J. Math. Kyoto Univ. (JMKYAZ)
30-4 (1990) 625-635

Functional central limit theorem and Strassen's law of the iterated logarithms for weakly multiplicative systems

Dedicated to Prof. N. Ikeda on his 60th birthday

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0. Introduction and Results

The notion of multiplicative systems was first introduced by Alexits [1], [2]. A sequence $\{\xi_n\}$ of random variables is called a uniformly bounded multiplicative system if there exists a constant K such that $|\xi_n|\leqq K$ for all n and $E(\xi_{n_1}\cdots\xi_{n_r})=0$ for all $r\in\boldsymbol{N}$ and $n_1<\cdots<n_r$. Of course sequences of independent random variables and martingale difference sequences are examples of this notion when they are uniformly bounded. But there are other important examples. Those are lacunary trigonometric sequences with Hadamard's gaps i.e. $\{\cos 2\pi n_k x\}$ on $([0, 1], dx)$ when the sequence $\{n_k\}$ of integers satisfies $n_{k+1}/n_k\geqq 2$ for all k. Kolmogorov [10] proved that lacunary trigonometric series having l_2-coefficients converge almost everywhere, and the converse theorem was proved by Zygmund [23]. The central limit theorem for lacunary trigonometric sequences with Hadamard's gaps was hardly studied as Kac [9] summarizes and was completely proved by Salem-Zygmund [20]. These works revealed the weak dependence property of these sequences, and this property was also realized for multiplicative systems by the following studies. Alexits-Sharma [3] proved the law of large numbers for uniformly bounded multiplicative systems (Cf. Preston [16]). The central limit theorem was proved by Révész [17] and the law of the iterated logarithm by Gaposkin [8], Takahashi [22] and Révész [18], [19] under some restrictive condition.

Recently, Móricz [14], [15] extended the notion of multiplicative systems to that of weakly multiplicative systems. Sequence $\{\xi_n\}$ of random variables is called weakly multiplicative system when $\boldsymbol{E}(\xi_{n_1}\cdots\xi_{n_r})$ is nearly 0 in some sense. Mainly we consider weakly multiplicative systems satisfying (0.3). For a sequence $\{\xi_n\}$ of random variables, we define an infinite dimensional vector $\boldsymbol{B}_r=(b_{i_1,\dots,i_r})_{i_1<\dots<i_r}$ for $r\in\boldsymbol{N}$ by $b_{i_1,\dots,i_r}=\boldsymbol{E}(\xi_{i_1}\cdots\xi_{i_r})$ and $\|\boldsymbol{B}_r\|_\delta$ indicates its l_δ-norm, i.e. $\|\boldsymbol{B}_r\|_\delta=(\sum_{i_1<\dots<i_r}|b_{i_1,\dots,i_r}|^\delta)^{1/\delta}$. Móricz [14] proved the following law of the iterated logarithm.

Theorem A. *Let $\{\xi_n\}$ be a sequence of random variables satisfying*

$$(0.1)\qquad |\xi_n|\leqq K \quad \text{for all } n,$$

Received December 13, 1988

(0.2) $\|\boldsymbol{B}_r\|_2<\infty$ *for all* r *and* $\limsup_{r\to\infty}\|\boldsymbol{B}_r\|_2^{1/r}=B<\infty$.

Then,

$$\limsup_{n\to\infty}\frac{S_n}{\sqrt{2(K^2+B^2)A_n^2\log\log A_n^2}}\leqq 1 \quad a.s.$$

where $S_n=a_1\xi_1+\cdots+a_n\xi_n$ *and* $A_n^2=a_1^2+\cdots+a_n^2$ *with* $A_n^2\uparrow\infty$ *as* $n\to\infty$.

Berkes [4] also proved Strassen's law of the iterated logarithms for weakly multiplicative systems satisfying much stronger conditions $\sum_{r=1}^{\infty}\|\boldsymbol{B}_r\|_1<\infty$ and $\sum_{r=1}^{\infty}\|\boldsymbol{B}_r'\|_1<\infty$ where $\boldsymbol{B}_r'$ is a vector defined in the same way as $\boldsymbol{B}_r$ using $\{\xi_j^2-1\}$ instead of $\{\xi_j\}$, i.e. $\boldsymbol{B}_r'=(b'_{i_1,\dots,i_r})_{i_1<\dots<i_r}$ where $b'_{i_1,\dots,i_r}=\boldsymbol{E}((\xi_{i_1}^2-1)\cdots(\xi_{i_r}^2-1))$. There are very important examples of weakly multiplicative systems which do not satisfy Berkes's conditions. For instance, nonharmonic trigonometric sequences with Hadamard's gaps, i.e. $\{\cos\lambda_k x\}$ on $([0, 1], dx)$ when the sequence $\{\lambda_k\}$ of real numbers satisfies $\lambda_k\uparrow\infty$ as $k\to\infty$ and $\lambda_{k+1}/\lambda_k\geqq 2$ for all k, are weakly multiplicative systems satisfying (0.3) as stated in Section 3.

We prove the following functional central limit theorem and Strassen's law of the iterated logarithms. Kôno [11] and Fukuyama [7] proved these theorems for some type of multiplicative systems. We extend these theorems to the case of weakly mutiplicative systems satisfying (0.3).

First we define $\boldsymbol{C}[0, 1]$-valued random variables X_n by $X_n(A_j^2/A_n^2)=S_j/A_n$ and is linear in $[A_j^2/A_n^2, A_{j+1}^2/A_n^2]$ where $S_n=a_1\xi_1+\cdots+a_n\xi_n$.

Theorem 1. *Let* $\{\xi_n\}$ *be a sequence of random variables satisfying* (0.1),

(0.3) $\sup_r\|\boldsymbol{B}_r\|_\delta^{1/r}<\infty$ *for some* $\delta\in[1, 2)$

and either

(0.4) $\lim_{\substack{i+j\to\infty\\ i\neq j}}\boldsymbol{E}((\xi_i^2-1)(\xi_j^2-1))=0$

or

(0.5) $\boldsymbol{E}((\xi_i^2-1)(\xi_j^2-1))\leqq\beta_{|i-j|}$ *for some sequence* $\{\beta_j\}$ *with* $\sum_{n=1}^{\infty}\beta_n<\infty$.

Let $\{a_n\}$ *satisfy*

(0.6) $A_n^2=a_1^2+\cdots+a_n^2\uparrow\infty$ *and* $a_n=o(A_n)$ *as* $n\to\infty$.

Then the distributions of $\{\boldsymbol{X}_n\}$ *converges weakly on* $\boldsymbol{C}[0, 1]$ *to the Wiener measure.*

Under the condition on $\boldsymbol{B}_r'$, we can weaken the condition that $\{\xi_n\}$ is uniformly bounded.

Theorem 2. *Let* $\{\xi_n\}$ *satisfy* (0.3).

(0.7) $\sup_{r\in\boldsymbol{N}}\|\boldsymbol{B}_r'\|_{\delta'}^{1/r}<\infty$ *for some* $\delta'\in[1, 2)$.

Let $\{\xi_n\}$ *and* $\{a_n\}$ *satisfy*

(0.8) $$A_n \uparrow \infty \quad \text{and} \quad a_n\|\xi_n\|_\infty = o(A_n) \quad \text{as } n\to\infty .$$

Then the distributions of $\{S_n/A_n\}$ *converges weakly to the standard normal distribution. And moreover, if we suppose*

(0.9) $$E\xi_n^4 \leqq K \quad \text{for all } n\in \boldsymbol{N},$$

then the distributions of $\{X_n\}$ *converges weakly on* $\boldsymbol{C}[0, 1]$ *to the Wiener measure.*

Theorem 3. (i) *Let* $\{\xi_n\}$ *satisfy* (0.1) *and* (0.3), *and* $\{a_n\}$ *satisfy*

(0.10) $$A_n^2 \uparrow \infty \quad \text{and} \quad a_n^2 = o\left(\frac{A_n^2}{\log\log A_n^2}\right) \quad \text{as } n\to\infty .$$

Then

$$\boldsymbol{P}(\{X_n/\sqrt{2\log\log A_n^2}\} \text{ is relatively compact in } \boldsymbol{C}[0, 1]) = 1 .$$

(ii) *Let* $\{\xi_n\}$ *satisfy* (0.1), (0.3) *and*

(0.11) $$\|\boldsymbol{B}_2'\|_2 < \infty .$$

Let $\{a_n\}$ *satisfy*

(0.12) $$A_n^2 \uparrow \infty \quad \text{and} \quad a_n^2 = o\left(\frac{A_n^2}{(\log A_n^2)^{1+\varepsilon}}\right) \quad \text{as } n\to\infty \text{ for some } \varepsilon>0.$$

Then

$$\boldsymbol{P}(\{\text{The cluster of } \{X_n/\sqrt{2\log\log A_n^2}\} \text{ in } \boldsymbol{C}[0, 1]\}\subset K) = 1 .$$

(iii) *Let* $\{\xi_n\}$ *satisfy* (0.1), (0.3) *and*

(0.13) $$\sup_{r\in \boldsymbol{N}} \|\boldsymbol{B}_r'\|_{\delta'}^{1/r} < \infty \quad \text{for some } \delta'\in[1, 2] .$$

Let $\{a_n\}$ *satisfy*

(0.14) $$A_n^2 \uparrow \infty \quad \text{and} \quad a_n^2 = o\left(\frac{A_n^2}{(\log\log A_n^2)^{\delta/(2-\delta)}}\right) \quad \text{as } n\to\infty .$$

Then

$$\boldsymbol{P}(\{\text{The cluster of } \{X_n/\sqrt{2\log\log A_n^2}\} \text{ in } \boldsymbol{C}[0, 1]\}\subset K) = 1 .$$

(iv) *Moreover if we suppose*

(0.15) $$A_n^2 \uparrow \infty \quad \text{and} \quad a_n = o(A_n^{1-\varepsilon}) \quad \text{as } n\to\infty \text{ for some } \varepsilon>0 ,$$

then we have

$$\boldsymbol{P}(\{\text{The cluster of } \{X_n/\sqrt{2\log\log A_n^2}\} \text{ in } \boldsymbol{C}[0, 1]\} = K) = 1 ,$$

where $K=\left\{x\in \boldsymbol{C}[0, 1] : x(0)=0,\ x \text{ is absolutely continuous and } \int_0^1\left(\frac{dx}{dt}\right)^2 dt \leqq 1\right\}$.

Most important part of proof of these theorems is to prove an estimate in lemma 1. It is very meaningful to prove these theorems in functional form, because the usual central limit theorem and the law of the iterated logarithms follow from these and moreover, other limit theorems can be derived out. (Cf. Billingsley [5] and Strassen [21].)

The author would like to express his heartily thanks to Prof. N. Kono and Prof. S. Watanabe for their advice and encouragement.

1. Proof of Theorem 1 and 2

We use the following lemma due to Móricz [14].

Lemma B. *Under the conditions* (0.1) *and* (0.2), *for all* $\gamma>0$ *and* $\{a_n\}$

$$P(|S_n|\geqq y)\leqq C_\gamma \exp\left(-\frac{y^2}{2(K^2+B^2+\gamma)A_n^2}\right).$$

Using this lemma, we can prove

$$P(|S_n|\geqq y)\leqq C \exp\left(-\frac{y^2}{C'A_n^2}\right) \tag{1.1}$$

under the conditions (0.1) and (0.3) for all $\{a_n\}$ for some constants C and C', because $\sup_{r\in N}\|B_r\|_\delta^{1/r}<\infty$ implies $\sup_{r\in N}\|B_r\|_2^{1/r}<\infty$. We say that sequence of random variables satisfying (1.1) is sub-gaussian. Discussion here asserts that uniformly bounded weakly multiplicative systems are sub-gaussian.

Using (1.1), tightness of sequence $\{X_n\}$ is easily proved. (Cf. Fukuyama [7].) Thus we only have to prove the weak convergence of finite dimensional distributions of $\{X_n\}$. We prove 1-dimensional case using next theorem due to McLeish [13]. After that, multidimensional case becomes trivial because of the well known Cramér-Wold theorem (Cf. Billingsley [5]).

Theorem C. *Let* $\{\zeta_{n,j}: 1\leqq j\leqq k_n\}$ *be a given triangular array of random variables and put* $T_n=\prod_{j\leq k_n}(1+it\zeta_{n,j})$. *Suppose for all real* t,

(a) $E(T_n)\to 1$, (b) $\{T_n\}$ *is uniformly integrable,*

(c) $\sum_{j\leq k_n}\zeta_{n,j}^2\xrightarrow{p}1$ *and* (d) $\max_{j\leq k_n}|\zeta_{n,j}|\xrightarrow{p}0$ *as* $n\to\infty$.

Then the distribution of $\sum_{j\leq k_n}\zeta_{n,j}$ *converges weakly to the standard normal distribution.*

Now we put $k_n=n$ and $\zeta_{n,j}=(a_j/A_n)\xi_j$. Then we have

$$|T_n|\leqq e^{t^2K^2/2} \quad\text{and}\quad \max_{j\leq n}|\zeta_{n,j}|\leqq\frac{K}{A_n}\max_{j\leq n}|a_j|\to 0\,.$$

We prove (a) in the following generalized form for the convenience of the later use.

Lemma 1. *Let* $\{\xi_n\}$ *satisfy* (0.3) *and* $\{a_n\}$ *satisfy* (0.6). *Then*

$$\left|E\prod_{j=1}^n\left(1+\frac{\lambda_{n,j}a_j\xi_j}{A_n}\right)-1\right|\leqq\frac{A_nTe^{1/\varepsilon}}{G_n^{1/2-1/\varepsilon}}\left(\frac{1}{2^\delta-1}\right)^{1/\delta}$$

for large enough n, *where* $\{G_n\}$ *is a real sequence satisfying*

$$G_n \max_{j\le n} a_j^2 \leqq A_n^2 \quad and \quad G_n \uparrow \infty \qquad as\ n\to\infty\,,$$

ε is the dual of δ i.e. $1/\delta+1/\varepsilon=1$ (in case $\delta=1$, $1/\varepsilon=0$.). $T=2\sup_{r\in N}\|B_r\|_\delta^{1/r}$, $\{\Lambda_n\}$ is a real sequence satisfying $\Lambda_n=o(G_n^{1/2-1/\varepsilon})$ as $n\to\infty$ and $\{\lambda_{n,j}\}$ is a triangular array of numbers satisfying $|\lambda_{n,j}|\leqq\Lambda_n$.

For the proof of (a), $\lambda_{n,j}=it$ for all n, j and $\Lambda_n=t$ for all n. Now we prove Lemma 1. In case $\delta=1$,

$$\begin{aligned}\left|E\prod_{j=1}^n\left(1+\frac{\lambda_{n,j}a_j\xi_j}{A_n}\right)-1\right| &= \left|\sum_{r=1}^n\sum_{j_1<\cdots<j_r\le n}\lambda_{n,j_1}\cdots\lambda_{n,j_r}\frac{a_{j_1}\cdots a_{j_r}}{A_n^r}b_{j_1\cdots j_r}\right|\\ &\leqq \sum_{r=1}^n\Lambda_n^r\frac{1}{G_n^{r/2}}\sum_{j_1<\cdots<j_r\le n}|b_{j_1\cdots j_r}|\\ &= \sum_{r=1}^n\left(\frac{T\Lambda_n}{G_n^{1/2}}\right)^r\left(\frac{\|B_r\|_1^{1/r}}{T}\right)^r\end{aligned}$$

Since $T\Lambda_n/G_n^{1/2}\to0$ as $n\to\infty$, for large enough n, $T\Lambda_n/G_n^{1/2}\leqq1$. Thus we have

$$\begin{aligned}&\leqq\frac{T\Lambda_n}{G_n^{1/2}}\sum_{r=1}^n 2^{-r}\\ &\leqq\frac{T\Lambda_n}{G_n^{1/2}}\end{aligned}$$

In case $\delta\in(1,2)$.

$$\begin{aligned}&\left|E\prod_{j=1}^n\left(1+\frac{\lambda_{n,j}a_j\xi_j}{A_n}\right)-1\right|\\ &=\left|\sum_{r=1}^n\sum_{j_1<\cdots<j_r\le n}T^r\lambda_{n,j_1}\cdots\lambda_{n,j_r}\frac{a_{j_1}\cdots a_{j_r}}{A_n^r}\frac{1}{T^r}b_{j_1\cdots j_r}\right|\\ &=\left(\sum_{r=1}^n\sum_{j_1<\cdots<j_r\le n}\left|\lambda_{n,j_1}\cdots\lambda_{n,j_r}T^r\frac{a_{j_1}\cdots a_{j_r}}{A_n^r}\right|^\varepsilon\right)^{1/\varepsilon}\left(\sum_{r=1}^n\sum_{j_1<\cdots<j_r\le n}\frac{1}{T^{\delta r}}|b_{j_1\cdots j_r}|^\delta\right)^{1/\delta}\\ &\leqq\left(\sum_{r=1}^n\sum_{j_1<\cdots<j_r\le n}(\Lambda_nT)^{r\varepsilon}\left|\frac{a_{j_1}\cdots a_{j_r}}{A_n^r}\right|^{\varepsilon-2}\left(\frac{a_{j_1}}{A_n}\right)^2\cdots\left(\frac{a_{j_r}}{A_n}\right)^2\right)^{1/\varepsilon}\left(\sum_{r=1}^n\left(\frac{\|B_r\|_\delta^{1/r}}{T}\right)^{r\delta}\right)^{1/\delta}\\ &\leqq\left(\sum_{r=1}^n\left(\frac{\Lambda_nT}{G_n^{1/2-1/\varepsilon}}\right)^{\varepsilon r}\sum_{j_1<\cdots<j_r\le n}\left(\frac{a_{j_1}}{A_n}\right)^2\cdots\left(\frac{a_{j_r}}{A_n}\right)^2\right)^{1/\varepsilon}\left(\frac{1}{2^\delta-1}\right)^{1/\delta}.\end{aligned}$$

Since $(T\Lambda_n/G_n^{1/2-1/\varepsilon})\to0$ as $n\to\infty$, for large enough n,

$$\begin{aligned}&\leqq\frac{\Lambda_nT}{G_n^{1/2-1/\varepsilon}}\left(\sum_{r=1}^n\sum_{j_1<\cdots<j_r\le n}\left(\frac{a_{j_1}}{A_n}\right)^2\cdots\left(\frac{a_{j_r}}{A_n}\right)^2\right)^{1/\varepsilon}\left(\frac{1}{2^\delta-1}\right)^{1/\delta}\\ &\leqq\frac{\Lambda_nT}{G_n^{1/2-1/\varepsilon}}\left(\frac{1}{2^\delta-1}\right)^{1/\delta}\left(\prod_{j=1}^n\left(1+\frac{a_j^2}{A_n^2}\right)\right)^{1/\varepsilon}.\end{aligned}$$

Making use of $1+x\leqq e$, $\prod_{j=1}^n\left(1+\frac{a_j^2}{A_n^2}\right)\leqq\exp\left(\sum_{j=1}^n\frac{a_j^2}{A_n^2}\right)=e$. Thus we have,

$$\leqq \frac{A_n T}{G_n^{1/2-1/\varepsilon}}\left(\frac{1}{2^\delta-1}\right)^{1/\delta} e^{1/\varepsilon}.$$

This completes the proof of Lemma 1. Next we prove (c). We prove

$$\frac{1}{A_n^2}\sum_{j=1}^n a_j^2\xi_j^2 \xrightarrow{L^2} 1 \quad \text{as } n\to\infty \tag{1.2}$$

when either (0.4) or (0.5) is assumed. From (0.4),

$$E\Big(\frac{1}{A_n^2}\sum_{j=1}^n a_j^2\xi_j^2-1\Big)^2=\frac{1}{A_n^2}\sum_{j=1}^n a_j^4E((\xi_j^2-1)^2)+\frac{2}{A_n^2}\sum_{1\le i<j\le n} a_i^2a_j^2E((\xi_i^2-1)(\xi_j^2-1))$$

$$=\Sigma_1+\Sigma_2,$$

$$\Sigma_1\leqq\frac{(K^2+1)^2}{A_n^4}\sum_{j=1}^n a_j^4\leqq\frac{(K^2+1)^2}{G_nA_n^2}\sum_{j=1}^n a_j^2=\frac{(K^2+1)^2}{G_n}\longrightarrow 0 \quad \text{as } n\to\infty.$$

By (0.4), for all $\varepsilon>0$, there exists N such that $i+j\geqq N$ implies $|E(\xi_i^2-1)(\xi_j^2-1)|<\varepsilon$, and

$$\Big|\frac{1}{A_n^4}\sum_{\substack{1\le i<j\le n\\ i+j\ge N}} a_i^2a_j^2E((\xi_i^2-1)(\xi_j^2-1))\Big|\leqq\frac{\varepsilon}{A_n^4}\sum_{1\le i<j<n} a_i^2a_j^2\leqq\varepsilon.$$

Thus we have $\limsup_{n\to\infty}|\Sigma_2|\leqq\varepsilon$. Since ε is arbitrary, we have proved (1.2) from (0.4). From (0.5)

$$E\Big(\frac{1}{A_n^2}\sum_{j=1}^n a_j^2\xi_j^2-1\Big)^2\leqq\frac{2}{A_n^4}\sum_{r=0}^{n-1}\beta_r\sum_{i=1}^{n-r}a_i^2a_{i+r}^2$$

$$\leqq\frac{2}{G_nA_n^2}\sum_{r=0}^{n-1}\beta_r\sum_{i=1}^{n-r}a_i^2$$

$$\leqq\frac{2}{G_n}\sum_{r=0}^{\infty}\beta_r\longrightarrow 0 \quad \text{as } n\to\infty.$$

This completes the proof of theorem 1.

Next we prove theorem 2. First we check the conditions of Theorem C. (a) and (c) are trivial because (0.11) implies (0.4). To prove (b), it is sufficient to show that $\sup_n E|T_n|^2<\infty$.

$$|T_n|^2=\prod_{j=1}^n\Big(1+\frac{\lambda^2a_j^2\xi_j^2}{A_n^2}\Big)$$

$$=\prod_{j=1}^n\Big(1+\frac{\lambda^2a_j^2}{A_n^2}\Big)\prod_{j=1}^n\Big(1+\frac{\lambda_{n,j}a_j^2(\xi_j^2-1)}{D_n}\Big)$$

where $D_n^2=a_1^4+\cdots+a_n^4$ and $\lambda_{n,j}=D_n/(\lambda^2a_j^2+A_n^2)$. Since $|\lambda_{n,j}|^2\leqq D_n^2/A_n^4\leqq 1/G_n$, we can apply Lemma 1 and prove that $E|T_n|^2$ is bounded. (d) is a direct consequence of (0.8). Tightness is proved by $E(S_n^4)\leqq CA_n^2$, but it is a consequence of (0.7) and (0.9) using the Theorem 1 of Móricz [14].

2. Proof of Theorem 3

The proof of (i) is the same as that of Theorem 7 (1) in Fukuyama [7], because it uses only the sub-gaussian property (1.1) of $\{\xi_n\}$. For the proofs of (ii), (iii) and (iv), we use the following theorem due to Kuelbs [12].

Theorem D. *Assume that*

$$P(\{X_n/\sqrt{2\log\log A_n^2}\} \text{ is relatively compact in } C[0,1]\})=1$$

and for all signed measure ν with bounded variation on [0, 1],

$$P\left(\limsup_{n\to\infty}\frac{\int_0^1 X_n(t)d\nu}{\sqrt{2\log\log A_n^2}}\leqq K_{\nu,1}\right)=1$$

holds. Then we have

$$P(\{\text{The cluster of } \{X_n/\sqrt{2\log\log A_n^2}\} \text{ in } C[0,1]\}\subset K)=1\,.$$

Further more suppose that

$$P\left(\limsup_{n\to\infty}\frac{\int_0^1 X_n(t)d\nu}{\sqrt{2\log\log A_n^2}}=K_{\nu,1}\right)=1\,.$$

then we have

$$P(\{\text{The cluster of } \{X_n/\sqrt{2\log\log A_n^2}\} \text{ in } C[0,1]\}=K)=1\,,$$

where

$$K_{\nu,\theta}^2=E\left[\left(\int_0^1 W(t\wedge\theta^{-1})d\nu(t)\right)^2\right]=\int_0^{\theta^{-1}}(\nu[x,1])^2dx\,.$$

($W(t)$ denotes the standard Brownian motion.)

First we prepare some notations.

Put $N=|\nu|([0,1])$,

$$\phi_{n,j}(t)=\begin{cases}0 & \text{for } t\in\left[0,\dfrac{A_{j-1}^2}{A_n^2}\right]\\[2ex] \dfrac{A_n^2}{a_j^2}\left(t-\dfrac{A_{j-1}^2}{A_n^2}\right) & \text{for } t\in\left[\dfrac{A_{j-1}^2}{A_n^2},\dfrac{A_j^2}{A_n^2}\right] \quad\text{and}\\[2ex] 1 & \text{otherwise}\end{cases}$$

$$c_{n,j}=\int_0^1\phi_{n,j}(t)d\nu(t)\quad\text{and}\quad A_{\nu,n}^2=\sum_{j=1}^n(a_jc_{n,j})^2$$

We have

$$X_n(t)=\frac{1}{A_n}\sum_{j=1}^n a_j\phi_{n,j}(t)\xi_j\quad\text{and}\quad\int_0^1 X_n(t)d\nu(t)=\frac{1}{A_n}\sum_{j=1}^n a_jc_{n,j}\xi_j\,.$$

The order of $A_{\nu,n}^2$ is calculated as follows.

(2.1) $$\lim_{n\to\infty}\frac{A_{\nu,n}^2}{A_n^2}=K_{\nu,1}^2.$$

This formula is easily proved as an application of the functional central limit theorem for the Rademacher sequence $\{r_n\}$. Let Y_n be a $C[0, 1]$-valued random variable defined in the same way as X_n using $\{r_n\}$ instead of $\{\xi_n\}$. Functional central limit theorem and uniform integrability imply

(2.2) $$\lim_{n\to\infty}\boldsymbol{E}\Big[\Big(\int_0^1 X_n(t\wedge\theta^{-1})d\nu(t)\Big)^2\Big]=\boldsymbol{E}\Big[\Big(\int_0^1 \boldsymbol{B}(t\wedge\theta^{-1})d\nu(t)\Big)^2\Big].$$

Putting $\theta=1$ and calculating the expectations, we have (2.1). Now we take $\theta>1$ and take $p(r)$ satisfying $A_{p(r)}^2\leqq\theta^r<A_{p(r)+1}^2$. We derive the conclusion of (ii) and (iii) from (0.1),

(2.3) $$\frac{1}{A_{p(r)}^2}\sum_{j=1}^{p(r)}(a_jc_{p(r),j}\xi_j)^2 \longrightarrow K_{\nu,1}^2 \qquad \text{a.s. as } r\to\infty$$

and

(2.4) $$\Big|\boldsymbol{E}\prod_{j=1}^{n}\Big(1+\frac{c_{n,j}a_j\xi_j}{A_n}\sqrt{2\log\log A_n^2}\Big)\Big|\leqq L \qquad \text{for all } n\in\boldsymbol{N}, \text{ for some } L>0.$$

Thus we first prove (2.3) and (2.4) under the condition of (ii) or (iii). First we asume the condition of (2). Since $|c_{n,j}|\leqq N$, (0.12) and (2.1) implies

$$(a_jc_{n,j})^2=o\Big(\frac{A_{\nu,n}^2}{(\log A_n^2)^{1+\varepsilon}}\Big) \qquad \text{as } n\to\infty.$$

Using this estimate,

$$\begin{aligned}
&\boldsymbol{E}\Big(\frac{1}{A_{\nu,p(r)}^2}\sum_{j=1}^{p(r)}(a_jc_{p(r),j})^2(\xi_j^2-1)^2\Big)^2\\
&=\frac{1}{A_{\nu,p(r)}^4}\sum_{j=1}^{p(r)}(a_jc_{p(r),j})^4\boldsymbol{E}((\xi_i^2-1)^2)\\
&\quad+\frac{2}{A_{\nu,p(r)}^4}\sum_{1\leq i<j\leq p(r)}(a_ic_{p(r),i})^2(a_jc_{p(r),j})^2b'_{i,j}\\
&\leqq\frac{((K^2+1)^2+2\|\boldsymbol{B}'_2\|_2)}{A_{\nu,p(r)}^2}\max_{i\leq p(r)}(a_ic_{p(r),i})^2=O\Big(\frac{1}{(\log A_{p(r)}^2)^{1+\varepsilon}}\Big)=O(r^{-1-\varepsilon}).
\end{aligned}$$

Since this is a term of a convergent series, by the Beppo-Levi theorem, (2.3) is proved.

Next we prove (2.3) under the conditions of (iii). Since $|c_{n,j}|\leqq N$, (0.14) and (2.1) implies

$$(a_jc_{n,j})^2=o\Big(\frac{A_{\nu,n}^2}{(\log\log A_n^2)^{\delta/(2-\delta)}}\Big) \qquad \text{as } n\to\infty.$$

Since $\{\xi^2-1\}$ is sub-gaussian, putting

$$H_n=\frac{A_{\nu,n}^2}{\Big(\max_{j\leq n}|a_jc_{n,j}|^2\Big)\log\log A_n^2}\uparrow\infty,$$

we have

$$P\left(\left|\frac{1}{A_{\nu,p(r)}^2}\sum_{j=1}^{p(r)}(a_j c_{p(r),j})^2(\xi_j^2-1)\right|\geqq\sqrt{2C'/H_{p(r)}}\right)$$

$$\leqq C\exp\left(-\frac{4A_{\nu,n}^4}{H_{p(r)}A_{\nu,p(r)}^2\left(\max_{j\leq p(r)}(a_j c_{p(r),j})^2\right)}\right)$$

$$\leqq C\exp(-4\log\log\theta^r).$$

Since this is a term of a convergent series, by the Borel-Canteil lemma, we have

$$\left|\frac{1}{A_{\nu,p(r)}^2}\sum_{j=1}^{p(r)}(a_j c_{p(r),j})^2(\xi_j^2-1)\right|\leqq\sqrt{2C'/H_{p(r)}}\qquad \text{f. e. a. s.}$$

Thus (2.3) is proved.

Next we derive (2.4) from (0.3) and (0.14). Put

$$c_{n,j}\sqrt{2\log\log A_n^2}=\lambda_{n,j}\quad\text{and}\quad \Lambda_n=N\sqrt{2\log\log A_n^2}\,.$$

By (0.14), we can take $1/G_n=o((\log\log A_n^2)^{-\delta/(2-\delta)})$ and this implies $(T\Lambda_n/G_n^{1/2-1/\varepsilon})=o(1)$. Thus we can apply Lemma 1 and prove (2.4).

Now we derive the final conclusion from (0.1), (2.3) and (2.4), using the method due to S. Takahashi. (Takahashi [22]) Put $\lambda_n=K_{\nu,1}^{-1}\sqrt{2\log\log A_n^2}$. Making use of $e^x\leqq(1+x)\exp(x^2/2+|x|^3)$ $(|x|\leqq 1)$, (2.3), (2.4) and uniform boundedness and taking large enough r, we can prove that

$$E\left[\exp\left(\frac{\lambda_{p(r)}}{A_{p(r)}}\sum_{j=1}^{p(r)}c_{p(r),j}a_j\xi_j-\frac{\lambda_{p(r)}^2}{2A_{p(r)}^2}\sum_{j=1}^{p(r)}\{c_{p(r),j}a_j\xi_j\}^2-(1+2\varepsilon)\frac{K_{\nu,1}^2\lambda_{p(r)}^2}{2}\right)\right]$$

$$\leqq E\prod_{j=1}^{p(r)}\left(1+\frac{c_{p(r),j}a_j\xi_j}{A_{p(r)}}\sqrt{2\log\log A_{p(r)}^2}\right)$$

$$\times\exp\left(\frac{\lambda_{p(r)}^3K^3}{A_{p(r)}^3}\sum_{j=1}^{p(r)}|c_{p(r),j}a_j|^3-(1+2\varepsilon)\frac{K_{\nu,1}^2\lambda_{p(r)}^2}{2}\right)$$

$$\leqq L\exp\left(\frac{\lambda_{p(r)}^3K^3}{A_{p(r)}}N^3\max_{j\leq p(r)}|a_j|-(1+2\varepsilon)\frac{K_{\nu,1}^2\lambda_{p(r)}^2}{2}\right)$$

$$\leqq L\exp(\log\log A_{p(r)}^2 o(1)-(1+2\varepsilon)\log\log A_{p(r)}^2)$$

$$\leqq K'r^{-1-\varepsilon}.$$

Since this is a term of convergent series, by Beppo-Levi's theorem, we have

$$\lim_{r\to\infty}\lambda_{p(r)}^2\left(\frac{1}{\lambda_{p(r)}}\int_0^1 X_{p(r)}d\nu-(1+\varepsilon)K_{\nu,1}^2\right)=-\infty$$

to conclude

$$\limsup_{r\to\infty}\frac{\int_0^1 X_{p(r)}d\nu}{\sqrt{2\log\log A_{p(r)}^2}}\leqq K_{\nu,1}\qquad\text{a. s.}.$$

For given n, take r as $p(r-1)<n\leqq p(r)$. Then

$$\int_0^1 \frac{X_n(t)}{\sqrt{2\log\log A_n^2}}\nu(dt) - \int_0^1 \frac{X_{p(r)}(t)}{\sqrt{2\log\log A_{p(r)}^2}}\nu(dt)$$

$$= \int_0^1 \frac{X_n(t)-X_{p(r)}(t)}{\sqrt{2\log\log A_p^2}}\nu(dt) + \Big(\frac{1}{\sqrt{2\log\log A_n^2}} - \frac{1}{\sqrt{2\log\log A_{p(r)}^2}}\Big)\int_0^1 X_{p(r)}\nu(dt).$$

$$= I_1 + I_2 .$$

$I_2 \to 0$ as $n \to \infty$ is trivial.

$$|I_1| \leq \frac{1}{\sqrt{2\log\log A_{p(r-1)}^2}}\Big(\frac{A_{p(r)}}{A_n}\Big|\int_0^1 \Big\{X_{p(r)}\Big(\frac{A_n^2}{A_{p(r)}^2}t\Big) - X_{p(r)}(t)\Big\}\nu(dt)\Big|$$

$$+ \Big(\frac{A_{p(r)}}{A_n} - 1\Big)\Big|\int_0^1 X_{p(r)}\nu(dt)\Big|\Big).$$

The first part tends to 0 a.s. as $\theta \downarrow 1$ by equi-continuity and the second part is trivial. Thus we have proved

$$\limsup_{n\to\infty} \frac{\int_0^1 X_n d\nu}{\sqrt{2\log\log A_n^2}} \leq K_{\nu,1} \qquad \text{a.s.}$$

Here we end the proof of (2).

The proof of

$$\limsup_{n\to\infty} \frac{\int_0^1 X_n d\nu}{\sqrt{2\log\log A_n^2}} \geq K_{\nu,1} \qquad \text{a.s.}$$

is the same as that in Fukuyama [7], because it use only the sub-gaussian property of $\{\xi_j^2 - 1\}$.

3. Examples

We consider on the sequence $\xi_n = \sqrt{2}\cos\lambda_n x$ on the probability space $([0, 1], dx)$ when the sequence $\{\lambda_k\}$ of real numbers satisfies $\lambda_k \to \infty$ as $k \to \infty$ and $\lambda_{k+1}/\lambda_k \geq 2$ for all k. $\{\xi_n\}$ is a uniformly bounded weakly multiplicative system such that (0.1), (0.3) and (0.7) hold with $\delta = \delta' = 1$. Since

$$|\boldsymbol{E}(\xi_{n_1}\cdots\xi_{n_r})| = \frac{\sqrt{2}^r}{2^{r-1}}\sum_{\pm,\cdots,\pm}\Big|\int \cos(\lambda_{n_r} \pm \cdots \pm \lambda_{n_1})x\,dx\Big| \leq \sqrt{2}^{\,r}(\lambda_{n_r} - \cdots - \lambda_{n_1})^{-1}.$$

$$\sum_{n_1<\cdots<n_r} |\boldsymbol{E}(\xi_{n_1}\cdots\xi_{n_r})| \leq \sum_{0\leq n_1<\cdots<n_r} \sqrt{2}^r(2^{n_r} - \cdots - 2^{n_1})^{-1}$$

$$\leq \sqrt{2}^r \sum_{n_1=1}^{\infty} 2^{-n_1} \sum_{1\leq n_2<\cdots<n_r} (2^{n_r} - \cdots - 2^{n_2} - 1)^{-1}$$

$$\leq \sqrt{2}^r \sum_{0\leq n_2<\cdots<n_r} (2^{n_r} - \cdots - 2^{n_2})^{-1}$$

$$\leq \cdots \leq \sqrt{2}^r .$$

Thus we have $\|\boldsymbol{B}_r\|_1^{1/r} \leq \sqrt{2}$ for all r. Similarly, we have $\|\boldsymbol{B}_r'\|_1^{1/r} \leq \sqrt{2}$.

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