J. Math. Kyoto Univ. (JMKYAZ)
26-4 (1986) 667-671

A criterion for spots

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A local ring $(S, \mathfrak{n})$ is a spot over a local ring $(R, \mathfrak{m})$ if S is the localization at a prime ideal of a finitely generated ring over R. If R is a one dimensional analytically unramified domain then, as is well known, every local domain birationally dominating R is a spot. The main result in this paper is a criterion for spots which generalizes this classical result. Onoda and Yoshida [O-Y] use similar techniques to give criteria for a subring of an affine domain over a field to be affine.

Recall that a local domain $(R, \mathfrak{m})$ is said to be analytically unramified (respectively analytically irreducible, respectively analytically normal) if its $\mathfrak{m}$-adic completion $\hat{R}$ is reduced (respectively a domain, respectively normal domain). $(R, \mathfrak{m})$ is said to be quasi-unmixed if dimension $\hat{R}/\hat{\mathfrak{p}} =$ dimension $\hat{R}$ for every minimal prime $\hat{\mathfrak{p}}$ of $\hat{R}$. Recall also, [C, Theorem 1], that a local domain $(R, \mathfrak{m})$ satisfies the dimension inequality:

$$\dim R + \mathrm{tr.\,d.}\, S : R \geqq \dim S + \mathrm{tr.\,d.}\, S/\mathfrak{n} : R/\mathfrak{m}$$

for all quasi-local domains $(S, \mathfrak{n})$ dominating R. If equality holds, S is said to satisfy the dimension equality with respect to R. If every S which is a spot over $R_{\mathfrak{P}}$, for some prime ideal $\mathfrak{P}$ of R, satisfies the dimension equality with respect to $R_{\mathfrak{P}}$, then R is said to satisfy the dimension formula. Ratliff [R] proved that R is quasi-unmixed if and only if R satisfies the dimension formula. Consequently, quasi-unmixedness implies that all maximal ideals in R', the integral closure of R, have the same height. (We will consistently use the notation R' for the integral closure of a ring R in its total quotient ring.) We will say that a local domain $(S, \mathfrak{n})$ birationally dominates a local domain $(R, \mathfrak{m})$ if $R \subseteqq S$, $\mathfrak{n} \cap R = \mathfrak{m}$ and S is contained in the quotient field of R.

For the proof of the theorem we rely on two well-known results. The first is Rees' criterion, [Re], for a local ring to be analytically unramified, namely: a reduced local ring $(R, \mathfrak{m})$ is analytically unramified if and only if for every finitely generated ring $S = R[s_1, \cdots, s_t]$ inside the total quotient ring of R, the integral closure S' is a finitely generated S module. The second result is a local form of Zariski's Main Theorem [Z; cf. also [N (37.4)]] which, when combined with a result of Cohen [C, Theorem 3], states that if $(R, \mathfrak{m})$ is a d-

Communicated by Prof. Nagata, August 9, 1985
* The authors are partially supported by grants from the NSF.

dimensional, normal, analytically irreducible local domain and $(S, \mathfrak{n})$ is a d-dimensional local domain birationally dominating R with $\mathfrak{m}S$ primary for $\mathfrak{n}$, then $S=R$.

The spot criterion involves the following property.

Definition. A d-dimensional local domain $(R, \mathfrak{m})$ has property L, respectively $\bar{L}$, if every d-dimensional normal spot birationally dominating R is analytically irreducible, respectively analytically normal.

All one dimensional local domains have $\bar{L}$, as do all excellent local domains. Lipman [L, Proposition, pg. 160] proved that a normal spot birationally dominating a 2-dimensional analytically normal domain is analytically normal. Thus all 2-dimensional analytically normal domains have $\bar{L}$. An example of a 2-dimensional analytically unramified domain not having property L is given at the end of this paper.

Theorem 1. *Let $(R, \mathfrak{m})$ be a d-dimensional analytically unramified local domain with property L. Let $(S, \mathfrak{n})$ be a d-dimensional local domain which birationally dominates R. If S is normal or quasi-unmixed, then S is a spot over R.*

Proof. We will give the proof for the case where S is quasi-unmixed. The case where S is normal is analogous and easier. Set

$$T=R[t_1, \cdots, t_r]_{\mathfrak{n}\cap R[t_1,\cdots,t_r]},$$

where $t_1, \cdots, t_r$ is a set of elements of S which generate $\mathfrak{n}$. Since R is analytically unramified T' is a finitely generated T-module. Let $\mathfrak{Q}_1, \cdots, \mathfrak{Q}_k$ be the maximal ideals—all of height d—in $S[T']\subseteq S'$. Each $\mathfrak{q}_i=\mathfrak{Q}_i\cap T'$ is a height d maximal ideal of T' and $(t_1, \cdots, t_r)T'\subseteq \mathfrak{q}_i$. Thus $\mathfrak{q}_i S[T']_{\mathfrak{Q}_i}$ is primary for the maximal ideal of $S[T']_{\mathfrak{Q}_i}$ and the local form of ZMT implies that $T'_{\mathfrak{q}_i}=S[T']_{\mathfrak{Q}_i}$. From this we conclude that there is at most one $\mathfrak{Q}_i$ over each maximal ideal of T'. It also follows that $S[T']=S'$, for $S[T']=\bigcap_{i=1}^{k} S[T']_{\mathfrak{Q}_i}=T'_W$, where $W=T'-\bigcup_{i=1}^{k}\mathfrak{q}_i$.

Let $\mathfrak{q}_{k+1}, \cdots, \mathfrak{q}_r$ be the remaining maximal ideals of T', and choose $t\in \mathfrak{q}_{k+1}\cap \cdots \cap\mathfrak{q}_r\setminus\bigcup_{i=1}^{k}\mathfrak{Q}_i$. Then t^{-1} is integral over S. Say

$$(t^{-1})^h+s_1(t^{-1})^{h-1}+\cdots+s_h=0, \qquad \text{with} \quad s_i\in S.$$

Set $A=T[s_1, \cdots, s_h]_{\mathfrak{n}\cap T[s_1,\cdots,s_h]}$. Since A' is a finitely generated A-module, the proof will be finished if we show that $S'=A'$. For then, $A\subseteq S\subseteq S'=A'$ will imply that S is a finitely generated A-module and hence a spot over R. Let $\mathfrak{P}$ be a maximal ideal of A'. Since $\mathfrak{P}\cap A$ contains $(t_1, \cdots, t_r)A$, $\mathfrak{P}\cap T'$ is maximal. Since $t^{-1}\in A'$, it follows that $\mathfrak{P}\cap T'=\mathfrak{q}_j$ for some j with $1\leqq j\leqq k$. Thus in A' there are exactly k maximal ideals $\mathfrak{P}_1, \cdots, \mathfrak{P}_k$ and $A'_{\mathfrak{P}_j}=T'_{\mathfrak{q}_j}$, so $A'=S'$.

Since excellent local domains [M, Theorem 79] and 2-dimensional regular local rings satisfy L [L Proposition, pg. 160], we obtain the following rather surprising corollaries.

Corollary 1. *Let $(R, \mathfrak{m})$ be a 2-dimensional regular local ring. Every 2-dimensional quasi-unmixed local domain $(S, \mathfrak{n})$ birationally dominating R is a spot over R.*

Corollary 2. *If $(R, \mathfrak{m})$ is a d-dimensional excellent local domain, then any d-dimensional normal or quasi-unmixed local domain $(S, \mathfrak{n})$ which birationally dominates R is a spot over R.*

Corollary 3. *Every d-dimensional normal local domain $(S, \mathfrak{n})$ which birationally dominates a d-dimensional excellent local domain $(R, \mathfrak{m})$ is quasi-unmixed.*

Remark. Localization arguments similar to those used in the proof of the theorem can be used to show that if $(R, \mathfrak{m})$ is an analytically unramified local domain of dimension d such that every d-dimensional normal local domain which birationally dominates R is a spot over R, then so is every d-dimensional quasi-unmixed local domain with Noetherian integral closure which birationally dominates R.

Zariski's analysis of the valuations birationally dominating a 2-dimensional regular local ring $(R, \mathfrak{m})$ shows that there may be discrete valuation rings birationally dominating R which are residually algebraic over R thus they are not spots over R. To say something about local rings $(S, \mathfrak{n})$ of dimension less then d, we make the following definition.

Definition. A local domain $(R, \mathfrak{m})$ has property L_k if every normal spot of dimension k birationally dominating R is analytically irreducible.

Theorem 2. *Let $(R, \mathfrak{m})$ be a d-dimensional analytically unramified local domain with property L_k. Let $(S, \mathfrak{n})$ be a quasi-unmixed local domain of dimension k which birationally dominates R. If S is dominated by a quasi-local domain $(W, \mathfrak{p})$ which is of finite transcendence degree over R and satisfies the dimension equality with respect to R, then S is a spot over R.*

Proof. We have

$$\begin{aligned}\dim R &\geqq \dim S + \text{tr. d. } S/\mathfrak{n} : R/\mathfrak{m} \\ &= \dim S + \text{tr. d. } W/\mathfrak{p} : R/\mathfrak{m} - \text{tr. d. } W/\mathfrak{p} : S/\mathfrak{n} \\ &= \dim S + \dim R - \dim W + \text{tr. d. } W : R - \text{tr. d. } W/\mathfrak{p} : S/\mathfrak{n} \\ &\geqq \dim R\end{aligned}$$

Hence, S satisfies the dimension equality with respect to R and $\text{tr. d. } S/\mathfrak{n} : R/\mathfrak{m} = d-k$. Let $s_1, \cdots s_{d-k}$ be elements of S which map onto a transcendence basis of $S/\mathfrak{n}$ over $R/\mathfrak{m}$ and set

$$T=R[s_1, \cdots, s_{d-k}]_{\mathfrak{n}\cap R[s_1,\cdots,s_{d-k}]}.$$

Then $R\subseteq T\subseteq S$ and, since T is an analytically unramified local domain of dimension k with property L_k, Theorem 1 may be applied to T and S.

We conclude with two examples.

1. Corollary 1 implies that a 2-dimensional local domain birationally dominating a 2-dimensional regular local ring is a spot precisely when it is quasi-unmixed. For the regular local ring $R=k[x, y]_{(x,y)}$ where k is a field, a Nagata non-quasi-unmixed "bad example" can be constructed to birationally dominate R. Let $R^{\sim}=R[y/x]$. Let (V, N) be a DVR in $k(x, y)$ such that $N\cap R^{\sim}=(x, y/x)$ and V/N is finite algebraic over $R^{\sim}/(x, y/x)$. Let $(W, M)=(R^{\sim}_{(x,y/x-1)}, (x, y/x-1))$ and set $S=R+M\cap N$. S birationally dominates R but is not a spot over R as it is not quasi-unmixed.

2. This is an example of a 2-dimensional analytically unramified, normal local domain that has a 2-dimensional regular local ring which birationally dominates it but is not a spot over it. The example is E 7.1 on page 210 of Nagata's *Local Rings*. David Shannon pointed out the existence of this example to us. The Nagata construction is as follows. K is a field with char $K\neq 2$, x and y are indeterminates, and $w=\sum_{i=1}^{\infty} a_i x^i$ is an element of $K[[x]]$ which is transcendental over $K(x)$. If $z_1=z=(y+w)^2$ and

$$z_{i+1}=[z-(y+\sum_{j<i} a_j x^j)^2]/x^i,$$

then Nagata proves that $R=K[x, y, z_1, z_2, \cdots]_{\mathfrak{m}}$, where $\mathfrak{m}$ is the ideal generated by $x, y, z_1, z_2, \cdots$, is a 2-dimensional regular local ring such that $K[[x, y]]$ is the completion of R. Moreover, z is an irreducible element of R such that z factors in $K[[x, y]]$ as a square, $z=(y+w)^2$. It is shown in [N] that $R[X]/(X^2-z)$ is a 2-dimensional normal local domain which is analytically reducible. Note that $R[X]/(X^2-z)\cong R[y+w]=R[w]\subseteq K[[x, y]]$.

Let $w=w_1$ and $w_{i+1}=(w-\sum_{j\leq i} a_j x^j)/x^i$, and consider $V=K[x, w_1, w_2, \cdots]_{\mathfrak{n}}$ where $\mathfrak{n}$ is the ideal generated by $x, w_1, w_2, \cdots$. Since $K[x, w_1, \cdots, w_j]_{(x,w,\cdots,w_j)}$ is a 2-dimensional regular local ring for each j, V is the union of a strictly ascending sequence of 2-dimensional regular local rings with a common quotient field and hence is a valuation ring [A, Lemma 17, pg. 346]. Since V is dominated by $K[[x]]\cap K(x, w)$, $V=K[[x]]\cap K(x, w)$, so V is rank one discrete with maximal ideal generated by x. Let $T=V[y]_{(x,y)}$. Then T is a 2-dimensional regular local ring with quotient field $K(x, y, w)$, maximal ideal $(x, y)T$, and coefficient field K. It follows that $K[[x, y]]$ is the completion of T, and $T=K[[x, y]]\cap K(x, y, w)$. Since R has completion $K[[x, y]]$, and quotient field $K(x, y, z)\subseteq K(x, y, w)$, we have $R=K[[x, y]]\cap K(x, y, z)=T\cap K(x, y, z)$. In particular, $R\subseteq T$, so T is a birational extension of the 2-dimensional normal local domain $R[w]$. Since $R[w]$ is analytically reducible and T is regular, $R[w] \neq T$. It follows from Zariski's Main Theorem as formulated by Peskine [P] and Evans [E] that T is not a spot over $R[w]$. For $R[w]$ is normal, and x, y in $R[w]$ implies that the extension of the maximal ideal of $R[w]$ to T is the maximal ideal of T.

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References

[A] S. S. Abhyankar, On the valuations centered in a local domain, Amer. J. Math., **78** (1956), 321-348.

[C] I. S. Cohen, Lengths of prime ideal chains, Amer. J. Math., **76** (1954), 654-668.

[E] E. G. Evans, A generalization of Zariski's main theorem, Proc. Amer. Math. Soc., **26** (1970), 45-48.

[L] J. Lipman, Desingularization of two-dimensional schemes, Annals of Math., **107** (1978), 151-207.

[M] H. Matsumura, Commutative Algebra, 2nd ed., Benjamin/Cummings, Mass. 1980.

[N] M. Nagata, Local Rings, J. Wiley & Sons, New York, 1962.

[O-Y] N. Onoda and K. Yoshida, On noetherian subrings of an affine domain, Hiroshima Math. J., **12** (1982), 377-384.

[P] C. Peskine, Une generalisation du "main theorem" de Zariski, Bull. Sci. Math., (2) **90** (1966), 119-127.

[R] L. R. Ratliff Jr., On quasi-unmixed local domains, the altitude formula, and the chain condition for prime ideals, I, Amer. J. Math., **91** (1969), 508-528.

[R] D. Rees, A note on analytically unramified local rings, J. London Math. Soc., **36** (1961), 24-28.

[Z] O. Zariski, A simple analytical proof of a fundamental property of birational transformations, Proc. Nat. Acad. Sciences, **35** (1949), 62-66.