the case (III) where the assumption (*D*) is removed. The study of this case will be divided into the following three cases:

(a) q/λ_i, $r/\lambda_i\in\boldsymbol{N}$ *for all* i

(b) $q/\lambda_i\in\boldsymbol{N}$ *and* $-r/\lambda_i\in\boldsymbol{N}$ *for all* i

(c) $-q/\lambda_i$, $-r/\lambda_i\in\boldsymbol{N}$ *for all* i.

Using the following notations:

$$A = \sum_{i=1}^{n} c_i x_i^{q/\lambda_i}, \quad B = \sum_{i=1}^{n} d_i x_i^{r/\lambda_i},$$

and putting $u=A^pB^sH$, we have from (1.1)

(7.1) $$ABL(H)+(sAL_a(B)+pBL_a(A))H = 0.$$

First, we consider the case (*a*). For the simplicity, we assume that A and B are irreducible. We note that λ_i $(i=1, 2 \cdots, n)$ q and r are as follows:

$$\lambda_i = \theta_i\gamma \qquad (i = 1, 2, \cdots,_1 n)$$
$$q = \tau\gamma$$
$$r = \sigma\gamma,$$

where γ is a complex number and θ_i $(i=1, 2, \cdots, n)$, τ and σ belong to $\boldsymbol{R}$.

Lemma 7.1. *It is necessary that*

(7.2) $$L_a(A)\equiv 0 \qquad (\text{mod } A)$$

(7.3) $$L_a(B)\equiv 0 \qquad (\text{mod } B),$$

in order that the equation (7.1) *has a non-trivial solution.*

Proof. We assume that $L_a(A)\not\equiv 0$ (mod A), then putting $H=A^{\nu}G$ as usual, we have $\nu+p=0$ and

$$(7.4)\qquad BL(G)+\nu qBG+sL_a(B)G=0\,.$$

If $L_a(B)\equiv 0$ (mod B), then we have for some $(\alpha_1, \cdots, \alpha_n)\in\bar{N}^n$

$$(7.5)\qquad \sum_{i=1}^{n}\alpha_i\lambda_i+\nu q=0\,.$$

However, this is a contradiction.

If $L_a(B)\not\equiv 0$ (mod B), we put $G=B^{\lambda}G_0$ as usual. Then we have $\mu+s=0$ and

$$(7.6)\qquad L(G_0)+(\nu q+\mu s)G_0=0\,.$$

But this leads us also to a contradiction. Q.E.D.

By virtue of Lemma 7.1, we can see easily that (1.12) is necessary for (1.1) to have a solution such as in (1.6).

Secondly we study the case (b). We put

$$(7.7)\qquad \Psi=x^m B\,,$$

where x^m denotes $\prod_{i=1}^{m} x_i^{m_i}$ with $m_i=-r/\lambda_i$ $(i=1, 2, \cdots, n)$. We notice that

$$(7.8)\qquad L(\Psi)=r(1-n)\Psi+x^m L_a(B)+\omega_m\Psi\,,$$

where $\omega_m=\sum_{i=1}^{n} m_i a_i/x_i$. Substituting (7.7) and (7.8) into (7.1), we obtain

$$(7.9)\qquad A\Psi-L(H)+\{(n-1)rsA-s\omega_m A+pL_a(A)\}\Psi H+sAL(\Psi)H=0\,.$$

For the simplicity, we assume that A and ω_m are mutually prime. By the usual way, we can shwo the

Lemma 7.2. *If* (7.8) *has a non-trivial solution, then it must holds*

$$(7.10)\qquad a_i\equiv 0 \quad (\text{mod } x_i) \quad \textit{for all } i$$

$$(7.11)\qquad L_a(A)\equiv 0 \quad (\text{mod } A)$$

$$(7.12)\qquad L(\Psi)\equiv 0 \quad (\text{mod } \Psi)\,.$$

By virtue of Lemma 7.2, ω_m becomes holomorphic and vanishes at the origin. Moreover, noticing the degree of Ψ as a polynomial, we obtain

$$(7.13)\qquad L(\Psi)=\{r(1-n)+h\}\Psi\,,$$

where h is a holomorphic function vanishing at the origin. From Lemma 7.2, we can show that $L_a(B)\equiv 0$ (mod B) is also necessary for (7.1) to have a non-trivial solution.

Finally we study the case (c). Let us introduce Φ by

$$\Phi = x^l A\,, \tag{7.14}$$

where x^l denotes $\prod_{i=1}^n x_i^{l_i}$ with $l_i = -q/\lambda_i$ $(i=1, 2, \cdots, n)$. We notice that

$$L(\Phi) = q(1-n)\Phi + x^l L_a(A) + \omega_l \Phi\,, \tag{7.15}$$

where $\omega_l = \sum_{i=1}^n l_i a_i / x_i$. We obtain from (7.1)

$$\Phi\Psi L(H) + \{(n-1)-\omega\}\Phi\Psi H + (s\Phi L(\Psi) + p\Psi L(\Phi))H = 0\,, \tag{7.16}$$

where ω denotes $\sum_{i=1}^n a_i/\lambda_i x_i$. To this case, by the same methods as those in the case (b) we can obtain the

Lemma 7.3. *If* (7.16) *has a non-trivial solution, then it must hold*

$$a_1 \equiv 0 \quad (\text{mod } x_i) \quad \textit{for all } i \tag{7.17}$$

$$L(\Phi) \equiv 0 \quad (\text{mod } \Phi)\,. \tag{7.18}$$

$$L(\Psi) \equiv 0 \quad (\text{mod } \Psi)\,. \tag{7.19}$$

In virtue of Lemma 7.3, becomes holomorphic and vanishes at the origin. Moreover, we have

$$L(\Phi) = \{q(1-n) + h_1\}\Phi \tag{7.20}$$

$$L(\Psi) = \{r(1-n) + h_2\}\Phi\,, \tag{7.21}$$

where h_1 and h_2 are holomorphic functions vanishing at the origin. From (7.18) and (7.19) we can easily obtain

$$L_a(A) \equiv 0 \quad (\text{mod } A) \tag{7.22}$$

$$L_a(B) \equiv 0 \quad (\text{mod } B)\,. \tag{7.23}$$

Summing up the results obtained above, we can have the following theorem.

Theorem 7.1. *To the case* (III), *the equation* (1.1) *has a solution such as in* (1.6), *if and only if* (1.12) *holds.*

Remark. the assertion of Theorem 7.1 is also valid to case where $u_0(x_1, \cdots, x_n)$ is given by

$$u_0(x_1, \cdots, x_n) = \prod_{j=1}^m \Big(\sum_{1=1}^n c_{ij} x_i^{q_j/\lambda_i}\Big)^{p_j}\,, \tag{7.24}$$

where $\sum_{j=1}^m p_j q_j = 1$.

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