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Some special cases of a conjecture of Sharp

Dedicated to Professor Masayoshi Nagata on his sixtieth birthday

By

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Introduction.

The purpose of this paper is to discuss a conjecture of Sharp on the existence of dualizing complexes:

Sharp's Conjecture ([30, 4.4]). A ring with a dualizing complex is a homomorphic image of a Gorenstein ring.

We have succeeded in giving an affirmative answer to the conjecture above in some special cases for local rings only. However the results are interesting, and further, the methods used attracted us and seem to point to one of the directions in commutative algebra. Therefore the writers would like to release this paper.

The notion of a dualizing complex was introduced by Grothendieck and Hartshorne to extend the duality theory to a large class of schemes and rings ([20]). The duality theory has a long history, and has been and will continue to be one of the most important themes in mathematics. We now have many versions of the duality theory in commutative algebra and algebraic geometry: for example, Macaulay's inverse systems ([22]), Serre's duality theory, Grothendieck's duality theory (cf. [1]), Matlis' duality theory ([23]), Grothendieck's local duality theory ([14], [15]), Herzog-Kunz's theory of the canonical module ([21]), Goto-Watanabe's duality theory for graded rings ([12]), and the theory of dualizing complexes ([20]). At the present time the theory of dualizing complexes seems to offer the vastest version of the duality theory in commutative algebra and algebraic geometry. In recent years it has become clear that the theory of dualizing complexes is very useful and powerful in commutative algebra, and some important results are shown by using it (cf. Introductions of [28], [30] and [6]). In regard to the existence of dualizing complexes, some necessary or sufficient conditions are known (cf. § 1, [8], [25], [33]). Classical-algebraic-geometric rings have dualizing complexes. However the class of rings with dualizing complexes is not yet clarified. Sharp's conjecture asserts that this class of rings coincides with the class of rings which are homomorphic images of finite-dimensional Gorenstein rings. In this paper we shall tackle this

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problem, give an affirmative answer in some special cases and present an interesting subject in commutative algebra.

Section one consists of preliminaries. We recall some facts which we need from the theory of dualizing complexes and state some results on finite local cohomologies which provided us with a useful tool. The theory of finite local cohomology (=(FLC)) has been generalized into a theory using a sequence property called unconditioned strong d-sequences. A comprehensive treatment of such sequences appears in the paper by the second author and K. Yamagishi [13]. In section two we show that an (FLC) local ring with the canonical module is a homomorphic image of a Gorenstein ring. We give similar results to [10, 5.3] in the case of dimension 1 or 2. In section three we give an affirmative answer to the conjecture for local rings of dimension $\leqq 4$ and an analogous result to [10, 5.3] in the case of dimension 3. Some of the ideas and techniques used in sections two and three are related to those employed by Ogoma [25] and [33] in his study of dualizing complexes and Sharp's conjecture. They go back ultimately, however, to Faltings' work [8] on the existence of dualizing complexes. In section four we study under some special conditions. A large part of this section is devoted to investigating the Rees algebra of a certain local ring. This investigation seems to present an interesting application of the theory of finite local cohomology (or unconditioned strong d-sequences). We give an affirmative answer to the conjecture for local rings of a special type. In the appendix we give an alternative proof of [3, 4.2], an important theorem in the theory of the canonical module.

1. Preliminaries.

In this section we first make conventions, give notations and state the definitions of the condition (S_t) and the canonical module. Then we state the definition of a dualizing complex and recall some facts which we need from the theory of dualizing complexes. We follow the treatment of R. Y. Sharp. Finally we summarize what we need concerning local rings of finite local cohomology =generalized Cohen-Macaulay local rings, which we call (FLC) local rings in this paper.

Throughout the paper a ring means a commutative noetherian ring with unit. R always denotes such a ring. Let M be a finitely generated R-module and N a submodule of M. We denote by $\mathrm{Min}_R(M)$ (resp. $\mathrm{Max}_R(M)$) the set of minimal (resp. maximal) elements in $\mathrm{Supp}_R(M)$. In the case where M is of finite dimension, we put $\mathrm{Assh}_R(M)=\{\mathfrak{p}\in\mathrm{Ass}_R(M) \mid \dim R/\mathfrak{p}=\dim M\}$ and $U_M(N)=\cap Q$ where Q runs through all the primary components of N in M such that $\dim M/Q=\dim M/N$. Let $\mathfrak{a}$ be an ideal of R and T an R-module. $E_R(T)$ denotes the injective envelope of T and $H_{\mathfrak{a}}^p(T)$ is the p-th local cohomology module of T with respect to $\mathfrak{a}$. For a system $a_1, \cdots, a_s$ of elements in R, $H_p(a_1, \cdots, a_s; -)$ denotes the p-th Koszul homology. We denote by $R(R, \mathfrak{a})$ the Rees algebra of R with respect to $\mathfrak{a}$, i.e., $R(R, \mathfrak{a})=\bigoplus_{n\geqq 0}\mathfrak{a}^n\cong R[\mathfrak{a}X]\subseteq R[X]$ with an indeterminate X. We put $\mathrm{V}(\mathfrak{a})=\{\mathfrak{p}\in\mathrm{Spec}(R)\,|\,\mathfrak{p}\supseteq\mathfrak{a}\}$, $\mathrm{CM}(R)=\{\mathfrak{p}\in\mathrm{Spec}(R)\,|\,R_{\mathfrak{p}}$ is Cohen-Macaulay$\}$ and $\mathrm{Gor}(R)=\{\mathfrak{p}\in\mathrm{Spec}(R)\,|\,R_{\mathfrak{p}}$ is Gorenstein$\}$. We denote by $\hat{}$ the maximal ideal adic completion over a local ring.

Definition 1.1. Let t be an integer. A finitely generated R-module M is said to be (S_t) if $\operatorname{depth} M_{\mathfrak{p}}\geqq\min\{t, \dim M_{\mathfrak{p}}\}$ for every $\mathfrak{p}$ in $\mathrm{Supp}_R(M)$.

Throughout the paper A denotes a d-dimensional local ring with the maximal ideal $\mathfrak{m}$, $\mathfrak{u}=U_A(0)$ and $E=E_A(A/\mathfrak{m})$.

Definition 1.2 ([21, 5.6]). (1) An A-module K is called the canonical module of A if $K\otimes_A\hat{A}\cong\mathrm{Hom}_A(H^d_{\mathfrak{m}}(A), E)$. The canonical module of A is usually denoted by K_A if it exists.

(2) Let M be a finitely generated A-module of dimension s. An A-module C is called the canonical module of M if $C\otimes_A\hat{A}\cong\mathrm{Hom}_A(H^s_{\mathfrak{m}}(M), E)$. The canonical module of M is usually denoted by K_M if it exists.

If the canonical module of M exists, it is unique up to isomorphisms and a finitely generated A-module of dimension $s(=\dim M)$ ([15, 6.4]). When A is complete, the canonical module K_M of M exists and is the module which represents the functor $\mathrm{Hom}_A(H^s_{\mathfrak{m}}(-\otimes_A M), E)$, i.e., $\mathrm{Hom}_A(H^s_{\mathfrak{m}}(T\otimes_A M), E)\cong\mathrm{Hom}_A(T, K_M)$ (functorial) for any A-module T ([21, 5.2]). For elementary properties of the canonical module, we refer the reader to [15, §6], [21, 5 und 6 Vorträge] and [3, §1]. Here we state the following important fact, an alternative proof of which will be given in the appendix.

(**1.3**) ([3, 4.3]) Suppose that the canonical module K of A exists, and let $\mathfrak{p}$ be in $\mathrm{Supp}_A(K)$. Then $K_{\mathfrak{p}}$ is the canonical module of $A_{\mathfrak{p}}$.

We also note the following fact.

(**1.4**) ([25, 4.1 and 2], [4, 1.1 and 2]) Suppose that A has the canonical module. If A is (S_2), then $\mathrm{Ass}(A)=\mathrm{Assh}(A)$ and $\hat{A}$ is also (S_2).

We briefly recall what we need about complexes from [28, §2]. For R-modules L, M and N, there is a natural homomorphism $L\otimes_R\mathrm{Hom}_R(M, N)\to\mathrm{Hom}_R(\mathrm{Hom}_R(L, M), N)$. From this homomorphism, for complexes $X^{\cdot}$, $Y^{\cdot}$ and $Z^{\cdot}$ over R, a natural homomorphism of complexes $X^{\cdot}\otimes_R\mathrm{Hom}_R(Y^{\cdot}, Z^{\cdot})\to\mathrm{Hom}_R(\mathrm{Hom}_R(X^{\cdot}, Y^{\cdot}), Z^{\cdot})$ is induced. In the case where $Y^{\cdot}$ and $Z^{\cdot}$ are bounded, we have natural isomorphisms of complexes $X^{\cdot}\otimes_R[\mathrm{Hom}_R(Y^{\cdot}, Z^{\cdot})]\xrightarrow{\sim}X^{\cdot}\otimes_R\mathrm{Hom}_R(Y^{\cdot}, Z^{\cdot})$ and $\mathrm{Hom}_R(\mathrm{Hom}_R(X^{\cdot}, Y^{\cdot}), Z^{\cdot})\xrightarrow{\sim}\mathrm{Hom}_R([\mathrm{Hom}_R(X^{\cdot}, Y^{\cdot})], Z^{\cdot})$. Hence we have a natural homomorphism of complexes $\eta(X^{\cdot}, Y^{\cdot}, Z^{\cdot})$: $X^{\cdot}\otimes_R[\mathrm{Hom}_R(Y^{\cdot}, Z^{\cdot})]\to\mathrm{Hom}_R([\mathrm{Hom}_R(X^{\cdot}, Y^{\cdot})], Z^{\cdot})$. Let $I^{\cdot}$ be a bounded complex. We define the map $\alpha(I^{\cdot})\colon R\to\mathrm{Hom}_R(I^{\cdot}, I^{\cdot})$ by $\alpha(I^{\cdot})^0\colon R\to\sum_p\mathrm{Hom}_R(I^p, I^p)$ which maps $x(\in R)$ to be the multiplication $e_p x$ in $\mathrm{Hom}_R(I^p, I^p)$ where $e_p=1, -1, -1, 1$ resp. according as $p\equiv 0, 1, 2, 3 \pmod 4$ resp. Then we define the map $\theta(X^{\cdot}, I^{\cdot})\colon X^{\cdot}\to$

$\mathrm{Hom}_R([\mathrm{Hom}_R(X^{\cdot}, I^{\cdot})], I^{\cdot})$ to be the composition map $X^{\cdot} \xrightarrow{\sim} X^{\cdot} \otimes_R R \xrightarrow{id \otimes \alpha(I^{\cdot})} X^{\cdot} \otimes_R [\mathrm{Hom}_R(I^{\cdot}, I^{\cdot})] \xrightarrow{\eta(X^{\cdot}, I^{\cdot}, I^{\cdot})} \mathrm{Hom}_R([\mathrm{Hom}_R(X^{\cdot}, I^{\cdot})], I^{\cdot})$. For an integer t, we denote by $X^{\cdot}[t]$ the complex such that $X^{\cdot}[t]^p = X^{p+t}$ and $d^p_{X^{\cdot}[t]} = (-1)^t d^{p+t}_{X^{\cdot}}$. A homomorphism of complexes $f^{\cdot}: X^{\cdot} \to Y^{\cdot}$ is said to be a quasi-isomorphism if it induces isomorphisms on their cohomology modules, i.e., $H^p(f^{\cdot}): H^p(X^{\cdot}) \to H^p(Y^{\cdot})$ is an isomorphism for every p.

Now we state the definition of a dualizing complex after Sharp.

Definition 1.5 ([28, 2.4], cf. [20, p. 258]). A complex $I^{\cdot}$ over R is called a dualizing complex of R if it satisfies the following four conditions:

(D1) $I^{\cdot}$ is bounded, i.e., $I^p = 0$ for $|p| \gg 0$.
(D2) $H^p(I^{\cdot})$ is finitely generated for every p.
(D3) Each I^p is an injective R-module.
(D4) Whenever $X^{\cdot}$ is a complex over R satisfying (D1) and (D2) for $X^{\cdot}$, the map $\theta(X^{\cdot}, I^{\cdot}): X^{\cdot} \to \mathrm{Hom}_R([\mathrm{Hom}_R(X^{\cdot}, I^{\cdot})], I^{\cdot})$ is a quasi-isomorphism.

(**1.6**) ([28, 3.6], [20, V. 2.1]) Under the conditions (D1), (D2) and (D3), the condition (D4) is equivalent to

(D4′) The map $\alpha(I^{\cdot}): R \to \mathrm{Hom}_R(I^{\cdot}, I^{\cdot})$ is a quasi-isomorphism.

A minimal injective resolution of a finite-dimensional Gorenstein ring is a typical example of a dualizing complex (cf. [28, 3.7]), and it has a special form (see [5, § 1]), which leads to the following

Definition 1.7 ([30, 1.1], cf. [20, p. 304]). A complex $I^{\cdot}$ over R is called a fundamental dualizing complex of R if it satisfies (D1), (D2) and

(D5) $\bigoplus_{p \in Z} I^p \cong \bigoplus_{\mathfrak{p} \in \mathrm{spec}(R)} E_R(R/\mathfrak{p})$.

A fundamental dualizing complex is a dualizing complex (cf. [29, 2.1 and 8], [20, V. 3.4], [28, 4.2], [20, V. 2.3]). If R has a dualizing complex, then R has a fundamental dualizing complex ([18, 3.6], cf. [20, V. 7.3 and VI. 1.1]).

For the elements of the theory of dualizing complexes, the reader is referred to [28], [29], [19], [18], [30], [6] and [20]. Here we recall some facts which we need. If a dualizing complex exists, it is unique in a certain sense, that is,

(**1.8**) ([28, 4.6], [20, V. 3.1]) Let $I^{\cdot}$ and $J^{\cdot}$ be dualizing complexes of R, and suppose that Spec(R) is connected. Then there exist an integer t, an invertible R-module P and a quasi-isomorphism $I^{\cdot} \to J^{\cdot}[t] \otimes_R P$.

(**1.9**) ([29, 3.2]) If R has a dualizing complex, then CM(R) and Gor(R) are open subsets of Spec(R).

(**1.10**) ([19, 3.3]) Suppose that $I^{\cdot}$ is a dualizing complex of R, and let S be a flat R-algebra of finite dimension. Then the following are equivalent:

(a) There exist a dualizing complex $J^{\cdot}$ of S and a quasi-isomorphism $I^{\cdot}\otimes_R S\to J^{\cdot}$.
(b) For every prime ideal $\mathfrak{q}$ of S, $S_{\mathfrak{q}}/(\mathfrak{q}\cap R)S_{\mathfrak{q}}$ is a Gorenstein ring.
(c) For each prime ideal $\mathfrak{p}$ of R being the contraction of a maximal ideal of S, $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ is a Gorenstein ring.

In connection with (1.10), we note that, if S is a finitely generated R-algebra and R has a dualizing complex, then S has a dualizing complex ([28, 3.9], [29, 3.5]), and that, if R has a dualizing complex, then every formal fibre of R is a Gorenstein ring ([28, 4.2], [29, 3.7], [20, p. 300]).

A ring which is a homomorphic image of a finite-dimensional Gorenstein ring has a dualizing complex (cf. [28, 3.7 and 9] and [20, V. 2.4]), and it is not known whether there is a ring with a dualizing complex which is not a homomorphic image of a Gorenstein ring. Sharp showed the following theorem.

(**1.11**) ([30, 4.3]) Suppose that R is a Cohen-Macaulay ring. If R has a dualizing complex, then R is a homomorphic image of a finite-dimensional Gorenstein ring.

And he posed the following conjecture ([30, 4.4]).

(**SC**) **Sharp's Conjecture:** If R has a dualizing complex, then R is a homomorphic image of a finite-dimensional Gorenstein ring.

In regard to the existence of dualizing complexes, we have two important papers, Faltings [8] and Ogoma [25], from which we recall the following result.

(**1.12**) ([8, Lemmata 3 und 5], [25, 3.7]) Let $R=S_1\times_T S_2$ be the fibre product of ring homomorphisms $f_i: S_i\to T$ $(i=1, 2)$, and suppose that f_1 is surjective and f_2 makes T a finitely generated S_2-module. Assume that $J_i^{\cdot}$ is a fundamental dualizing complex of S_i $(i=1, 2)$ and $\mathrm{Hom}_{S_1}(T, J_1^{\cdot})\cong\mathrm{Hom}_{S_2}(T, J_2^{\cdot})$ as complexes. Then there is a fundamental dualizing complex $I^{\cdot}$ of R such that $\mathrm{Hom}_R(S_i, I^{\cdot})\cong J_i^{\cdot}$ as complexes for $i=1, 2$. In particular, if T is a local ring and both S_1 and S_2 have dualizing complexes, then R has a dualizing complex.

For an A-module T, we define $D_A^p(T)=\mathrm{Hom}_A(H_{\mathfrak{m}}^p(T), E)$ for every integer p. When M is a finitely generated A-module, $D_A^p(M)$ is a finitely generated $\hat{A}$-module and $D_A^p(M)\cong D_{\hat{A}}^p(\hat{M})$ naturally ([15, 6.4]).

Let $I^{\cdot}$ be a dualizing complex of R and $\mathfrak{p}$ a prime ideal of R. Then there is a unique integer t such that $H^t(\mathrm{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, I_{\mathfrak{p}}^{\cdot}))\neq 0$ ([29, 2.1 and 8], [20, V. 3.4]). We denote this integer by $t(\mathfrak{p}; I^{\cdot})$. ([29, p. 218])

Let M be a finitely generated A-module. Suppose that A has a dualizing complex. Let $I^{\cdot}$ and $J^{\cdot}$ be dualizing complexes of A, $t=t(\mathfrak{m}; I^{\cdot})$ and $u=t(\mathfrak{m}; J^{\cdot})$. Then we have an isomorphism $H^{t-p}(\mathrm{Hom}_A(M, I^{\cdot}))\cong H^{u-p}(\mathrm{Hom}_A(M, J^{\cdot}))$ for every p (cf. (1.8)). We denote this module by $K_A^p(M)$, that is, $K_A^p(M)\cong H^{t-p}(\mathrm{Hom}_A(M, I^{\cdot}))\cong H^{u-p}(\mathrm{Hom}_A(M, J^{\cdot}))$. $K_A^p(M)$ is a finitely generated A-module ([28, 3.4]).

(**1.13**) ([6, 2.5], [20, V. 6.3]) Suppose that A has a dualizing complex. Then there is a functorial isomorphism $H_{\mathfrak{m}}^{p}(M) \xrightarrow{\sim} \mathrm{Hom}_A(K_A^{p}(M), E)$ for every integer p. Consequently $K_A^{p}(M) \otimes_A \hat{A} \cong D_A^{p}(M)$ (functorial) for every p and M has the canonical module $K_M \cong K_A^{s}(M)$ where $s = \dim M$.

We now assume that A has a fundamental dualizing complex $I^{\cdot}$ such that $t(\mathfrak{m}; I^{\cdot}) = 0$ (cf. [18, 3.6] and [28, 4.3]). We consider the double complex $C^{\cdot\cdot} = \{C^{pq} = \mathrm{Hom}_A(\mathrm{Hom}_A(M, I^{-q}), I^{p})\}$. As $C^{pq} \cong M \otimes_A \mathrm{Hom}_A(I^{-q}, I^{p})$, we have $C^{pq} = 0$ for $p+q<0$. We also have $C^{pq}=0$ if $p<-s$ or $q>s$ where $s = \dim M$. Since $\theta(M, I^{\cdot}): M \to \mathrm{Hom}_A([\mathrm{Hom}_A(M, I^{\cdot})], I^{\cdot})$ is a quasi-isomorphism, we have $H^{n}(C^{\cdot}) = 0$ for $n \neq 0$ and $H^{0}(C^{\cdot}) \cong M$ where $C^{\cdot}$ is the total complex associated with $C^{\cdot\cdot}$. We have $H_{\mathrm{I}}^{p} H_{\mathrm{II}}^{q}(C^{\cdot\cdot}) \cong K_A^{-p}(K_A^{q}(M))$. Hence, from the filtration $F^{r}C^{n} = \sum_{p+q=n,\, p \geq r} C^{pq}$, we obtain the following spectral sequences.

(**1.14**) (1) Suppose that A has a dualizing complex. Then there is a spectral sequence $K_A^{p}(K_A^{q}(M)) \underset{-p}{\Longrightarrow} M^{q-p}$ where $M^{0} = M$ and $M^{n} = 0$ for $n \neq 0$.

(2) There is a spectral sequence $D_A^{p} D_A^{q}(M) \underset{-p}{\Longrightarrow} \hat{M}^{q-p}$ where $\hat{M}^{0} = \hat{M}$ and $\hat{M}^{n} = 0$ for $n \neq 0$.

We note that we have $D_A^{0}(D_A^{1}(M)) = 0$ and an exact sequence $0 \to D_A^{0}(D_A^{0}(M)) \to \hat{M} \to D_A^{1}(D_A^{1}(M)) \to 0$ if $\dim M = 1$, and $D_A^{0}(D_A^{s}(M)) \cong D_A^{1}(D_A^{s}(M)) \cong 0$ if $\dim M = s \geqq 2$.

Now we state the definition of (FLC).

Definition 1.15. A finitely generated A-module M is said to be (FLC) if $H_{\mathfrak{m}}^{p}(M)$ is finitely generated (equivalently, of finite length) for $p \neq \dim M$.

It is obvious that M is (FLC) if and only if so is $\hat{M}$.

(**1.16**) Suppose that M is an (FLC) A-module of dimension $s \geqq 2$. Then there is an exact sequence $0 \to H_{\mathfrak{m}}^{0}(M) \to \hat{M} \to D_A^{s}(D_A^{s}(M)) \to H_{\mathfrak{m}}^{1}(M) \to 0$ and $H_{\mathfrak{m}}^{p}(D_A^{s}(M)) \cong D_A^{s-p+1}(M)$ for $2 \leqq p < s$. Consequently, if M has the canonical module K_M, K_M is also (FLC) and $H_{\mathfrak{m}}^{p}(K_M) \cong D_A^{s-p+1}(M)$ for $2 \leqq p < s$.

Proof. We have $D_A^{p}(D_A^{q}(M)) = 0$ if $q \neq s$ and $p \neq 0$ (in fact $H_{\mathrm{II}}^{q}(C^{p\cdot}) = 0$ if $q \neq s$ and $p \neq 0$ in the notation before (1.14)) and $D_A^{0}(D_A^{q}(M)) = \mathrm{Hom}_A(H_{\mathfrak{m}}^{0}(D_A^{q}(M)), E) \cong \mathrm{Hom}_A(D_A^{q}(M), E) \cong H_{\mathfrak{m}}^{q}(M)$ for $q \neq s$. Hence, by the standard spectral sequence argument, we have the assertion from (1.14). q.e.d.

(**1.17**) ([27]) Let M be a finitely generated A-module.

(1) If M is (FLC), then $M_{\mathfrak{p}}$ is a Cohen-Macaulay $A_{\mathfrak{p}}$-module and $\dim M_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim M$ for every $\mathfrak{p}$ in $\mathrm{Supp}_A(M) \backslash \{\mathfrak{m}\}$.

(2) When A has a dualizing complex or A is a homomorphic image of a Cohen-Macaulay ring, the converse to (1) holds.

(**1.18**) ([27], [31, Lemma 3]) Let M be a finitely generated A-module of dimension s. Then the following are equivalent:

(a) M is (FLC).
(b) There is an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ such that, for every system $x_1, \cdots, x_s$ of parameters for M contained in $\mathfrak{q}$, $(x_1, \cdots, x_i)M : x_{i+1} = (x_1, \cdots, x_i)M : \mathfrak{q}$ holds for $0 \leqq i < s$.

When this is the case, $\mathfrak{q}H_{\mathfrak{m}}^p(M) = 0$ for $p \neq s$.

In the remainder of this section, we assume that A is (FLC) and $d \geqq 1$. Then we have $\mathfrak{u} = H_{\mathfrak{m}}^0(A)$. Let $\mathfrak{q}$ be such an $\mathfrak{m}$-primary ideal as described in (1.18) (b) for A, i.e., for every system $x_1, \cdots, x_d$ of parameters for A contained in $\mathfrak{q}$, $(x_1, \cdots, x_i) : x_{i+1} = (x_1, \cdots, x_i) : \mathfrak{q}$ for $0 \leqq i < d$.

(1.19) Suppose depth $A > 0$. Let $a_1, \cdots, a_d$ be a system of parameters for A contained in $\mathfrak{q}$ and $\mathfrak{a} = (a_1, \cdots, a_d)$. Then $R(A, \mathfrak{a}^n)$ is a Cohen-Macaulay ring for $n \geqq d-1$.

Proof. We put $R = R(A, \mathfrak{a}^n)$, $\mathfrak{M} = \mathfrak{m}R + R_+$, $S = R(A, \mathfrak{a})$ and $\mathfrak{N} = \mathfrak{m}S + S_+$. By virtue of [7, 5.1] or [26, 4.1] (cf. [26, 3.8]), we have $H_{\mathfrak{N}}^p(S) = 0$ for $0 \leqq p \leqq \min\{2, d\}$ and $H_{\mathfrak{N}}^p(S) = \bigoplus_{i=2-p}^{-1} [H_{\mathfrak{N}}^p(S)]_i$ for $3 \leqq p \leqq d$. We denote by $-^{(v)}$ the Veronesean functor of order v for graded objects, i.e., $[-^{(v)}]_i = [-]_{iv}$. Then we have $H_{\mathfrak{M}}^p(R) \cong H_{\mathfrak{N}^{(n)}}^p(S^{(n)}) \cong H_{\mathfrak{N}}^p(S)^{(n)}$ by [12, 3.1.1] if $n > 0$. Hence we have $H_{\mathfrak{M}}^p(R) = 0$ for $0 \leqq p \leqq d$ and R is a Cohen-Macaulay ring. q. e. d.

(1.20) Let $(B, \mathfrak{n}_1, \cdots, \mathfrak{n}_t)$ be a semi-local ring. Assume that each $B_{\mathfrak{n}_i}$ is (FLC) and depth $B_{\mathfrak{n}_i} > 0$ if $\dim B_{\mathfrak{n}_i} > 0$. Then there is an ideal $\mathfrak{b}$ such that $R(B, \mathfrak{b})$ is a Cohen-Macaulay ring and $\sqrt{\mathfrak{b}} = \mathfrak{n}_1 \cdots \mathfrak{n}_t$.

Proof. By (1.19), there is an $\mathfrak{n}_i B_{\mathfrak{n}_i}$-primary ideal $\mathfrak{a}_i$ such that $R(B_{\mathfrak{n}_i}, \mathfrak{a}_i)$ is Cohen-Macaulay for every i. Let $\mathfrak{b}_i = \mathfrak{a}_i \cap B$ for $i = 1, \cdots, t$ and $\mathfrak{b} = \mathfrak{b}_1 \cdots \mathfrak{b}_t$. Then $R(B, \mathfrak{b})_{\mathfrak{N}}$ is Cohen-Macaulay for every graded maximal ideal $\mathfrak{N}$. Hence we have the assertion. q. e. d.

(1.21) Assume depth $A = d-1$ and let $\mathfrak{a} = \operatorname{ann}(H_{\mathfrak{m}}^{d-1}(A))$. Then, for every system $x_1, \cdots, x_d$ of parameters for A contained in $\mathfrak{a}$, $(x_1, \cdots, x_i) : x_{i+1} = (x_1, \cdots, x_i) : \mathfrak{a}$ holds for $0 \leqq i < d$. Consequently $R(A, \mathfrak{b}^n)$ is a Cohen-Macaulay ring for every parameter ideal $\mathfrak{b}$ contained in $\mathfrak{a}$ and every integer $n \geqq d-1$ if $d \geqq 2$.

Proof. It is not difficult to see that $x_1, \cdots, x_{d-1}$ is an A-regular sequence (cf. (1.17)). Since $(x_1, \cdots, x_{d-1}) : x_d/(x_1, \cdots, x_{d-1}) \subseteqq H_{\mathfrak{m}}^0(A/(x_1, \cdots, x_{d-1})) \cong H_{\mathfrak{m}}^{d-1}(A)$, we have $(x_1, \cdots, x_{d-1}) : x_d \subseteqq (x_1, \cdots, x_{d-1}) : \mathfrak{a}$. The opposite inclusion is obvious. q. e. d.

2. The case of (FLC).

In this section we show that an (FLC) local ring with the canonical module is a homomorphic image of a Gorenstein ring. We also consider the case of dimension one or two.

Theorem 2.1. *Suppose that A is (FLC). Then the following are equivalent:*

(a) *A is a homomorphic image of a Gorenstein ring.*
(b) *A has a dualizing complex.*
(c) *A has the canonical module.*

Proof. (c)$\Rightarrow$(b)*): Let $L^{\cdot}$ be a minimal injective resolution of the canonical module K of A, $I^{\cdot}=H_{\mathfrak{m}}^{0}(L^{\cdot})$ and $J^{\cdot}=L^{\cdot}/I^{\cdot}$. From the exact sequence of complexes $0\to I^{\cdot}\to L^{\cdot}\to J^{\cdot}\to 0$, we have a long exact sequence $\cdots\to H^{p-1}(J^{\cdot})\to H^{p}(I^{\cdot})\to H^{p}(L^{\cdot})\to H^{p}(J^{\cdot})\to\cdots$. Recall $H^{0}(L^{\cdot})\cong K$, $H^{p}(L^{\cdot})=0$ for $p>0$, $H^{p}(I^{\cdot})\cong H_{\mathfrak{m}}^{p}(K)$ for every p and $I^{p}=0$ for $p<\min\{2, d\}$ ($\leqq \operatorname{depth} K$). For every $\mathfrak{p}$ in $\operatorname{Spec}(A)\backslash\{\mathfrak{m}\}$, $A_{\mathfrak{p}}$ is Cohen-Mocaulay by (1.17) and $K_{\mathfrak{p}}$ is the canonical module of $A_{\mathfrak{p}}$ by (1.3) (cf. (1.17) and [3, 1.7]). Hence, by virtue of [21, 6.1], $J^{p}\cong\bigoplus_{\operatorname{ht}\mathfrak{p}=p} E_{A}(A/\mathfrak{p})$ for $p<d$ and $J^{p}=0$ for $p\geqq d$ because $J_{\mathfrak{p}}^{\cdot}\cong L_{\mathfrak{p}}^{\cdot}$ is a minimal injective resolution of $K_{\mathfrak{p}}$. By (1.16), K is (FLC).

The case of $d\geqq 2$. Applying the functor $\operatorname{Hom}_{A}(\ ,E)$ to the exact sequence $0\to\mathfrak{u}\to A\to\operatorname{Hom}_{A}(K, K)\to H_{\mathfrak{m}}^{1}(A)\to 0$ (cf. (1.16)), we have an exact sequence $0\to D_{A}^{1}(A)\to H_{\mathfrak{m}}^{d}(K)\to E\to D_{A}^{0}(A)\to 0$. Let us define a map $J^{d-1}\to E$ to be the composition map $J^{d-1}\twoheadrightarrow H^{d-1}(J^{\cdot})\xrightarrow{\sim} H^{d}(I^{\cdot})\xrightarrow{\sim} H_{\mathfrak{m}}^{d}(K)\to E$. Then we have a complex $D^{\cdot}=0\to D^{0}=J^{0}\to\cdots\to D^{d-1}=J^{d-1}\to D^{d}=E\to 0$. Since $H^{p}(D^{\cdot})\cong H^{p}(J^{\cdot})$ for $p<d-1$, $H^{0}(J^{\cdot})\cong K$, $H^{p}(J^{\cdot})\cong H^{p+1}(I^{\cdot})\cong H_{\mathfrak{m}}^{p+1}(K)$ for $0<p<d-1$, $H^{d-1}(D^{\cdot})\cong D_{A}^{1}(A)$ and $H^{d}(D^{\cdot})\cong D_{A}^{0}(A)$, the complex $D^{\cdot}$ satisfies (D1), (D2) and (D5), that is, $D^{\cdot}$ is a fundamental dualizing complex of A.

The case of $d=1$. Applying the functor $\operatorname{Hom}_{A}(\ ,E)$ to the exact sequence $0\to\mathfrak{u}\to A\to\operatorname{Hom}_{A}(K, K)\to 0$, we have an exact sequence $0\to H_{\mathfrak{m}}^{1}(K)\to E\to D_{A}^{0}(A)\to 0$. There is an exact sequence $0\to H^{0}(L^{\cdot})\cong K\to H^{0}(J^{\cdot})\cong J^{0}\to H^{1}(I^{\cdot})\cong H_{\mathfrak{m}}^{1}(K)\to H^{1}(L^{\cdot})=0$. Let us define a map $J^{0}\to E$ to be the composition map $J^{0}\twoheadrightarrow H_{\mathfrak{m}}^{1}(K)\hookrightarrow E$. Let $D^{\cdot}=0\to D^{0}=J^{0}\to D^{1}=E\to 0$. Since $H^{0}(D^{\cdot})\cong K$ and $H^{1}(D^{\cdot})\cong D_{A}^{0}(A)$, $D^{\cdot}$ is a fundamental dualizing complex of A.

(b)$\Rightarrow$(a): The case of $\operatorname{depth} A>0$. By (1.19), there is an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ such that $R(A, \mathfrak{q})$ is a Cohen-Macaulay ring. As $R(A, \mathfrak{q})$ is a finitely generated A-algebra, $R(A, \mathfrak{q})$ has a dualizing complex. Hence $R(A, \mathfrak{q})$ is a homomorphic image of a Gorenstein ring by (1.11). As $A\cong R(A, \mathfrak{q})/R(A, \mathfrak{q})_{+}$, we have the assertion.

The case of $\operatorname{depth} A=0$. We have an expression $(0)=\mathfrak{u}\cap\mathfrak{q}$ in A with some $\mathfrak{m}$-primary ideal $\mathfrak{q}$. Since $A/\mathfrak{u}$ has a dualizing complex, $H_{\mathfrak{m}}^{0}(A/\mathfrak{u})=0$ and $H_{\mathfrak{m}}^{p}(A/\mathfrak{u})\cong H_{\mathfrak{m}}^{p}(A)$ for $p>0$, $A/\mathfrak{u}$ is a homomorphic image of a Gorenstein local ring R

*) The proof of [25, 5.5] remains valid also to prove the implication (c)$\Rightarrow$(b) in our Theorem 2.1 (see [25, 5.8]), but it demands more preliminary knowledge than ours. On the other hand, our proof given here gives a concrete construction of a fundamental dualizing complex using only the characteristic properties of the canonical module and (FLC) local rings.

by the case of depth>0. As $A/\mathfrak{q}$ is artinian, $A/\mathfrak{q}$ is a homomorphic image of a Gorenstein local ring S. We may assume $\dim R=\dim S=d$. Let f be the surjective ring homomorphism from $R\times S$ to $A/\mathfrak{n}\times A/\mathfrak{q}$ and $B=f^{-1}(A)$, the inverse image of the unit subring A of $A/\mathfrak{n}\times A/\mathfrak{q}$. We have the following commutative diagram of B-modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & R\oplus S & \longrightarrow & R\oplus S/B & \longrightarrow & 0\\ & & \downarrow & & \downarrow f & & \downarrow\wr & & \\ 0 & \longrightarrow & A & \longrightarrow & A/\mathfrak{n}\oplus A/\mathfrak{q} & \longrightarrow & A/\mathfrak{n}+\mathfrak{q} & \longrightarrow & 0\,. \end{array}$$

Since $R\oplus S/B\cong A/\mathfrak{n}+\mathfrak{q}$ is of finite length, $R\oplus S$ is finitely generated as a B-module and therefore B is a local ring with the maximal ideal $\mathfrak{n}=f^{-1}(\mathfrak{m})$. Since $A/\mathfrak{n}+\mathfrak{q}$ is a local ring and both $R\to A/\mathfrak{n}+\mathfrak{q}$ and $S\to A/\mathfrak{n}+\mathfrak{q}$ are surjective, B has a dualizing complex by (1.12). We have $H^p_{\mathfrak{n}}(B)=0$ for $p\neq 1$, d and $H^1_{\mathfrak{n}}(B)\cong R\oplus S/B$ if $d>1$ from the above exact sequence. Hence, by the case of depth>0, B is a homomorphic image of a Gorenstein ring and therefore so is A.

q. e. d.

Corollary 2.2. *Let $d=2$. Then A has the canonical module if and only if $A/\mathfrak{n}$ is a homomorphic image of a Gorenstein ring.* (cf. [3, 1.12])

Corollary 2.3. *Let $d=3$ and suppose A is (S_2). If A has the canonical module, then A is a homomorphic image of a Gorenstein ring.* (cf. (1.4) and (1.17))

The next proposition gives a generalization of [10, 5.3].

Proposition 2.4. *Let $d=1$. Then the following are equivalent:*

(a) *A is a homomorphic image of a Gorenstein ring.*
(b) *A has a dualizing complex.*
(c) *A has the canonical module.*
(d) *Every formal fibre of A is a Gorenstein ring.*

Proof. It is sufficient to show (d)$\Rightarrow$(b). For each prime ideal $\mathfrak{p}$ of A, $E_A(A/\mathfrak{p})\otimes_A\hat{A}\cong\oplus E_{\hat{A}}(\hat{A}/\mathfrak{P})$ where $\mathfrak{P}$ runs through all the prime ideals of $\hat{A}$ such that $\mathfrak{P}\cap A=\mathfrak{p}$ by the assumption (cf. [25, 2.5]). Let $D^{\cdot}=0\to D^0\to D^1\to 0$ be a fundamental dualizing complex of $\hat{A}$. Since $\bigoplus_{\mathfrak{P}\neq\hat{\mathfrak{m}}} E_{\hat{A}}(\hat{A}/\mathfrak{P})\cong(\bigoplus_{\mathfrak{p}\neq\mathfrak{m}} E_A(A/\mathfrak{p}))\otimes_A\hat{A}$ and $E_{\hat{A}}(\hat{A}/\hat{\mathfrak{m}})\cong E\otimes_A\hat{A}\cong E$, we have a compelx $I^{\cdot}=0\to I^0=\bigoplus_{\mathfrak{p}\neq\mathfrak{m}} E_A(A/\mathfrak{p})\to I^1=E\to 0$ over A and an isomorphism of complexes $I^{\cdot}\otimes_A\hat{A}\cong D^{\cdot}$. Then $I^{\cdot}$ is a fundamental dualizing complex of A.

q. e. d.

Corollary 2.5. *Suppose that A has the canonical module and $d\geq 1$, and let $\mathfrak{a}$ be an ideal of height $d-1$. Then $A/\mathfrak{a}$ is a homomorphic image of a Gorenstein ring.*

Proof. Let $\mathfrak{P}$ be in $V(\mathfrak{a}\hat{A})\setminus\{\hat{\mathfrak{m}}\}$ and $\mathfrak{p}=\mathfrak{P}\cap A$. Then $\mathfrak{p}\supseteqq\mathfrak{a}$ and height $\mathfrak{p}=d-1$. By [3, 1.9] and (1.3), $\mathfrak{p}$ is in $\mathrm{Supp}_A(K_A)$ and $A_\mathfrak{p}$ has the canonical module $(K_A)_\mathfrak{p}$. Since $(K_A)_\mathfrak{p}\otimes_{A_\mathfrak{p}}\hat{A}_\mathfrak{P}\cong K_{\hat{A}}\otimes_{\hat{A}}\hat{A}_\mathfrak{P}\cong K_{\hat{A}_\mathfrak{P}}$ and $\hat{A}_\mathfrak{P}/\mathfrak{p}\hat{A}_\mathfrak{P}$ is artinian, $\hat{A}_\mathfrak{P}/\mathfrak{p}\hat{A}_\mathfrak{P}$ is Gorenstein by [3, 4.1] (see also Lemma 5.1). Hence the assertion follows from Proposition 2.4 (d)$\Rightarrow$(a). q. e. d.

Remark 2.6. By virtue of [9], there exists a one-dimensional local domain whose generic formal fibre is not Gorenstein. Hence there is a one-dimensional local domain which does not have the canonical module.

In the case of $d=2$, we have the following analogous result to Proposition 2.4.

Proposition 2.7. *Let $d=2$. Then the following are equivalent:*

(a) *A is a homomorphic image of a Gorenstein ring.*
(b) *A has a dualizing complex.*
(c) *A has the canonical module and every formal fibre of A is Gorenstein.*
(d) *For every ideal $\mathfrak{a}(\neq A)$, $A/\mathfrak{a}$ has the canonical module.*

Proof. It is sufficient to show (c)$\Rightarrow$(d)$\Rightarrow$(a).

(c)$\Rightarrow$(d): If $\dim A/\mathfrak{a}=2$, then $\mathrm{Hom}_A(A/\mathfrak{a}, K_A)$ is the canonical module of $A/\mathfrak{a}$ ([21, 5.14]). If $\dim A/\mathfrak{a}=1$, the Gorensteinness of formal fibres guarantees that $A/\mathfrak{a}$ has the canonical module (Proposition 2.4). If $\dim A/\mathfrak{a}=0$, it is obvious that $A/\mathfrak{a}$ has the canonical module.

(d)$\Rightarrow$(a): If $\mathfrak{u}=0$, the assertion follows from Corollary 2.2. Suppose $\mathfrak{u}\neq 0$ and let $\mathfrak{b}$ be an ideal such that $\mathfrak{u}\cap\mathfrak{b}=(0)$ and $\dim A/\mathfrak{b}\leqq 1$. Then $A/\mathfrak{u}$ is a homomorphic image of a Gorenstein local ring R by Corollary 2.2, and $A/\mathfrak{b}$ is a homomorphic image of a Gorenstein local ring S by Proposition 2.4 or the case of $\dim A/\mathfrak{b}=0$. We may assume $\dim R=\dim S=2$. Let f be the surjective ring homomorphism from $R\times S$ to $A/\mathfrak{u}\times A/\mathfrak{b}$ and $B=f^{-1}(A)$, the inverse image of the unit subring A of $A/\mathfrak{u}\times A/\mathfrak{b}$. We have the following commutative diagram of B-modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & R\oplus S & \longrightarrow & R\oplus S/B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow f & & \wr\!\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & A/\mathfrak{u}\oplus A/\mathfrak{b} & \longrightarrow & A/\mathfrak{u}+\mathfrak{b} & \longrightarrow & 0\,. \end{array}$$

Since $R\oplus S/B\cong A/\mathfrak{u}+\mathfrak{b}$ is a finitely generated B-module, $R\oplus S$ is finitely generated as a B-module. Hence B is a local ring with the maximal ideal $\mathfrak{n}=f^{-1}(\mathfrak{m})$. Since $A/\mathfrak{u}+\mathfrak{b}$ is a local ring and both $R\to A/\mathfrak{u}+\mathfrak{b}$ and $S\to A/\mathfrak{u}+\mathfrak{b}$ are surjective, B has a dualizing complex by virtue of (1.12). As $H_\mathfrak{n}^0(B)=0$ and $H_\mathfrak{n}^1(B)\cong H_\mathfrak{n}^0(R\oplus S/B)$ from the above exact sequcnce, B is a homomorphic image of a Gorenstein ring by Theorem 2.1, and therefore so is A. q. e. d.

Corollary 2.8. *Let $d=2$, and assume that A has the canonical module and* $\mathrm{Min}(A)=\mathrm{Assh}(A)$. *Then A is a homomorphic image of a Gorenstein ring.*

Proof. Let $\mathfrak{a}$ be any ideal ($\neq A$). If $\dim A/\mathfrak{a}=2$, then $\mathrm{Hom}_A(A/\mathfrak{a}, K_A)$ is the canonical module of $A/\mathfrak{a}$ ([21, 5.14]). Let $\dim A/\mathfrak{a}=1$. Then height $\mathfrak{a}=1$ by the assumption, hence $A/\mathfrak{a}$ has the canonical module by Corollary 2.5. If $\dim A/\mathfrak{a}=0$, it is obvious that $A/\mathfrak{a}$ has the canonical module. Hence A is a homomorphic image of a Gorenstein ring by Proposition 2.7 (d)⇒(a). q. e. d.

Remark 2.9. Let B be a one-dimensional local domain which does not have the canonical module (cf. Remark 2.6). Let $\mathfrak{n}$ be the maximal ideal of B and $k=B/\mathfrak{n}$. Put $C=B[\![X, Y]\!]/\mathfrak{p}\cap(X, Y)$ with indeterminates X, Y and $\mathfrak{p}=\mathfrak{n}B[\![X, Y]\!]$. Then $\dim C=2$, $U_C(0)=\mathfrak{p}/\mathfrak{p}\cap(X, Y)$ and $C/U_C(0)\cong k[\![X, Y]\!]$. Hence C has the canonical module by Corollary 2.2 or [3, 1.12], but does not have a dualizing complex by Proposition 2.7 as $C/\mathfrak{a}\cong B$ where $\mathfrak{a}=(X, Y)/\mathfrak{p}\cap(X, Y)$.

Remark 2.10. By virtue of [24, Appendix Example 2], there exists a two-dimensional local domain B which satisfies (i) the multiplicity of B is equal to 1, (ii) B is not regular, (iii) the derived normal ring C of B is regular, and (iv) $C/B\cong$the residue field of B as B-modules. If B has the canonical module, then B is unmixed by [3, 1.8] and therefore B is regular by (i) and [24, 40.6], which contradicts (ii). By (iii) and (iv), every formal fibre of B is Gorentein. To sum up, there is a two-dimensional local domain B such that every formal fibre of B is a Gorenstein ring but B does not have the canonical module.

3. The case of dimension $\leqq 4$.

In this section we first show that an affirmative answer to (SC) for (S_2) local rings implies one for general local rings. Then we give an affirmative answer for local rings of dimension $\leqq 4$. In the case of $d=3$, we show an analogous result to Propositions 2.4 and 7.

Lemma 3.1. *Assume that (SC) is affirmative for local rings of dimension $<n$ and (S_2) local rings of dimension n. Then (SC) is affirmative for local rings of dimension n.*

Proof. Let B be an n-dimensional local ring with a dualizing complex and suppose $\mathrm{Ass}(B)\neq\mathrm{Assh}(B)$. Then we have an expression $(0)=\mathfrak{a}\cap\mathfrak{b}$ with $\mathfrak{a}=U_B(0)$ and some ideal $\mathfrak{b}$ such that $\dim B/\mathfrak{b}<n$. Suppose that $B/\mathfrak{a}$ is a homomorphic image of a Gorenstein local ring R. As $\dim B/\mathfrak{b}<n$, $B/\mathfrak{b}$ is a homomorphic image of a Gorenstein local ring S by the assumption. We may assume $\dim R=\dim S=n$. Let f be the surjective ring homomorphism from $R\times S$ to $B/\mathfrak{a}\times B/\mathfrak{b}$ and $C=f^{-1}(B)$, the inverse image of the unit subring B of $B/\mathfrak{a}\times B/\mathfrak{b}$. Then by the same argument as in Proof of Proposition 2.7 (d)⇒(a), C is an n-dimensional local ring with a dualizing complex. B is a homomorphic image of C and

$\mathrm{Ass}(C)=\mathrm{Assh}(C)$. Now assume that the assertion is false. Then there is an n-dimensional local ring B such that B has a dualizing complex and is not a homomorphic image of a Gorenstein ring and $\mathrm{Ass}(B)=\mathrm{Assh}(B)$ by the argument above. By the assumption, B is not (S_2). Put $T(B)=\{\mathfrak{p}\in\mathrm{Spec}(B)\mid\mathrm{depth}\,B_{\mathfrak{p}}=1$ and $\dim B_{\mathfrak{p}}>1\}$. Then $T(B)$ is not empty. Let Z be the cokernel of the natural map $B\to\mathrm{Hom}_B(K_B, K_B)$. (Note that the kernel is (0) as $\mathrm{Ass}(B)=\mathrm{Assh}(B)$ (cf. [3, 1.8])) Let $\mathfrak{p}$ be in $T(B)$. As $\mathrm{depth}\,B_{\mathfrak{p}}=1$ and $\mathrm{depth}\,\mathrm{Hom}_B(K_B, K_B)\geqq 2$, $\mathrm{depth}\,Z_{\mathfrak{p}}=0$, whence $T(B)\subseteqq\mathrm{Ass}(Z)$. We put $s(B)=\max\{\dim B_{\mathfrak{p}}\mid\mathfrak{p}\in T(B)\}$, $T_0(B)=\{\mathfrak{p}\in T(B)\mid\dim B_{\mathfrak{p}}=s(B)\}$ and $T_1(B)=T(B)\backslash T_0(B)$. Consider all such local rings stated above, and take a local ring B from them whose $s(B)$ is the smallest. Put $s=s(B)$. If a local ring $(R, \mathfrak{n})$ with a dualizing complex has the property $\mathrm{Ass}(R)=\mathrm{Assh}(R)$ and $\dim R\geqq 2$, then $H^1_{\mathfrak{n}}(R)$ is of finite length. Hence there is a non zero divisor $x\in\bigcap_{\mathfrak{p}\in T_0(B)}\mathfrak{p}\backslash\bigcup_{\mathfrak{p}\in T_1(B)}\mathfrak{p}$ such that $xH^1_{\mathfrak{p}B_{\mathfrak{p}}}(B_{\mathfrak{p}})=0$ for every $\mathfrak{p}$ in $T_0(B)$. By the assumption, $B/U_B(xB)$ is a homomorphic image of an n-dimensional Gorenstein local ring G. Let C be the fibre product of the surjective ring homomorphisms $B\to B/U_B(xB)$ and $G\to B/U_B(xB)$. We have an exact sequence of C-modules $0\to C\to B\oplus G\to B/U_B(xB)\to 0$. By the same argument as in Proof of Proposition 2.7 (d)$\Rightarrow$(a), we have that C is an n-dimensional local ring with a dualizing complex. We have $\mathrm{Ass}(C)=\mathrm{Assh}(C)$. C is not a homomorphic image of a Gorenstein ring since B is a homomorphic image of C. Hence C is not (S_2) and $T(C)\neq\emptyset$. Put $t=s(C)$. We have $t\geqq s$ by the choice of B. Let $\mathfrak{P}$ be in $T_0(C)$. If $(B/U_B(xB))_{\mathfrak{P}}=0$, $C_{\mathfrak{P}}\cong B_{\mathfrak{P}}$ as $G_{\mathfrak{P}}$ is Gorenstein or trivial. Hence $\mathfrak{P}B$ is in $T_0(B)$ and $\mathfrak{P}B\ni x$, a contradiction. Hence we have $(B/U_B(xB))_{\mathfrak{P}}\neq 0$. Put $\dim(B/U_B(xB))_{\mathfrak{P}}=r$. Then $\dim C_{\mathfrak{P}}=\dim G_{\mathfrak{P}}=\dim B_{\mathfrak{P}}=r+1=t\geqq s>1$. From the exact sequence $0\to C_{\mathfrak{P}}\to B_{\mathfrak{P}}\oplus G_{\mathfrak{P}}\to(B/U_B(xB))_{\mathfrak{P}}\to 0$, we have $\mathrm{depth}\,B_{\mathfrak{P}}=1$ as $\mathrm{depth}\,C_{\mathfrak{P}}=1$, $\mathrm{depth}\,G_{\mathfrak{P}}=r+1>1$ and $\mathrm{depth}(B/U_B(xB))_{\mathfrak{P}}>0$. Therefore $\mathfrak{P}B\in T_0(B)$ and $t=s$. Hence $xH^1_{\mathfrak{P}B_{\mathfrak{P}}}(B_{\mathfrak{P}})=0$ and $H^1_{\mathfrak{P}B_{\mathfrak{P}}}(B_{\mathfrak{P}})\to H^1_{\mathfrak{P}B_{\mathfrak{P}}}(B_{\mathfrak{P}}/xB_{\mathfrak{P}})$ is injective. Suppose that $(U_B(xB)/xB)_{\mathfrak{P}}$ is not of finite length. Then there is a prime ideal $\mathfrak{Q}$ of C such that $\mathfrak{Q}\subsetneqq\mathfrak{P}$ and $(U_B(xB)/xB)_{\mathfrak{Q}}\neq 0$. $\mathfrak{Q}B\ni x$ is obvious. By the definition of $U_B(xB)$, there is a prime ideal $\mathfrak{q}$ of B such that $\mathfrak{q}B_{\mathfrak{Q}}\in\mathrm{Ass}(B_{\mathfrak{Q}}/xB_{\mathfrak{Q}})\backslash\mathrm{Assh}(B_{\mathfrak{Q}}/xB_{\mathfrak{Q}})$. Then we have $\mathrm{depth}\,B_{\mathfrak{q}}=1$ and $\dim B_{\mathfrak{q}}\leqq\dim B_{\mathfrak{Q}}<\dim B_{\mathfrak{P}}=s$, therefore $\mathfrak{q}\notin T(B)$ as $\mathfrak{q}\ni x$. Hence we have $\dim B_{\mathfrak{q}}=1$ and $\mathfrak{q}B_{\mathfrak{Q}}\in\mathrm{Assh}(B_{\mathfrak{Q}}/xB_{\mathfrak{Q}})$, a contradiction. Therefore $(U_B(xB)/xB)_{\mathfrak{P}}$ is of finite length and $H^1_{\mathfrak{P}B_{\mathfrak{P}}}((U_B(xB)/xB)_{\mathfrak{P}})=0$. Hence $H^1_{\mathfrak{P}B_{\mathfrak{P}}}(B_{\mathfrak{P}}/xB_{\mathfrak{P}})\to H^1_{\mathfrak{P}B_{\mathfrak{P}}}(B_{\mathfrak{P}}/U_B(xB)_{\mathfrak{P}})$ is injective. Therefore the composition map $H^1_{\mathfrak{P}B_{\mathfrak{P}}}(B_{\mathfrak{P}})\to H^1_{\mathfrak{P}B_{\mathfrak{P}}}(B_{\mathfrak{P}}/xB_{\mathfrak{P}})\to H^1_{\mathfrak{P}B_{\mathfrak{P}}}(B_{\mathfrak{P}}/U_B(xB)_{\mathfrak{P}})$ is injective. From the exact sequence $0=H^0_{\mathfrak{P}B_{\mathfrak{P}}}(B_{\mathfrak{P}}/U_B(xB)_{\mathfrak{P}})\to H^1_{\mathfrak{P}C_{\mathfrak{P}}}(C_{\mathfrak{P}})\to H^1_{\mathfrak{P}C_{\mathfrak{P}}}(B_{\mathfrak{P}}\oplus G_{\mathfrak{P}})\cong H^1_{\mathfrak{P}B_{\mathfrak{P}}}(B_{\mathfrak{P}})\to H^1_{\mathfrak{P}B_{\mathfrak{P}}}(B_{\mathfrak{P}}/U_B(xB)_{\mathfrak{P}})$, we have $H^1_{\mathfrak{P}C_{\mathfrak{P}}}(C_{\mathfrak{P}})=0$, which contradicts $\mathrm{depth}\,C_{\mathfrak{P}}=1$. Hence we obtain the assertion.

q. e. d.

Theorem 3.2. *If A has a dualizing complex and $d\leqq 4$, then A is a homomorphic image of a Gorenstein ring.*

Proof. If $d\leqq 2$, the assertion is already shown in section two. If $d=3$, the assertion follows from Corollary 2.3 and Lemma 3.1. Let $d=4$. We may assume that A is (S_2) by virtue of Lemma 3.1. Furthermore we assume that A is not (FLC) by virtue of Theorem 2.1. There is an ideal $\mathfrak{a}$ such that $\mathrm{Spec}(A)\backslash \mathrm{CM}(A) = \mathrm{V}(\mathfrak{a})$ by (1.9). Since A is (S_2) and not (FLC), height $\mathfrak{a}=3$ and $\mathrm{V}(\mathfrak{a})$ is a finite set. For every $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})\backslash\{\mathfrak{m}\}$, $A_{\mathfrak{p}}$ is a three-dimensional (S_2) local ring with a dualizing complex, therefore (FLC). Hence there is a non zero divisor $a\in\mathfrak{a}$ such that $aH^2_{\mathfrak{p}A_{\mathfrak{p}}}(A_{\mathfrak{p}})=0$ for every $\mathfrak{p}$ in $\mathrm{V}(\mathfrak{a})\backslash\{\mathfrak{m}\}$. Let $C=\mathrm{Hom}_{A/aA}(K_{A/aA}, K_{A/aA})$. As A is (S_2), $\mathrm{Ass}(A/aA)=\mathrm{Assh}(A/aA)$ and $A/aA\subseteqq C$ (cf. [3, 1.8]). Since every maximal chain of prime ideals in C has length three, C is an (S_2) semi-local ring ([3, 3.2]) and C has a dualizing complex, C is a homomorphic image of a Gorenstein ring G by (1.20) and Proof of Theorem 2.1 (b)⇒(a). We may assume that $\mathrm{Max}(G)=\{\mathfrak{p}\cap G\,|\,\mathfrak{p}\in\mathrm{Max}(C)\}$ and $\dim G_{\mathfrak{n}}=4$ for every $\mathfrak{n}$ in $\mathrm{Max}(G)$. Let B be the fibre product of the ring homomorphisms $A\to C$ and $G\twoheadrightarrow C$. We have an exact sequence of B-modules $0\to B\to A\oplus G\to C\to 0$. Since $B\to A$ is surjective and C is finitely generated as an A-module, C is finitely generated as a B-module. By the same argument as in Proof of Proposition 2.7 (d)⇒(a), it is known that B is a four-dimensional local ring with the maximal ideal $\mathfrak{m}\cap B$. Let $I^{\cdot}=0\to I^0 \to\cdots\to I^4\to 0$ and $J^{\cdot}=0\to J^0\to\cdots\to J^4\to 0$ be fundamental dualizing complexes of A and G, respectively (cf. [18, 3.6] and [28, 4.3]). Both $\mathrm{Hom}_A(C, I^{\cdot})$ and $\mathrm{Hom}_G(C, J^{\cdot})$ are fundamental dualizing complexes of C (cf. [8, Lemma 1]) and $\mathrm{Hom}_A(C, I^0)\cong\mathrm{Hom}_G(C, J^0)\cong 0$. Let $C=C_1\times\cdots\times C_t$ be a decomposition such that each $\mathrm{Spec}(C_i)$ is connected. Then $\mathrm{Hom}_A(C_i, I^{\cdot})\cong\mathrm{Hom}_G(C_i, J^{\cdot})$ as complexes for $i=1,\cdots,t$ by (1.8) and [18, 4.2] (note that C is a semi-local ring). Hence we have $\mathrm{Hom}_A(C, I^{\cdot})\cong\mathrm{Hom}_G(C, J^{\cdot})$ as complexes. Therefore B has a dualizing complex by (1.12). Let $\mathfrak{P}$ be a non maximal prime ideal of B. If $C_{\mathfrak{P}}=0$, $A_{\mathfrak{P}}$ is Cohen-Macaulay as $\mathfrak{P}A\not\ni a$ or $A_{\mathfrak{P}}=0$. Hence $B_{\mathfrak{P}}$ is Cohen-Macaulay as $B_{\mathfrak{P}}\cong A_{\mathfrak{P}}$ or $G_{\mathfrak{P}}$. Let $C_{\mathfrak{P}}\neq 0$ and put $\dim C_{\mathfrak{P}}=r$. Then $0\leqq r\leqq 2$ and $\dim B_{\mathfrak{P}}=\dim A_{\mathfrak{P}}=\dim G_{\mathfrak{P}}=r+1$. If $r=0$ or 1, from the exact sequence $0\to B_{\mathfrak{P}}\to A_{\mathfrak{P}}\oplus G_{\mathfrak{P}}\to C_{\mathfrak{P}}\to 0$, we have $B_{\mathfrak{P}}$ is Cohen-Macaulay as $\mathrm{depth}\,A_{\mathfrak{P}}=\mathrm{depth}\,G_{\mathfrak{P}}=r+1$ and $\mathrm{depth}\,C_{\mathfrak{P}}=r$. If $r=2$, we have $\mathrm{depth}\,B_{\mathfrak{P}}\geqq 2$ from the same exact sequence as $\mathrm{depth}\,A_{\mathfrak{P}}\geqq 2$, $\mathrm{depth}\,G_{\mathfrak{P}}=3$ and $\mathrm{depth}\,C_{\mathfrak{P}}>0$. As $aH^2_{\mathfrak{P}A_{\mathfrak{P}}}(A_{\mathfrak{P}})=0$, $H^2_{\mathfrak{P}A_{\mathfrak{P}}}(A_{\mathfrak{P}})\to H^2_{\mathfrak{P}A_{\mathfrak{P}}}(A_{\mathfrak{P}}/aA_{\mathfrak{P}})$ is injective. Since $\dim A_{\mathfrak{P}}/aA_{\mathfrak{P}}=2$, $\mathrm{Coker}(A_{\mathfrak{P}}/aA_{\mathfrak{P}}\to C_{\mathfrak{P}})$ is of finite length (cf. (1.3) and [4, 0.5.2]), therefore $H^2_{\mathfrak{P}B_{\mathfrak{P}}}(A_{\mathfrak{P}}/aA_{\mathfrak{P}})\to H^2_{\mathfrak{P}B_{\mathfrak{P}}}(C_{\mathfrak{P}})$ is injective. Hence the composition map $H^2_{\mathfrak{P}B_{\mathfrak{P}}}(A_{\mathfrak{P}})\to H^2_{\mathfrak{P}B_{\mathfrak{P}}}(A_{\mathfrak{P}}/aA_{\mathfrak{P}})\to H^2_{\mathfrak{P}B_{\mathfrak{P}}}(C_{\mathfrak{P}})$ is injective. Since C is (S_2), we have $\mathrm{depth}\,C_{\mathfrak{P}}=2$, hence $H^1_{\mathfrak{P}B_{\mathfrak{P}}}(C_{\mathfrak{P}})=0$. From the exact sequence $0=H^1_{\mathfrak{P}B_{\mathfrak{P}}}(C_{\mathfrak{P}})\to H^2_{\mathfrak{P}B_{\mathfrak{P}}}(B_{\mathfrak{P}}) \to H^2_{\mathfrak{P}B_{\mathfrak{P}}}(A_{\mathfrak{P}}\oplus G_{\mathfrak{P}})\cong H^2_{\mathfrak{P}B_{\mathfrak{P}}}(A_{\mathfrak{P}})\to H^2_{\mathfrak{P}B_{\mathfrak{P}}}(C_{\mathfrak{P}})$, we have $H^2_{\mathfrak{P}B_{\mathfrak{P}}}(B_{\mathfrak{P}})=0$ and $B_{\mathfrak{P}}$ is Cohen-Macaulay. Hence B is (FLC) as $\mathrm{Ass}(B)=\mathrm{Assh}(B)$. Therefore B is a homomorphic image of a Gorenstein ring by Theorem 2.1. As A is a homomorphic image of B, we obtain the assertion. q. e. d.

Proposition 3.3. *Let $d=3$. Assume that A has the canonical module and* $\mathrm{Min}(A)=\mathrm{Assh}(A)$. *Then A is a homomorphic image of a Gorenstein ring.*

Proof. It is sufficient to show that A has a dualizing complex by Theorem 3.2. By virtue of [8, Satz 2], it is sufficient to show that $A/\sqrt{(0)}$ has a dualizing complex. By the assumption, we have $\sqrt{(0)}=\sqrt{\mathfrak{u}}$. Hence it is sufficient to show that $A/\mathfrak{u}$ has a dualizing complex. Since K_A is the canonical module of $A/\mathfrak{u}$ ([3, 1.8]), we may assume $\mathfrak{u}=0$. If A is (FLC), the assertion follows from Theorem 2.1. Suppose A is not (FLC). Let $H=\mathrm{Hom}_A(K_A, K_A)$. By [3, 3.2], H is a semi-local ring which contains A and is finitely generated as an A-module, every maximal chain of prime ideals in H has length three, K_A is the canonical module of H, and H is (S_2). Hence, by the same argument as in Proof of Theorem 2.1 (c)⇒(b), H has a fundamental dualizing complex $I^{\cdot}=0\to I^0\to\cdots\to I^3\to 0$. Put $\mathfrak{c}=\{a\in A\mid aH\subseteq A\}$. Since $\mathfrak{u}=0$ and A is not (FLC), we have height $\mathfrak{c}=2$ (cf. [4, p. 26]). By Corollary 2.5, $A/\mathfrak{c}$ has a dualizing complex $J^{\cdot}$ such that $J^p=0$ for $p\neq 2, 3$. Both $\mathrm{Hom}_H(H/\mathfrak{c}, I^{\cdot})$ and $\mathrm{Hom}_{A/\mathfrak{c}}(H/\mathfrak{c}, J^{\cdot})$ are fundamental dualizing complexes of $H/\mathfrak{c}$ (cf. [8, Lemma 1]) and concentrate on degrees 2 and 3. By the same argument as in Proof of Theorem 3.2, we have $\mathrm{Hom}_H(H/\mathfrak{c}, I^{\cdot})\cong\mathrm{Hom}_{A/\mathfrak{c}}(H/\mathfrak{c}, J^{\cdot})$ as complexes. As $A\cong H\times_{H/\mathfrak{c}}A/\mathfrak{c}$ ([25, 3.2]), A has a dualizing complex by (1.12). q. e. d.

Corollary 3.4. *Let $d=3$. Then A has the canonical module if and only if $A/\mathfrak{u}$ is a homomorphic image of a Gorenstein ring.* (cf. [3, 1.12])

The following gives an analogous result to Propositions 2.4 and 7 in the case of $d=3$.

Corollary 3.5. *Let $d=3$. Then the following are equivalent:*

(a) *A is a homomorphic image of a Gorenstein ring.*
(b) *A has a dualizing complex.*
(c) *For every ideal $\mathfrak{a}$ $(\neq A)$, $A/\mathfrak{a}$ has the canonical module.*

Proof. It is sufficient to show (c)⇒(b) in the case of $\mathfrak{u}\neq 0$. We have an expression $(0)=\mathfrak{u}\cap\mathfrak{b}$ with some ideal $\mathfrak{b}$ such that $\dim A/\mathfrak{b}\leqq 2$. By Proposition 3.3, $A/\mathfrak{u}$ has a dualizing complex. Since every factor ring of $A/\mathfrak{b}$ has the canonical module by the assumption, $A/\mathfrak{b}$ has a dualizing complex by Proposition 2.7 or 4 or the case of $\dim A/\mathfrak{b}=0$. As $A\cong A/\mathfrak{u}\times_{A/\mathfrak{u}+\mathfrak{b}}A/\mathfrak{b}$ and $A/\mathfrak{u}+\mathfrak{b}$ is a local ring, A has a dualizing complex by (1.12). q. e. d.

Remark 3.6. Let B be a two-dimensional local domain such that every formal fibre of B is Gorenstein but B does not have the canonical module (cf. Remark 2.10). Let $\mathfrak{n}$ be the maximal ideal of B and $k=B/\mathfrak{n}$. Put $C=B[\![X, Y, Z]\!]/\mathfrak{p}\cap(X, Y, Z)$ with indeterminates X, Y, Z and $\mathfrak{p}=\mathfrak{n}B[\![X, Y, Z]\!]$. Then $\dim C=3$, $U_C(0)=\mathfrak{p}/\mathfrak{p}\cap(X, Y, Z)$ and $C/U_C(0)\cong k[\![X, Y, Z]\!]$. Hence C has the canonical module by Corollary 3.4 or [3, 1.12], and it is obvious that every formal fibre of C is Gorenstein, but C does not have a dualizing complex by Corollary 3.5 as $C/\mathfrak{a}\cong B$ where $\mathfrak{a}=(X, Y, Z)/\mathfrak{p}\cap(X, Y, Z)$.

Remark 3.7. By virtue of [25, §6. Example 2], there exists a four-dimensional factorial local domain B such that B has the canonical module but does not have a dualizing complex. Furhter it is shown that, for every non-zero ideal $\mathfrak{a}$ ($\neq B$, the local ring given in [25, §6. Example 2]), $B/\mathfrak{a}$ is a localization of a finitely generated algebra over a field. Hence every factor ring of B has the canonical module.

Remark 3.8. A local ring A has a dualizing complex if and only if every formal fibre of A is a Gorenstein ring and every factor ring of A has the canonical module. ([25, 5.5 or 6])

Remark 3.9. Ogoma proved that, if a ring R (not necessarily a local ring) has a dualizing complex and w. K. dim $R \leqq 2$ (see [33, p. 4] for the definition of w. K. dim), then R is a homomorphic image of a Gorenstein ring. ([33, 3.7])

4. A special case of (S_{d-2}).

To Theorem 4.10, we assume that the following four conditions are satisfied:

(a1) A has a fundamental dualizing complex $D^{\cdot}=0\to D^0\to \cdots \to D^d\to 0$.
(a2) $d\geqq 5$.
(a3) A is (S_{d-2}).
(a4) A is not (FLC).

Lemma 4.1. *$H^p(D^{\cdot})=0$ for $p>2$, $H^2(D^{\cdot})$ is of finite length, and* $\dim H^1(D^{\cdot})=1$.

Proof. We first recall $\mathrm{Hom}_A(H^p(D^{\cdot}), E)\cong H_{\mathfrak{m}}^{d-p}(A)$ for every p ((1.13)). As depth $A\geqq d-2$, we have $H^p(D^{\cdot})=0$ for $p>2$. By the condition (S_{d-2}), we have $H^2(D^{\cdot}_{\mathfrak{p}})=0$ for every $\mathfrak{p}$ in $\mathrm{Spec}(A)\backslash\{\mathfrak{m}\}$ and $H^1(D^{\cdot}_{\mathfrak{p}})=0$ for every $\mathfrak{p}\in\mathrm{Spec}(A)$ with $\dim A/\mathfrak{p}\geqq 2$ (note that $D^{\cdot}_{\mathfrak{p}}$ is a fundamental dualizing complex of $A_{\mathfrak{p}}$). Hence $H^2(D^{\cdot})$ is of finite length and $\dim H^1(D^{\cdot})=1$ as A is not (FLC). q. e. d.

Lemma 4.2. *$H_{\mathfrak{m}}^p(K_A)=0$ for $p\neq 2, 3, d$, $H_{\mathfrak{m}}^2(K_A)\cong H_{\mathfrak{m}}^0(H^1(D^{\cdot}))$, and $H_{\mathfrak{m}}^3(K_A)\neq 0$. Consequently* depth $K_A=2$ *or* 3, *and* depth $K_A=3\Leftrightarrow$depth $H^1(D^{\cdot})>0\Leftrightarrow H^1(D^{\cdot})$ *is Cohen-Macaulay.*

Proof. Recall $K_A\cong H^0(D^{\cdot})$. Let $B^p=\mathrm{Im}(D^{p-1}\to D^p)$, $Z^p=\mathrm{Ker}(D^p\to D^{p+1})$ and $H^p=H^p(D^{\cdot})=Z^p/B^p$. We have the following five exact sequences (note $H^p(D^{\cdot})=0$ for $p>2$):

$$0\longrightarrow K_A\longrightarrow D^0\longrightarrow B^1\longrightarrow 0 \qquad (1)$$

$$0\longrightarrow B^1\longrightarrow Z^1\longrightarrow H^1\longrightarrow 0 \qquad (2)$$

$$0\longrightarrow Z^1\longrightarrow D^1\longrightarrow B^2\longrightarrow 0 \qquad (3)$$

$$0\longrightarrow B^2\longrightarrow Z^2\longrightarrow H^2\longrightarrow 0 \qquad (4)$$

$$0\longrightarrow Z^2\longrightarrow D^2\longrightarrow \cdots\longrightarrow D^d\longrightarrow 0 \qquad (5)$$

As $D^0\cong \bigoplus_{\mathrm{ht}\,\mathfrak{p}=0} E_A(A/\mathfrak{p})$, we have $H_{\mathfrak{m}}^p(D^0)=0$ for every p, hence $H_{\mathfrak{m}}^p(K_A)\cong H_{\mathfrak{m}}^{p-1}(B^1)$

for every p from the exact sequence (1). As $D^p \cong \bigoplus_{\mathrm{ht}\,\mathfrak{p}=p} E_A(A/\mathfrak{p})$, we have $H^p_\mathfrak{m}(Z^2)=0$ for $p\neq d-2$ and $H^{d-2}_\mathfrak{m}(Z^2)\cong E$ from the exact sequence (5). Hence we have $H^p_\mathfrak{m}(B^2)=0$ for $p\neq 1, d-2$, $H^1_\mathfrak{m}(B^2)\cong H^2$ and $H^{d-2}_\mathfrak{m}(B^2)\cong E$ from the exact sequence (4) as H^2 is of finite length. From the exact sequence (3), we have $H^p_\mathfrak{m}(Z^1)=0$ for $p\neq 2, d-1$, $H^2_\mathfrak{m}(Z^1)\cong H^2$ and $H^{d-1}_\mathfrak{m}(Z^1)\cong E$ as $H^p_\mathfrak{m}(D^1)=0$ for every p. Hence, from the exact sequence (2), we have $H^p_\mathfrak{m}(B^1)=0$ for $p\neq 1, 2, d-1$, $H^1_\mathfrak{m}(B^1)\cong H^0_\mathfrak{m}(H^1)$, $H^2_\mathfrak{m}(B^1)\supsetneq H^1_\mathfrak{m}(H^1)\neq 0$ and $H^{d-1}_\mathfrak{m}(B^1)\cong E$ as $\dim H^1=1$. From these we obtain the assertion. q.e.d.

We put $\mathfrak{a}=\mathrm{ann}(H^1(D^\cdot))\cap\mathrm{ann}(H^2(D^\cdot))$. By Lemma 4.1, $\mathrm{V}(\mathfrak{a})=\mathrm{Spec}(A)\backslash\mathrm{CM}(A)$, $\mathfrak{a}H^p(D^\cdot)=0$ for $p\neq 0$, i.e., $\mathfrak{a}H^p_\mathfrak{m}(A)=0$ for $p\neq d$, height $\mathfrak{a}=d-1$, and $\mathrm{V}(\mathfrak{a})$ is a finite set. Let $x_1, \cdots, x_{d-1}$ be any subsystem of parameters for A contained in $\mathfrak{a}$ and $\mathfrak{b}=(x_1, \cdots, x_{d-1})A$ to Lemma 4.9. By our assumptions (cf. (1.4)), it is not difficult to see the following lemma.

Lemma 4.3. *$x_1, \cdots, x_{d-2}$ is an A-regular sequence.*

Lemma 4.4. *$(x_1, \cdots, x_{d-2}) : x^n_{d-1}=(x_1, \cdots, x_{d-2}) : x_{d-1}$ for every integer $n\geqq 2$ if $x_{d-1}\in\mathfrak{a}^2$ or* depth $A=d-1$.

Proof. Put $L=(x_1, \cdots, x_{d-2}) : x^n_{d-1}/(x_1, \cdots, x_{d-2})\subset A/(x_1, \cdots, x_{d-2})$, and let $\mathfrak{p}$ be in $\mathrm{V}(x_1, \cdots, x_{d-1})\backslash\{\mathfrak{m}\}$. If $\mathfrak{p}\not\supseteq\mathfrak{a}$, $A_\mathfrak{p}$ is Cohen-Macaulay and $L_\mathfrak{p}=0$. Let $\mathfrak{p}\supseteq\mathfrak{a}$. Then $\dim A_\mathfrak{p}=d-1$ and $A_\mathfrak{p}$ is (S_{d-2}). As $\mathfrak{a}H^1(D^\cdot)=0$ and $H^{d-2}_{\mathfrak{p}A_\mathfrak{p}}(A_\mathfrak{p})\cong \mathrm{Hom}_{A_\mathfrak{p}}(H^1(D^\cdot_\mathfrak{p}), E_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}))$, we have $(\mathfrak{a}L)_\mathfrak{p}=0$ by (1.21). Hence $\mathfrak{a}L$ is of finite length and $\mathfrak{a}L\subseteq H^0_\mathfrak{m}(A/(x_1, \cdots, x_{d-2}))\cong H^{d-2}_\mathfrak{m}(A)$ (recall $\mathfrak{a}H^p_\mathfrak{m}(A)=0$ for $p\neq d$ and Lemma 4.3). Since $\mathfrak{a}H^{d-2}_\mathfrak{m}(A)=0$ and $H^{d-2}_\mathfrak{m}(A)=0$ in the case of depth $A=d-1$, we have $x_{d-1}L=0$, i.e., $(x_1, \cdots, x_{d-2}) : x^n_{d-1}\subseteq(x_1, \cdots, x_{d-2}) : x_{d-1}$. The opposite inclusion is obvious. q.e.d.

Lemma 4.5. *For every integer $n>0$, $(x_1, \cdots, x_t)\cap\mathfrak{b}^n=(x_1, \cdots, x_t)\mathfrak{b}^{n-1}$ for $0\leqq t\leqq d-1$ and $((x_1, \cdots, x_{d-2}) : x_{d-1})\cap\mathfrak{b}^n=(x_1, \cdots, x_{d-2})\mathfrak{b}^{n-1}$ if $x_{d-1}\in\mathfrak{a}^2$ or* depth $A=d-1$.

Proof. $(x_1, \cdots, x_{d-1})\cap\mathfrak{b}^n=(x_1, \cdots, x_{d-1})\mathfrak{b}^{n-1}$ is trivial. Let $y\in((x_1, \cdots, x_{d-2}) : x_{d-1})\cap\mathfrak{b}^n$. Then $y=u+x^n_{d-1}w$ with $u\in(x_1, \cdots, x_{d-2})\mathfrak{b}^{n-1}$ and $w\in A$. As $x_{d-1}y\in(x_1, \cdots, x_{d-2})$, $w\in(x_1, \cdots, x_{d-2}) : x^{n+1}_{d-1}=(x_1, \cdots, x_{d-2}) : x_{d-1}$ by Lemma 4.4. Hence $x^n_{d-1}w\in(x_1, \cdots, x_{d-2})x^{n-1}_{d-1}$ and $y\in(x_1, \cdots, x_{d-2})\mathfrak{b}^{n-1}$. Therefore we have $((x_1, \cdots, x_{d-2}) : x_{d-1})\cap\mathfrak{b}^n=(x_1, \cdots, x_{d-2})\mathfrak{b}^{n-1}$ if $x_{d-1}\in\mathfrak{a}^2$ or depth $A=d-1$. We have $(x_1, \cdots, x_{d-2})\cap\mathfrak{b}^n\subseteq((x_1, \cdots, x_{d-2}) : x_{d-1})\cap\mathfrak{b}^n=(x_1, \cdots, x_{d-2})\mathfrak{b}^{n-1}\subseteq(x_1, \cdots, x_{d-2})\cap\mathfrak{b}^n$. Let $t<d-2$ and suppose $(x_1, \cdots, x_{t+1})\cap\mathfrak{b}^n=(x_1, \cdots, x_{t+1})\mathfrak{b}^{n-1}$. As $(x_1, \cdots, x_t)\cap\mathfrak{b}=(x_1, \cdots x_t)$ is obvious, let $n>1$ and suppose $(x_1, \cdots, x_t)\cap\mathfrak{b}^{n-1}=(x_1, \cdots, x_t)\mathfrak{b}^{n-2}$. Take $y\in(x_1, \cdots, x_t)\cap\mathfrak{b}^n$. Then $y\in(x_1, \cdots, x_{t+1})\cap\mathfrak{b}^n=(x_1, \cdots, x_{t+1})\mathfrak{b}^{n-1}$, hence $y=u+x_{t+1}w$ with $u\in(x_1, \cdots, x_t)\mathfrak{b}^{n-1}$ and $w\in\mathfrak{b}^{n-1}$. Since $x_1, \cdots, x_{t+1}$ is an A-regular sequence by Lemma 4.3, $w\in(x_1, \cdots, x_t)\cap\mathfrak{b}^{n-1}=(x_1, \cdots, x_t)\mathfrak{b}^{n-2}$. Hence $x_{t+1}w\in(x_1, \cdots, x_t)\mathfrak{b}^{n-1}$ and $y\in(x_1, \cdots, x_t)\mathfrak{b}^{n-1}$. $(x_1, \cdots, x_t)\cap\mathfrak{b}^n\supseteq(x_1, \cdots, x_t)\mathfrak{b}^{n-1}$

is obvious. q. e. d.

Lemma 4.6. *$x_1, x_2/x_1, \cdots, x_{d-1}/x_1$ is a regular sequence in $A[x_2/x_1, \cdots, x_{d-1}/x_1]$ if $x_1 \in \mathfrak{a}^2$ or* depth $A=d-1$.

Proof. Put $B=A[x_2/x_1, \cdots, x_{d-1}/x_1]$. First note that x_1 is not a zero divisor. Take $f \in (x_1, x_2/x_1, \cdots, x_t/x_1)B : x_{t+1}/x_1$ $(1 \leqq t \leqq d-2)$, and write $(x_{t+1}/x_1)f = x_1 f_1 + (x_2/x_1)f_2 + \cdots + (x_t/x_1)f_t$ with $f_1, \cdots, f_t \in B$. We can write $f=y/x_1^n$ and $f_i = y_i/x_1^n$ $(i=1, \cdots, t)$ with $y, y_i \in \mathfrak{b}^n$ and $n \gg 0$. Then $x_{t+1}y = x_1^2 y_1 + x_2 y_2 + \cdots + x_t y_t$. Hence $y_1 \in ((x_2, \cdots, x_{t+1}) : x_1^2) \cap \mathfrak{b}^n = (x_2, \cdots, x_{t+1})\mathfrak{b}^{n-1}$ by the preceding lemmas, and $y_1 = x_2 z_2 + \cdots + x_{t+1} z_{t+1}$ with $z_i \in \mathfrak{b}^{n-1}$. Therefore $x_{t+1}(y - x_1^2 z_{t+1}) = x_2(y_2 + x_1^2 z_2) + \cdots + x_t(y_t + x_1^2 z_t)$ and $y - x_1^2 z_{t+1} \in ((x_2, \cdots, x_t) : x_{t+1}) \cap \mathfrak{b}^n = (x_2, \cdots, x_t) \cap \mathfrak{b}^n = (x_2, \cdots, x_t)\mathfrak{b}^{n-1}$ by Lemmas 4.3 and 5. Hence $y = x_1^2 z_{t+1} + x_2 u_2 + \cdots + x_t u_t$ with $u_i \in \mathfrak{b}^{n-1}$ and $f = y/x_1^n = x_1(z_{t+1}/x_1^{n-1}) + (x_2/x_1)(u_2/x_1^{n-1}) + \cdots + (x_t/x_1)(u_t/x_1^{n-1})$.
q. e. d.

Lemma 4.7. (1) $U_A(x_1, \cdots, x_{d-2}) = (x_1, \cdots, x_{d-2}) : x_{d-1}$ *if $x_{d-1} \in \mathfrak{a}^2$ or* depth $A = d-1$.

(2) $\mathrm{Supp}_A(H_1(x_1, \cdots, x_{d-1}; A)) = \mathrm{V}(\mathfrak{a})$.

Proof. (1) Let $(x_1, \cdots, x_{d-2}) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ be a primary decomposition such that $\dim A/\mathfrak{q}_i = 2$ if and only if $i \leqq s$ $(1 \leqq s \leqq t)$, and $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. For $i \leqq s$, $\mathfrak{p}_i \not\ni x_{d-1}$. For $s < i \leqq t$, $\dim A_{\mathfrak{p}_i} > d-2 = \mathrm{depth}\, A_{\mathfrak{p}_i}$, hence $\mathfrak{p}_i \supseteqq \mathfrak{a} \ni x_{d-1}$. Take an integer $n>0$ such that $x_{d-1}^n \in \mathfrak{q}_i$ for $s < i \leqq t$. Then, by Lemma 4.4, $(x_1, \cdots, x_{d-2}) : x_{d-1} = (x_1, \cdots, x_{d-2}) : x_{d-1}^n = \bigcap_{i=1}^{t} (\mathfrak{q}_i : x_{d-1}^n) = \bigcap_{i=1}^{s} \mathfrak{q}_i = U_A(x_1, \cdots, x_{d-2})$.

(2) If $\mathfrak{p} \in \mathrm{Supp}(H_1(x_1, \cdots, x_{d-1}; A))$, $x_1, \cdots, x_{d-1}$ is a subsystem of parameters for $A_{\mathfrak{p}}$ and not an $A_{\mathfrak{p}}$-regular sequence. Hence $\mathfrak{p} \supseteqq \mathfrak{a}$. Let $\mathfrak{p}$ be a minimal prime ideal of $\mathfrak{a}$. Then $\dim A_{\mathfrak{p}} = d-1$ and $x_1, \cdots, x_{d-1}$ is a system of parameters for $A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}$ is not Cohen-Macaulay, $x_1, \cdots, x_{d-1}$ is not an $A_{\mathfrak{p}}$-regular sequence and therefore $H_1(x_1, \cdots, x_{d-1}; A)_{\mathfrak{p}} \neq 0$. q. e. d.

Lemma 4.8. *Suppose* depth $A = d-1$. *Then* depth $K_A = 3$ *if and only if $A/U_A(x_1, \cdots, x_{d-2})$ is Cohen-Macaulay.*

Proof. We put $U = U_A(x_1, \cdots, x_{d-2})/(x_1, \cdots, x_{d-2})$, $B = A/(x_1, \cdots, x_{d-2})$ and $C = A/U_A(x_1, \cdots, x_{d-2})$. We have an exact sequence $0 \to U \to B \to C \to 0$. Since $U = (x_1, \cdots, x_{d-2}) : x_{d-1}/(x_1, \cdots, x_{d-2}) \cong H_1(x_1, \cdots, x_{d-2}, x_{d-1}; A)$ (Lemmas 4.3 and 7), we have $\dim U = 1$. As depth $B = 1$, U is Cohen-Macaulay. As $\dim C = 2$ and $U_C(0) = 0$, C is (FLC) (note that C has a dualizing complex). We have an exact sequence $0 \to D_A^1(C) \to D_A^1(B) \to D_A^1(U) \to 0$. $D_A^1(U)$ is a Cohen-Macaulay module of dimension one as $\dim U = 1$. As $H_{\mathfrak{m}}^1(B) \cong H_{\mathfrak{m}}^{d-1}(A)$, we have $D_A^1(B) \cong H^1(D^{\cdot}) \otimes_A \hat{A}$. Since $D_A^1(C)$ is of finite length, we obtain depth $K_A = 3 \Leftrightarrow$ depth $H^1(D^{\cdot}) > 0 \Leftrightarrow$ depth $D_A^1(B) > 0 \Leftrightarrow D_A^1(C) = 0 \Leftrightarrow C$ is Cohen-Macaulay. q. e. d.

Corollary 4.9. (1) *If* depth $A = d-1$ *and* depth $K_A = 3$, *then* $A/(x_1, \cdots, x_{d-1})$

is Cohen-Macaulay.

(2) *If* $A/(x_1, \cdots, x_{d-1})$ *is Cohen-Macaulay and* $x_i \in \mathfrak{a}^2$ *for some* i, *then* depth $A = d-1$ *and* depth $K_A = 3$.

Proof. We put $U = U_A(x_2, \cdots, x_{d-1})/(x_2, \cdots, x_{d-1})$, $B = A/(x_2, \cdots, x_{d-1})$ and $C = A/U_A(x_2, \cdots, x_{d-1})$. In (2) we may assume $x_1 \in \mathfrak{a}^2$. By Lemma 4.7 (1), $U_A(x_2, \cdots, x_{d-1}) = (x_2, \cdots, x_{d-1}) : x_1$. We have an exact sequence $0 \to U \to B/x_1B \to C/x_1C \to 0$ from the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow x_1 & & \downarrow x_1 & & \downarrow x_1 & & \\ 0 & \longrightarrow & U & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

(1) By Lemma 4.8, depth $C/x_1C > 0$. As depth $B > 0$, we have depth $U > 0$. Hence we have depth $B/x_1B > 0$.

(2) As depth $B/x_1B > 0$, we have depth $U > 0$. As depth $C > 0$, we have depth $B > 0$, i.e., depth $A = d-1$. As $H^0_\mathfrak{m}(C) = 0$, $H^1_\mathfrak{m}(U) \to H^1_\mathfrak{m}(B)$ is injective. Since $H^1_\mathfrak{m}(B) \cong H^{d-1}_\mathfrak{m}(A)$ and $x_1 H^{d-1}_\mathfrak{m}(A) = 0$, $H^1_\mathfrak{m}(B) \to H^1_\mathfrak{m}(B/x_1B)$ is injective. Hence, from the exact sequence $0 = H^0_\mathfrak{m}(B/x_1B) \to H^0_\mathfrak{m}(C/x_1C) \to H^1_\mathfrak{m}(U) \to H^1_\mathfrak{m}(B/x_1B)$, we have $H^0_\mathfrak{m}(C/x_1C) = 0$, which shows depth $C > 1$. By Lemma 4.8, we have depth $K_A = 3$.

q. e. d.

Let $a_1, \cdots, a_{d-1}$ be a subsystem of parameters for A contained in $\mathfrak{a}$ and $\mathfrak{q} = (a_1, \cdots, a_{d-1})A$. We put $R = R(A, \mathfrak{q}^n)$ with $n \geqq d-2$ and $\mathfrak{N} = \mathfrak{m}R + R_+$. Then we have the following

Theorem 4.10. *If* depth $A = d-1$ *and* depth $K_A = 3$, *then* $H^p_\mathfrak{N}(R)$ *is finitely generated for* $p \neq d+1$.

Proof. Let k be an algebraic closure of the residue field $A/\mathfrak{m}$. By virtue of [16, 0.10.3.1], there exists a flat local A-algebra B such that $B/\mathfrak{m}B \cong k$. By (1.10), B has a dualizing complex $I^\cdot$ with a quasi-isomorphism $D^\cdot \otimes_A B \to I^\cdot$ and $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is Gorenstein for every $\mathfrak{p}$ in Spec(A). Hence B also satisfies (a1), (a2), (a3) and (a4), depth $B = d-1$, depth $K_B = 3$ ($K_B \cong K_A \otimes_A B$), and $\mathfrak{a}B = \mathrm{ann}(H^1(I^\cdot)) \cap \mathrm{ann}(H^2(I^\cdot))$. Since $R(B, \mathfrak{q}^nB) \cong R \otimes_A B$ is faithfully flat over R, it is sufficient to show the theorem for $R(B, \mathfrak{q}^nB)$. Hence we may assume that $A/\mathfrak{m}$ is algebraically closed. Since R has a dualizing complex and dim $R/\mathfrak{p} = d+1$ for every $\mathfrak{p}$ in Min(R) (cf. [32, §1]), it is sufficient to show that $R_\mathfrak{P}$ is Cohen-Macaulay for every homogeneous prime ideal $\mathfrak{P} \neq \mathfrak{N}$. Put $\mathfrak{p} = \mathfrak{P} \cap A$. First suppose $\mathfrak{p} \neq \mathfrak{m}$. If $\mathfrak{p} \supseteq \mathfrak{q}$, $R_\mathfrak{p} \cong R(A_\mathfrak{p}, \mathfrak{q}^n A_\mathfrak{p})$ is Cohen-Macaulay by (1.19) (cf. (1.21)). If $\mathfrak{p} \not\supseteq \mathfrak{q}$, $R_\mathfrak{p} \cong A_\mathfrak{p}[X]$ and $A_\mathfrak{p}$ is Cohen-Macaulay. Now let $\mathfrak{p} = \mathfrak{m}$. As $\mathfrak{q}^{n(d-1)} = (a_1^n, \cdots, a_{d-1}^n)\mathfrak{q}^{n(d-2)}$ and $\mathfrak{P} \not\supseteq R_+$, $a_i^n X \notin \mathfrak{P}$ for some i. We may assume $a_1^n X \notin \mathfrak{P}$. Put $t = a_1^n X$, $S = R[1/t]$, $B = S_0$ and $\mathfrak{Q} = \mathfrak{P}S \cap B$ ($\supseteq \mathfrak{m}B$). Since $S = B[t, 1/t]$ and t is algebraically independent over B, $S_{\mathfrak{P}S}$ is Cohen-Macaulay if and only if so is $B_\mathfrak{Q}$. Hence it is sufficient to show that $B_\mathfrak{M}$ is Cohen-Macaulay for every maximal ideal $\mathfrak{M}$ of B containing $\mathfrak{m}B$. $B=S_0=A[x/a_1^n \mid x\in\mathfrak{q}^n]=A[x/a_1 \mid x\in\mathfrak{q}]$ $=A[a_2/a_1, \cdots, a_{d-1}/a_1]\subset A[1/a_1]$. Because $A/\mathfrak{m}$ is algebraically closed, $\mathfrak{M}=$ $\mathfrak{m}B+((a_2/a_1)-c_2, \cdots, (a_{d-1}/a_1)-c_{d-1})B$ with $c_i\in A$. Put $b_1=a_1$ and $b_i=a_i-c_ia_1$ for $i=2, \cdots, d-1$. Then $(b_1, \cdots, b_{d-1})A=\mathfrak{q}$. By Lemma 4.6, $b_1, b_2/b_1, \cdots, b_{d-1}/b_1$ is a B-regular sequence. It is obvious that the natural map $A\to B/(b_1, b_2/b_1, \cdots,$ $b_{d-1}/b_1)B$ is surjective. Let $x\in U_A(b_2, \cdots, b_{d-1})$. By Lemma 4.7 (1), $b_1x=b_2x_2$ $+\cdots+b_{d-1}x_{d-1}$ with $x_i\in A$. Hence $x=(b_2/b_1)x_2+\cdots+(b_{d-1}/b_1)x_{d-1}$ in B and $b_1A+U_A(b_2, \cdots, b_{d-1})\subseteq(b_1, b_2/b_1, \cdots, b_{d-1}/b_1)B$. Take $f\in A\cap(b_1, b_2/b_1, \cdots, b_{d-1}/$ $b_1)B$. Then we can write $f=b_1f_1+(b_2/b_1)f_2+\cdots+(b_{d-1}/b_1)f_{d-1}$ with $f_i=x_i/b_1^e$, $x_i\in\mathfrak{q}^e$ and $e\gg 0$. Then $b_1^{e+1}f=b_1^2x_1+b_2x_2+\cdots+b_{d-1}x_{d-1}$ in A. As $\mathfrak{q}^e=b_1^eA+$ $(b_2, \cdots, b_{d-1})\mathfrak{q}^{e-1}$, $x_1=b_1^ey+b_2y_2+\cdots+b_{d-1}y_{d-1}$ with $y\in A$ and $y_i\in\mathfrak{q}^{e-1}$. Hence $b_1^{e+1}f=b_1^{e+2}y+b_2(x_2+b_1^2y_2)+\cdots+b_{d-1}(x_{d-1}+b_1^2y_{d-1})$ and $f-b_1y\in(b_2, \cdots, b_{d-1})$: $b_1^{e+1}=U_A(b_2, \cdots, b_{d-1})$ by Lemmas 4.4 and 7. Hence we have $A\cap(b_1, b_2/b_1, \cdots,$ $b_{d-1}/b_1)B=b_1A+U_A(b_2, \cdots, b_{d-1})$ and therefore $B/(b_1, b_2/b_1, \cdots, b_{d-1}/b_1)B\cong A/b_1A$ $+U_A(b_2, \cdots, b_{d-1})$. By Lemma 4.8, $A/U_A(b_2, \cdots, b_{d-1})$ is Cohen-Macaulay, therefore so are $B/(b_1, b_2/b_1, \cdots, b_{d-1}/b_1)$ and $B_{\mathfrak{M}}$. q. e. d.

Now we can give an affirmative answer to (SC) for local rings of a special type as following.

Theorem 4.11. *Assume that A has a dualizing complex, A is (S_{d-2}),* depth $A\geqq$ $d-1$ *and* depth $K_A\geqq 3$. *Then A is a homomorphic image of a Gorenstein ring.*

Proof. We may assume $d\geqq 5$ and depth $A=d-1$ (cf. Theorem 3.2 and (1.11)). In this case A is not (FLC) as depth $K_A\geqq 3$ (cf. [2, Lemma 1] or (1.16)). Hence A satisfies (a1), (a2), (a3) and (a4). Then there is an ideal $\mathfrak{a}$ such that $H^p_{\mathfrak{N}}(R)$ is finitely generated for $p\neq d+1$ where $R=R(A, \mathfrak{a})$ and $\mathfrak{N}=\mathfrak{m}R+R_+$ by virtue of Theorem 4.10. $R_{\mathfrak{N}}$ is an (FLC) local ring which has a dualizing complex. By Theorem 2.1, $R_{\mathfrak{N}}$ is a homomorphic image of a Gorenstein ring, therefore so is A as $A\cong R_{\mathfrak{N}}/(R_+)_{\mathfrak{N}}$. q. e. d.

A local ring is said to be quasi-Gorenstein if it has a free canonical module (Platte and Storch).

Remark 4.12. Suppose that $d\geqq 6$, A is (S_{d-2}) and A has a dualizing complex. If A is quasi-Gorenstein, then A is Gorenstein.

Proof. It is sufficient to show that $\hat{A}$ is Cohen-Macaulay, hence we may assume that A is complete. (Hence it is sufficient to assume that every formal fibre of A is (S_{d-2}) instead of assuming A has a dualizing complex.) Suppose that A is not Cohen-Macaulay. Since $\dim D_A^{d-1}(A)\leqq 1$ by Proof of Lemma 4.1, we have depth $K_A\leqq 3$ (cf. [2, Lemma 1]). On the other hand depth $K_A\geqq d-2\geqq 4$ as $K_A\cong A$. q. e. d.

Remark 4.13. Suppose that $d=5$, A is (S_3) and A has a dualizing complex. If A is quasi-Gorenstein, then A is (FLC), consequently a homomorphic image of a Gorenstein ring. A itself is not necessarily a Gorenstein ring.

Proof. Suppose that A is not (FLC). In our case A is (FLC) if and only if $A_{\mathfrak{p}}$ is Cohen-Macaulay for every $\mathfrak{p}$ in $\mathrm{Spec}(A)\backslash\{\mathfrak{m}\}$ (cf. (1.17)). Therefore there is a prime ideal $\mathfrak{p}$ such that $\dim A_{\mathfrak{p}}=4$ and depth $A_{\mathfrak{p}}=3$. As $A_{\mathfrak{p}}$ is (S_3), $A_{\mathfrak{p}}$ is (FLC) and depth $K_{A_{\mathfrak{p}}}=2$ (cf. [2, Lemma 1] or (1.16)). On the other hand depth $K_{A_{\mathfrak{p}}}=3$ as $K_{A_{\mathfrak{p}}}\cong A_{\mathfrak{p}}$. By virtue of [11, 1.1], there is a five-dimensional complete local ring $(B, \mathfrak{n})$ such that $H_{\mathfrak{n}}^{3}(B)\cong B/\mathfrak{n}$ and $H_{\mathfrak{n}}^{p}(B)=0$ for $p\neq 3, 5$. Let $A=B\ltimes K_B$, the idealization. Then A satisfies the assumption (cf. (1.16) and [3, 2.11]), but A is not Gorenstein. q. e. d.

Remark 4.14. Let n, s and t be integers such that $n\geqq 5$, $n-2\leqq s<n$ and $2\leqq t\leqq 3$. We construct a local ring B such that $\dim B=n$, depth $B=s$, B is (S_{n-2}), S is not (FLC), B has a dualizing complex and depth $K_B=t$.

Let $(R, \mathfrak{p})$ be an n-dimensional regular local ring, $a_1, \cdots, a_{n-1}$ a subsystem of parameters for R, $\mathfrak{a}=(a_1, \cdots, a_{n-1})R$ and L a non-zero R-module of finite length. Let $0\to X_{n-1}\to\cdots\to X_0\to R/\mathfrak{a}\to 0$, $0\to Y_n\to\cdots\to Y_0\to L\to 0$ and $0\to Z_n\to\cdots\to Z_0\to R/\mathfrak{p}\to 0$ be minimal free resolutions of $R/\mathfrak{a}$, L and $R/\mathfrak{p}$, respectively. Put $X=\mathrm{Ker}(X_{n-3}\to X_{n-4})$, $Y=\mathrm{Ker}(Y_{n-3}\to Y_{n-4})$ and $Z=\mathrm{Ker}(Z_{n-2}\to Z_{n-3})$. Then we have: $H_{\mathfrak{p}}^{p}(X)=0$ for $p\neq n-1$, n and $H_{\mathfrak{p}}^{n-1}(X)\cong H_{\mathfrak{p}}^{1}(R/\mathfrak{a})$; $H_{\mathfrak{p}}^{p}(Y)=0$ for $p\neq n-2$, n and $H_{\mathfrak{p}}^{n-2}(Y)\cong L$; $H_{\mathfrak{p}}^{p}(Z)=0$ for $p\neq n-1$, n and $H_{\mathfrak{p}}^{n-1}(Z)\cong R/\mathfrak{p}$. Let $\mathfrak{q}$ be in $\mathrm{Spec}(R)\backslash\{\mathfrak{p}\}$. Then $Y_{\mathfrak{q}}$ and $Z_{\mathfrak{q}}$ are free. If $\mathfrak{q}\not\supseteq\mathfrak{a}$, $X_{\mathfrak{q}}$ is free. If $\mathfrak{q}\supseteq\mathfrak{a}$, $(R/\mathfrak{a})_{\mathfrak{q}}$ is of finite length and $H_{\mathfrak{q}R_{\mathfrak{q}}}^{p}(X_{\mathfrak{q}})=0$ for $p\neq n-2$, $n-1$ and $H_{\mathfrak{q}R_{\mathfrak{q}}}^{n-2}(X_{\mathfrak{q}})=(R/\mathfrak{a})_{\mathfrak{q}}$. Let $B=R\ltimes X$ if $(s, t)=(n-1, 3)$, $B=R\ltimes(X\oplus Y)$ if $(s, t)=(n-2, 3)$, $B=R\ltimes(X\oplus Z)$ if $(s, t)=(n-1, 2)$, and $B=R\ltimes(X\oplus Y\oplus Z)$ if $(s, t)=(n-2, 2)$. Then it is not difficult to see that B is the required example.

Appendix.

In this appendix we give an alternative proof of [3, 4.2] which is an important theorem in the theory of canonical modules.

In the following let B be a faithfully flat local A-algebra with the canonical module K_B.

Lemma 5.1. *Suppose that there is an A-module T such that $T\otimes_A B\cong K_B$ and* $\dim B/\mathfrak{m}B=0$. *Then $B/\mathfrak{m}B$ is a Gorenstein ring.*

Proof. We may assume that both A and B are complete. Let $\mathfrak{p}$ be a prime ideal of A such that $\dim A/\mathfrak{p}=d$. Then $\mathrm{Hom}_A(A/\mathfrak{p}, T)\otimes_A B\cong\mathrm{Hom}_B(B/\mathfrak{p}B, T\otimes_A B)\cong\mathrm{Hom}_B(B/\mathfrak{p}B, K_B)$ is the canonical module of $B/\mathfrak{p}B$ ([21, 5.14]). Hence we may further assume that A is a domain, considering $A/\mathfrak{p}$, $B/\mathfrak{p}B$ and $\mathrm{Hom}_A(A/\mathfrak{p}, T)$. Let R be the derived normal ring of A and $S=R\otimes_A B$. R is a finite local A-algebra. Put $I=\mathrm{Hom}_A(R, T)$ and $J=\mathrm{Hom}_B(S, K_B)$. As $I\otimes_A B\cong J$, $\mathrm{Hom}_S(J, J)\cong\mathrm{Hom}_R(I, I)\otimes_A B$. I is isomorphic to an ideal of R because I is torsionfree and $\mathrm{rank}_A T=1$. Therefore $\mathrm{Hom}_R(I, I)$ is contained in the quotient field of A. Since R is normal, $\mathrm{Hom}_R(I, I)=R$ and $\mathrm{Hom}_S(J, J)\cong S$. Let $\mathfrak{M}$ be the maximal ideal of R and $\mathfrak{N}$ a maximal ideal of S. Then $J_{\mathfrak{N}}$ is the canonical module of $S_{\mathfrak{N}}$ ([21, 5.12]) and $S_{\mathfrak{N}}\cong\mathrm{Hom}_{S_{\mathfrak{N}}}(J_{\mathfrak{N}}, J_{\mathfrak{N}})$. Hence $H^d_{\mathfrak{M}}(I)\otimes_R S_{\mathfrak{N}}\cong H^d_{\mathfrak{N}S_{\mathfrak{N}}}(J_{\mathfrak{N}})\cong E_{S_{\mathfrak{N}}}(S_{\mathfrak{N}}/\mathfrak{N}S_{\mathfrak{N}})$ by the definition of the canonical module. Since $R/\mathfrak{M}$ is contained in $H^d_{\mathfrak{M}}(I)$, $S_{\mathfrak{N}}/\mathfrak{M}S_{\mathfrak{N}}$ is contained in $E_{S_{\mathfrak{N}}}(S_{\mathfrak{N}}/\mathfrak{N}S_{\mathfrak{N}})$. Therefore $\mathrm{Hom}_{S_{\mathfrak{N}}}(S_{\mathfrak{N}}/\mathfrak{N}S_{\mathfrak{N}}, S_{\mathfrak{N}}/\mathfrak{M}S_{\mathfrak{N}})\cong S_{\mathfrak{N}}/\mathfrak{N}S_{\mathfrak{N}}$, that is, $S_{\mathfrak{N}}/\mathfrak{M}S_{\mathfrak{N}}$ is Gorenstein. As $B/\mathfrak{m}B\otimes_{A/\mathfrak{m}}R/\mathfrak{M}\cong S/\mathfrak{M}S$, we have the assertion. q. e. d.

Theorem 5.2 ([3, 4.2]). *Let T be an A-module. If $T\otimes_A B\cong K_B$, then A has the canonical module and T is the canonical module of A.*

Proof. First we note that T is a finitely generated A-module. It is sufficient to show that $\hat{T}$ is the canonical module of $\hat{A}$. Hence we may assume that both A and B are complete. Let $\mathfrak{q}$ be a minimal prime ideal of $\mathfrak{m}B$. Then $T\otimes_A B_{\mathfrak{q}}\cong(K_B)_{\mathfrak{q}}$ is the canonical module of $B_{\mathfrak{q}}$ by [21, 5.22] as B is complete (cf. [3, 1.9]). Hence we may further assume $\dim B/\mathfrak{m}B=0$ (considering $\widehat{B_{\mathfrak{q}}}$). Then $B/\mathfrak{m}B$ is Gorenstein by Lemma 5.1. By [3, 4.1], $K_A\otimes_A B\cong K_B$. Hence we have $T\cong K_A$ by virtue of [17, IV. 2.5.8]. q. e. d.

Corollary 5.3 ([3, 4.3]). *Suppose that A has the canonical module K, and let $\mathfrak{p}$ be in* $\mathrm{Supp}_A(K)$. *Then $K_{\mathfrak{p}}$ is the canonical module of $A_{\mathfrak{p}}$.*

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