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On the behavior of non-negative finely superharmonic functions at the Martin boundary

By

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Introduction

By F. Riesz [13] and M. Parreau [11], a non-negative superharmonic function is decomposed uniquely into the sum of a Green potential, a quasibounded harmonic function and a singular harmonic function. L. Naïm [10] and J. L. Doob (cf. [5]) connected this decomposition theorem with the Fatou boundary theorem on the Martin boundary for non-negative superharmonic function as the following:

Theorem A. *Let u be a non-negative superharmonic function in a Green space R. Then it holds that*

(i) *u is decomposed uniquely into the sum of a Green potential p in R, a quasibounded harmonic function f_1 in R, and a singular harmonic function f_2 in R, and that*

(ii) *u and f_1 have a common fine limit at almost every minimal point with respect to the harmonic measure $\omega_z(z\in R)$.*

B. Fuglede [7] introduced the notion of finely superharmonic functions to potential theory on harmonic spaces. The main purpose of this paper is to extend Theorem A to the case of a non-negative finely superharmonic function and to obtain the next theorem:

Main Theorem. *Let U be a finely open subset of a Green space R in the sense of Doob* [6, §4] *which admits the Green function, and u be a non-negative finely superharmonic function in U. Then it holds that*

(i) *u is decomposed uniquely into the sum of a fine potential p in U, a non-negative finely superharmonic function f_1 in U which is quasibounded finely harmonic in $U-\{z\in U: u(z)=+\infty\}$ and a non-negative finely superharmonic function f_2 in U which is singular finely harmonic in $U-\{z\in U: u(z)=+\infty\}$*, and that

(ii) *u and f_1 have a common fine limit at almost every point of $\Delta_1(U)$ (see §4) with respect to ω_z.*

The definitions of notations and terminologies used in this theorem are stated in §§ 1, 2 and 4, and the word "almost every" will be later used with respect to ω_z unless otherwise stated. Roughly speaking, the idea of the proof of Main Theorem is to

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modify u in order to extend its modification to R as a bounded subharmonic function. In the process of its proof, the behavior of solutions of the generalized fine Dirichlet problem (see § 3) at the Martin boundary will play an important role.

As an important application of Main Theorem, a Plessner type theorem for finely harmonic functions is obtained.

In § 1 we provide some definitions and results from the fine potential theory and state in § 2 a Parreau type decomposition theorem for finely harmonic functions. The generalized fine Dirichlet problem is stated in § 3 and the minimum principle for finely superharmonic functions is shown in § 4. Using the above results, we give in § 5 the proof of Main Theorem and a Plessner type theorem for finely harmonic functions is proved in § 6.

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§ 1. Preliminaries

First we introduce the notations which will be used throughout this paper.

R: a Green space in the sense of Doob [6, § 4] (cf. [4]) which admits the Green function,

R_M^*: the Martin compactification of R,

$\Delta(=R_M^*-R)$: the Martin boundary of R,

Δ_1: the totality of minimal points in Δ,

k_ζ: the Martin function with pole at $\zeta\in R_M^*$,

ω_z: the harmonic measure on Δ_1 relative to $z\in R$ and R, and

χ: the canonical measure on Δ_1 representing 1.

We refer to [5, Ch. 13] for the notion of Martin's compactification. In [5], the Martin compactification of a hyperbolic Riemann surface is considered, but the results in [5] for superharmonic functions are valid even if we observe a Green space in place of a hyperbolic Riemann surface.

Next we state some notions and results from fine potential theory. For that purpose we introduce into R the weakest topology which makes all positive superharmonic functions in R continuous. Such a topology is called the fine topology (cf. [3, Ch. 1]). In this paper, as for the topological terminologies in fine topology, we put the word "fine" or "finely" before those of the original topology in R, for example, finely open, a fine neighborhood etc.

Definition 1.1 ([7, 8.1~8.3]). If an extended real valued function u defined in a finely open subset U of R satisfies the following conditions, u is called to be *finely harmonic* (resp. *finely hyperharmonic*, *finely hypoharmonic*) in U.

(i) $-\infty<u<+\infty$ (resp. $-\infty<u\leqq+\infty$, $-\infty\leqq u<+\infty$);

(ii) u is finely continuous (resp. finely lower semi-continuous, finely upper semi-continuous) in U;

(iii) for every $x\in U$, there exists a compact fine neighborhood V of x such that u is bounded (resp. bounded below, bounded above) in V $(\subset U)$ and that for every $z\in V'$

(=the fine interior of V),

$$u(z)=\int u\,d\varepsilon_z^{CV}\,(\text{resp. } u(z)\geqq\int u\,d\varepsilon_z^{CV},\ u(z)\leqq\int u\,d\varepsilon_z^{CV}),$$

where ε_z^{CV} is the balayage of the Dirac measure ε_z at z on CV $(:=R-V)$ (cf. [1]).

Moreover, if a finely hyperharmonic (resp. finely hypoharmonic) function u in U is not equal to $+\infty$ (resp. $-\infty$) identically, u is called to be finely superharmonic (resp. finely subharmonic) in U.

We state the fundamental theorem for finely hyperharmonic functions:

Theorem 1.1 (Fuglede [7, Theorem 9.10]). *Every finely hyperharmonic function is finely continuous.*

Definition 1.2 ([7, 10.5]). A non-negative finely superharmonic function p in a finely open subset U of R is called a *fine potential* in U if every finely subharmonic function s in U dominated by p is non-positive in U.

The next theorem is considered as a Riesz type decomposition theorem for finely superharmonic functions.

Theorem 1.2 (Fuglede [7, Theorem 10.7]). *For a non-negative finely superharmonic function u in a finely open subset U of R, u is decomposed uniquely into the sum of a fine potential p in U and a non-negative finely superharmonic function h in U which is finely harmonic in $U-\{z\in U: u(z)=+\infty\}$.*

§ 2. A Parreau type theorem for finely harmonic functions

In [9], by using a Brownian motion on $\boldsymbol{R}^d (d\geqq 2)$, we obtained a Parreau type theorem for finely harmonic functions in a fine subdomain of $\boldsymbol{R}^d$ (cf. [11]). In the same way as in [9] we conclude that such a result is valid for finely harmonic functions in a finely open subset of R since we can construct a Brownian motion on R (cf. [8, Ch. 5], in particular for Riemann surface see [14, § 2]). Thus we only state the result without proof.

Definition 2.1. Let U be a finely open subset of R and f a non-negative finely harmonic function in U. Then f is called to be *quasibounded* in U if there exists a sequence $\{f_n\}_{n=1}^{+\infty}$ of bounded non-negative finely harmonic functions in U such that $\{f_n\}_{n=1}^{+\infty}$ converges increasingly to f on U.

Definition 2.2. Let U be a finely open subset of R and f a non-negative finely harmonic function in U. Then f is called to be *singular* in U if a bounded non-negative finely harmonic function h in U dominated by f is equal to zero identically in U.

Theorem 2.1. *Let U be a finely open subset of R and f a non-negative finely harmonic function in U. Then f is decomposed uniquely into the sum of a quasibounded* *finely harmonic function f_1 in U and a singular finely harmonic function f_2 in U.*

We refer to [9] for the proof of this theorem.

§ 3. The generalized fine Dirichlet problem

First we introduce the generalized fine Dirichlet problem considered by Fuglede [7, 14.3].

Definition 3.1 ([2, p. 41]). A Green potential p in R is called to be *semi-bounded* in R if $\liminf_{x\to z}(\inf_{\lambda>0} \hat{R}_p^{\{p>\lambda\}}(x))=0$ for every $z\in R$, where $\hat{R}_p^{\{p>\lambda\}}$ is the balayage of p on $\{x\in R: p(x)>\lambda\}$, that is $\hat{R}_p^{\{p>\lambda\}}(z):=\inf\{s(z): s$ is non-negative superharmonic in R and $s\geqq p$ q.e. (=quasi-everywhere) on $\{x\in R: p(x)>\lambda\}\}$ (cf. [1]).

Definition 3.2. Let U be a finely open subset of R and f an extended real valued function on the fine boundary $\partial_f U$ of U. Then we denote by $\bar{\mathfrak{Q}}_f^U$ the totality of functions u satisfying the following conditions:

(i) u is finely hyperharmonic in U;

(ii) $-u$ is dominated in U by some semi-bounded Green potential p in R, where p depends on u;

(iii) $\operatorname{fine}\liminf_{z(\in U)\to\zeta} u(z)\geqq f(\zeta)$ for every $\zeta\in\partial_f U$,

and we define the functions $\bar{H}_f^U$ and $\underline{H}_f^U$ in U as follows:

$$\bar{H}_f^U(z):=\inf\{u(z): u\in\bar{\mathfrak{Q}}_f^U\}, \qquad \underline{H}_f^U:=-\bar{H}_{-f}^U.$$

Moreover, if $-\infty<\bar{H}_f^U=\underline{H}_f^U<+\infty$ in U, f is called to be *finely resolutive relatively to* U (for the generalized fine Dirichlet problem), we denote $\bar{H}_f^U$ by H_f^U, and we call H_f^U the *solution of the generalized fine Dirichlet problem* of f on U.

The next theorem gives us a necessary and sufficient condition for the fine resolutivity of boundary functions.

Theorem 3.1 (Fuglede [7, Theorem 14.6]). *Let U be a finely open subset of R and f an extended real valued function on $\partial_f U$. Then it holds that*

(i) $\bar{H}_f^U(z)=\int^* f\,d\varepsilon_z^{CU}$ *and* $\underline{H}_f^U(z)=\int_* f\,d\varepsilon_z^{CU}$ *for every $z\in U$, where* $\int^* f\,d\varepsilon_z^{CU}$ *and* $\int_* f\,d\varepsilon_z^{CU}$ *is the upper and the lower integral of f respectively;*

(ii) *f is finely resolutive if and only if f is integrable w.r.t.(=with respect to) ε_z^{CU} for every $z\in U$;*

(iii) *if f is finely resolutive, $H_f^U(z)=\int f\,d\varepsilon_z^{CU}$ for every $z\in U$ and H_f^U is finely harmonic in U.*

Next we introduce notions of regular and irregular points. For a subset A of R, we denote by $b(A)$ and $i(A)$ the totality of finely accumulated points of A and the totality of finely isolated points of A respectively. We call $b(A)$ the *base* of A. Then it is well-known that $i(A)=A-b(A)$ and $i(A)$ is a polar set (cf. [3, Theorem VII, 7]).

Definition 3.3 (cf. [7, 12.5]). Let U be a finely open subset of R. Then the points of $b(\partial_f U)=b(CU)\cap\partial_f U$ are called the *regular points* of $\partial_f U$ or the *regular* (*fine boundary*) *points* for U, the remaining points of $\partial_f U$, forming the polar set $i(\partial_f U)=i(CU)$, are called the *irregular points* of $\partial_f U$ or the *irregular* (*fine boundary*) *points* for U.

Lemma 3.1. *Let U be a finely open subset of R and u a non-negative finely superharmonic function in U such that u is finite in U and is finely continuous on the fine closure $\tilde{U}$ of U. Then it holds that*

(i) *u is finely resolutive relatively to U;*

(ii) *for every regular points $\zeta\in\partial_f U$,*

$u(\zeta)=\operatorname{fine}\lim_{z(\in U)\to\zeta} H_u^U(z)$.

Proof. Let u be a non-negative finely superharmonic function such that u is finite in U and is finely continuous on $\tilde{U}$. From the definition of $\bar{H}_u^U$, $u\geqq\bar{H}_u^U$ in U. Therefore we obtain (i) from (i) and (ii) of Theorem 3.1. Next we show (ii). Since u is finely continuous on $\tilde{U}$ and $\varepsilon_z^{CU}=\varepsilon_z^{b(\partial_f U)}$ for every $z\in U$ (cf. [7, Lemma 12.3]), by Theorem 3.1 and [7, Remark of Theorem 9.13] we have, for every regular point $\zeta\in\partial_f U$,

$$\begin{aligned}\operatorname{fine}\liminf_{z(\in U)\to\zeta} H_u^U(z)&=\operatorname{fine}\liminf_{z(\in U)\to\zeta}\int u\,\mathrm{d}\varepsilon_z^{CU}\\&=\operatorname{fine}\liminf_{z(\in U)\to\zeta}\int u\,\mathrm{d}\varepsilon_z^{b(\partial_f U)}\\&\geqq\operatorname{fine}\liminf_{z(\in R)\to\zeta}\int u\,\mathrm{d}\varepsilon_z^{b(\partial_f U)}\\&=\operatorname{fine}\liminf_{z(\in b(\partial_f U))\to\zeta} u(z)\\&=u(\zeta).\end{aligned}$$

On the other hand, for every $\zeta\in\partial_f U$,

$$u(\zeta)=\operatorname{fine}\limsup_{z(\in U)\to\zeta} u(z)\geqq\operatorname{fine}\limsup_{z(\in U)\to\zeta} H_u^U(z),$$

since u is finely continuous on $\tilde{U}$ and $u\geqq H_u^U$ in U. Thus the proof of (ii) is completed q. e. d.

Lemma 3.2. *Let U be a finely open subset of R and u a non-negative superharmonic function in R. Then*

$$\underline{H}_u^U=\bar{H}_u^U=\hat{R}_u^{CU} \text{ in } U.$$

Moreover, if $u:=\int_{R_M^*} k_\zeta\,\mathrm{d}\mu(\zeta)$, where μ is a regular and positive Borel measure supported on R_M^*,

$$\underline{H}_u^U=\bar{H}_u^U=\hat{R}_u^{CU}=\int_{R_M^*}\hat{R}_{k_\zeta}^{CU}\,\mathrm{d}\mu(\zeta) \text{ in } U.$$

Proof. By (i) of Theorem 3.1 and the general theory of balayage (cf. [1, Theorem 6]), for every $z\in U$,

$$\underline{H}_u^U(z)=\overline{H}_u^U(z)=\int u\,\mathrm{d}\varepsilon_z^{CU}=\hat{R}_u^{CU}(z),$$

since u is measurable w. r. t. ε_z^{CU}(cf. [7, 4.9]). Here we suppose that $u=\int_{R_M^*}k_\eta\,\mathrm{d}\mu(\zeta)$, where μ is a regular and positive Borel measure supported on R_M^*. By Fubini's Theorem and [1, Theorem 6], for every $z\in U$,

$$\begin{aligned}\underline{H}_u^U(z)=\overline{H}_u^U(z)&=\int u\,\mathrm{d}\varepsilon_z^{CU}=\iint_{R_M^*}k_\zeta\,\mathrm{d}\mu(\zeta)\,\mathrm{d}\varepsilon_z^{CU}\\&=\int_{R_M^*}\int k_\zeta\,\mathrm{d}\varepsilon_z^{CU}\,\mathrm{d}\mu(\zeta)=\int_{R_M^*}\hat{R}_{k_\zeta}^{CU}(z)\,\mathrm{d}\mu(\zeta).\end{aligned}$$

q. e. d.

§4. Minimum principle

First we state several definitions.

Definition 4.1 (cf. [5, p. 145]). For a point $\zeta\in\Delta_1$ and a finely open subset U of R, $U\cup\{\zeta\}$ is called a *finely open neighborhood* of ζ if $\hat{R}_{k_\zeta}^{CU}\neq k_\zeta$. We denote by $\mathcal{G}_\zeta$ the totality of finely open subsets U of R such that $U\cup\{\zeta\}$ is a fine neighborhood of ζ.

Definition 4.2. For a finely open subset U of R, we define

$$\Delta_1(U):=\{\zeta\in\Delta_1:\ U\in\mathcal{G}_\zeta\}$$

and

$$\Delta_1(U)^*:=\{\zeta\in\Delta_1:\ V\cap U\neq\emptyset\ \text{for every}\ V\in\mathcal{G}_\zeta\}.$$

Definition 4.3. Let U be a finely open subset of R and f an extended real valued function defined in U. Then, for a point $\zeta\in\Delta_1(U)^*$, we define

$$f^*\text{-}\limsup_{z(\in U)\to\zeta}f(z):=\inf_{V\in\mathcal{G}_\zeta}\sup_{z\in U\cap V}f(z)$$

and

$$f^*\text{-}\liminf_{z(\in U)\to\zeta}f(z):=-\{f^*\text{-}\limsup_{z(\in U)\to\zeta}(-f(z))\}.$$

In particular, if $f^*\text{-}\limsup_{z(\in U)\to\zeta}f(z)=f^*\text{-}\liminf_{z(\in U)\to\zeta}f(z)$, we denote this common value by $f^*\text{-}\lim_{z(\in U)\to\zeta}f(z)$. Moreover, if $\zeta\in\Delta_1(U)$, we denote $f^*\text{-}\limsup_{z(\in U)\to\zeta}f(z)$, $f^*\text{-}\liminf_{z(\in U)\to\zeta}f(z)$ and $f^*\text{-}\lim_{z(\in U)\to\zeta}f(z)$ by $f\text{-}\limsup_{z(\in U)\to\zeta}f(z)$, $f\text{-}\liminf_{z(\in U)\to\zeta}f(z)$ and $f\text{-}\lim_{z(\in U)\to\zeta}f(z)$ respectively. $f\text{-}\lim_{z(\in U)\to\zeta}f(z)$ is called the *fine limit* of f at ζ.

Theorem 4.1 (Minimum principle). *Let U be a finely open subset of R and u a finely superharmonic function bounded below in U. If, for a.e. (=almost every) $\zeta\in\Delta_1(U)^*$, $f^*\text{-}\liminf_{z(\in U)\to\zeta}u(z)\geqq 0$ and for q.e. $\zeta\in\partial_f U$, $\operatorname{fine}\liminf_{z(\in U)\to\zeta}u(z)\geqq 0$, then $u\geqq 0$ in U.*

Proof. Assume that u is a finely superharmonic function bounded below in U, $f^*\text{-}\liminf_{z(\in U)\to\zeta}u(z)\geqq 0$ for a. e. $\zeta\in\Delta_1(U)^*$, and $\operatorname{fine}\liminf_{z(\in U)\to\zeta}u(z)\geqq 0$ for q. e. $\zeta\in\partial_f U$. Let

$u_1 := \max(-u, 0)$,

$$u_2(z) := \begin{cases} u_1(z) & \text{for} \quad z \in U \\ 0 & \text{for} \quad z \in CU, \end{cases}$$

$N := \{\zeta \in \partial_f U : \operatorname*{fine\,lim\,inf}_{z(\in U) \to \zeta} u(z) < 0\}$,

p: a Green potential in R such that $p = +\infty$ on N and

$u_\eta := u_2 - \eta p \quad (\eta > 0)$.

In order to obtain the desired fact, it is sufficient to prove that u_η is finely subharmonic in R. In fact, by the definition of u_η and [7, Theorem 9.8], u_η is bounded above in R and subharmonic in R. Furthermore, by the assumption, $f-\liminf_{z(\in U) \to \zeta} u_\eta(z) \leqq 0$ for a. e. $\zeta \in \Delta_1$. By [10, Principe de maximum (p. 261)], $u_\eta \leqq 0$ in R. Since η is arbitrary, $u_2 \leqq 0$ q. e. in U and hence $u \geqq 0$ q. e. in U. This implies the assertion of this theorem. Thus we must prove that u_η is finely subharmonic in R. To do this, we have only to check the conditions (i)～(iii) of Definition 1.1 at every point of $\partial_f U$ because of the definition of u_η.

First it is obvious from the definition of u_η that (i) holds.

Next we show (ii). At every point of $\partial_f U - N$, u_η is finely continuous because p is finely continuous in R, $u_2 = 0$ on CU and for every $\zeta \in \partial_f U - N$,

$$0 \leqq \operatorname*{fine\,lim\,sup}_{z(\in U) \to \zeta} u_2(z) = \operatorname*{fine\,lim\,sup}_{z(\in U) \to \zeta} (\max(-u(z), 0)) \leqq 0,$$

which shows that u_2 is finely continuous at every point of $\partial_f U - N$. At every point of N, u_η is finely upper semi-continuous because u_2 is bounded above in R and $p = +\infty$ on N. Thus (ii) is shown.

Finally we show (iii). To do this, it is sufficient to prove that u_2 satisfies (iii) since $-\eta p$ satisfies (iii). For every $\zeta \in \partial_f U$, we take an open neighborhood V of ζ. For every $z \in V \cap CU$, $u_2(z) = 0 \leqq \int u_2 \, d\varepsilon_z^{CV}$, for $u_2 \geqq 0$ on R. To prove that, for every $z \in V \cap U$, $u_2(z) \leqq \int u_2 \, d\varepsilon_z^{CV}$, we set

$$u_3(z) = \begin{cases} H_{u_2}^V(z) & \text{for} \quad z \in \partial_f U \cap V, \\ u_2(z) & \text{for} \quad z \in \partial_f(U \cap V) - \partial_f U \cap V. \end{cases}$$

Since $u_2 \leqq u_3$ on $\partial_f(V \cap U)$, $H_{u_2}^{V \cap U} \leqq H_{u_3}^{V \cap U}$ in $V \cap U$. Thus by Theorem 3.1 it is sufficient to prove that $u_2 \leqq H_{u_2}^{V \cap U}$ on $V \cap U$ since $H_{u_2}^V = H_{u_3}^{V \cap V}$ on $V \cap U$. Considering the generalized fine Dirichlet problem of u_η on $V \cap U$, we find that $u_\eta \leqq H_{u_\eta}^{V \cap U} \leqq H_{u_2}^{V \cap U}$ on $V \cap U$, because u_η is finely subharmonic in R. Since η is arbitrary, $u_2 \leqq H_{u_2}^{V \cap U}$ q. e. on $V \cap U$. By Theorem 3.1 $u_2 \leqq H_{u_2}^{V \cap U}$ on $V \cap U$. q. e. d.

§ 5. Proof of Main Theorem

1. Here we provide several lemmas from the need for proofs of propositions and theorems stated later. To start with, from [7, Theorems 9.14 and 9.15] we obtain

Lemma 5.1 (Fuglede). *Let U be a finely open subset of R such that $i(\partial_f U)\neq\emptyset$, u a finely superharmonic (resp. finely harmonic) function bounded below (resp. bounded) in U, and $V:=Cb(CU)=U\cup i(CU)$. Then u is uniquely extended to V as the finely superharmonic (resp. finely harmonic) function bounded below (resp. bounded) in V.*

Lemma 5.2. *Let U be a finely open subset of R, u a non-negative finely superharmonic function in U such that* (i) *u is finite in U;* (ii) *u is finely continuous in $\tilde{U}$;* *and* (iii) *u is dominated in U by some non-negative superharmonic function s in R. Then H_u^U has a fine limit* 0 *at almost every point of $\varDelta_1(U)$.*

Proof. Let u be a finely superharmonic function in U satisfying the above properties (i)~(iii). By Lemmas 3.1 and 3.2,

$$H_u^U \leqq \bar{H}_s^U = \hat{R}_s^{CU} \quad \text{in } U.$$

By [5, Theorem 4.6 and Corollaries 2.1 and 14.2] we find that s is decomposed uniquely into the sum of a Green potential p in R, a quasibounded harmonic function $\int_{\varDelta_1} f_1(\zeta) k_\zeta \, d\chi(\zeta)$ in R, where f_1 is the Fatou boundary function of s on $\varDelta_1$, and a singular harmonic function f_2 in R, that is

$$s = p + \int_{\varDelta_1} f_1(\zeta) k_\zeta \, d\chi(\zeta) + f_2 \quad \text{in } R.$$

Thus by Lemma 3.2 and the definition of $\varDelta_1(U)$ we have

$$(0\leqq) H_u^U \leqq \hat{R}_p^{CU} + \int_{\varDelta_1} f_1(\zeta) \hat{R}_{k_\zeta}^{CU} \, d\chi(\zeta) + \hat{R}_{f_2}^{CU}$$

$$\leqq p + \int_{\varDelta_1 - \varDelta_1(U)} f_1(\zeta) k_\zeta \, d\chi(\zeta) + \int_{\varDelta_1(U)} f_1(\zeta) \hat{R}_{k_\zeta}^{CU} \, d\chi(\zeta) + f_2 \quad \text{in } U.$$

Hence by [10, Theorem 21] and [5, Corollary 14.2] we have, for a. e. $\zeta \in \varDelta_1(U)$,

$$(0\leqq) \mathscr{f}\text{-}\limsup_{z(\in U)\to\zeta} H_u^U(z) \leqq \mathscr{f}\text{-}\limsup_{z(\in U)\to\zeta} \int_{\varDelta_1(U)} f_1(\zeta) \hat{R}_{k_\zeta}^{CU} \, d\chi(\zeta).$$

Thus by [10, Theorem 21] it is sufficient to prove that $\int_{\varDelta_1(U)} f_1(\zeta) \hat{R}_{k_\zeta}^{CU} \, d\chi(\zeta)$ is a Green potential in R in order to prove this lemma. By the definition of $\varDelta_1(U)$ and [10, Lemma 1], for every $\zeta \in \varDelta_1(U)$, $\hat{R}_{k_\zeta}^{CU}$ is a Green potential in R and hence by [5, Corollary 4.7] $\int_{\varDelta_1(U)} f_1(\zeta) \hat{R}_{k_\zeta}^{CU} \, d\chi(\zeta)$ is a Green potential in R. q. e. d.

Lemma 5.3. *For a finely open subset U of R, there exists finely open set U_1 such that* (i) *$\tilde{U}_1 \subset U \cup i(CU)$;* (ii) *$i(\partial_f U_1) = \emptyset$ and* (iii) *$\omega_z(\varDelta_1(U) - \varDelta_1(U_1)) = 0$.*

Proof. Let $f(z) := \omega_z(\varDelta_1(U)) = \int_{\varDelta_1(U)} k_\zeta(z) \, d\chi(\zeta)$ (cf. [4, Theorem 13.4]), $W := U \cup i(CU)$, $f_1 := f - H_f^W$ and $U_1 := \{z \in W : f_1(z) > \varepsilon\}$ $(0 < \varepsilon < 1)$. By Lemma 3.1 $\operatorname{fine\,lim}_{z(\in W)\to\zeta} f_1(z) = 0$ for every $\zeta \in \partial_f W$. This means the statement (i). By [7, Theorem 12.6] we get (ii). To show (iii) it is sufficient to prove that $\mathscr{f}\text{-}\lim_{z(\in W)\to\zeta} f_1(z) = 1$ for a. e. $\zeta \in \varDelta_1(U)$. By [5, Corollary 14.2] $f-\lim_{z(\in W)\to\zeta} f(z)=1$ for a. e. $\zeta\in\Delta_1$. Moreover by Lemma 5.2 we find that $f-\lim_{z(\in W)\to\zeta} H_f^W(z)=0$ for a. e. $\zeta\in\Delta_1(U)$ because $f\leqq 1$ in R. Therefore the statement (iii) is proved. q. e. d.

2. We study the behavior of fine potentials at the Martin boundary. For that purpose we show the following:

Proposition 5.1. *Let U be a finely open subset of R such that $i(\partial_f U)=\emptyset$ and u a non-negative finely superharmonic function in U dominated by some non-negative superharmonic function s in R. Then u has a fine limit at almost every point of $\Delta_1(U)$.*

Proof. Let u be a non-negative finely superharmonic function in U dominated by some non-negative superharmonic function s in R. By Riesz' Theorem [5, Theorem 4.6] s is uniquely represented as the sum of a Green potential p in R and a non-negative harmonic function h in R. From the Riesz' decomposition property [7, Lemma 11.14] we see that there exist non-negative finely superharmonic functions u_1 and u_2 in U such that (i) $u_1\leqq p$ in U and $u_2\leqq h$ in U; and (ii) $u=u_1+u_2$ in U. By (i) and [10, Theorem 21] we find that for a. e. $\zeta\in\Delta_1(U)$, $(0\leqq) f-\limsup_{z(\in U)\to\zeta} u_1(z)\leqq f-\lim_{z(\in U)\to\zeta} p(z)=0$ and hence $f-\lim_{z(\in U)\to\zeta} u_1(z)=0$ for a. e. $\zeta\in\Delta_1(U)$. Thus it is sufficient to prove that u_2 has a fine limit at almost every point of $\Delta_1(U)$ because of (ii). To do this, we take a finely open subset U_1 of U as in Lemma 5.3 and let

$$f:=h-u_2-H_{h-u_2}^{U_1},\quad\text{and}$$

$$g:=\begin{cases}\max(f,0) & \text{in } U_1\\ 0 & \text{in } CU_1.\end{cases}$$

First we show that g has a fine limit at almost every point of Δ_1. In the same way as in the proof of Theorem 4.1, we see from Lemma 3.1 that g is a non-negative finely subharmonic function in R and hence we see from [7, Theorem 9.8] that g is a non-negative subharmonic function in R. Hence by [5, Lemma 14.3] we find that g has a fine limit at almost every point of Δ_1, for $g\leqq h$ in R.

Next we prove that u_2 has a fine limit at almost every point of $F_1:=\{\zeta\in\Delta_1(U): f-\lim_{z(\in U)\to\zeta} g(z)>0\}$. Setting $G:=\{z\in U: f(z)>0\}$, we find that $\Delta_1(G)\supset F_1$. Therefore by Theorem 1.1, Lemma 5.2 and [5, Corollary 14.2] we find that u_2 has a fine limit at almost every point of F_1 because $u_2=h-g-H_{h-u_2}^{U_1}$ in G.

Finally we prove that u_2 has a fine limit at almost every point of $F_2:=\Delta_1(U)-F_1$. By the definition of F_2, Theorem 1.1 and Lemma 5.2, for a. e. $\zeta\in F_2$,

$$\begin{aligned}0&\leqq f-\limsup_{z(\in U_1)\to\zeta}(h(z)-u_2(z))=f-\limsup_{z(\in U_1)\to\zeta} f(z)\\ &\leqq f-\lim_{z(\in U_1)\to\zeta} g(z)=0\end{aligned}$$

and so $f-\lim_{z(\in U_1)\to\zeta}(h(z)-u_2(z))=0$ for a. e. $\zeta\in F_2$. Therefore we find that u_2 has a fine limit at almost every point of F_2, for h has a fine limit at almost every point of Δ_1 ([5, Corollary 14.2]). q. e. d.

By the above proposition we immediately can prove the following:

Lemma 5.4. *Let U be a finely open subset of R such that $i(\partial_f U)=\emptyset$ and p a fine potential in U such that* (i) *p is bounded in U;* (ii) *p is finely continuous in $\tilde{U}$; and* (iii) *p is zero on $\partial_f U$. Then p has a fine limit* 0 *at almost every point of $\Delta_1(U)$.*

Proof. Let p be a fine potential in U satisfying the above conditions (i)~(iii). By Proposition 5.1 we find that p has a fine limit at almost every point of $\Delta_1(U)$. Thus we have only to prove that $f\text{-}\liminf_{z(\in U)\to\zeta} p(z)=0$ for a. e. $\zeta\in\Delta_1(U)$. Setting

$$F_\varepsilon:=\{\zeta\in\Delta_1(U):\ f\text{-}\liminf_{z(\in U)\to\zeta} p(z)\geqq\varepsilon\}(\varepsilon>0),$$

we suppose that there exists a positive number η such that $\omega_z(F_\eta)>0$. Let $f(z):=\eta\omega_z(F_\eta)=\eta\int_{F_\eta} k_\zeta(z)\,d\chi(\zeta)$ and $f_1:=f-H_f^U$. Then we see from (ii) and Lemma 3.1 that $\text{fine}\lim_{z(\in U)\to\zeta}(p(z)-f_1(z))=0$ for every $\zeta\in\partial_f U$. By Lemma 5.2 and [5, Corollary 14.2] we have $f\text{-}\liminf_{z(\in U)\to\zeta}(p(z)-f_1(z))\geqq f\text{-}\liminf_{z(\in U)\to\zeta} p(z)-\eta\geqq 0$ for a. e. $\zeta\in F_\eta$ and $f^*\text{-}\liminf_{z(\in U)\to\zeta}(p(z)-f_1(z))\geqq f^*\text{-}\liminf_{z(\in U)\to\zeta} p(z)\geqq 0$ for a. e. $\zeta\in\Delta_1(U)^*-F_\eta$, for $f\geqq H_f^U$ in U. Then we see from Theorem 4.1 that $p\geqq f_1$ in U. This implies that $f_1=0$ in U because p is a fine potential in U. On the other hand Lemma 5.2 and [5, Corollary 14.2] give us that $f\text{-}\lim_{z(\in U)\to\zeta} f_1(z)=\eta$ for a. e. $\zeta\in F_\eta$. This is a contradiction. q. e. d.

We can omit the conditions (i)~(iii) from the above lemma. To see this, we show a property of fine potentials.

Lemma 5.5. *Let U be a finely open subset of R such that $i(\partial_f U)=\emptyset$ and p a finite potential in U. If we take a finely open set V such that $\tilde{V}\subset U$ and $i(\partial_f V)=\emptyset$, then $p_1:=p-H_p^V$ is a fine potential in V such that $\text{fine}\lim_{z(\in U)\to\zeta} p_1(z)=0$ for every $\zeta\in\partial_f V$.*

Proof. Let p be a finite fine potential in U, V a finely open set such that $\tilde{V}\subset U$ and $i(\partial_f V)=\emptyset$, and $p_1:=p-H_p^V$. By Theorem 1.1 and Lemma 3.1 $\text{fine}\lim_{z(\in V)\to\zeta} p_1(z)=0$ for every $\zeta\in\partial_f V$. To show that p_1 is a fine potential in V, suppose that h is a non-negative finely harmonic function in V such that $h\leqq p_1$ in V and we set

$$h_1:=\begin{cases} h & \text{in } V \\ 0 & \text{in } U-V. \end{cases}$$

In the same way as in the proof of Theorem 4.1 we find that h_1 is a non-negative finely subharmonic function in U because $\text{fine}\lim_{z(\in V)\to\zeta} h(z)=0$ for every $\zeta\in\partial_f V$. Thus we find that $h_1=0$ in U, for $h_1\leqq p$ in U. Hence $h=0$ in V. Therefore by [7, Lemma 10.9] p_1 is a fine potential in V. q. e. d.

Proposition 5.2. *Let U be a finely open subset of R and p a fine potential in U. Then p has a fine limit* 0 *at almost every point of $\Delta_1(U)$.*

Proof. We choose a finely open set U_1 as in Lemma 5.3. Let p be a fine potential in U. By Lemma 5.1 p is extended uniquely to $U \cup i(CU)$ as a fine potential in $U \cap i(CU)$. Setting $p_1 := \min(p, 1)$ and $p_2 := p_1 - H_{p_1}^{U_1}$, we see from Theorem 1.1 and Lemmas 3.1 and 5.5 that p_2 is a fine potential in U_1 such that (i) p_2 is bounded in U_1; (ii) p_2 is finely continuous in $\tilde{U}_1$; and (iii) $p_2=0$ on $\partial_f U_1$. Hence by Theorem 1.1 and Lemmas 5.2 and 5.4 we find that p_1 has fine limit 0 at almost every point of $\Delta_1(U)$. Therefore this implies the assertion of this proposition. q. e. d.

The above proposition gives us the behavior of singular finely harmonic functions at the Martin boundary.

Proposition 5.3. *Let U be a finely open subset of R and h a singular finely harmonic function in U. Then h has a fine limit* 0 *at almost every point of $\Delta_1(U)$.*

Proof. Let h be a singular finely harmonic function in U. It is obvious that $h_1 := \min(h, 1)$ is a fine potential in U. Thus by Proposition 5.2 we find that h_1 has a fine limit 0 at almost every point of $\Delta_1(U)$. Therefore this shows the assertion of this proposition.

3. **Proof of Main Theorem.** From Theorems 1.2 and 2.1 we obtain (i). To show (ii), by Propositions 5.2 and 5.3 it is sufficient to prove that u has a fine limit at almost every point of $\Delta_1(U)$ under the assumption of this theorem. Let

$u_n := \min(u, n)$, where n is a positive integer,

$E_1 := \bigcap_{n=1}^{+\infty} E_1^n$, where $E_1^n := \{\zeta \in \Delta_1(U) : f\text{-}\lim_{z(\in U) \to \zeta} u_n(z) = n\}$, and

$E_2 := \bigcap_{n=1}^{+\infty} E_2^n$, where $E_2^n := \{\zeta \in \Delta_1(U) : u_n$ has a fine limit at $\zeta\}$,

Then Proposition 5.1 shows that u_n has a fine limit at almost every point of $\Delta_1(U)$, that is $\omega_z(\Delta_1(U) - E_2^n) = 0$ for every n. Hence

$$\omega_z(\Delta_1(U) - E_2) = \omega_z\Big(\bigcup_{n=1}^{+\infty} (\Delta_1(U) - E_2^n)\Big)$$

$$\leqq \sum_{n=1}^{+\infty} \omega_z(\Delta_1(U) - E_2^n) = 0 \quad \text{and so}$$

$$\omega_z(\Delta_1(U) - E_2) = 0.$$

Thus we have only to prove that u has a fine limit at every point of E_2. First we consider a point $\zeta \in E_2 - E_1$. Since $E_2 - E_1 = \bigcup_{n=1}^{+\infty} (E_2 - E_1^n)$, there exists a positive integer n_0 such that $\zeta \in E_2 - E_1^{n_0}$, that is $f\text{-}\lim_{z(\in U) \to \zeta} u_{n_0}(z) < n_0$. This implies that u has a fine limit at $\zeta \in E_2 - E_1$. Next we consider a point $\zeta \in E_1$. For all n, $n = f\text{-}\lim_{z(\in U) \to \zeta} u_n(z) \leqq$

$f\text{-}\liminf_{z(\in U)\to\zeta} u(z)$, and hence $f\text{-}\lim_{z(\in U)\to\zeta} u(z)=+\infty$. q. e. d.

§6 A Plessner type theorem for finely harmonic functions

First we introduce the notion of fine cluster sets.

Definition 6.1. Let U be a finely open subset of R and f an extended real valued function defined in U. Then we define the fine cluster set $f^{\wedge}(\zeta)$ at $\zeta\in\Delta_1(U)$ as follows:

$$f^{\wedge}(\zeta):=\bigcap_{V\in\mathcal{G}_\zeta}\overline{f(V\cap U)},$$

where $\overline{f(V\cap U)}$ is the closure of $f(V\cap U)$ in $\boldsymbol{R}\cup\{+\infty\}\cap\{-\infty\}$.

Next we state a Lemma on fine open neighborhoods.

Lemma 6.1. *Let ζ be a point of Δ_1 and U a finely open subset of R such that $U\in\mathcal{G}_\zeta$. Then there exists a fine subdomain (=finely connected and finely open subset) V of U such that $V\in\mathcal{G}_\zeta$.*

Proof. Let ζ be a point of Δ_1 and U a finely open subset of R such that $U\in\mathcal{G}_\zeta$. Then there exists a point z of U such that $k_\zeta(z)>\hat{R}^{CU}_{k_\zeta}(z)$. Let V be a fine component of U containing z. We see from Lemma 3.2 and Definition 3.2 that

$$\hat{R}^{CV}_{k_\zeta}=H^V_{k_\zeta}=H^U_{k_\zeta}=\hat{R}^{CU}_{k_\zeta} \quad \text{in } V.$$

Therefore we find that $k_\zeta\neq\hat{R}^{CV}_{k_\zeta}$, that is $V\in\mathcal{G}_\zeta$. q. e. d.

From the above lemma we can get the following:

Lemma 6.2. *Let U be a finely open subset of R and f a finely continuous function in U. Then, for every $\zeta\in\Delta_1(U)$, $f^{\wedge}(\zeta)$ is a closed interval in $\boldsymbol{R}\cup\{+\infty\}\cup\{-\infty\}$.*

Proof. Let ζ be a point of $\Delta_1(U)$ and f a finely continuous function in U. Setting $\mathcal{G}^*_\zeta:=\{V\in\mathcal{G}_\zeta:\ V\subset U$ and V is finely connected$\}$, we see from Lemma 6.1 that $f^{\wedge}(\zeta)=\bigcap_{V\in\mathcal{G}^*_\zeta}\overline{f(V)}$. The general theory of topology says that $f(V)$ is connected for every $V\in\mathcal{G}^*_\zeta$ because f is finely continuous in U. Hence $\overline{f(V)}$ is a closed interval in $\boldsymbol{R}\cup\{+\infty\}\cup\{-\infty\}$ for every $V\in\mathcal{G}^*_\zeta$. This means that $f^{\wedge}(\zeta)$ is a closed interval in $\boldsymbol{R}\cup\{+\infty\}\cup\{-\infty\}$. q. e. d.

Finally we show a Plessner type theorem for finely harmonic functions as an application of Main Theorem (cf. [12])

Theorem 6.1. *Let U be a finely open subset of R and h a finely harmonic function in U. Then for a.e. $\zeta\in\Delta_1(U)$, $h^{\wedge}(\zeta)$ is a singleton or $\boldsymbol{R}\cup\{+\infty\}\cup\{-\infty\}$.*

Proof. Let h be a finely harmonic function in U. We set
$\mathcal{G}^*_\zeta:=\{V\in\mathcal{G}_\zeta:\ V\subset U$ and V is finely connected$\}$ for $\zeta\in\Delta_1(U)$,
$E_1:=\bigcup_{n=1}^{+\infty}\Delta_1(G^n_2)$, where $G^n_2:=\{z\in U:\ h(z)>-n\}$($n$: a positive integer),

$E_2:=\bigcup_{n=1}^{+\infty}\Delta_1(G_2^n)$, where $G_2^n:=\{z\in U\colon h(z)<n\}$, and

$E_3:=\Delta_1(U)-E_1-E_2$.

First we consider the behavior of h at $E_1\cup E_2$. Main theorem states that h has a fine limit at almost every point of $\Delta_1(G_1^n)$. Hence h has a fine limit at almost every point of E_1. In the same way as above, h has a fine limit at almost every point of E_2. Thus we find that $f^{\wedge}(\zeta)$ is a singleton for a. e. $\zeta\in E_1\cup E_2$.

Next we consider the behavior of h at E_3. Taking a point $\zeta\in E_3$, we see from the definition of E_3 that for every $V\in\mathcal{G}_\zeta^*$ and every n, $V\cap CG_1^n\neq\emptyset$ and $V\cap CG_2^n\neq\emptyset$, that is there exist two points z_1 and z_2 of V such that $h(z_1)\leqq -n$ and $h(z_2)\geqq n$. By Lemma 6.2 we find that $h(V)\supset\{x\in\boldsymbol{R}\colon -n\leqq x\leqq n\}$ for every $V\in\mathcal{G}_\zeta^*$ and every n. Hence $\overline{h(V)}\supset\boldsymbol{R}\cup\{+\infty\}\cup\{-\infty\}$ for every $V\in\mathcal{G}_\zeta^*$ since n is arbitrary. Therefore from Lemma 6.1 we conclude that $h^{\wedge}(\zeta)=\boldsymbol{R}\cup\{+\infty\}\cup\{-\infty\}$. q. e. d.

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