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Lyapunov-like functions and geodesic flows

By

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Introduction.

As is well known, Anosov systems (or hyperbolic systems) play important roles in the theory of dynamical systems. Once a given system is proved to be Anosov, one knows that it is both structurally stable and topologically stable, and it has many ergodic properties. (See, for instance, Anosov [1] and Walters [19].) Moreover small deviation from Anosov systems gives wider classes of interesting systems. Therefore, in this paper, we are interested mainly in obtaining good criteria for a given system to be Anosov.

Here, we shall give general comments for Anosov systems. Already at the starting point of the investigation, Anosov systems were considered analogous to hyperbolic ordinary linear differential equations with constant coefficients. We explain this more exactly. Let A be an n-square matrix over $\boldsymbol{R}$ and let us consider an ordinary differential equation: $\mathrm{d}x/\mathrm{d}t=Ax$. The fixed point $x=0$ is called hyperbolic if there exists an invariant hyperbolic splitting of $\boldsymbol{R}^n : \boldsymbol{R}^n = E^s \oplus E^u$. This corresponds to the definition of Anosov systems. There are several equivalent criteria for this as follows:

(A) The real parts of every eigenvalues of A are non-zero.

(B) There exists no non-zero bounded orbit.

(C) There exists a non-degenerate quadratic form $Q(x)$ such that $(\mathrm{d}/\mathrm{d}t)Q(x) > 0$ for any non-zero solution x.

The criterion (A) corresponds to the spectral theory of vector bundle systems. This point of view goes bach at least to the work of Mather [13], and important results have been obtained by many other authors. (See Chicone-Swanson [3], [4], Churchill-Franke-Selgrade [5], Hirsch-Pugh-Shub [8] and Ôtsuki [14], [15].) The criterion (B) corresponds to the theory of quasi-hyperbolic systems. We cite the contributions of Mañé [12], Sacker-Sell [16] and Selgrade [18] which are especially important for our present work. The criterion (C) corresponds to the work of Lewowicz [10], [11]. He proved that the hyperbolicity of a vector bundle system is equivalent to the existence of a non-degenerate quadratic form with adequate properties.

In the present article, we prove first a proposition which gives a criterion for vector bundle homeomorphisms to be quasi-hyperbolic. By applying this result to geodesic flows, we give a sufficient condition for geodesic flows to be

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Anosov. This criterion may be applied to Riemannian manifolds with small paches of small positive curvatures. Our method enables us also to obtain a generalization of Chicone'criterion in [2] for geodesic flows to be Aonsov.

The contents of this paper is devided into five sections. In § 1, we define notations of vector bundle systems and prove Theorem 1 which gives a criterion for vector bundle homeomorphisms to be quasi-hyperbolic. In § 2, applying Theorem 1 to diffeomorphisms, we get Proposition 2 which states that the quasi-hyperbolicity is stable in a certain sense.

In § 3, we prove one of our main results (Theorem 4) in this paper, which gives a sufficient condition for geodesic flows to be Anosov. The key point of the proof of this theoerem lies in finding an appropriate quadratic form used in Theorem 1 for geodesic flows. This theorem may be meaningful in geometry; that is, Theorem 4 seems to us to suggest relations between geodesic flows of Anosov type and indices of geodesic curves. It may be also interesting to compare Theorem 4 with the work of Eberlein [7]. As a consequence of Theorem 4, Proposition 5 is obtained in § 4. In the 2-dimensional case, this proposition is known (Lewowicz [11]), but it seems to us that it is new for general case. In this section, we define "asymptotic curvature" and prove some result (Proposition 6). We shall not discuss it in detail in this paper.

In § 5, we prove Theorem 9 which gives a generalization of Chicone's criterion. This theorem is another main result in this paper. Our proof is different from Chicone's one, and is based on two key ideas, that is, on Lemma 8 which is suggested by our Theorem 1 and on finding the quadratic form in L^2-setting which is analogous to that used in Theorem 4.

In this paper, we denote the sets of integers, real numbers and complex numbers by $\boldsymbol{Z}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ respectively.

§ 1. Lyaupnov-like functions and quasi-hyperbolicity of vector bundle homeomorphisms.

We begin with notations of vector bundle systems. Let M be a compact metric space and E a real or complex vector bundle on M with an inner product. We call a pair (Φ, ϕ) a *vector bundle homeomorphism* if ϕ is a homeomorphism on M and Φ is a bundle automorphism which intertwines with ϕ. A vector bundle homeomorphism (Φ, ϕ) is called *quasi-hyperbolic* if there exists no non-zero $v \in E$ such that $\{\|\Phi^n v\| : n \in \boldsymbol{Z}\}$ is bounded. Here $\|\cdot\|$ denotes the norm induced from the inner product on E. Further (Φ, ϕ) is called *hyperbolic* if there exists a Φ-invariant hyperbolic splitting $E = E^s \oplus E^u$, where E^s and E^u are the stable and unstable subbundles of E respectively, namely there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|\Phi^n v\| \leqq C\lambda^n \|v\| \quad \text{for} \quad v \in E^s,\ n \geqq 0$$

and

$$\|\Phi^{-n} v\| \leqq C\lambda^n \|v\| \quad \text{for} \quad v \in E^u,\ n \geqq 0.$$

A real valued function Q on E is called a *quadratic form* on E if Q is continuous and $Q_x \equiv Q|E_x$ is a quadratic form on the fibre space E_x for every $x \in M$.

The following theorem is essentially due to J. Lewowicz, but we prove it here for convenience of readers.

Theorem 1. *A vector bundle homeomorphism* (Φ, ϕ) *is quasi-hyperbolic if and only if there exists a quadratic form* Q *on* E *such that* $Q(\Phi v)-Q(v)>0$ *for every non-zero* $v \in E$.

Proof. (i) Necessity. If (Φ, ϕ) is quasi-hyperbolic, then there exists a positive integer N such that for every non-zero $v \in E$, there exists at least an integer n, $|n| \leqq N$, for which $\|\Phi^n v\| > 2\|v\|$ holds. We prove this by contradiction. Assume, for every positive integer n, there exists $v_n \in E$ with $\|v_n\|=1$ uch that

$$\max\{\|\Phi^m v_n\| : |m| \leqq n\} \leqq 2.$$

Then by taking a subsequence of $\{v_n\}$, if necessary, we may assume that $\{v_n\}$ converges to some $v \in E$ as $n \to +\infty$. So we obtain that $\max\{\|\Phi^m v\| : m \in \boldsymbol{Z}\} \leqq 2$ and $\|v\|=1$, which contradicts the quasi-hyperbolicity of (Φ, ϕ).

By the technique of Lewowicz [10, Lemma 2.3], combined with above fact, we see that there exists a positive integer m such that $\|\Phi^m v\| > 2\|v\|$ or $\|\Phi^{-m} v\| > 2\|v\|$, for every non-zero $v \in E$.

Define $Q(v)$ as follows:

$$Q(v)=\sum_{i=0}^{m-1}\{\|\Phi^{m+i}v\|^2-\|\Phi^i v\|^2\}.$$

Then $Q(v)$ is obiously a quadratic form on E, and we have

$$\begin{aligned} Q(\Phi v)-Q(v) &= \|\Phi^{2m}v\|^2-2\|\Phi^m v\|^2+\|v\|^2 \\ &> 2\|\Phi^m v\|^2>0, \end{aligned}$$

for non-zero $v \in E$, because of

$$\|\Phi^{2m}v\|^2+\|v\|^2>4\|\Phi^m v\|^2.$$

(ii) Sufficiency. Conversely, let Q be a quadratic form on E such that $Q(\Phi v)-Q(v)>0$ for every non-zero $v \in E$. Then the compactness of M and the continuity of Q imply that there exist positive constants C_1 and C_2 such that

$$|Q(v)| \leqq C_1\|v\|^2, \quad Q(\Phi v)-Q(v) \geqq C_2\|v\|^2 \quad (v \in E).$$

Assume that there exists non-zero $v \in E$ such that $\{\|\Phi^n v\| : n \in \boldsymbol{Z}\}$ is bounded. Then $\{Q(\Phi^n v) : n \in \boldsymbol{Z}\}$ is bounded because $|Q(v)| \leqq C_1\|v\|^2$.

On the other hand, for every positive integer n, we have

$$Q(\Phi^n v)-Q(v)=\sum_{i=0}^{n-1}\{Q(\Phi^{i+1}v)-Q(\Phi^i v)\} \geqq C_2\sum_{i=0}^{n-1}\|\Phi^i v\|^2,$$

$$Q(v)-Q(\Phi^{-n}v)=\sum_{i=1}^{n}\{Q(\Phi^{-i+1}v)-Q(\Phi^{-i}v)\} \geqq C_2\sum_{i=1}^{n}\|\Phi^{-i}v\|^2.$$

Since $\{Q(\Phi^n v):\ n\in \boldsymbol{Z}\}$ is bounded, we see that

$$\|\Phi^n v\| \longrightarrow 0 \quad \text{as} \quad n\to\pm\infty .$$

Therefore $Q(\Phi^n v)\to 0$ as $n\to\pm\infty$ because $|Q(v)|\leqq C_1\|v\|^2$.

Note that for positive integer n,

$$Q(\Phi^n v)>Q(\Phi^{n-1}v)> \cdots >Q(\Phi v)>Q(v),$$

then we must have $Q(v)<Q(\Phi v)\leqq 0$ because $Q(\Phi^n v)\to 0$ as $n\to+\infty$. Similarly we get $Q(v)>0$ because

$$Q(v)>Q(\Phi^{-1}v)> \cdots >Q(\Phi^{-n+1}v)>Q(\Phi^{-n}v),$$

for positive integer n and $Q(\Phi^{-n}v)\to 0$ as $n\to+\infty$. This is a contradiction. Hence there exists no non-zero $v\in E$ such that $\{\|\Phi^n v\|:\ n\in \boldsymbol{Z}\}$ is bounded. This completes the proof.

Remark 1. Theorem 1 remains to be true even if Q is a continuous homogeneous function of degree 2, not necessary a quadratic form. This can be seen directly from the above proof of Theorem 1.

§ 2. An application for diffeomorphisms.

Let us now consider the case where ϕ is a diffeomorphism f of a manifold M, Φ its differential Tf and E the tangent bundle TM of M.

Let f be a C^1-diffeomorphism on a compact C^r-manifold M $(r\geqq 1)$. The diffeomorphism f is called *quasi-Anosov* (resp. *Anosov*) if the vector bundle homeomorphism (Tf, f) on $E\equiv TM$ is quasi-hyperbolic (resp. hyperbolic) on E. By definition of quasi-Anosov diffeomorphisms, one can apply Theorem 1 directly to diffeomorphisms.

By the way, we give the definition of Anosov flows here. Let f_t be a flow on M and X the vector field on M generating f_t. The flow f_t is called *Anosov* if there exists a Tf_t-invariant continuous splitting of TM: $TM=\bar{X}\oplus E^s\oplus E^u$, where $\bar{X}$ is one dimensional subbundle of TM defined by the vector field X and E^s is exponentially contracted by Tf_t in positive time while E^u is exponentially contracted by Tf_t in negative time, for some Riemannian metric on M.

We can prove the following proposition by the same argument in the proof of Corollary 2.2 in [10].

Proposition 2. *Let f be a C^1-quasi-Ansov diffeomorphism on a compact C^r-manifold M $(r\geqq 1)$. Then there exists a C^1-neighbourhood U of f, in the space of diffeomorphisms of M, such that any finite composition of elements of U is quasi-Anosov.*

Proof. By Theorem 1, there exists a quadratic form Q such that $Q(Tfv)-Q(v)>0$ for every non-zero $v\in TM$. Let U be the set of C^1-diffeomorphism g of M such that $Q(Tgv)-Q(v)>0$ for every non-zero $v\in TM$. Then U is a C^1-

neighbourhood of f because M is compact.

Take g, $h\in U$ and a non-zero $v\in TM$, then we have

$$Q(T(h\circ g)v)-Q(v)=\{Q(Th(Tgv))-Q(Tgv)\}+\{Q(Tgv)-Q(v)\}>0,$$

which completes the proof by Theorem 1.

§ 3. Geodesic flows of Anosov type.

In this section, we consider geodesic flows. General references for geodesic flows are [1], [7], [9] and [14].

Let Y be an n-dimensional compact connected Riemannian C^r-manifold ($r\geqq 2$) without boundary, and TY the tangent bundle of Y. We can naturally interpret the double tangent bundle $T(TY)\equiv T^2Y$ as the vector bundle on Y as follows. Let $K: T^2Y\to TY$ be the Riemannian connector (cf. [6] P. 74), and let $\pi_Y: TY\to Y$ and $\pi_{TY}: T^2Y\to TY$ be natural projections for tangent bundles on Y and TY respectively. Further let π_* be the differential of π_Y. Then $\pi_{TY}\oplus\pi_*\oplus K$ maps T^2Y to $TY\oplus TY\oplus TY$ isomorphically as vector bundle on Y. (About this fact, see [6] and [17] for details.) From now on, we identify a tangent vector ξ on TY with a pair $(\pi^*\xi, K\xi)$ of tangent vectors on Y.

It is well known that the tangent bundle is a Riemannian manifold with Sasakian metric: $\langle\xi, \eta\rangle_{TY}=\langle\pi_*\xi, \pi_*\eta\rangle_Y+\langle K\xi, K\eta\rangle_Y$, where $\langle\,,\rangle_Y$ is the Riemnnian metric on Y. We will omit the suffices Y and TY of $\langle\,,\rangle_Y$ and $\langle\,,\rangle_{TY}$ in the following.

Let $M\equiv SY$ be the sphere bundle of Y and E the vector bundle on M defined as follows: For $v\in M$, the fibre E_v is given by

$$E_v=\{\xi\in T_vM: \langle\pi_*\xi, v\rangle=\langle K\xi, v\rangle=0\}.$$

Let $\phi_t: M\to M$ be the geodesic flow on Y and $\Phi_t\equiv T\phi_t$ the tangent cocycle of ϕ_t. We proved in [14, Lemma 3] that (Φ_t, ϕ_t) is a vector bundle flow on E and obtained the following result which enables us to apply Theorem 1 for geodesic flows.

Proposition 3 [14, Theorem 1].

The geodesic flow ϕ_t is Anosov if and only if for some $T>0$ (and hence for all $T>0$), the vector bundle homeomorphism (Φ_T, ϕ_T) on E is hyperbolic.

Remark 2. Actually we gave there the criterion with $T=1$. But the above generalization can be obtained easily.

Now we define a bundle map $A: E\to E$ covering the identity map of M as follows: for $\xi\in E_v$,

$$\begin{cases} \pi_*(A\xi)\equiv\pi_*A\xi=-R(v, \pi_*\xi)v, \\ K(A\xi)\equiv KA\xi=K\xi, \end{cases} \tag{3.1}$$

where R is the curvature tensor of the Riemannian metric on Y. It is easy to check that $\langle A\xi, \eta\rangle=\langle\xi, A\eta\rangle$, for $\xi, \eta\in E_v$.

From Theorem 1 and Proposition 3, we obtain the following sufficient condition for a geodesic flow to be Anosov, which is one of our main results.

Theorem 4. *Let Y be a compact connected C^r-Riemannian manifold ($r\geqq 2$) without boundary. Let ϕ_t be the geodesic flow on Y and Φ_t its differential, and let $M\equiv SY$, E and A be as above. Assume that there exists $T>0$ such that*

$$\int_0^T \langle A\xi(t), \xi(t)\rangle \mathrm{d}t>0,$$

for every non-zero $\xi\in E_v$ and $v\in M$, where $\xi(t)=\Phi_t\xi$. Then the geodesic flow ϕ_t is Anosov.

Proof. Define a quadratic form Q on E as $Q(\xi)=\langle\pi_*\xi, K\xi\rangle$, and put $\xi(t)=\Phi_t\xi$, $v(t)=\phi_t v$ for $v\in M$, $\xi\in E_v$. In [14, Lemma 2], we gave the following equations:

$$\left\{\begin{array}{l}\dfrac{\mathrm{D}}{\mathrm{d}t}\pi_*\xi(t)=K\xi(t), \\ \dfrac{\mathrm{D}}{\mathrm{d}t}K\xi(t)=-R(v(t), \pi_*\xi(t))v(t),\end{array}\right. \tag{3.2}$$

where $\mathrm{D}/\mathrm{d}t$ is the covariant derivative for Riemannian connection on Y.

By (3.1) and (3.2), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Q(\xi(t))&=\left\langle\frac{\mathrm{D}}{\mathrm{d}t}\pi_*\xi(t), K\xi(t)\right\rangle+\left\langle\pi^*\xi(t), \frac{\mathrm{D}}{\mathrm{d}t}K\xi(t)\right\rangle \\ &=\langle K\xi(t), K\xi(t)\rangle-\langle\pi_*\xi(t), R(v(t), \pi_*\xi(t))v(t)\rangle \\ &=\langle A\xi(t), \xi(t)\rangle.\end{aligned} \tag{3.3}$$

Hence

$$Q(\Phi_T\xi)-Q(\xi)=Q(\xi(T))-Q(\xi)=\int_0^T\langle A\xi(t), \xi(t)\rangle\mathrm{d}t,$$

which is positive for every non-zero $\xi\in E_v$ by assumption. Therefore it follows from Theorem 1 that the vector bundle homeomorphism (Φ_T, ϕ_T) on E is quasi-hyperbolic.

On the other hand, since the geodesic flow preserves the Riemannian measure induced on M, the homeomorphism ϕ_T is chain recurrent on M. Hence we see from the theory of Selgrade [18] and Sacker-Sell [16] that the quasi-hyperbolic vector bundle homeomorphism (Φ_T, ϕ_T) is hyperbolic. Then Proposition 3, combining with this fact, implies that the geodesic flow is Anosov.

Remark 3. We can not yet prove the converse of Theorem 4. But it is likely to be true under a certain appropriate condition.

§ 4. Simple consequences of Theorem 4.

From Theorem 4, we obtain the following proposition which enables us to prove easily the familiar fact that geodesic flows on Riemannian manifolds with negative curvature are Anosov.

Proposition 5. *Let Y be a compact connected Riemannian manifold without boundary and with non-positive sectional curvatures. Let Λ be the set of points at which every sectional curvatures are zero. If Λ contains no full geodesic curve, then the geodesic flow on Y is Anosov.*

Proof. Note that Λ is compact, being a closed set in the compact set Y. The compactness of Λ implies that there exists $T>0$ such that the geodesic curve with the length T is not completely contained in Λ. Therefore for every $v\in M$ and non-zero $\xi\in E_v$ (M and E_v being the same as in §3), we have

$$\int_0^T \langle A\xi(t),\ \xi(t)\rangle \mathrm{d}t=\int_0^T \{\|K\xi(t)\|^2-\langle \pi_*\xi(t),\ R(v(t),\ \pi_*\xi(t))v(t)\rangle\}\mathrm{d}t>0,$$

because $\langle \pi_*\xi(t),\ R(v(t),\ \pi_*\xi(t))v(t)\rangle \leqq 0$, and for some t $(0\leqq t\leqq T)$, $\langle \pi_*\xi(t),\ R(v(t),\ \pi_*\xi(t))v(t)\rangle<0$, by assumption. This completes the proof.

We define $\widetilde{K(v)}$ as follows and call it the *asymptotic curvature* of the direction v: for $v\in M$,

$$\widetilde{K(v)}=\limsup_{T\to+\infty}\left\{\sup_{\xi\in E_v,\|\xi\|=1}\frac{1}{T}\int_0^T\langle -A\xi(t),\ \xi(t)\rangle \mathrm{d}t\right\}, \tag{4.1}$$

where $\xi(t)=\Phi_t\xi$.

Untill now we do not know exactly what $\widetilde{K(v)}$ means. However Theorem 4 combining with the familiar technique of analysis induces the following.

Proposition 6. *Assume that*

$$\frac{1}{T}\int_0^T\langle -A\xi(t),\ \xi(t)\rangle \mathrm{d}t$$

converges uniformly in ξ, $\|\xi\|=1$, as $T\to+\infty$. If $\widetilde{K(v)}<0$ for every $v\in M$, then the geodesic flow is Anosov.

Proof. The assertion follows easily from the following lemma.

Lemma. *Let X and Y be compact metric spaces and $f: X\times Y\times \boldsymbol{R}\to\boldsymbol{R}$ a continuous function. Put $g(x, t)=\max\{f(x, y, t): y\in Y\}$, and assume $f(x, y, t)$ converges uniformly in x, y, as $t\to+\infty$. If $\limsup_{t\to+\infty} g(x, t)<0$ for every $x\in X$, then there exists $T>0$ such that $g(x, T)<0$ for every $x\in X$.*

Remark 4. In [11], Lewowicz has proven Theorem 4 and Proposition 5 in the case of 2-dimensional manifolds.

§5. Generalization of Chicone's criterion.

We keep the same notations as in §3. Let $C(E)$ denote the real Banach space of continuous sections of E with the supremum norm $\|\xi\|=\sup\{\langle\xi(x),\xi(x)\rangle^{1/2}: x\in M\}$.

On the other hand, as is well known the geodesic flow ϕ_t preserves the Riemannian measure μ on M. The space $C(E)$ is equipped with the inner product

$$(\xi, \eta)=\int_M\langle\xi(x), \eta(x)\rangle d\mu(x), \qquad \text{for} \quad \xi, \eta\in C(E).$$

Let $L^2(E)$ denote the completion of $C(E)$ with L^2-norm induced from this inner product. $L^2(E)$ is a Hilbert space over $\boldsymbol{R}$. We denote the complexifications of $C(E)$ and $L^2(E)$ by $\Gamma(E)$ and $\Gamma^2(E)$ respectively. We extend π_*, K, A and Φ_t so as to commute with complex conjugation. And we also extend inner products $\langle , \rangle$ and $(,)$ to Hermitian inner products.

For a fixed $t\in\boldsymbol{R}$, the vector bundle homeomorphism (Φ_t, ϕ_t) induces a bounded linear operator $\Phi_t^\#$, so called the adjoint representation of (Φ_t, ϕ_t), on $\Gamma(E)$ (resp. $\Gamma^2(E)$) as follows:

$$\Phi_t^\#\xi=\Phi_t\circ\xi\circ\phi_{-t} \qquad \text{for} \quad \xi\in\Gamma(E) \text{ (resp. } \Gamma^2(E)).$$

It is easily checked that the vector bundle map $A: E\to E$ also induces a bounded linear operator $A^\#$ on $\Gamma(E)$ (resp. $\Gamma^2(E)$), in particular, the operator $A^\#$ on $\Gamma^2(E)$ is selfadjoint.

Let T be a bounded linear operator on a Banach space H over $\boldsymbol{C}$. We denote by $\sigma(T:H)$ and $\sigma_{ap}(T:H)$ the sets of spectra and approximate point spectra of T on H respectively.

J.N. Mather [13] has already proven that for a fixed $t\in\boldsymbol{R}$, the vector bundle homeomorphism (Φ_t, ϕ_t) is hyperbolic if and only if $\sigma(\Phi_t^\#:\Gamma(E))$ is disjoint from the unit circle. (Mather has proven this theorem in some restricted situation. For a proof in general setting as above, see [15], for instance.) Recently, C. Chicone and R.C. Swanson proved the following powerful result.

Proposition 7 [3, Proposition 1.4 and Theorem 1.5].
There hold the following equalities for any $t\in\boldsymbol{R}$,

$$\sigma(\Phi_t^\#:\Gamma(E))=\sigma_{ap}(\Phi_t^\#:\Gamma(E))=\sigma_{ap}(\Phi_t^\#:\Gamma^2(E))=\sigma(\Phi_t^\#:\Gamma^2(E)).$$

Proposition 7 combining with the following lemma enables us to obtain a theorem corresponding to Theorem 4.

Lemma 8. *Let H be a Hilbert space over $\boldsymbol{C}$, T a bounded linear operator on H, and Q a quadratic form on H: $Q(\xi)=(S\xi, \xi)$, $\xi\in H$, where S is a bounded selfadjoint operator. If* $\inf\{Q(T\xi)-Q(\xi): \|\xi\|=1\}>0$, *then $\sigma_{ap}(T:H)$ is disjoint from the unit circle.*

Proof. Let $\lambda\in\boldsymbol{C}$ with $|\lambda|=1$ and $\xi_n\in H$ with $\|\xi_n\|=1$, and assume $\|(T-\lambda I)\xi_n\|\to 0$ as $n\to+\infty$. Then $(S(T-\lambda I)\xi^u, T\xi_n)\to 0$ as $n\to+\infty$, because

$$|(S(T-\lambda I)\xi_n, T\xi_n)|\leqq\|S\|\cdot\|(T-\lambda I)\xi_n\|\cdot\|T\|.$$

On the other hand, we have

$$(5.1)\qquad (S(T-\lambda I)\xi_n,\ T\xi_n)=(T^*S(T-\lambda I)\xi_n,\ \xi_n)$$
$$=(T^*ST\xi_n,\ \xi_n)-\lambda(T^*S\xi_n,\ \xi_n)$$
$$=(T^*ST\xi_n,\ \xi_n)-(S\xi_n,\ \xi_n)+\lambda\bar{\lambda}(S\xi_n,\ \xi_n)-\lambda(S\xi_n,\ T\xi_n)$$
$$=(T^*ST\xi_n,\ \xi_n)-(S\xi_n,\ \xi_n)-\lambda(S\xi_n,\ (T-\lambda I)\xi_n).$$

By assumption, there exists $C>0$ such that

$$(5.2)\qquad (T^*ST\xi_n,\ \xi_n)-(S\xi_n,\ \xi_n)=Q(T\xi_n)-Q(\xi_n)\geqq C>0.$$

The above equalities (5.1) and (5.2) contradict that

$$(S(T-\lambda I)\xi_n,\ T\xi_n),\quad (S\xi_n,\ (T-\lambda I)\xi_n)\longrightarrow 0\quad \text{as } n\to+\infty.$$

Thus we have shown that $\lambda\notin\sigma_{ap}(T:H)$.

Theorem 9. *Let ϕ_t be the geodesic flow on a compact connected Riemannian manifold. Assume that there exists $T>0$ such that*

$$\inf\left\{\int_0^T(A^\#\xi(t),\ \xi(t))\mathrm{d}t:\ \xi\in\Gamma^2(E),\ \|\xi\|=1\right\}$$

is positive, where $\xi(t)=\Phi_t^\#\xi$. Then the geodesic flow is Anosov.

Proof. Let us define a quadratic form Q on the Hilbert space $\Gamma^2(E)$ as

$$Q(\xi)=\int_M\langle\pi_*\xi(x),\ K\xi(x)\rangle\mathrm{d}\mu(x)\qquad(\xi\in\Gamma^2(E)).$$

We see easily that $Q(\xi)$ gives actually a quadratic form on the Hilbert space.

Since ϕ_t preserves the measure μ, we have

$$Q(\Phi_T^\#\xi)=\int_M\langle\pi_*\Phi_T\xi(\phi_{-T}x),\ K\Phi_T\xi(\phi_{-T}x)\rangle\mathrm{d}\mu(x)$$
$$=\int_M\langle\pi_*\Phi_T\xi(x),\ K\Phi_T\xi(x)\rangle\mathrm{d}\mu(x)$$

Therefore, from (3.3) and Fubini's theorem, we have

$$(5.2)\qquad Q(\Phi_T^\#\xi)-Q(\xi)=\int_M\{\langle\pi_*\Phi_T\xi(x),\ K\Phi_T\xi(x)\rangle-\langle\pi_*\xi(x),\ K\xi(x)\rangle\}\mathrm{d}\mu(x)$$
$$=\int_M\left\{\int_0^T\langle A\Phi_t\xi(x),\ \Phi_t\xi(x)\rangle\mathrm{d}t\right\}\mathrm{d}\mu(x)$$
$$=\int_0^T\left\{\int_M\langle A\Phi_t\xi(\phi_{-t}x),\ \Phi_t\xi(\phi_{-t}x)\rangle\mathrm{d}\mu(x)\right\}\mathrm{d}t$$
$$=\int_0^T(A^\#\xi(t),\ \xi(t))\mathrm{d}t,\qquad \text{where}\quad \xi(t)=\Phi_t^\#\xi.$$

By using (5.3) and Lemma 8, we see that $\sigma_{ap}(\Phi_T^\#:\Gamma^2(E))$ is disjoint from the unit circle, and hence, so is $\sigma(\Phi_T^\#:\Gamma(E))$ because of Proposition 7.

By Proposition 3, combined with Mather's theorem, we conclude that the geodesic flow ϕ_t is Anosov.

C. Chicone [2] defined $K_0(Y)$, the so-called H^0-curvature of Y, as

$$K_0(Y)=\sup\{(-A^{\#}\xi,\ \xi):\ \xi\in\Gamma^2(E),\ \|\xi\|=1\},$$

in our notation. We can obtain the Chicone's criterion as a corollary of Theorem 9.

Corollary 10 [2, Corollary 5.9]
If $K_0(Y)<0$, then the geodesic flow on Y is Anosov.

Proof. Assume $K_0(Y)<0$. Then we can get

$$\inf\left\{\int_0^T (A^{\#}\xi(t),\ \xi(t))\mathrm{d}t:\ \xi\in\Gamma^2(E),\ \|\xi\|=1\right\}>0$$

for any $T>0$, where $\xi(t)=\Phi_t^{\#}\xi$.

In fact, Put $K_0(Y)=-C<0$, then

$$\inf\{(A^{\#}\xi,\ \xi):\ \|\xi\|=1\}=C>0,$$

whence $(A^{\#}\xi,\ \xi)\geqq C\|\xi\|^2$ for every $\xi\in\Gamma^2(E)$.

Since $(A^{\#}\Phi_t^{\#}\xi,\ \Phi_t^{\#}\xi)\geqq C\|\Phi_t^{\#}\xi\|^2$, we have

$$(A^{\#}\Phi_t^{\#}\xi,\ \Phi_t^{\#}\xi)\geqq C\inf_{\|\xi\|=1}\|\Phi_t^{\#}\xi\|^2\geqq C\|\Phi_{-t}^{\#}\|^{-2},$$

because

$$1=\|\xi\|=\|\Phi_{-t}^{\#}\circ\Phi_t^{\#}\xi\|\leqq\|\Phi_{-t}^{\#}\|\cdot\|\Phi_t^{\#}\xi\|.$$

Since $\{\Phi_t^{\#}\}$ is a strongly continuous group of bounded linear operators, there are $k>0$ and $\alpha>0$ such that

$$\|\Phi_{-t}^{\#}\|\leqq ke^{\alpha t} \qquad \text{for} \quad t\geqq 0,$$

hence

$$\|\Phi_{-t}^{\#}\|^{-1}\geqq\frac{1}{k}e^{-\alpha t}.$$

Therefore

$$\inf_{\|\xi\|=1}\left\{\int_0^T (A^{\#}\xi(t),\ \xi(t))\mathrm{d}t\right\}\geqq(C/k^2)\int_0^T e^{-2\alpha t}\mathrm{d}t$$
$$=(C/2k^2\alpha)(1-e^{-2\alpha T})>0.$$

So we can apply Theorem 9 and get the assertion.

We can consider the object analogous to $\widetilde{K(v)}$ in (4.1) in L^2-setting. We call the following $\widetilde{K_0(Y)}$ the *asymptotic H^0-curvature* of Y:

$$\widetilde{K_0(Y)}=\limsup_{T\to+\infty}\left\{\sup_{\xi\in\Gamma^2(E),\|\xi\|=1}\frac{1}{T}\int_0^T(-A^{\#}\xi(t),\ \xi(t))\mathrm{d}t\right\},$$

where $\xi(t)=\Phi_t^{\#}\xi$. We suspect whether it has some relations with the ergodic properties of the geodesic flow ϕ_t on a Riemannian manifold Y.

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