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Notes on the resolvent set

Dedicated to Professor Sigeru Mizohata on his sixtieth birthday

By

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The resolvent set $\rho(A)$ of a linear operator A in a normed linear space X is defined as follows: A complex number λ is in $\rho(A)$ if and only if $(A-\lambda)^{-1}$ exists and is a bounded, densely defined operator (Stone [2], p. 129; Taylor-Lay [3], p. 264). As is mentioned in most functional analysis textbooks, if X is a Banach space and A is closed, then $\mathrm{Ran}(A-\lambda)=X$ (Ran=range) for $\lambda\in\rho(A)$, and $\rho(A)$ is open. This statement for general, not necessarily closable operators is treated, as far as we know, in Akhiezer-Glazman [1] (pp. 349-351) and Taylor-Lay [3] (p. 273). Akhiezer and Glazman treat the Hilbert space case and exploit the fact that if $A-\lambda$ has a bounded inverse, then $\overline{\mathrm{Ran}(A-\mu)}$ ($\overline{}$=closure) has the same codimension as $\overline{\mathrm{Ran}(A-\lambda)}$ for μ close to λ, which follows from an observation on the aperture or opening between two closed subspaces. Taylor and Lay work in a normed linear space and use the fact that if $A-\lambda$ has a bounded inverse, then $\overline{\mathrm{Ran}(A-\mu)}$ is not a proper subset of $\overline{\mathrm{Ran}(A-\lambda)}$ for μ close to λ, which is based on Riesz' lemma (Taylor-Lay [3], p. 64).

We want to add here another two proofs that the resolvent set of a linear operator is open. The first proof depends on the Hahn-Banach theorem and the second on the Neumann series construction of a linear isomorphism between $\mathrm{Ran}(A-\lambda)$ and $\mathrm{Ran}(A-\mu)$.

Theorem 1. *Let X be a normed linear space and A a linear operator in X. Then $\rho(A)$ is open.*

Proof. Let $\lambda\in\rho(A)$ so that $(A-\lambda)^{-1}$ exists and is a densely defined bounded operator in X. The boundedness of $(A-\lambda)^{-1}$ implies that there exists a constant $k>0$ such that $\|(A-\lambda)u\|\geqq k\|u\|$ for $u\in\mathrm{Dom}(A)$ ($\|\ \|$=norm, Dom=domain). Therefore, for μ sufficiently near λ, $\|(A-\mu)u\|\geqq\|(A-\lambda)u\|-|\lambda-\mu|\,\|u\|\geqq(k-|\lambda-\mu|)\|u\|$, $u\in\mathrm{Dom}(A)$, which implies that $(A-\mu)^{-1}$ exists and is bounded. So, it remains to show that $\mathrm{Ran}(A-\mu)$ is dense. Suppose the contrary. Then by the Hahn-Banach theorem (Taylor-Lay [3], Theorem 3.4, p. 136) there should exist a nonzero bounded linear functional $f(\ )$ on X such that $f(x)=0$, $x\in\overline{\mathrm{Ran}(A-\mu)}$. Thus for any $u\in\mathrm{Dom}(A)$, $f((A-\mu)u)=0$, i. e., $f((A-\lambda)u)=(\mu-\lambda)f(u)$. By the definition of the norm $\|f\|$ of f and the density of

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$\mathrm{Ran}(A-\lambda)$, for any $\varepsilon>0$ there exists a $u\in\mathrm{Dom}(A)$, $u\neq 0$, such that $|f((A-\lambda)u)|\geqq(\|f\|-\varepsilon)\|(A-\lambda)u\|\geqq k(\|f\|-\varepsilon)\|u\|$. Therefore, $|\mu-\lambda|\|f\|\|u\|\geqq k(\|f\|-\varepsilon)\|u\|$. But this leads to a contradiction if we choose ε small enough. q. e. d.

Theorem 2. *Let X and A be as in Theorem 1. Let $\hat{X}$ be the completion of X. Assume that $A-\lambda$ has a bounded inverse, and that there exist bounded projections (idempotents) P_λ and Q_λ of $\hat{X}$ onto $\overline{\mathrm{Ran}(A-\lambda)}$ (closure in $\hat{X}$) and its complementary subspace, respectively, such that $P_\lambda+Q_\lambda=1$. Then for μ sufficiently close to λ, there exists a linear isomorphism $T_{\lambda\mu}$ on $\hat{X}$ such that $(A-\lambda)^{-1}T_{\lambda\mu}$ (whose domain is $T_{\lambda\mu}^{-1}\ \mathrm{Ran}(A-\lambda)$ contained in X) is the bounded inverse of $A-\mu$, and $P_\mu=T_{\lambda\mu}^{-1}P_\lambda T_{\lambda\mu}$ and $Q_\mu=T_{\lambda\mu}^{-1}Q_\lambda T_{\lambda\mu}$ are bounded projections on $\hat{X}$ onto $\overline{\mathrm{Ran}(A-\mu)}$ and its complementary subspace, respectively, verifying $P_\mu+Q_\mu=1$, so that $\overline{\mathrm{Ran}(A-\lambda)}$ and $\overline{\mathrm{Ran}(A-\mu)}$ have the same codimension.*

Proof. Let us put $R_\lambda=(A-\lambda)^{-1}$. Define an operator $T_{\lambda\mu}$ in $\hat{X}$ by $T_{\lambda\mu}=\sum_{n=0}^{\infty}(\mu-\lambda)^n(\tilde{R}_\lambda P_\lambda)^n$ ($\tilde{\ }$=closure (smallest closed extension) in $\hat{X}$), where we note that the Neumann series converges to a limit in the operator norm topology (since $\hat{X}$ is complete) for μ close enough to λ. $T_{\lambda\mu}$ is an everywhere defined (in $\hat{X}$) bounded linear operator inverse to $1-(\mu-\lambda)\tilde{R}_\lambda P_\lambda$, and thus is an isomorphism of $\hat{X}$ onto $\hat{X}$. Define $S_\mu=R_\lambda T_{\lambda\mu}$ so that $\mathrm{Dom}(S_\mu)=T_{\lambda\mu}^{-1}\mathrm{Dom}(R_\lambda)=T_{\lambda\mu}^{-1}\mathrm{Ran}(A-\lambda)$ and $\mathrm{Ran}(S_\mu)=\mathrm{Ran}(R_\lambda)=\mathrm{Dom}(A)$.

Now for $f\in\mathrm{Dom}(S_\mu)$ we have (♯) $(A-\mu)S_\mu f=(A-\lambda)R_\lambda T_{\lambda\mu}f-(\mu-\lambda)R_\lambda T_{\lambda\mu}f=T_{\lambda\mu}f-(\mu-\lambda)\tilde{R}_\lambda P_\lambda T_{\lambda\mu}f=T_{\lambda\mu}f-(T_{\lambda\mu}f-f)=f$. Here, we have used the Neumann series definition of $T_{\lambda\mu}$ and the facts that $T_{\lambda\mu}f\in\mathrm{Dom}(R_\lambda)$ and that $R_\lambda=\tilde{R}_\lambda P_\lambda$ in $\mathrm{Dom}(R_\lambda)$. Next, take $u\in\mathrm{Dom}(A)$. Then $T_{\lambda\mu}(A-\mu)u=\sum_{n=0}^{\infty}(\mu-\lambda)^n(\tilde{R}_\lambda P_\lambda)^n(A-\lambda)u+(\lambda-\mu)T_{\lambda\mu}u=(A-\lambda)u+(\mu-\lambda)\sum_{n=1}^{\infty}(\mu-\lambda)^{n-1}(\tilde{R}_\lambda P_\lambda)^{n-1}\tilde{R}_\lambda P_\lambda(A-\lambda)u+(\lambda-\mu)T_{\lambda\mu}u=(A-\lambda)u+(\mu-\lambda)T_{\lambda\mu}u+(\lambda-\mu)T_{\lambda\mu}u=(A-\lambda)u$, where we have used $\tilde{R}_\lambda P_\lambda(A-\lambda)u=R_\lambda(A-\lambda)u=u$ for $u\in\mathrm{Dom}(A)$. Thus we have $T_{\lambda\mu}(A-\mu)u\in\mathrm{Ran}(A-\lambda)=\mathrm{Dom}(R_\lambda)$ and (♯♯) $S_\mu(A-\mu)u=R_\lambda T_{\lambda\mu}(A-\mu)u=u$, $u\in\mathrm{Dom}(A)$. From (♯) and (♯♯) we can conclude that for μ close enough to λ, $(A-\mu)^{-1}$ exists and equals $S_\mu=(A-\lambda)^{-1}T_{\lambda\mu}$ which is bounded on $\mathrm{Ran}(A-\mu)$, and $T_{\lambda\mu}$ serves as an isomorphism between $\mathrm{Ran}(A-\lambda)$ and $\mathrm{Ran}(A-\mu)$. ($T_{\lambda\mu}$ is defined on $\hat{X}$. But when we restrict it to $\mathrm{Ran}(A-\mu)$, its values lie in $\mathrm{Ran}(A-\lambda)$, as we have shown above.)

Now if we define P_μ and Q_μ as stated in the theorem, they are easily seen to be idempotent. If $f=P_\mu f$, then $T_{\lambda\mu}f=P_\lambda T_{\lambda\mu}f\in\overline{\mathrm{Ran}(A-\lambda)}$ and hence $f\in T_{\lambda\mu}^{-1}\overline{\mathrm{Ran}(A-\lambda)}=\overline{\mathrm{Ran}(A-\mu)}$. Conversely, if $f\in\overline{\mathrm{Ran}(A-\mu)}$, then $T_{\lambda\mu}f\in\overline{\mathrm{Ran}(A-\lambda)}$ and, by the definition of P_μ, $P_\mu f=f$. Therefore, P_μ is a projection onto $\overline{\mathrm{Ran}(A-\mu)}$. Since $Q_\mu=1-P_\mu$, the rest of the assertion is obvious. q. e. d.

Remarks. 1) Theorem 1 is an immediate consequence of Theorem 2. For, if $\lambda\in\rho(A)$, we can put $P_\lambda=1$ in Theorem 2.

2) Although R_λ is invertible, $\tilde{R}_\lambda$ may have a nontrivial null space. $\tilde{R}_\lambda$ is

invertible if and only if A is closable (Taylor-Lay [3], Problem 5, p. 276). More generally: Suppose T is closable and invertible. Then $\tilde{T}$ is invertible if and only if T^{-1} is closable. Indeed, let $\tilde{T}$ be invertible. Let $u_n \in \mathrm{Dom}(T^{-1})$, $u_n \to 0$ and $T^{-1}u_n = v_n \to v$. Then $Tv_n \to 0$. Since T is closable, $v \in \mathrm{Dom}(\tilde{T})$ and $\tilde{T}v=0$. Since $\tilde{T}$ is invertible, $v=0$, which shows that T^{-1} is closable. Conversely, let T^{-1} be closable. Let $\tilde{T}v=0$. Then there exist $v_n \in \mathrm{Dom}(T)$ such that $v_n \to v$ and $Tv_n \to 0$. Put $u_n = Tv_n$. Then $u_n \to 0$. Since T^{-1} is closable, $T^{-1}u_n = v_n \to 0$ and hence $v=0$.

3) For $\lambda, \mu \in \rho(A)$ the resolvent equation holds: $\tilde{R}_\lambda - \tilde{R}_\mu = (\lambda-\mu)\tilde{R}_\lambda \tilde{R}_\mu$. But, if we assume only the boundedness of $(A-\lambda)^{-1}$ and $(A-\mu)^{-1}$, we cannot expect it to hold either for $\tilde{R}_\lambda$ and $\tilde{R}_\mu$ or for $\tilde{R}_\lambda P_\lambda$ and $\tilde{R}_\mu P_\mu$.

4) If one defines $\rho(A)$ as the totality of λ such that $(A-\lambda)^{-1}$ exists and is an everywhere defined bounded operator, then every nonclosed operator has empty resolvent set. Ordinarily, this definition is not adopted, but $\rho(A)$ of a closable but nonclosed operator A is *defined* to be $\rho(\tilde{A})$. According to our definition there exists a nonclosable operator with nonempty resolvent set (Taylor-Lay [3], Problem 6, p. 276).

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