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Geometric aspects of Malliavin's calculus on vector bundles

By

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1. Introduction.

Various analytical researches have been made concerning Malliavin's stochastic calculus of variations ([9], [10]): cf. Shigekawa [13], Bismut [1], Stroock [14], Ikeda and Watanabe [6]. In Euclidean case, Watanabe [17] has defined the composite of "smooth and non-degenerate" Wiener functionals and Schwartz's tempered distributions to obtain probabilistic expressions of heat kernels (cf. [7], [18]). In this direction, Taniguchi [16] has studied the case of manifold-valued Wiener functionals. The integration by parts over a Wiener space plays a fundamental role in their calculus.

In this paper, we give differential geometric aspects to the integration by parts over a Wiener space and to the above Watanabe's idea by working on vector bundles. Since tensor distributions, such as currents of de Rham, can be grasped as section distributions (Schwartz [12], cf. [2]) and as generalized sections (Guillemin and Sternberg [5]) of certain vector bundles, the use of vector bundles will be suitable for geometric comprehension of Watanabe's idea.

Now let $\pi_E: E\to M$ be a C^∞ real vector bundle over a σ-compact manifold M; we admit the case where M is not orientable (or not oriented). First we derive integration by parts formulas for the composite of M-valued "smooth and non-degenerate" Wiener functionals and covariant derivatives of C^∞ cross sections of E in the case where a linear connection is given in E (Theorem 4.1). To do this, we develop an intrinsic (coordinate free) formulation, which will be adapted to the modern differential geometric style ([8]). Next, we establish the composite of M-valued "smooth and non-degenerate" Wiener functionals and section distributions [resp. generalized sections] of E (Theorem 6.1).

When E is a Riemannian vector bundle and M is compact and Riemannian, this result is applied to probabilistic expressions of heat kernels for E-valued differential forms. We roughly illustrate our idea of getting the heat kernel of the heat equation

$$\frac{\partial\theta}{\partial t}=\frac{1}{2}\Delta_E\theta \qquad \left(:=\frac{1}{2}\ \mathrm{Trace}\ \nabla\nabla\theta\right) \tag{1.1}$$

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for E-valued differential p-forms. Consider the vector bundle $\pi_\xi: \xi=(\Lambda^p TM)\otimes E^* \to M$, so that $\xi^*=(\Lambda^p T^*M)\otimes E$. Let v be the Riemannian volume density, τ_t be the stochastic parallel displacement of fibers of ξ along the Brownian motion X_t on M starting from the point $x\in M$ at time $t=0$, and $|\Lambda|^\alpha(M)$ be the (real) bundle of densities of order $\alpha\in\boldsymbol{R}$. Then $\tau_t\zeta\otimes|v|^{-1}(X_t)$ is a $\xi\otimes|\Lambda|^{-1}(M)$-valued Wiener functional covering X_t for each $\zeta\in\xi_x=\pi_\xi^{-1}(x)$. Let δ_y^{ξ}, $(y\in M)$, be the ξ_y-valued generalized section of $\xi^*\otimes|\Lambda|(M)$ defined by $\delta_y^\xi[\Xi]=\Xi(y)$ for every C^∞ cross section Ξ of ξ. For each $t>0$ and $x, y\in M$, we would obtain a linear mapping $\xi_x\to\xi_y$ which assigns to each $\zeta\in\xi_x$ the ξ_y-valued "generalized expectation" $\langle\delta_y^\xi(X_t)\{\tau_t\zeta\otimes|v|^{-1}(X_t)\}\rangle$ of the "pairing" $(\delta_y^\xi(X_t), \tau_t\zeta\otimes|v|^{-1}(X_t))$ between $\xi^*\otimes|\Lambda|(M)$ and $\xi\otimes|\Lambda|^{-1}(M)$. Let $f_p(t, x, y)\in\mathrm{Hom}(\xi_y^*, \xi_x^*)$ be its dual mapping. Then $f_p(t, x, y)$, $(t>0; x, y\in M)$, would be the heat kernel of (1.1) with respect to v. We give a precise meaning to $f_p(t, x, y)$ in the final section. A probabilistic expression of the heat kernel of the heat equation involving $-\square_E=-(d_E d_E^*+d_E^* d_E)$ (the Laplacian) [resp. $-(d_E+d_E^*)^2$] on E-valued differential forms (cf. [3], [11]) is also obtained in a similar manner.

This paper is organized as follows. In §2, we prepare necessary notations about Sobolev spaces of Wiener functionals and about operators on Wiener functionals following [7] and [18]. In §3, E-valued Wiener functionals are considered. Also the operators in §2 are extended (in a sense) in treating E-valued Wiener functionals. Section 4 establishes integration by parts formulas in our intrinsic way. In §5, we fix some terminology and notations about section distributions, density bundles, and generalized sections according to [2], [4], [5], [12] for later use. [When there is no need to fix any volume measure on M, the density bundle $|\Lambda|(M)$ of order one is useful; for, unlike d-forms ($\dim M=d$), C^∞ densities ($=C^\infty$ cross sections of $|\Lambda|(M)$) with compact support can be always integrated over M even if M is not orientable (or not oriented) and thus C^∞ densities become distributions on M.] We state Theorem 6.1 in §6 and give a proof in §7. Extending the concept of the composition in Theorem 6.1 to the case of section distributions with values in a finite dimensional real (topological) vector space, we obtain useful cross sections of some vector bundles in §8. Finally, we apply our results to heat kernels in §9.

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2. Preliminaries.

In this section, we give some notational preliminaries following [7] and [18] (cf. [16]).

We fix an abstract Wiener space (W, H, μ). For a separable real Hilbert space S, let $\boldsymbol{D}_p^r(S)$, $(1\leq p<\infty, r\in\boldsymbol{R})$, denote the system of Sobolev spaces of Wiener functionals obtained by completing $\mathcal{P}(S)=\{S$-valued polynomial Wiener functionals$\}$ with respect to the norm

$$\|F\|_{p,r}=\|(I-L)^{r/2}F\|_{L^p}, \qquad F\in\mathcal{P}(S).$$

Here $\|\cdot\|_{L^p}$ is the L^p-norm of the L^p-space $L^p(S)$ of S-valued Wiener functionals and L is the Ornstein-Uhlenbeck operator. Set

$$\boldsymbol{D}^{\infty}(S)=\bigcap_{\substack{1<p<\infty\\ r\in\boldsymbol{R}}}\boldsymbol{D}_p^r(S), \qquad \boldsymbol{D}^{-\infty}(S)=\bigcup_{\substack{1<p<\infty\\ r\in\boldsymbol{R}}}\boldsymbol{D}_p^r(S).$$

The weak (Fréchet) derivative, its dual operator, and the (extended) Ornstein-Uhlenbeck operator are denoted by D, $D^*=-\delta$ and L, respectively, so that

$$D:\boldsymbol{D}_p^r(S)\longrightarrow\boldsymbol{D}_p^{r-1}(\Lambda(H,S)),$$

$$D^*:\boldsymbol{D}_p^r(\Lambda(H,S))\longrightarrow\boldsymbol{D}_p^{r-1}(S),$$

$$L:\boldsymbol{D}_p^r(S)\longrightarrow\boldsymbol{D}_p^{r-2}(S)$$

are continuous linear mappings ($1<p<\infty$, $r\in\boldsymbol{R}$), where $\Lambda(H,S)$ denotes the Hilbert space of all continuous linear mappings $f:H\to S$ of Hilbert-Schmidt class. We write $\boldsymbol{D}_p^r$, $\boldsymbol{D}^{\infty}$, $\boldsymbol{D}^{-\infty}$ for $\boldsymbol{D}_p^r(\boldsymbol{R})$, $\boldsymbol{D}^{\infty}(\boldsymbol{R})$, $\boldsymbol{D}^{-\infty}(\boldsymbol{R})$, respectively. For details, see [7], [18] (cf. [6], [15]).

From now on, M will denote a σ-compact d-dimensional C^{∞} manifold. Let $C^{\infty}(M)$ be the set of all C^{∞} functions on M and $C_0^{\infty}(M)$ be the elements of $C^{\infty}(M)$ with compact support.

Definition 2.1 (cf. [16]). A μ-measurable mapping $F:W\to M$ is said to be of class $\boldsymbol{D}^{\infty}(M)$ if $f\circ F\in\boldsymbol{D}^{\infty}$ for all $f\in C_0^{\infty}(M)$.

For $F\in\boldsymbol{D}^{\infty}(M)$, we define DF and $\langle DF, DF\rangle$ to be

$$(DF(w)[h])(f)=D(f\circ F)(w)[h],$$

$$\langle DF, DF\rangle(w)(df, dg)=\langle D(f\circ F)(w), D(g\circ F)(w)\rangle_H$$

for every $f, g\in C_0^{\infty}(M)$, $w\in W$, $h\in H$, where $\langle\,,\rangle_H$ denotes the inner product in H. Thus $DF(w)$ is an $\boldsymbol{R}$-linear mapping from H into the tangent space $TM_{F(w)}$ of M at $F(w)$, and $\langle DF, DF\rangle$ is a mapping from W into the tensor bundle $T_0^2(M)$ of order (2, 0) with $\tilde{\pi}\circ\langle DF, DF\rangle=F$, where $\tilde{\pi}:T_0^2(M)\to M$ stands for the projection.

3. Wiener functionals with values in a vector bundle.

Let $\pi_E:E\to M$ be a C^{∞} real vector bundle of rank (=fiber dimension) m and E^* be the dual bundle of E. Let $\Gamma(E)$ be the real vector space of all C^{∞} cross sections of E and $\Gamma_0(E)$ be the elements of $\Gamma(E)$ with compact support. Denote by E_x the fiber $\pi_E^{-1}(x)$ of E over $x\in M$. In this section, we develop an intrinsic formulation to treat E-valued Wiener functionals.

Definition 3.1. A μ-measurable mapping $G:W\to E$ is said to belong to $\mathrm{Dom}(D, E)$ if it satisfies the following conditions:

(1) $\pi_E \circ G \in \boldsymbol{D}^\infty(M)$,

(2) ${}_E(G, \xi \circ \pi_E \circ G)_{E^*} \in \boldsymbol{D}^\infty$ for every $\xi \in \Gamma_0(E^*)$, where ${}_E(\ ,\ )_{E^*}$ denotes the canonical pairing between E and E^*.

Example 3.1. If $F \in \boldsymbol{D}^\infty(M)$, $\xi \in \Gamma(E)$ and $G \in \boldsymbol{D}^\infty$, then $G \cdot (\xi \circ F) \in \mathrm{Dom}(D, E)$.

Let $\mathrm{Map}(H, E)$ be the set of all mappings from H into E.

Definition 3.2. A mapping $F: W \to \mathrm{Map}(H, E)$ is said to belong to $\mathrm{Dom}(D^*, E)$ if it satisfies the following conditions:

(1) There exists an element $\tilde{F} \in \boldsymbol{D}^\infty(M)$ such that $\pi_E(F(w)[h]) = \tilde{F}(w)$ for every $w \in W$ and $h \in H$. [This condition will be written briefly as $\pi_E \circ F = \tilde{F}$.]

(2) The mapping $H \to E_{\tilde{F}(w)}$ $(= \pi_E^{-1}(\tilde{F}(w)))$ defined by $h \to F(w)[h]$ is $\boldsymbol{R}$-linear.

(3) For each $\xi \in \Gamma_0(E^*)$, the mapping ${}_E(F, \xi \circ \pi_E \circ F)_{E^*}$: $W \to \mathrm{Lin}(H, \boldsymbol{R}) = \{$linear mappings from H into $\boldsymbol{R}\}$ given by

$$w \longrightarrow [h \longrightarrow {}_E(F(w)[h], \xi \circ \pi_E \circ F(w))_{E^*}]$$

belongs to $\boldsymbol{D}^\infty(H)$.

Example 3.2. In the case $E = M \times \boldsymbol{R}$ (the trivial line bundle), we see that

$$\boldsymbol{D}^\infty(M) \times \boldsymbol{D}^\infty \subset \mathrm{Dom}(D, M \times \boldsymbol{R}), \qquad \boldsymbol{D}^\infty(M) \times \boldsymbol{D}^\infty(H) \subset \mathrm{Dom}(D^*, M \times \boldsymbol{R}).$$

Remark 3.1. Let TM, T^*M and $TM \circledcirc TM$ be respectively the tangent bundle over M, the cotangent bundle and the symmetrization of $TM \otimes TM$. Suppose $F \in \boldsymbol{D}^\infty(M)$. Then $DF \in \mathrm{Dom}(D^*, TM)$ and $\langle DF, DF \rangle \in \mathrm{Dom}(D, TM \circledcirc TM)$, by setting

$$_{TM}(DF, (df) \circ F)_{T^*M} = DF(f) (= D(f \circ F)),$$

$$_{TM \circledcirc TM}(\langle DF, DF \rangle, (df \circledcirc dg) \circ F)_{T^*M \circledcirc T^*M} = \langle DF, DF \rangle (df, dg)$$

for every $f, g \in C_0^\infty(M)$.

Example 3.3. If $F \in \boldsymbol{D}^\infty(M)$ and $\xi \in \Gamma(T^*M \otimes E)$, then ${}_{TM}(DF, \xi \circ F)_{T^*M} \in \mathrm{Dom}(D^*, E)$.

Definition 3.3. Let $\pi_{E_i}: E_i \to M$ $(i=1, 2)$ be C^∞ real vector bundles of finite rank. For $F_i \in \mathrm{Dom}(D^*, E_i)$ $(i=1, 2)$ such that $\pi_{E_1} \circ F_1 = \pi_{E_1} \circ F_2 = \tilde{F}$ $(\in \boldsymbol{D}^\infty(M))$, define $\langle F_1 \otimes F_2 \rangle \in \mathrm{Dom}(D, E_1 \otimes E_2)$ by

$$\begin{aligned} &{}_{E_1 \otimes E_2}(\langle F_1 \otimes F_2 \rangle, (\xi_1 \otimes \xi_2) \circ \tilde{F})_{E_1^* \otimes E_2^*} \\ &\qquad = \langle {}_{E_1}(F_1, \xi_1 \circ \tilde{F})_{E_1^*}, {}_{E_2}(F_2, \xi_2 \circ \tilde{F})_{E_2^*} \rangle_H \end{aligned}$$

for every $\xi_i \in \Gamma_0(E_i^*)$, $i=1, 2$. (Use $E_1^* \otimes E_2^* \cong (E_1 \otimes E_2)^*$.)

Example 3.4. If $F \in \boldsymbol{D}^\infty(M)$, then $DF \in \mathrm{Dom}(D^*, TM)$ and $\langle DF \otimes DF \rangle = \langle DF, DF \rangle$.

Now suppose a linear connection ∇ is given in E. Let ∇^* be the dual connection in E^* induced naturally from ∇.

Definition 3.4 (*definition of D_E and D_E^**).

(1) For $F\in \mathrm{Dom}(D, E)$, define $D_E F: W\to \mathrm{Map}(H, E)$, $D_E F(w)[h]\in \pi_E^{-1}(\pi_E\circ F(w))$ $(w\in W,\ h\in H)$ by

$$
\begin{aligned}
&{}_E((D_E F(w)[h],\ \xi\circ\pi_E\circ F(w))_{E^*}\\
&\quad = D({}_E(F,\ \xi\circ\pi_E\circ F)_{E^*})(w)[h]\\
&\qquad -{}_{E\otimes T^*M}(F(w)\otimes(D(\pi_E\circ F)(w)[h]),\ (\nabla^*\xi)\circ\pi_E\circ F(w))_{E^*\otimes T^*M}
\end{aligned}
$$

for every $\xi\in\Gamma_0(E^*)$.

(2) For $F\in\mathrm{Dom}(D^*, E)$, define $D_E^* F: W\to E$, $D_E^* F(w)\in\pi_E^{-1}(\pi_E\circ F(w))$ by

$$
\begin{aligned}
&{}_E(D_E^* F,\ \xi\circ\pi_E\circ F)_{E^*}\\
&\quad = D^*({}_E(F,\ \xi\circ\pi_E\circ F)_{E^*}) + {}_{E\otimes T^*M}(\langle F\otimes D(\pi_E\circ F)\rangle,\ (\nabla^*\xi)\circ\pi_E\circ F)_{E^*\otimes T^*M}
\end{aligned}
$$

for every $\xi\in\Gamma_0(E^*)$. [Note that the left hand sides of the formulas in (1) and (2) become $C_0^\infty(M)$-linear with respect to ξ.]

Remark 3.2. If $F\in\mathrm{Dom}(D, E)$, then the mapping $H\to E_{\pi_E\circ F(w)}$ defined by $h\to D_E F(w)[h]$ is $\boldsymbol{R}$-linear.

From the definition, we have $D_E(\mathrm{Dom}(D, E))\subset\mathrm{Dom}(D^*, E)$ and $D_E^*(\mathrm{Dom}(D^*, E))\subset\mathrm{Dom}(D, E)$.

Definition 3.5. Put $L_E=-D_E^*\circ D_E$: $\mathrm{Dom}(L_E)=\mathrm{Dom}(D, E)\to\mathrm{Map}(W, E)$.

In the following proposition, let E_1 and E_2 be C^∞ real vector bundles of finite rank over M. Suppose that a linear connection is given in each of E_1 and E_2, and consider the tensor product connection in $E_1\otimes E_2$. In the case of a trivial bundle, we consider the trivial connection. We shall use the letter Z indifferently to denote E, E^*, E_1, E_2, $E_1\otimes E_2$, and so on. Recall Example 3.2.

Proposition 3.1 (*properties of D_Z, D_Z^* and L_Z*). *The following hold.*

(1) *If* $F\in\mathrm{Dom}(D, E_1)$, $G\in\mathrm{Dom}(D, E_2)$ *and* $\pi_{E_1}\circ F=\pi_{E_2}\circ G$, *then* $F\otimes G\in \mathrm{Dom}(D, E_1\otimes E_2)$ *and*

$$D_{E_1\otimes E_2}(F\otimes G)=(D_{E_1}F)\otimes G+F\otimes(D_{E_2}G).$$

(2) *If* $F\in\mathrm{Dom}(D, E_1)$, $G\in\mathrm{Dom}(D^*, E_2)$ *and* $\pi_{E_1}\circ F=\pi_{E_2}\circ G$, *then* $F\otimes G\in\mathrm{Dom}(D^*, E_1\otimes E_2)$ *and*

$$D_{E_1\otimes E_2}^*(F\otimes G)=-\langle(D_{E_1}F)\otimes G\rangle+F\otimes(D_{E_2}^*G)$$

where $(F\otimes G)(w)[h]=F(w)\otimes(G(w)[h])$ *for every* $w\in W$ *and* $h\in H$.

(3) *If* $F\in\mathrm{Dom}(L_{E_1})$, $G\in\mathrm{Dom}(L_{E_2})$ *and* $\pi_{E_1}\circ F=\pi_{E_2}\circ G$, *then* $F\otimes G\in\mathrm{Dom}(L_{E_1\otimes E_2})$ *and*

$$L_{E_1\otimes E_2}(F\otimes G)=(L_{E_1}F)\otimes G+F\otimes(L_{E_2}G)+2\langle(D_{E_1}F)\otimes(D_{E_2}G)\rangle.$$

(4) *The operation of* D_Z [*resp.* D_Z^*, L_Z] *commutes with every pairing.* [*Here, for example, for* $F\in\mathrm{Dom}(D, E)$ *and* $G\in\mathrm{Dom}(D, E^*)$ *with* $\pi_E\circ F=\pi_{E^*}\circ G=\tilde{F}$, *regard* ${}_E(F, G)_{E^*}\in\boldsymbol{D}^\infty$ *as* $(\tilde{F}, {}_E(F, G)_{E^*})\in\mathrm{Dom}(D, M\times\boldsymbol{R})$.]

(5) *If* $F\in\mathrm{Dom}(D, E)$, $G\in\mathrm{Dom}(D^*, E^*)$, $\pi_E\circ F=\pi_{E^*}\circ G$ *and* ${}_E(F, G)_{E^*}\in\boldsymbol{D}^\infty(H)$, *then*

$$\int_W {}_E(\langle D_E F\otimes G\rangle)_{E^*}d\mu=\int_W {}_E(F, D_{E^*}^*G)_{E^*}d\mu.$$

(6) *If* $F\in\mathrm{Dom}(L_E)$, $G\in\mathrm{Dom}(L_{E^*})$, $\pi_E\circ F=\pi_{E^*}\circ G$ *and* ${}_E(D_E F, G)_{E^*}$, ${}_E(F, D_{E^*}G)_{E^*}\in\boldsymbol{D}^\infty(H)$, *then*

$$\int_W {}_E(L_E F, G)_{E^*}d\mu=\int_W {}_E(F, L_{E^*}G)_{E^*}d\mu.$$

Proof. By choosing an adequate C^∞ partition $\{\chi_i\}$ of unity, every $\xi\in\Gamma_0(E_1^*\otimes E_2^*)$ can be expressed in the form $\xi=\sum_i\chi_i\xi=\sum_i\eta_{(1)i}\otimes\eta_{(2)i}$ (finite sum) with $\eta_{(1)i}\in\Gamma_0(E_1^*)$ and $\eta_{(2)i}\in\Gamma_0(E_2^*)$. Then (1)-(3) are easily shown. To verify (4), let $\tau\in\Gamma(E^*\otimes E)$ be such that $\tau_x\in(E^*\otimes E)_x\cong\mathrm{Hom}(E_x, E_x)$ equals the identity mapping of E_x for every $x\in M$. Using

$${}_E(F, G)_{E^*}={}_{E\otimes E^*}(F\otimes G, \tau\circ\tilde{F})_{E^*\otimes E}$$

and $\nabla^{E^*\otimes E}(u\tau)=du\otimes\tau$ $(u\in C_0^\infty(M))$, we can get (4). From (2), (4) and the relation

$$\int_W D^*J\,d\mu=\int_W\langle D1, J\rangle_H d\mu=0, \qquad (J\in\boldsymbol{D}^\infty(H)),$$

we obtain (5). Finally, (6) follows from (5).

4. Integration by parts.

In this section we give vector bundle version of integration by parts formulas. We use the same notations as in § 3. Let $F: W\to M$ satisfy the following assumptions of "smoothness" (A1) and "non-degeneracy" ((A2)-(A3)):

(A1) $F\in\boldsymbol{D}^\infty(M)$.

(A2) $\sigma(w)=\langle DF, DF\rangle(w)$ is positive definite μ-a.e. Then for almost all w, an inner product $\gamma(w)$ is induced in $TM_{F(w)}$ from $\sigma(w)$. That is, defining Φ: $T^*M_{F(w)}\to TM_{F(w)}$ by $\Phi(\theta)=\sigma(w)(\theta, \cdot)$ $(\theta\in T^*M_{F(w)})$ and setting $\Psi=\Phi^{-1}: TM_{F(w)}\to T^*M_{F(w)}$, we obtain $\gamma: W\to T^*M\circledcirc T^*M$ such that for almost all w,

$$\gamma(w)(X_1, X_2)=\sigma(w)(\Psi(X_1), \Psi(X_2)), \qquad (X_1, X_2\in TM_{F(w)}).$$

(A3) $\gamma\in\mathrm{Dom}(D, T^*M\circledcirc T^*M)$.

Example 4.1. Consider the case $M=\boldsymbol{R}^d$. Let $F=(F^1, \cdots, F^d): W\to\boldsymbol{R}^d$ satisfy $F^i\in\boldsymbol{D}^\infty$, $(i=1, \cdots, d)$. Then F satisfies (A1). If F satisfies (A2) and if $\gamma=(\gamma_{ij})=\langle DF, DF\rangle^{-1}$ satisfies $\gamma_{ij}\in L^p$ for all $p>1$ and $i, j=1, 2, \cdots, d$, then $\gamma_{ij}\in\boldsymbol{D}^\infty$ and we see that F satisfies (A3) (cf. [7], [18]).

In the next theorem, we use the following notation: For each $i=1, \cdots, k$, let $A_i \in \mathrm{Dom}(D, E_{(i)})$ or $A_i \in \mathrm{Dom}(D^*, E_{(i)})$ for some vector bundle $E_{(i)}$ of the form

$$E_{(i)} = E_{i,1} \otimes E_{i,2} \otimes \cdots \otimes E_{i,j(i)}, \qquad (j(i) \geqq 2 \text{ for } i \neq 1, k)$$

where $E_{i,1}, \cdots, E_{i,j(i)}$, $(i=1, \cdots, k)$, are vector bundles such that $E^*_{i,j(i)} = E_{i+1,1}$ $(i=1, \cdots, k-1)$. Denote by $[\![A_1 \otimes A_2 \otimes \cdots \otimes A_k]\!]$ the evaluations of the nearest-neighbor-(vector bundle)-components (that is, $E_{i,j(i)}$- and $E_{i+1,1}$-components) of all pairs (A_i, A_{i+1}), $i=1, \cdots, k-1$. Namely

$$[\![A_1 \otimes A_2 \otimes \cdots \otimes A_k]\!] = {}_{E_{k-1,j(k-1)}}(\cdots{}_{E_{2,j(2)}}({}_{E_{1,j(1)}}(A_1, A_2)_{E_{2,1}}, A_3)_{E_{3,1}} \cdots)_{E_{k,1}}.$$

Theorem 4.1 (*integration by parts*). *Let* $F: W \to M$ *satisfy* (A1)-(A3). *Let* $\Xi \in \Gamma(E)$, $\eta \in \mathrm{Dom}(D, TM \otimes E^*)$, $\pi_{TM \otimes E^*} \circ \eta = F$.

(1) *If* ${}_E(\Xi, [\![DF \otimes \gamma \otimes \eta]\!])_{E^*} \in \boldsymbol{D}^\infty(H)$, *then*

$$\int_W {}_{T^*M \otimes E}((\nabla \Xi) \circ F, \eta)_{TM \otimes E^*} d\mu = \int_W {}_E(\Xi \circ F, D^*_{E^*}[\![DF \otimes \gamma \otimes \eta]\!])_{E^*} d\mu. \tag{4.1}$$

(2) *If moreover a linear connection is given in M, then the right hand side of* (4.1) *can be rewritten as*

$$\int_W {}_E(\Xi \circ F, [\![-L_M F \otimes \gamma \otimes \eta]\!] - [\![\langle DF \otimes D_{T^*M \otimes E^*}[\![\gamma \otimes \eta]\!]\rangle]\!])_{E^*} d\mu \tag{4.2}$$

where $L_M F = -D^*_{TM} DF: W \to TM$.

Proof. We first note that ${}_{TM}(DF, \gamma)_{T^*M} \in \mathrm{Dom}(D^*, T^*M)$ and thus $[\![DF \otimes \gamma \otimes \eta]\!]$ $= {}_{T^*M}({}_{TM}(DF, \gamma)_{T^*M}, \eta)_{TM} \in \mathrm{Dom}(D^*, E^*)$. (Local components of $[\![DF \otimes \gamma \otimes \eta]\!]$ are expressed as $\sum_{i,j=1}^d \eta^i_\lambda \gamma_{ij} DF^j$, $(\lambda=1, \cdots, m)$.) Now for every $\xi \in \Gamma_0(E^*)$ and $h \in H$ we have

$$\begin{aligned}
&{}_E((D_E(\Xi \circ F)[h], \xi \circ F)_{E^*} \\
&\quad = D({}_E(\Xi \circ F, \xi \circ F)_{E^*})[h] - {}_{E \otimes TM}((\Xi \circ F) \otimes (DF[h]), (\nabla^* \xi) \circ F)_{E^* \otimes T^*M} \\
&\quad = {}_{TM}(DF[h], d({}_E(\Xi, \xi)_{E^*}) \circ F)_{T^*M} \\
&\qquad - {}_{TM}(DF[h], \{d({}_E(\Xi, \xi)_{E^*}) - {}_E(\nabla \Xi, \xi)_{E^*}\} \circ F)_{T^*M} \\
&\quad = {}_E({}_{T^*M}(\{(\nabla \Xi) \circ F\} \otimes (DF[h]))_{TM}, \xi \circ F)_{E^*},
\end{aligned}$$

that is,

$$D_E(\Xi \circ F) = [\![\{(\nabla \Xi) \circ F\} \otimes DF]\!].$$

If ${}_E(\Xi \circ F, [\![DF \otimes \gamma \otimes \eta]\!])_{E^*} \in \boldsymbol{D}^\infty(H)$, then (4.1) follows from (5) of Proposition 3.1 and the relation

$$\begin{aligned}
&{}_E(\langle [\![\{\nabla \Xi \circ F\} \otimes DF]\!] \otimes [\![DF \otimes \gamma \otimes \eta]\!] \rangle)_{E^*} \\
&\quad = {}_{T^*M \otimes E}((\nabla \Xi) \circ F, [\![\langle DF \otimes DF \rangle \otimes \gamma \otimes \eta]\!])_{TM \otimes E^*} \\
&\quad = {}_{T^*M \otimes E}((\nabla \Xi) \circ F, \eta)_{TM \otimes E^*}.
\end{aligned}$$

As for (2), since

$$D^*_{E^*}[\![DF\otimes\gamma\otimes\eta]\!]=[\![D^*_{TM}DF\otimes[\![\gamma\otimes\eta]\!]]\!]-[\![\langle DF\otimes D_{T^*M\otimes E^*}[\![\gamma\otimes\eta]\!]\rangle]\!],$$

(4.2) is also verified.

Remark 4.1. If a linear connection is given in each of E and M, then using Theorem 4.1 repeatedly, we can rewrite an integral of the form

$$\int_W {}_{\underbrace{T^*M\otimes\cdots\otimes T^*M}_{k}\otimes E}((\nabla^k\Xi)\circ F,\ \cdots)_{\underbrace{TM\otimes\cdots\otimes TM}_{k}\otimes E^*}d\mu$$

into the form

$$\int_W {}_E(\Xi\circ F,\ \cdots)_{E^*}d\mu$$

where $\nabla^k=\underbrace{\nabla\cdots\nabla}_{k}$.

Corollary 4.2. *Let* $\Xi\in\Gamma(E)$, $u\in C^\infty(M)$, $\Theta\in\mathrm{Dom}\,(D,\ TM\otimes E^*)$ *and* $\pi_{TM\otimes E^*}\circ\Theta=F$. *Suppose* $\mathrm{supp}\,\Xi\cap\mathrm{supp}\,u$ *is compact. Then*

$$\int_W {}_{T^*M\otimes E}((\nabla\Xi)\circ F,\ (u\circ F)\Theta)_{TM\otimes E^*}d\mu$$

$$=\int_W {}_E(\Xi\circ F,\ D^*_{E^*}[\![(u\circ F)DF\otimes\gamma\otimes\Theta]\!])_{E^*}d\mu. \tag{4.3}$$

Proof. Since ${}_E(\Xi\circ F,\ [\![DF\otimes\gamma\otimes(u\circ F)\Theta]\!])_{E^*}={}_E((u\Xi)\circ F,\ [\![DF\otimes\gamma\otimes\Theta]\!])_{E^*}\in\boldsymbol{D}^\infty(H)$, (4.3) follows from (4.1) with $\eta=(u\circ F)\Theta$.

Remark 4.2. Let X be a C^∞ vector field on M, so that $X\in\Gamma(TM)$. Let $\Xi\in\mathrm{Dom}\,(D,\ E)$, $\theta\in\mathrm{Dom}\,(D,\ E^*)$ and $\pi_{E^*}\circ\theta=F$. If $\mathrm{supp}\,\Xi\cap\mathrm{supp}\,X$ is compact, then setting $\eta=(X\circ F)\otimes\theta$ in (4.1) we have

$$\int_W {}_E((\nabla_X\Xi)\circ F,\ \theta)_{E^*}d\mu=\int_W {}_E(\Xi\circ F,\ D^*_{E^*}([\![DF\otimes\gamma\otimes(X\circ F)]\!]\theta))_{E^*}d\mu.$$

5. Section distributions and generalized sections.

We shall prepare some terminology and notations about section distributions and generalized sections for later use. (For details, see [2], [5], [12]. cf. [4].)

Let $\pi_E: E\to M$ be as in § 3. We admit the case where M is not oriented (or not orientable). Here, we need not consider a linear connection in E, unless otherwise stated. Give $\Gamma(E)$ the topology of uniform convergence together with each finite number of derivatives on each compact subset of M. For each compact set K of M, put

$$\Gamma_K(E)=\{\rho\in\Gamma(E)\,;\ \mathrm{supp}\,\rho\subset K\}.$$

Since M is σ-compact, there exists a sequence of compact sets $K_1\subset K_2\subset\cdots\subset K_n\subset K_{n+1}\subset\cdots$ such that $\bigcup_n K_n=M$. Then $\Gamma_{K_n}(E)(\subset\Gamma_{K_{n+1}}(E))$ are topological subspaces of $\Gamma(E)$. Take the inductive limit $\mathcal{D}(M,\ E)=\varinjlim\Gamma_{K_n}(E)(=\Gamma_0(E)$ as sets). Each element of the (topological) dual space $\mathcal{D}'(M.\ E)$ of $\mathcal{D}(M,\ E)$ is called a *section distribution of* E in this paper.

Let $|\Lambda|^{\alpha}(M)$ be the C^{∞} real bundle of densities of order α ($\in \boldsymbol{R}$) over M; it is the line bundle associated with the bundle $L(M)$ of linear frames over M, with standard fiber $\boldsymbol{R}$, the structure group $GL(d, \boldsymbol{R})$ whose element A acts on $\boldsymbol{R}$ in such a way that $A: v \to |\det A|^{-\alpha} v$, ($v \in \boldsymbol{R}$). If (x^i) is a local coordinate system valid on a coordinate neighborhood U of M, we denote by $|dx^1 \wedge \cdots \wedge dx^d|^{\alpha}(p)$, ($p \in U$), the image of $([(\partial/\partial x^1)_p, \cdots, (\partial/\partial x^d)_p], 1)$ under the natural projection $L(M) \times \boldsymbol{R} \to |\Lambda|^{\alpha}(M)$. Then $|dx^1 \wedge \cdots \wedge dx^d|^{\alpha}$ is a local frame for $|\Lambda|^{\alpha}(M)$ over U, and under a change of local coordinates $(x^i) \to (y^i)$, we have

$$|dy^1 \wedge \cdots \wedge dy^d|^{\alpha} = \left|\det\left(\frac{\partial y^i}{\partial x^j}\right)\right|^{\alpha} |dx^1 \wedge \cdots \wedge dx^d|^{\alpha}.$$

Each $\rho \in \Gamma(E^* \otimes |\Lambda|(M))$ ($|\Lambda|(M) = |\Lambda|^1(M)$) is regarded as an element $T^{\rho} \in \mathscr{D}'(M, E)$ by

$$T^{\rho}[\eta] = \int_M \eta \cdot \rho, \qquad \eta \in \mathscr{D}(M, E)$$

where $\eta \cdot \rho$ is obtained from the evaluation map $\eta \otimes \rho \to \eta \cdot \rho = {}_E(\eta, \rho)_{E^*} (= [\![\eta \otimes \rho]\!])$, $E \otimes E^* \otimes |\Lambda|(M) \to |\Lambda|(M)$. Each section distribution of $E^* \otimes |\Lambda|(M)$ is called also a *generalized section of E*.

As usual, we identify $\Gamma(M \times \boldsymbol{R})$ [resp. $\mathscr{D}(M, M \times \boldsymbol{R})$] with $C^{\infty}(M)$ [resp. $\mathscr{D}(M) = \{f \in C^{\infty}(M)$; supp f is compact$\} = C_0^{\infty}(M)$] and denote the dual space of $\mathscr{D}(M)$ by $\mathscr{D}'(M)$. For each relatively compact open set U of M, we define $\mathscr{D}(U)$ and $\mathscr{D}'(U)$ in the same manner. We set $\mathscr{E}'(U) = \{T \in \mathscr{D}'(U);\ \operatorname{supp} T \subset U\}$, $\mathscr{E}'(M) = \{T \in \mathscr{D}'(M);\ \operatorname{supp} T$ is compact$\}$.

We shall give local expressions of $T \in \mathscr{D}'(M, E)$. Let (U, ϕ) be a chart of M such that U is relatively compact and $E|_U = \pi_E^{-1}(U)$ is trivial. Let $(e_{\kappa})_{\kappa=1,\cdots,m}$, ($e_{\kappa}: U \to E$), be a C^{∞} local frame for E over U and (f^{κ}) be its dual frame ($f^{\kappa}: U \to E^*$). Define $T_{\kappa} \in \mathscr{D}'(\phi(U))$ by

$$T_{\kappa}[v] = T[(v \circ \phi) e_{\kappa}]$$

for every $v \in \mathscr{D}(\phi(U))$. (Note that $v \circ \phi \in \mathscr{D}(U)$, $(v \circ \phi) e_{\kappa} \in \mathscr{D}(U, E) (= \mathscr{D}(U, E|_U)) \subset \mathscr{D}(M, E)$.) Moreover, define $((\phi^{-1})_* T_{\kappa}) f^{\lambda} \in \mathscr{D}'(U, E) (= \{$section distributions of $E|_U\})$ by

$$((\phi^{-1})_* T_{\kappa}) f^{\lambda}[\xi] = ((\phi^{-1})_* T_{\kappa})[{}_{E^*}(f^{\lambda}, \xi)_E] = T_{\kappa}[{}_{E^*}(f^{\lambda}, \xi)_E \circ \phi^{-1}]$$

for every $\xi \in \mathscr{D}(U, E)$, where $(\phi^{-1})_* T_{\kappa}$ stands for the push-forward of T_{κ} by $\phi^{-1}: \phi(U) \to U$. Then T is expressed on U as

$$T = \sum_{\kappa=1}^{m} ((\phi^{-1})_* T_{\kappa}) f^{\kappa}, \qquad (T_{\kappa} \in \mathscr{D}'(\phi(U))). \tag{5.1}$$

Take a C^{∞} partition $\{\chi_i\}$ of unity subordinate to a countable, locally finite atlas $\{(U_i, \phi_i);\ i \in I\}$ of M such that each U_i is relatively compact and each $E|_{U_i}$ is trivial. (Only such atlases will be considered in the following.) Then for some $T_{i\kappa} \in \mathscr{D}'(\phi_i(U_i))$,

$$T[\xi] = \sum_i T[\chi_i \xi] = \sum_i \sum_{\kappa=1}^{m} T_{i\kappa}[\xi_i^{\kappa}], \qquad (\xi \in \mathscr{D}(M, E))$$

where $\xi_i^\kappa = {}_{E^*}(f_i^\kappa, \chi_i\xi)_E \circ \phi_i^{-1} \in \mathcal{D}(\phi_i(U_i))$. (For each i, (f_i^κ) is a local frame for E^* over U_i.)

Let $\mathcal{S}=\mathcal{S}(\boldsymbol{R}^d)$ be the Schwartz space of all rapidly decreasing C^∞ (real) functions on $\boldsymbol{R}^d$ and let $\mathcal{S}'=\mathcal{S}'(\boldsymbol{R}^d)$ be the Schwartz space of all tempered distributions on $\boldsymbol{R}^d$. For $u\in\mathcal{S}$ and $k=0, \pm1, \pm2, \cdots$, put

$$\|u\|_{2k}=\|(1+|x|^2-\Delta)^k u\|_\infty$$

where $\Delta=\sum_{i=1}^d(\partial/\partial x^i)^2$ is the d-dimensional Laplacian and $\|f\|_\infty=\max\{|f(x)|\,;\,x\in\boldsymbol{R}^d\}$. Let $\mathcal{S}_{2k}$ be the completion of $\mathcal{S}$ by the norm $\|\cdot\|_{2k}$. Then (cf. [7], [15], [18])

$$\mathcal{S}=\bigcap_{k>0}\mathcal{S}_{2k}\,,\qquad \mathcal{S}'=\bigcup_{k>0}\mathcal{S}_{-2k}\,.$$

Let K be a compact set in M. Set $\mathcal{D}'_K(M, E)=\{T\in\mathcal{D}'(M, E)\,;\,\operatorname{supp} T\subset K\}$. Let $\{\chi_i\}$ be as above. Using a local frame (f_i^κ) $(\kappa=1, \cdots, m)$ for E^* over U_i, express $\chi_i\rho$ $(\rho\in\Gamma_K(E^*\otimes|\Lambda|(M)))$ locally as

$$\chi_i\rho=\sum_\kappa(\rho_{i\kappa}\circ\phi_i)f_i^\kappa\otimes|dx^1\wedge\cdots\wedge dx^d|\quad \text{on } U_i\,,$$

where $\rho_{i\kappa}\in\mathcal{D}(\phi_i(U_i))\subset\mathcal{D}(\boldsymbol{R}^d)\subset\mathcal{S}$. Define a norm in $\Gamma_K(E^*\otimes|\Lambda|(M))$ by

$$\|\rho\|_{(2k)}=\sum_i\sum_\kappa\|\rho_{i\kappa}\|_{2k}\,,\qquad \rho\in\Gamma_K(E^*\otimes|\Lambda|(M))\,.$$

For each integer k, let $\mathcal{D}_{K,2k}(M, E^*\otimes|\Lambda|(M))$ be the topological vector space obtained by completing $\Gamma_K(E^*\otimes|\Lambda|(M))$ with respect to this norm. The topology on $\mathcal{D}_{K,2k}(M, E^*\otimes|\Lambda|(M))$ does not depend on the choice of $\{\chi_i\}$ (and the local trivialization). It holds that $\mathcal{D}_{K,2k}(M, E^*\otimes|\Lambda|(M))\subset\mathcal{D}'(M, E)$ for every integer k. Using the increasing sequence $\{K_n\}$ $(\bigcup_n K_n=M)$ of compact sets, we have

$$\mathcal{D}'_K(M, E)\subset\bigcup_{n=1}^\infty\bigcup_{k=1}^\infty\mathcal{D}_{K_n,-2k}(M, E^*\otimes|\Lambda|(M))\,.$$

6. The composite of manifold-valued Wiener functionals and section distributions or generalized sections.

We shall first state the composite of manifold-valued Wiener functionals and section distributions. Let $\pi_E: E\to M$ be as in §5. For an M-valued Wiener functional $F\in\boldsymbol{D}^\infty(M)$, set

$$\mathcal{K}(F, E\otimes|\Lambda|^{-1}(M))=\{\eta\in\operatorname{Dom}(D, E\otimes|\Lambda|^{-1}(M))\,;\,\pi_{E\otimes|\Lambda|^{-1}(M)}\circ\eta=F\}.$$

Theorem 6.1. *Let F be an M-valued Wiener functional satisfying the assumptions* (A1)-(A3) *in* §4. *Then there exists a unique $\boldsymbol{R}$-linear mapping*

$$\Psi_F:\mathcal{D}'(M, E)\longrightarrow\operatorname{Map}(C_0^\infty(M)\times\mathcal{K}(F, E\otimes|\Lambda|^{-1}(M)), \boldsymbol{D}^{-\infty}),$$
$$T\longrightarrow T_F=\Psi_F(T)\qquad (T\in\mathcal{D}'(M, E))$$

satisfying the following conditions:

(1) *$T_F(\cdot, \eta): C_0^\infty(M)\to\boldsymbol{D}^{-\infty}$ is $\boldsymbol{R}$-linear for every $\eta\in\mathcal{K}(F, E\otimes|\Lambda|^{-1}(M))$.*

(2) *$T_F(u, \cdot): \mathcal{K}(F, E\otimes|\Lambda|^{-1}(M))\to \boldsymbol{D}^{-\infty}$ is $\boldsymbol{R}$-linear for every $u\in C_0^\infty(M)$.*

(3) *If $T\in\mathcal{D}(M, E^*\otimes|\Lambda|(M))(\subset\mathcal{D}'(M, E))$, then*

$$T_F(u, \cdot)=(u\cdot T)\circ F$$

in the sense that

$$T_F(u, \eta)={}_{E^*\otimes|\Lambda|(M)}((uT)\circ F, \eta)_{E\otimes|\Lambda|^{-1}(M)}\in \boldsymbol{D}^\infty\subset \boldsymbol{D}^{-\infty}$$

for every $\eta\in\mathcal{K}(F, E\otimes|\Lambda|^{-1}(M))$.

(4) *If $u_{(1)}, u_{(2)}\in C_0^\infty(M)$ and $\eta\in\mathcal{K}(F, E\otimes|\Lambda|^{-1}(M))$, then*

$$T_F(u_{(1)}\cdot u_{(2)}, \eta)=T_F(u_{(1)}, u_{(2)}(F)\eta)=u_{(2)}(F)\cdot T_F(u_{(1)}, \eta)=(u_{(2)}T)_F(u_{(1)}, \eta)$$
$$(=\Psi_F(u_{(2)}T)(u_{(1)}, \eta)).$$

(5) *Let $u\in C_0^\infty(M)$ and $\eta\in\mathcal{K}(F, E\otimes|\Lambda|^{-1}(M))$. Then for each compact set K in M and for each positive integer k, the mapping $T\to T_F(u, \eta)$ gives the continuous $\boldsymbol{R}$-linear mapping*

$$T\in\mathcal{D}_{K,-2k}(M, E^*\otimes|\Lambda|(M)) \longrightarrow T_F(u, \eta)\in \boldsymbol{D}_p^{-2k}$$

for every $1<p<\infty$.

The proof will be given in §7.

We shall call T_F in the above theorem the *composite of F and T.* To define the *composite of F* (satisfying (A1)-(A3)) *and a generalized section of E*, we have only to replace E, $E\otimes|\Lambda|^{-1}(M)$, $E^*\otimes|\Lambda|(M)$ by $E^*\otimes|\Lambda|(M)$, E^*, E, respectively in Theorem 6.1, since $|\Lambda|(M)\otimes|\Lambda|^{-1}(M)$ and $(|\Lambda|(M))^*\otimes|\Lambda|(M)$ are isomorphic to the trivial real line bundle over M.

7. Proof of Theorem 6.1.

For the proof, we shall prepare some lemmas obtainable with slight changes from Watanabe's discussion [18] about $\boldsymbol{R}^d$-valued Wiener functionals (see also [7], [17]).

Lemma 7.1. *Let $F=(F^1, \cdots, F^d): W\to\boldsymbol{R}^d$ be an $\boldsymbol{R}^d$-valued Wiener functional such that $F^i\in\boldsymbol{D}^\infty$, $i=1, \cdots, d$. Set $\sigma^{ij}(w)=\langle DF^i, DF^j\rangle_H(w)$, $(i, j=1, \cdots, d)$, for $w\in W$. Assume that there exist elements $\gamma_{ij}\in\boldsymbol{D}^\infty$, $(i, j=1, \cdots, d)$, and a μ-measurable set A such that the matrix $(\sigma^{ij}(w))$ has the inverse for almost all w in A and for such $w\in A$*

$$(\gamma_{ij}(w))=(\sigma^{ij}(w))^{-1}.$$

Furthermore, let $\beta\in\boldsymbol{D}^\infty$ be such that $\beta(w)=0$ for $w\notin A$. Then for every $u\in\mathcal{S}$ and $G\in\boldsymbol{D}^\infty$,

$$\int_W\left(\frac{\partial u}{\partial x^i}\circ F\right)\beta G\, d\mu=\sum_{k=1}^d\int_W(u\circ F)\cdot D^*(\gamma_{ik}\beta G\cdot DF^k)d\mu. \tag{7.1}$$

Moreover, the right hand side of (7.1) can also be written in the form

$$\int_W (u\circ F)\Big\{\beta G_0+\sum_{i=1}^{d}\langle DF^i, D\beta\rangle_H G_i\Big\}d\mu$$

for some $G_0, G_1, \cdots, G_d\in \boldsymbol{D}^\infty$.

Proof. For almost all w in A, we have

$$\frac{\partial u}{\partial x^i}\circ F(w)=\sum_k \gamma_{ik}\langle D(u(F)), DF^k\rangle_H(w).$$

Thus letting I_A be the indicator function of A, we obtain

$$\begin{aligned}\int_W\Big(\frac{\partial u}{\partial x^i}\circ F\Big)\beta G\,d\mu&=\int_W I_A\Big(\frac{\partial u}{\partial x^i}\circ F\Big)\beta G\,d\mu\\ &=\sum_k\int_W\langle D(u(F)), \gamma_{ik}\beta G\cdot DF^k\rangle_H d\mu\\ &=\sum_k\int_W u(F)\cdot D^*(\gamma_{ik}\beta G\cdot DF^k)d\mu\\ &=\int_W u(F)\cdot\{(\sum_k(-LF^k)\gamma_{ik}G-\sum_k\langle DF^k, D(\gamma_{ik}G)\rangle_H)\beta\\ &\quad-\sum_k\langle DF^k, D\beta\rangle_H\cdot(\gamma_{ik}G)\}d\mu.\end{aligned}$$

This proves the lemma.

Lemma 7.2. *Let* F *and* β *be as in Lemma* 7.1. *Suppose for* $w\notin A$,

$$\langle DF^{i_1}, D\langle DF^{i_2}, D\langle DF^{i_3}, \cdots, D\langle DF^{i_l}, D\beta\rangle_H\cdots\rangle_H\rangle_H\rangle_H(w)=0 \qquad (7.2)$$

holds for every positive integer l *and* $i_1, i_2, \cdots, i_l\in\{1, 2, \cdots, d\}$. *Then for every* $G\in\boldsymbol{D}^\infty$ *and positive integer* k, *there exists* $\eta_{2k}(\beta G)\in\boldsymbol{D}^\infty$ *such that*

$$\int_W(\{(1+|x|^2-\Delta)^k f\}\circ F)\cdot\beta G\,d\mu=\int_W f(F)\cdot\eta_{2k}(\beta G)d\mu$$

for every $f\in\mathcal{S}$.

Proof. This is an immediate consequence of Lemma 7.1.

From this lemma, we get the following fundamental proposition analogous to [7, p. 21, Corollary 4.2], [18, pp. 57-58].

Proposition 7.3. *Let* F, β *be as in Lemma* 7.2. *Then the mapping* $f\in\mathcal{S}\to\beta\cdot f(F)\in\boldsymbol{D}^\infty$ *extends to a unique continuous* $\boldsymbol{R}$*-linear mapping* $\Theta(F, \beta): T\in\mathcal{S}_{-2k}\to\Theta(F, \beta)[T]\in\boldsymbol{D}_p^{-2k}$ *for every positive integer* k *and* $p\in(1, \infty)$. (*Thus* $\Theta(F, \beta): \mathcal{S}'\to\boldsymbol{D}^{-\infty}$ *is defined*.)

Proof. For $f\in\mathcal{S}$ and $G\in\boldsymbol{D}^\infty$, it holds from Lemma 7.2 that

$$\begin{aligned}\Big|\int_W\beta f(F)G\,d\mu\Big|&=\Big|\int_W\{(1+|x|^2-\Delta)^k(1+|x|^2-\Delta)^{-k}f\circ F\}\beta G\,d\mu\Big|\\ &\leqq\|(1+|x|^2-\Delta)^{-k}f\|_\infty\cdot\|\eta_{2k}(\beta G)\|_{L^1}.\end{aligned}$$

Take positive numbers q and r such that $(1/p)+(1/q)=1$, $1<r<q<\infty$. Let q' be such that $(1/q)+(1/q')=1/r$. Then by [18, Proposition 1.10] there exists a constant $C_{q,q',2k}$ such that

$$\|\beta G\|_{r,2k}\leqq C_{q,q',2k}\|\beta\|_{q',2k}\|G\|_{q,2k}.$$

Therefore, we have

$$C=\sup\{\|\eta_{2k}(\beta G)\|_{L^1};\ G\in\boldsymbol{D}_q^{2k},\ \|G\|_{q,2k}\leqq 1\}<\infty$$

in the same way as in [18]. Thus we have

$$\begin{aligned}\|\beta\cdot f(F)\|_{p,-2k}&=\sup\left\{\left|\int_W\beta\cdot f(F)G\,d\mu\right|;\ G\in\boldsymbol{D}_q^{2k},\ \|G\|_{q,2k}\leqq 1\right\}\\&\leqq C\|f\|_{-2k}.\end{aligned}$$

This shows that $\Theta(F,\beta):\mathcal{S}\to\boldsymbol{D}^\infty$ given by $f(\in\mathcal{S})\to\beta\cdot f(F)$ extends uniquely to a continuous $\boldsymbol{R}$-linear mapping $\mathcal{S}_{-2k}\to\boldsymbol{D}_p^{-2k}$ and thus we obtain $\Theta(F,\beta):\mathcal{S}'\to\boldsymbol{D}^{-\infty}$.

Lemma 7.4. *Let F, (γ_{ij}) and A be as in Proposition 7.3 and let $G,\beta\in\boldsymbol{D}^\infty$. Suppose β satisfies the assumption in Lemma 7.2. Then $\Theta(F,G\beta)[T]=G\cdot(\Theta(F,\beta)[T])$ for every $T\in\mathcal{S}'$.*

Proof. For $T\in\mathcal{S}'$, take a positive integer k such that $T\in\mathcal{S}_{-2k}$. Then there exists a sequence $\{f_n\}\subset\mathcal{S}$ such that $f_n\to T$ in $\mathcal{S}_{-2k}$ as $n\to\infty$. Note that $G\beta$ has the same property as β in Lemma 7.2. Since $\Theta(F,G\beta)[f_n]=G\beta\cdot f_n(F)=G\cdot(\Theta(F,\beta)[f_n])$, we have $\Theta(F,G\beta)[T]=G\cdot(\Theta(F,\beta)[T])$ if we let $n\to\infty$.

Proof of Theorem 6.1. Let $F\in\boldsymbol{D}^\infty(M)$ satisfy (A1)-(A3). For simplicity, we write $\tilde{\mathcal{K}}$ for $\mathcal{K}(F,E\otimes|\Lambda|^{-1}(M))$.

Step 1. Let (U,ϕ) be a chart of M such that U is relatively compact and $E|_U$ is trivial, so that $(E\otimes|\Lambda|^{-1}(M))|_U$ and $(E^*\otimes|\Lambda|(M))|_U$ are also trivial. Take an open set $V\subset U$ such that the closure $\bar{V}$ of V is included in U. Suppose $u\in C_0^\infty(M)$ satisfies $\operatorname{supp}u\subset V$. Choose $\kappa\in C_0^\infty(M)$ such that $\operatorname{supp}\kappa\subset U$, $0\leqq\kappa\leqq 1$ and $\kappa(p)=1$ for $p\in\bar{V}$. Since $\kappa|_U\cdot\phi$ is an $\boldsymbol{R}^d$-valued C^∞ function on U whose support is contained in U, it can be extended to an $\boldsymbol{R}^d$-valued C^∞ function $v=(v^1,\cdots,v^d)$ on M in an obvious manner, so that $\operatorname{supp}v\subset U$, $v|_U=\kappa|_U\cdot\phi$ and $v|_V=\phi|_V$. Moreover $\hat{F}=v\circ F=(\hat{F}^1,\cdots,\hat{F}^d)$ is an $\boldsymbol{R}^d$-valued Wiener functional such that $\hat{F}^1,\cdots,\hat{F}^d\in\boldsymbol{D}^\infty$ and $\hat{F}(w)=\phi(F(w))$ for every $w\in F^{-1}(V)$.

Let (x^a) be a local coordinate system in (U,ϕ). Then there exist C^∞ vector fields X_a $(a=1,\cdots,d)$ on M such that each X_a is expressed on V as $X_a=\partial/\partial x^a$ and satisfies $(X_a)_p=0$ if $p\notin U$. By the assumption (A3), it holds that $\gamma_{ab}:=\gamma(X_a,X_b)\in\boldsymbol{D}^\infty$ $(a,b=1,\cdots,d)$. Furthermore

$$(\gamma_{ab}(w))=(\langle D\hat{F}^a,D\hat{F}^b\rangle_H(w))^{-1}$$

for every $w\in F^{-1}(V)$. In fact, since

$${}_{TM}(X_b,dv^a)_{T^*M}|_V=\delta_b^a,$$

we have, for $w\in F^{-1}(V)$,

$$\langle D\hat{F}^a, D\hat{F}^b\rangle_H(w) = {}_{TM\circledcirc TM}(\langle DF, DF\rangle(w), (dv^a \circledcirc dv^b)\circ F(w))_{T^*M\circledcirc T^*M}$$
$$= \text{"the } (a, b)\text{-component of } \langle DF, DF\rangle(w) \text{ with respect to}$$
$$((\partial/\partial x^1)_{F(w)}, \cdots, (\partial/\partial x^d)_{F(w)})\text{"}.$$

Take a C^∞ local frame (e_λ) for E over U and let (f^λ) be its dual local frame. Then $(\tilde{f}^\lambda)$ defined by

$$\tilde{f}^\lambda = f^\lambda \otimes |dx^1 \wedge \cdots \wedge dx^d| : U \longrightarrow E^* \otimes |\Lambda|(M), \qquad \lambda = 1, \cdots, m$$

becomes a local frame for $E^*\otimes|\Lambda|(M)$ over U. Making use of the above κ, we extend $\kappa|_U \cdot \tilde{f}^\lambda$ to $f^{*\lambda}$ on the whole M by

$$f^{*\lambda}(p) = \begin{cases} \kappa(p)\tilde{f}^\lambda(p) & \text{if } p \in U, \\ 0 & \text{if } p \notin U, \end{cases}$$

so that $\operatorname{supp} f^{*\lambda} \subset U$. (Of course, each $f^{*\lambda}$ is a (global) C^∞ cross section of $E^*\otimes|\Lambda|(M)$, while $(f^{*\lambda})_{\lambda=1,\cdots,m}$ does not become a global frame.)

Next, for $\eta \in \tilde{\mathcal{K}}$, define $\tilde{\eta}^\lambda \in \boldsymbol{D}^\infty$ by

$$\tilde{\eta}^\lambda = {}_{E\otimes|\Lambda|^{-1}(M)}(\eta, f^{*\lambda}\circ F)_{E^*\otimes|\Lambda|(M)}.$$

Recall that $T \in \mathcal{D}'(M, E)$ is expressed on U as $T = \sum_\lambda ((\phi^{-1})_* T_\lambda) f^\lambda$, $(T_\lambda \in \mathcal{D}'(\phi(U)))$. Choose $\chi \in C_0^\infty(\boldsymbol{R}^d)$ such that $\operatorname{supp}\chi \subset \phi(U)$, $0 \leqq \chi \leqq 1$, and $\chi(p) = 1$ for $p \in \phi(\bar{V})$. Since $\chi|_{\phi(U)} \cdot T_\lambda \in \mathcal{D}'(\phi(U))$ has its support in $\phi(U)$ for $\lambda = 1, \cdots, m$, it can be interpreted as an element, which we write simply as χT_λ, of $\mathcal{E}'(\phi(U)) \subset \mathcal{E}'(\boldsymbol{R}^d) \subset \mathcal{S}'$. Now set $\beta = u\circ F (\in \boldsymbol{D}^\infty)$. We shall show the following lemma.

Lemma 7.5. *Let V, U, u, β, X_a, v^a, $\hat{F} = (\hat{F}^1, \cdots, \hat{F}^d)$ be as above. Then β satisfies the assumption in Lemma 7.2 with respect to $\hat{F}$ and $A = F^{-1}(V)$.*

Proof. Since $\operatorname{supp}(du) \subset V$, setting $u_a = {}_{TM}(X_a, du)_{T^*M}$, we have $u_a \in C_0^\infty(M)$ with $\operatorname{supp} u_a \subset V$ and $du = \sum_a u_a dv^a$. Therefore

$$\langle D\hat{F}^a, D\beta\rangle_H = {}_{TM\circledcirc TM}(\langle DF, DF\rangle, (dv^a \circledcirc du)\circ F)_{T^*M\circledcirc T^*M}$$
$$= \sum_b (u_b\circ F)\cdot {}_{TM\circledcirc TM}(\langle DF, DF\rangle, (dv^a \circledcirc dv^b)\circ F)_{T^*M\circledcirc T^*M}$$
$$= \sum_b (u_b\circ F)\cdot \langle D\hat{F}^a, D\hat{F}^b\rangle_H$$

which is of the form $\sum_b (u_b\circ F) G^{ab}$ with $G^{ab} \in \boldsymbol{D}^\infty$. Set $\beta_a = u_a\circ F$. Then $\beta_a(w) = 0$ for $w \notin A$. Since u_a also satisfies $u_a \in C_0^\infty(M)$ and $\operatorname{supp} u_a \subset V$ (as well as u does), we can deduce by the derivation property of D that β satisfies (7.2) for $w \notin A$ with respect to $\hat{F}$ and $A = F^{-1}(V)$.

Let us continue to prove Theorem 6.1. Taking $A = F^{-1}(V)$ and using $\hat{F}$, we can construct $\Theta(\hat{F}, \beta\tilde{\eta}^\lambda)[\chi T_\lambda]$ (Proposition 7.3). Then by Lemma 7.4,

$$\sum_\lambda {}_{D^{-\infty}}\langle\Theta(\hat{F},\ \beta)[\chi T_\lambda],\ G\tilde{\eta}^\lambda\rangle_{D^\infty}=\sum_\lambda {}_{D^{-\infty}}\langle\Theta(\hat{F},\ \beta\tilde{\eta}^\lambda)[\chi T_\lambda],\ G\rangle_{D^\infty}$$

$$=\sum_\lambda {}_{D^{-\infty}}\langle\Theta(\hat{F},\ \beta G\tilde{\eta}^\lambda)[\chi T_\lambda],\ 1\rangle_{D^\infty} \qquad (7.3)$$

for every $G\in \boldsymbol{D}^\infty$.

We shall show that (7.3) does not depend on the choice of κ. Let k be a positive integer such that $\chi T_\lambda\in\mathcal{S}_{-2k}$ for every λ. Let $\{f_{n,\lambda}\}_{n=1}^\infty\subset\mathcal{S}$ be such that $f_{n,\lambda}\rightarrow\chi T_\lambda$ in $\mathcal{S}_{-2k}$ as $n\rightarrow\infty$. Then we have

$$\sum_\lambda {}_{D^{-\infty}}\langle\Theta(\hat{F},\ \beta G\tilde{\eta}^\lambda)[f_{n,\lambda}],\ 1\rangle_{D^\infty}=\sum_\lambda\int_W(f_{n,\lambda})(\hat{F})\cdot\beta G\tilde{\eta}^\lambda d\mu$$

$$=\sum_\lambda\int_{F^{-1}(V)}f_{n,\lambda}(\hat{F})\beta G\tilde{\eta}^\lambda d\mu. \qquad (7.4)$$

We see that the last side of (7.4) does not depend on the choice of κ in actual fact. Therefore, if we let $n\rightarrow\infty$, the left hand side of (7.4) converges to

$${}_{D^{-\infty}}\langle\sum_\lambda\Theta(\hat{F},\ \beta G\tilde{\eta}^\lambda)[\chi T_\lambda],\ 1\rangle_{D^\infty}={}_{D^{-\infty}}\langle\sum_\lambda\Theta(\hat{F},\ \beta\tilde{\eta}^\lambda)[\chi T_\lambda],\ G\rangle_{D^\infty}$$

which must also be independent of the choice of κ.

Step 2. Let $T\in\mathcal{D}'(M,\ E)$. Let $(U',\ \phi')$ be another chart of M such that U' is relatively compact and $E|_{U'}$ is trivial. Take V', χ', $(e_{\nu'})$, $\tilde{\eta}^{\nu'}$, $\hat{F}'$, $f_{n,\nu'}$ ($\nu=1,\cdots,m$) in the same manner as in step 1. Suppose that $V\cap V'$ is not empty. Assume that $\operatorname{supp} u\subset V\cap V'$, so that $\beta=u\circ F$ satisfies $\beta(w)=0$ for $w\notin F^{-1}(V\cap V')$. Since there exists a $GL(m,\ \boldsymbol{R})$-valued C^∞ function $(\tilde{g}_{\nu'}^\lambda)$ on $U\cap U'$ such that $e_{\nu'}(p)=\sum_\lambda\tilde{g}_{\nu'}^\lambda(p)e_\lambda(p)$, $p\in U\cap U'$, it holds that

$$\tilde{\eta}^\lambda(w)=\sum_\nu(|\det(\phi'\circ\phi^{-1})|^{-1}\cdot g_{\nu'}^\lambda)(\phi(F(w)))\cdot\tilde{\eta}^{\nu'}(w)$$

for every $w\in F^{-1}(V\cap V')$ where $g_{\nu'}^\lambda=\tilde{g}_{\nu'}^\lambda\circ\phi^{-1}$. Here, we may assume that

$$f_{n,\nu'}(\hat{F}'(w))=\sum_\lambda(|\det(\phi'\circ\phi^{-1})|^{-1}\cdot g_{\nu'}^\lambda f_{n,\lambda})(\hat{F}(w))$$

for $n=1,\ 2,\cdots$ and $w\in F^{-1}(V\cap V')$. Construct $\sum_\nu\Theta(\hat{F}',\ \beta\tilde{\eta}^{\nu'})[f_{n,\nu'}]$ in the same way as in step 1. Then for every $G\in\boldsymbol{D}^\infty$ we have

$$\sum_\nu\int_{F^{-1}(V\cap V')}f_{n,\nu'}(\hat{F}')\beta\tilde{\eta}^{\nu'}G\,d\mu=\sum_\lambda\int_{F^{-1}(V\cap V')}f_{n,\lambda}(\hat{F})\beta\tilde{\eta}^\lambda G\,d\mu,$$

which implies (cf. (7.4))

$$\sum_\nu {}_{D^{-\infty}}\langle\Theta(\hat{F}',\ \beta\tilde{\eta}^{\nu'})[f_{n,\nu'}],\ G\rangle_{D^\infty}=\sum_\lambda {}_{D^{-\infty}}\langle\Theta(\hat{F},\ \beta\tilde{\eta}^\lambda)[f_{n,\lambda}],\ G\rangle_{D^\infty}.$$

Letting $n\rightarrow\infty$, we obtain

$$\sum_\nu {}_{D^{-\infty}}\langle\Theta(\hat{F}',\ \beta\tilde{\eta}^{\nu'})[\chi' T_{\nu'}],\ G\rangle_{D^\infty}=\sum_\lambda {}_{D^{-\infty}}\langle\Theta(\hat{F},\ \beta\tilde{\eta}^\lambda)[\chi T_\lambda],\ G\rangle_{D^\infty}$$

or

$$\sum_\nu\Theta(\hat{F}',\ \beta\tilde{\eta}^{\nu'})[\chi' T_{\nu'}]=\sum_\lambda\Theta(\hat{F},\ \beta\tilde{\eta}^\lambda)[\chi T_\lambda].$$

(This implies in particular that (7.3) does not depend on the choice of χ.)

Step 3 (general case). Let $u\in C_0^\infty(M)$. Let $\{(U_i,\ \phi_i)\}$ be a countable, locally finite atlas of M such that each U_i is relatively compact and $E|_{U_i}$ is trivial. Then $(E\otimes|\Lambda|^{-1}(M))|_{U_i}$ and $(E^*\otimes|\Lambda|(M))|_{U_i}$ are also trivial. We can choose a countable, locally finite open covering $\{V_i\}$ of M, such that $\overline{V}_i\subset U_i$ for each i. Take a C^∞ partition $\{\phi_i\}$ of unity subordinate to $\{V_i\}:\phi_i\in C_0^\infty(M)$, $\operatorname{supp}\phi_i\subset V_i$, $0\leqq\phi_i\leqq 1$ and $\sum_i\phi_i=1$.

Set $u_i=\phi_i u$ and $\beta_i=u_i\circ F$. Use the same notations as in steps 1 and 2 with the subscripts i's. Then we can construct $\Theta(\hat{F}_i, \beta_i\tilde{\eta}_i^\lambda)[\chi_i T_{i\lambda}]$. Put

$$T_F(u, \eta)=\sum_i\sum_{\lambda=1}^m\Theta(\hat{F}_i, \beta_i\tilde{\eta}_i^\lambda)[\chi_i T_{i\lambda}], \tag{7.5}$$

namely,

$${}_{D^{-\infty}}\langle T_F(u, \eta), G\rangle_{D^\infty}=\sum_i\sum_{\lambda=1}^m {}_{D^{-\infty}}\langle\Theta(\hat{F}_i, \beta_i\tilde{\eta}_i^\lambda)[\chi_i T_{i\lambda}], G\rangle_{D^\infty}, \quad (G\in\boldsymbol{D}^\infty).$$

The above summation with respect to i is essentially a finite sum, since $\operatorname{supp} u$ is compact. By Lemma 7.4 and step 2, it is easily shown that (7.5) does not depend on the choice of $\{(U_i, \phi_i)\}$, $\{V_i\}$ and $\{\phi_i\}$; (7.5) depends only on T, F, u and η. Moreover, we see from the construction of $T_F(u, \eta)$ that the conditions (1), (2), (3), (4) and (5) are satisfied (cf. Proposition 7.3) and that the mapping $\Psi_F: T\rightarrow T_F$ is $\boldsymbol{R}$-linear. Uniqueness follows from the way of the construction.

This completes the proof.

8. Some cross sections obtained by Theorem 6.1.

Let $\pi_E: E\rightarrow M$ be as in §6 and let $F: W\rightarrow M$ satisfy (A1)-(A3). Let V be a k-dimensional real (topological) vector space (k: finite). We denote by $\mathcal{D}'(M, E)\otimes V$ the space of *V-valued section distributions of E*. For $T\in\mathcal{D}'(M, E)\otimes V$, define

$$T_F: C_0^\infty(M)\times\mathcal{K}(F, E\otimes|\Lambda|^{-1}(M))\longrightarrow\boldsymbol{D}^{-\infty}\otimes V$$

as follows: Take a basis $\{e_1, \cdots, e_k\}$ of V, express T as $T=\sum_{\alpha=1}^k T^\alpha\otimes e_\alpha$ with $T^\alpha\in\mathcal{D}'(M, E)$ and set

$${}_{D^{-\infty}}\langle T_F(u, \eta), G\rangle_{D^\infty}=\sum_{\alpha=1}^k {}_{D^{-\infty}}\langle(T^\alpha)_F(u, \eta), G\rangle_{D^\infty}e_\alpha(\in V)$$

for every $u\in C_0^\infty(M)$, $\eta\in\mathcal{K}(F, E\otimes|\Lambda|^{-1}(M))$ and $G\in\boldsymbol{D}^\infty$.

In the following of this section, we assume for simplicity that M is compact. (Othewise, a function $u\in C_0^\infty(M)$ will be needed. When $\operatorname{supp} T$ is compact, we take $u\in C_0^\infty(M)$ such that $u=1$ on $\operatorname{supp} T$; cf. Theorem 6.1, (4).) From now on, we write $\langle T(F)\eta\rangle$ for ${}_{D^{-\infty}}\langle T_F(1, \eta), 1\rangle_{D^\infty}$. For each $x\in M$, we define $\tilde{\delta}_x\in\mathcal{D}'(M, |\Lambda|(M))\otimes(|\Lambda|(M)_x)$ by

$$\tilde{\delta}_x[\zeta]=\zeta(x), \qquad \zeta\in\mathcal{D}(M, |\Lambda|(M)).$$

Moreover, for each $\phi\in\Gamma(E)$, define

$$\phi\tilde{\delta}_{(\cdot)}: x\in M\longrightarrow\phi\tilde{\delta}_x\in\mathcal{D}'(M, E^*\otimes|\Lambda|(M))\otimes(|\Lambda|(M)_x)$$

by

$$(\psi\tilde{\delta}_x)[\phi]=\tilde{\delta}_x[\psi\cdot\phi]\in|\Lambda|(M)_x, \qquad \phi\in\mathcal{D}(M, E^*\otimes|\Lambda|(M)).$$

The relation

$$\langle(\psi\tilde{\delta}_x)(F)\eta\rangle=\langle\tilde{\delta}_x(F)_E(\psi(F), \eta)_{E^*}\rangle(\in|\Lambda|(M)_x), \qquad (\eta\in\mathcal{K}(F, E^*)) \tag{8.1}$$

is easily shown by expressing both sides of (8.1) locally and using Theorem 6.1, (5). The quantity (8.1) is $\boldsymbol{R}$-linear in ψ and depends only on F, η and $\psi(x)$. (Note that $\psi\tilde{\delta}_x=0$ if $\psi(x)=0$.) By using the local triviality of E and taking local coordinates, it is seen from the well-known Euclidean case (cf. [7], [15], [18]) that (8.1) is C^∞ in the x-variable.

On the other hand, define

$$\tilde{\delta}_x^{E^*}\in\mathcal{D}'(M, E^*\otimes|\Lambda|(M))\otimes((E^*\otimes|\Lambda|(M))_x)$$

by

$$\tilde{\delta}_x^{E^*}[\Xi]=\Xi(x), \qquad \Xi\in\mathcal{D}(M, E^*\otimes|\Lambda|(M)).$$

Then we get an element $\langle\tilde{\delta}_{(\cdot)}^{E^*}(F)\eta\rangle\in\Gamma(E^*\otimes|\Lambda|(M))$ defined by: $x\in M\rightarrow\langle\tilde{\delta}_x^{E^*}(F)\eta\rangle\in(E^*\otimes|\Lambda|(M))_x$, $\eta\in\mathcal{K}(F, E^*)$. Now let $v\in E_x$. Assume $\psi\in\Gamma(E)$ satisfies $\psi(x)=v$. We shall show that

$$_{E_x^*}(\langle\tilde{\delta}_x^{E^*}(F)\eta\rangle, v)_{E_x}=\langle\tilde{\delta}_x(F)\{_E(\psi(F), \eta)_{E^*}\}\rangle(\in|\Lambda|(M)_x). \tag{8.2}$$

Choose $e_\alpha\in\Gamma(E)$, $(\alpha=1, \cdots, m)$, in such a way that $(e_\alpha(x))$ is a basis of the vector space E_x (x: fixed). Let (f^α) be the dual basis of $(e_\alpha(x))$. Since

$$(\sum_\alpha(e_\alpha\tilde{\delta}_x)\otimes f^\alpha)[\Xi]:=\sum_\alpha\tilde{\delta}_x[_{E^*}(\Xi, e_\alpha)_E]\otimes f^\alpha=\Xi(x)$$

for every $\Xi\in\mathcal{D}(M, E^*\otimes|\Lambda|(M))$, it holds that

$$\tilde{\delta}_x^{E^*}=\sum_\alpha(e_\alpha\tilde{\delta}_x)\otimes f^\alpha.$$

Letting $v=\sum_\alpha v^\alpha e_\alpha(x)$, $(v^\alpha\in\boldsymbol{R})$, we have

$$\begin{aligned}_{E_x^*}(\langle\tilde{\delta}_x^{E^*}(F)\eta\rangle, v)_{E_x}(\in|\Lambda|(M)_x)&={}_{E_x^*}(\langle\sum_\alpha(e_\alpha\tilde{\delta}_x)(F)\eta\rangle\otimes f^\alpha, v)_{E_x}\\&=\langle\tilde{\delta}_x(F)_{E^*}(\eta, \sum_\alpha v^\alpha e_\alpha(F))_E\rangle.\end{aligned} \tag{8.3}$$

If we use the conditional expectation $\boldsymbol{E}^\mu[\cdot|F=x]$, (8.3) is equal to

$$\begin{aligned}\boldsymbol{E}^\mu[_E(\sum_\alpha v^\alpha e_\alpha(F), \eta)_{E^*}|F=x]\langle\tilde{\delta}_x(F)1\rangle&=\boldsymbol{E}^\mu[_E(\psi(F), \eta)_{E^*}|F=x]\langle\tilde{\delta}_x(F)1\rangle\\&=\langle\tilde{\delta}_x(F)\{_E(\psi(F), \eta)_{E^*}\}\rangle,\end{aligned}$$

(cf. Euclidean case [7], [18]). This proves (8.2).

Since $T^\psi[\phi]=\int_M(\psi\tilde{\delta}_{(\cdot)})[\phi]$ for $\phi\in\mathcal{D}(M, E^*\otimes|\Lambda|(M))$, Theorem 6.1 implies (cf. Euclidean case [7], [17], [18])

$$\int_W {}_E(\psi\circ F, \eta)_{E^*}d\mu=\langle T^\psi(F)\eta\rangle=\int_M\langle(\psi\tilde{\delta}_{(\cdot)})(F)\eta\rangle=\int_M {}_{E^*}(\langle\tilde{\delta}_{(\cdot)}^{E^*}(F)\eta\rangle, \psi(\cdot))_E.$$

Thus we have the following.

Theorem 8.1. *Assume that the base manifold M of the vector bundle E is compact. Let F satisfy* (A1)-(A3) *in* § 4 *and let* $\eta\in\mathcal{K}(F, E^*)$. *Then the mapping* $\mathcal{D}(M, E)\to\boldsymbol{R}$ *defined by* $\psi\to\int_W {}_E(\psi\circ F, \eta)_{E^*}d\mu$ *is a section distribution of E obtainable from* $\langle\tilde{\delta}^{E^*}_{(\cdot)}(F)\eta\rangle\in\Gamma(E^*\otimes|\Lambda|(M))$, *namely*

$$\int_W {}_E(\psi\circ F, \eta)_{E^*}d\mu=\int_M \psi\cdot\langle\tilde{\delta}^{E^*}_{(\cdot)}(F)\eta\rangle.$$

Combining Theorem 4.1 with the above theorem, we get the following.

Corollary 8.2. *Let M and F be as in Theorem* 8.1. *If a linear connection* ∇^E *is given in E, then*

$$\int_W {}_{T^*M\otimes E}((\nabla^E\sigma)\circ F, \eta)_{TM\otimes E^*}d\mu=\int_M \sigma\cdot\langle\tilde{\delta}^{E^*}_{(\cdot)}(F)D^*_{E^*}[\![DF\otimes\gamma\otimes\eta]\!]\rangle$$

for every $\sigma\in\mathcal{D}(M, E)$ *and* $\eta\in\mathcal{K}(F, TM\otimes E^*)$.

Remark 8.1. Let M and F be as in Theorem 8.1. Take a positive density $\rho\in\Gamma(|\Lambda|(M))$; that is, in each coordinate neighborhood, ρ is expressed as $\rho(x)=f(x)\cdot|dx^1\wedge\cdots\wedge dx^d|$ with $f(x)>0$. (Such ρ exists even if M is not orientable.) Then $|\rho|^{-1}$, which has a local expression $|\rho|^{-1}=(1/f(x))|dx^1\wedge\cdots\wedge dx^d|^{-1}$, is in $\Gamma(|\Lambda|^{-1}(M))$. Since

$$\tilde{\delta}^{E^*}_x=(|\rho|^{-1}\delta^{E^*}_x)\otimes\rho(x);$$

$$\begin{aligned}\tilde{\delta}^{E^*}_x[\zeta]&=\{(|\rho|^{-1}\delta^{E^*}_x)[\zeta]\}\otimes\rho(x)\\&=(\delta^{E^*}_x[{}_{|\Lambda|^{-1}(M)}(|\rho|^{-1}, \zeta)_{|\Lambda|(M)}])\otimes\rho(x)(=\zeta(x)),\\&\qquad\qquad(\zeta\in\mathcal{D}(M, E^*\otimes|\Lambda|(M)))\end{aligned}$$

where $\delta^{E^*}_x\in\mathcal{D}'(M, E^*)\otimes(E^*_x)$ is defined by $\delta^{E^*}_x[\Xi]=\Xi(x)$ for all $\Xi\in\Gamma(E^*)$, we have, for $\eta\in\mathcal{K}(F, E^*)$,

$$\langle\tilde{\delta}^{E^*}_x(F)\eta\rangle=\langle\delta^{E^*}_x(F)(\eta\otimes(|\rho|^{-1}\circ F))\rangle\otimes\rho(x),\qquad(x\in M).$$

This implies in particular that $\langle\delta_{(\cdot)}(F)(|\rho|^{-1}\circ F)\rangle$: $M\to\boldsymbol{R}$ *is the* C^∞ *density function of the probability law of F* (satisfying (A1)-(A3)) *with respect to the positive density* ρ, where $\delta_x\in\mathcal{D}'(M)$ is defined by $\delta_x[f]=f(x)$ for all $f\in C^\infty(M)$. That is,

$$\int_W f(F)\,d\mu=\int_M f(x)\langle\delta_x(F)(|\rho|^{-1}\circ F)\rangle\rho(x)$$

for every $f\in C^\infty(M)$. Note, however, that there is no "canonical" way to choose a positive density for a general manifold.

9. Applications to heat kernels for differential forms with values in a Riemannian vector bundle.

Vector bundle valued differential forms appear in several contexts (cf. [3], [11]). [For example, if $f: M\to M'$ is an isometric immersion from a Riemannian manifold into another, then the differential of f can be regarded as a (differential) 1-form on M with values in the pull-back bundle $f^{-1}TM'$ of TM' by f, and the second fundamental form of (M, f) can be treated as a $\mathrm{Hom}(TM, NM)$-valued 1-form where NM is the normal bundle of M in M'. If, moreover, E is a Riemannian vector bundle over M, the curvature operator of the induced connection of E is regarded as a $\mathrm{Hom}(E, E)$-valued 2-form on M. Note that each of $f^{-1}TM'$, $\mathrm{Hom}(TM, NM)$ and $\mathrm{Hom}(E, E)$ becomes a Riemannian vector bundle.] In this section, we give probabilistic expressions of the heat kernels of two heat equations ((9.1) and (9.2)) for Riemannian vector bundle valued differential forms.

Let $\pi: E\to M$ be a C^∞ Riemannian vector bundle of rank m (with a metric linear connection in E) over a compact, connected, d-dimensional Riemannian manifold (M, g) (cf. [11]). Consider the vector bundle $\pi_\xi: \xi=(\Lambda^p TM)\otimes E^*\to M$. By definition, the C^∞ cross sections of the dual bundle $\xi^*=(\Lambda^p T^*M)\otimes E$ of ξ are the E-valued p-forms on M. We use the symbol ∇ indifferently to denote various covariant differentiations induced naturally from the Riemannian metric of M and the given metric connection in E. Set $\Lambda=\bigoplus_{p=0}^{d}\{(\Lambda^p TM)\otimes E^*\}$ (Whitney sum) and $\Lambda^*=\bigoplus_{p=0}^{d}\{(\Lambda^p T^*M)\otimes E\}$. Let $d_E: \Gamma(\Lambda^*)\to\Gamma(\Lambda^*)$ be the exterior differential operator on E-valued differential forms and $d_E^*: \Gamma(\Lambda^*)\to\Gamma(\Lambda^*)$ be the codifferential operator ([3]). Consider the following two heat equations (up to the time $t_1\in(0, \infty)$) for E-valued p-forms on M;

$$\frac{\partial\theta}{\partial t}=\frac{1}{2}\Delta_E\theta\Big(:=\frac{1}{2}\,\mathrm{Trace}\,\nabla\nabla\theta\Big),\qquad \lim_{t\downarrow 0}\theta(t, x)=\sigma(x), \tag{9.1}$$

$$\frac{\partial\theta}{\partial t}=-\frac{1}{2}\square_E\theta,\qquad \lim_{t\downarrow 0}\theta(t, x)=\sigma(x) \tag{9.2}$$

where $\square_E=d_E d_E^*+d_E^* d_E$ stands for the Laplacian ([3], [11]) acting on E-valued differential forms on M, and $\sigma\in\Gamma(\xi^*)$. From now on, we take the d-dimensional Wiener space (cf. [6]; with the time parameter t being considered up to t_1) as (W, H, μ).

We shall construct the solutions of (9.1) and (9.2). Let $\pi_1: O(M)\to M$ be the bundle of orthonormal frames for TM over M and let $\pi_2: OF(E)\to M$ be the bundle of orthonormal frames for E over M. Let $P=\{r=(r_1, r_2)\in O(M)\times OF(E); \pi_1(r_1)=\pi_2(r_2)\}$ denote the fiber product of π_1 and π_2. Then we obtain the principal $O(d)\times O(m)$-bundle $\pi_P: P\to M$ with $\pi_P(r)=\pi_1(r_1)$. Obviously the vector bundles ξ and ξ^* are associated with P. Let $p_1: O(M)\times OF(E)\to O(M)$ [resp. $p_2: O(M)\times OF(E)\to OF(E)$] be the projection onto the first [resp. second] factor. Let $\iota: P\to O(M)\times OF(E)$ be the inclusion mapping. Let ω_1 [resp. ω_2] be the connection from on $O(M)$ [resp. $OF(E)$] giving the Riemannian connection on M [resp. the metric connection in E]. Then P has a connection with the connection form ([8])

$$\omega=\iota^* p_1^*\omega_1\oplus\iota^* p_2^*\omega_2:\ TP\longrightarrow \mathfrak{o}(d)\oplus\mathfrak{o}(m)\qquad\text{(direct sum)}.$$

Define a 1-form β on P with values in $\boldsymbol{R}^d$ by

$$\beta(X)=r_1^{-1}(\pi_{P*}X), \quad X\in TP_r, \quad r=(r_1, r_2)\in P.$$

As is easily seen, there exist uniquely C^∞ vector fields L_k, $(k=1, 2, \cdots, d)$, on P such that

$$\beta(L_k)={}^t(0, \cdots, 0, \overset{k}{\widetilde{1}}, 0, \cdots, 0)\in \boldsymbol{R}^d, \qquad \omega(L_k)=0.$$

Setting $\Delta_P=\sum_{k=1}^{d}(L_k)^2$ and defining $f_\sigma: P\to(\Lambda^p\boldsymbol{R}^d)\otimes\boldsymbol{R}^m$ by $f_\sigma(r)=r^{-1}\sigma_{\pi_P(r)}$ for $\sigma\in\Gamma(\xi^*)$, we have

$$(\Delta_P f_\sigma)(r)=f_{\Delta_E\sigma}(r).$$

Consider the stochastic differential equation (in the Stratonovich form)

$$dr_t=\sum_{k=1}^{d}(L_k)_{r_t}\circ dw_t^k$$

for the process $\{r_t\}$ on P, where $w_t=(w_t^k)$ is the canonical realization of a d-dimensional Wiener process. Let $r_t(r)=(r(t, r, w))$ denote the solution such that $r_0=r$. Set $X_t(r)=\pi_P(r_t(r))$ and write $X_t(r)=(X(t, r, w))$. Then $\{r_t\}$ is the (1/2) Δ_P-diffusion and

$$\tau_t=r_t\circ r_0^{-1}: \xi_x \longrightarrow \xi_{X_t(r)} \qquad [\text{resp. } \xi_x^*\to\xi_{X_t(r)}^*]$$

is the stochastic parallel displacement of fibers of ξ [resp. ξ^*] along $X_t(r)$ with $x=\pi_P(r)$. It is easy to see that

$$r(t, ra, w)=r(t, r, a_1w)a, \qquad X(t, ra, w)=X(t, r, a_1w)$$

for $t\geqq 0$, $a=(a_1, a_2)\in O(d)\times O(m)$ and $w\in W$. Since a_1w is another d-dimensional Wiener process, the probability law of $X(\cdot, r, w)$ depends only on $x=\pi_P(r)$. Moreover $X_t(r)$ becomes the Brownian motion on M such that $X_0(r)=x$. Recall Weitzenböck's formula ([3], [11]):

$$-\Box_E\sigma=\Delta_E\sigma+\Phi\sigma, \quad \Phi(x)\in\mathrm{End}(\xi_x^*), \quad (\sigma\in\Gamma(\xi^*)).$$

Set

$$\tilde{\Phi}(r):=r^{-1}\circ\frac{1}{2}\Phi(\pi_P(r))\circ r\in\mathrm{End}((\Lambda^p\boldsymbol{R}^d)\otimes\boldsymbol{R}^m), \qquad r\in P$$

and define a stochastic process $\tilde{K}_t$ with values in $\mathrm{End}((\Lambda^p\boldsymbol{R}^d)\otimes\boldsymbol{R}^m)$ by the equation

$$\frac{d}{dt}\tilde{K}_t=\tilde{K}_t\tilde{\Phi}(r_t), \qquad \tilde{K}_0=\text{identity mapping}.$$

Then $K_t:=r\circ\tilde{K}_t\circ r^{-1}$, with values in $\mathrm{End}(\xi_{\pi_P(r)}^*)$, is the solution of

$$\frac{d}{dt}K_t=\frac{1}{2}K_t\tau_t^{-1}\Phi(X_t(r))\tau_t, \qquad K_0=\text{identity mapping}.$$

It is easy to show that

$$\theta(t, x)=\int_W \tau_t^{-1}\sigma(X_t(r))\,d\mu$$

$$\left[\text{resp. } \theta(t, x)=\int_W K_t\tau_t^{-1}\sigma(X_t(r))\,d\mu\right] \qquad (\pi_P(r)=x)$$

is the solution of (9.1) [resp. (9.2)]. The proof is similar to the case of $\boldsymbol{R}$-valued differential forms and so we omit it (cf. [6]; note that our sign convention in $\square_E$ differs from that of [6] in the case $E=M\times\boldsymbol{R}$).

To study the heat kernel of (9.1) [resp. (9.2)], for each $t>0$ and $x, y\in M$, we want to take T, F and η as $T=\delta_y^{\xi}\in\mathcal{D}'(M, \xi)\otimes\xi_y$, $F=X_t(r)$ $(=X(t, r, \cdot))$, and $\eta=\tau_t\zeta\otimes|v|^{-1}(X_t(r))$ [resp. $\eta=\tau_t K_t^*\zeta\otimes|v|^{-1}(X_t(r))$], where v is the Riemannian volume density on M and $K_t^*\in\mathrm{End}(\xi_{\pi_P(r)})$ is the dual (stochastic) mapping of K_t.

Lemma 9.1. *Let $t>0$ and $x, y\in M$. Then $X(t, r, \cdot)$, $(r\in\pi_P^{-1}(x))$, satisfies the conditions* (A1)-(A3). *Moreover*

$$\tau_t\zeta\otimes|v|^{-1}(X_t(r))\in\mathcal{K}(X_t(r), \xi\otimes|\Lambda|^{-1}(M)), \tag{9.3}$$

$$\tau_t K_t^*\zeta\otimes|v|^{-1}(X_t(r))\in\mathcal{K}(X_t(r), \xi\otimes|\Lambda|^{-1}(M)) \tag{9.4}$$

for every $\zeta\in\xi_x$.

The proof will be given later.

Once this is established, by Theorem 6.1 (cf. §8), we can define a linear mapping $\xi_x\to\xi_y$ by

$$\zeta(\in\xi_x)\longrightarrow\langle\delta_y^{\xi}(X_t(r))\{\tau_t\zeta\otimes|v|^{-1}(X_t(r))\}\rangle, \qquad r\in\pi_P^{-1}(x).$$

We write the dual mapping of this mapping by $f_p(t, x, y)$, which is an element of $\mathrm{Hom}(\xi_y^*, \xi_x^*)$. Similarly, we can define an $\boldsymbol{R}$-linear mapping $\xi_x\to\xi_y$ by

$$\zeta\longrightarrow\langle\delta_y^{\xi}(X_t(r))\{\tau_t K_t^*\zeta\otimes|v|^{-1}(X_t(r))\}\rangle, \qquad r\in\pi_P^{-1}(x).$$

Let $k_p(t, x, y)\in\mathrm{Hom}(\xi_y^*, \xi_x^*)$ denote its dual mapping. Then we have the following.

Theorem 9.2. *The above $f_p(t, x, y)$ [resp. $k_p(t, x, y)$], $(t>0;\ x, y\in M)$, is the heat kernel of* (9.1) [*resp.* (9.2)] *relative to the Riemannian volume density v in the sense that $\theta(t, x)=\int_M f_p(t, x, y)\sigma(y)v(y)$ [resp. $\theta(t, x)=\int_M k_p(t, x, y)\sigma(y)v(y)$] satisfies* (9.1) [*resp.* (9.2)].

Assuming that Lemma 9.1 holds, we prove Theorem 9.2 with the use of the following lemma.

Lemma 9.3. *Let $\phi\in\xi_y^*$, $\zeta\in\xi_x$, $r\in\pi_P^{-1}(x)$. If $\tilde{\phi}\in\Gamma(\xi^*)$ satisfies $\tilde{\phi}(y)=\phi$, then for $t>0$,*

$$_{\xi_x^*}(k_p(t, x, y)\phi, \zeta)_{\xi_x}v(y)=\langle\tilde{\delta}_y(X_t(r))\{_{\xi^*}(\tilde{\phi}(X_t(r)), \tau_t K_t^*\zeta)_{\xi}\}\rangle.$$

Proof. The assertion follows from the discussion in §8. (Replace E^* by ξ in §8.)

Proof of Theorem 9.2. Let $t>0$. For every $\zeta\in\xi_x$,

$$
\begin{aligned}
&{}_{\xi_x^*}\Big(\int_W K_t\tau_t^{-1}\sigma(X_t(r))\,d\mu,\ \zeta\Big)_{\xi_x}\\
&=\int_W {}_{\xi_x^*}(K_t\tau_t^{-1}\sigma(X_t(r)),\ \zeta)_{\xi_x}d\mu\\
&=\int_W {}_{\xi^*}(\sigma(X_t(r)),\ \tau_t K_t^*\zeta)_{\xi}d\mu\\
&=\int_M {}_{\xi_x^*}(k_p(t,\ x,\ y)\sigma(y),\ \zeta)_{\xi_x}v(y)\\
&\qquad\text{(by Theorem 8.1, Remark 8.1 and Lemma 9.3)}\\
&={}_{\xi_x^*}\Big(\int_M k_p(t,\ x,\ y)\sigma(y)v(y),\ \zeta\Big)_{\xi_x}.
\end{aligned}
$$

This shows that $\int_W K_t\tau_t^{-1}\sigma(X_t(r))\,d\mu=\int_M k_p(t, x, y)\sigma(y)v(y)$. The proof for $f_p(t, x, y)$ is similar.

Now we prove Lemma 9.1. Let $t>0$. Observe that $\mathrm{Span}\{\pi_{P*}(L_1)_r, \cdots, \pi_{P*}(L_d)_r\}=TM_x$, $(\pi_P(r)=x)$. As is seen from the proof of Taniguchi [16, Theorem 2.1], it holds that

$$
\begin{aligned}
&r(t, r, \cdot)\in \boldsymbol{D}^\infty(P), \qquad \tilde{K}_t\in \boldsymbol{D}^\infty\otimes\mathrm{End}\,((\Lambda^p\boldsymbol{R}^d)\otimes\boldsymbol{R}^m),\\
&X(t, r, \cdot)\in \boldsymbol{D}^\infty(M),
\end{aligned}
$$

and moreover $\sigma(w):=\langle DX(t, r, w), DX(t, r, w)\rangle$ is positive definite a.s., (cf. [16]). Set $F=X(t, r, \cdot)$ $(t, x:\text{fixed})$. To show that the condition (A3) is satisfied, since $\pi_{T^*M\circledcirc T^*M}\circ\gamma=X(t, r, \cdot)\in\boldsymbol{D}^\infty(M)$, it is sufficient to show that $\gamma(w)(Y_1, Y_2)\in\boldsymbol{D}^\infty$ holds for every C^∞ vector fields Y_1, Y_2 on M. To show this, using the same notations as in step 1 in the proof of Theorem 6.1, we have only to verify that

$$
u(F)\gamma(X_a, X_b)\in\boldsymbol{D}^\infty, \qquad (a, b=1, \cdots, d) \tag{9.5}
$$

holds for every $u\in C_0^\infty(M)$ such that $\mathrm{supp}\,u\subset\bar{V}\subset U$. Now for almost all w,

$$
[\![\sigma(w)\otimes g_{F(w)}]\!]\in\mathrm{End}\,(TM_{F(w)})
$$

(cf. §4 for the definition of $[\![\ ]\!]$) and thus $\det[\![\sigma(w)\otimes g_{F(w)}]\!]$ can be defined. Introduce $d\times d$-matrices

$$
\tilde{\sigma}(w):=(\langle D\hat{F}^a, D\hat{F}^b\rangle(w)), \qquad \tilde{g}_{F(w)}:=(g_{F(w)}(X_a, X_b)).
$$

Then $\det[\![\sigma(w)\otimes g_{F(w)}]\!]=\det(\tilde{\sigma}(w)\tilde{g}_{F(w)})$ for $w\in F^{-1}(V)$. Therefore it holds that

$$
(\gamma(w)(X_a, X_b))=\tilde{g}_{F(w)}(\tilde{\sigma}(w)\tilde{g}_{F(w)})^{-1}
$$

for $w\in F^{-1}(V)$. Thus for almost all $w\in W$, we have

$$
u(F(w))(\gamma(w)(X_a, X_b))=u(F(w))\{\det[\![\sigma(w)\otimes g_{F(w)}]\!]\}^{-1}\tilde{g}_{F(w)}\tilde{C}(w),
$$

where $\tilde{C}(w)=(C_{ij}(w))$, $C_{ij}(w)$ being the (j, i)-th cofactor of $\tilde{\sigma}(w)\tilde{g}_{F(w)}$. From the discussion in [16, § 3], we see that $\{\det[\![\sigma\otimes g_F]\!]\}^{-1}\in L^p$, $(1<p<\infty)$, so that we can show in the same manner as in [18, pp. 52-53] that each component of $\{1/\det[\![\sigma\otimes g_F]\!]\}\tilde{C}$ belongs to $\boldsymbol{D}^\infty$. Since $u(F)\in\boldsymbol{D}^\infty$ and since each component of $\tilde{g}_F$ is in $\boldsymbol{D}^\infty$, we have (9.5). Since

$$\pi_\xi(\tau_t K_t^*\zeta)=\pi_\xi(r_t\tilde{K}_t^*(r_0^{-1}\zeta))=\pi_P(r_t)=X_t$$

where $\tilde{K}_t^*$ is the dual mapping of $\tilde{K}_t$, and since

$$\begin{aligned}{}_{\xi^*}(\Xi_{X_t},\ \tau_t K_t^*\zeta)_\xi&=(r_t^{-1}\Xi_{X_t})\cdot(r_0^{-1}K_t^*\zeta)\\ &\qquad\text{(the standard inner product in } (\Lambda^p\boldsymbol{R}^d)\otimes\boldsymbol{R}^m)\\ &=f_\Xi(r_t)\cdot\{\tilde{K}_t r_0^{-1}\zeta\}\in\boldsymbol{D}^\infty,\qquad \Xi\in\Gamma(\xi^*),\end{aligned}$$

we have $\tau_t K_t^*\zeta\in\mathcal{K}(X(t,\ r,\ \cdot),\ \xi)$. This yields (9.4). Similarly, we can get (9.3). The proof is completed.

Remark. We obtain

$$\operatorname{Trace} k_p(t,\ x,\ x)=\boldsymbol{E}^\mu[\operatorname{Trace}(\tau_t K_t^*)\,|\,X_t(r)=x]k_0^0(t,\ x,\ x).\qquad (r\in\pi_P^{-1}(x))$$

from Lemma 9.3, denoting k_0 by k_0^0 when $E=M\times\boldsymbol{R}$.

Next, consider the heat equation

$$\frac{\partial\theta}{\partial t}=-\frac{1}{2}(d_E+d_E^*)^2\theta,\quad \lim_{t\downarrow 0}\theta(t,\ x)=\sigma(x),\quad (\sigma\in\Gamma(\Lambda^*)).\tag{9.6}$$

Note that the relation $(d_E)^2\sigma=R\sigma$ holds, where $R(\in\Gamma(\operatorname{End}(\Lambda^*))):\Gamma(\Lambda^*)\to\Gamma(\Lambda^*)$, $((R\sigma)(x)=R(x)[\sigma(x)],\ R(x)\in\operatorname{Hom}(((\Lambda^pT^*M)\otimes E)_x,\ ((\Lambda^{p+2}T^*M)\otimes E)_x),\ x\in M)$, is defined by

$$(R\sigma)(X_1,\ \cdots,\ X_{p+2})=-\sum_{i<j}(-1)^{i+j}R(X_i,\ X_j)[\sigma(X_1,\ \cdots,\ \hat{X}_i,\ \cdots,\ \hat{X}_j,\ \cdots,\ X_{p+2})],$$

$$R(X_i,\ X_j)=\nabla_{X_i}\nabla_{X_j}-\nabla_{X_j}\nabla_{X_i}-\nabla_{[X_i,X_j]}:\ \Gamma(E)\longrightarrow\Gamma(E),$$

$$(\hat{}\ :\ \text{omitted}),\qquad X_1,\ \cdots,\ X_{p+2}\in\Gamma(TM).$$

Let $R^*\in\Gamma(\operatorname{End}(\Lambda^*))$ be the adjoint of R with respect to the global inner product

$$\langle\sigma_1,\ \sigma_2\rangle_{\Lambda^*}=\int_M(\sigma_1,\ \sigma_2)_x v(x),\qquad (\sigma_1,\ \sigma_2\in\Gamma(\Lambda^*)),$$

where $(\ ,\ )_x$ is the pointwise inner product in Λ^* induced naturally from g and the fiber metric in E. Then $-(d_E+d_E^*)^2\sigma=-\{\Box_E+(d_E)^2+(d_E^*)^2\}\sigma=\Delta_E\sigma+(\Phi-R-R^*)\sigma$ holds and $\Phi-R-R^*\in\Gamma(\operatorname{End}(\Lambda^*))$. Therefore, to obtain the heat kernel $e(t,\ x,\ y)$ of (9.6) relative to v, we have only to replace ξ, ξ^*, ξ_x, ξ_x^*, $(\Lambda^p\boldsymbol{R}^d)\otimes\boldsymbol{R}^m$, Φ, δ_y^ξ and $k_p(t,\ x,\ y)$ by Λ, Λ^*, Λ_x, Λ_x^*, $\bigoplus_{p=0}^{d}((\Lambda^p\boldsymbol{R}^d)\otimes\boldsymbol{R}^m)$, $\Phi-R-R^*$, δ_y^Λ and $e(t,\ x,\ y)$, respectively, in the preceding context.

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