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KO-theory of complex Grassmannians

By

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§ 0. Introduction

Let $M_{m,n}$ be the complex Grassmann manifold $G_m(\mathbf{C}^{m+n})$ of m-planes in $\mathbf{C}^{m+n}$. There is a homeomorphism:

$$M_{m,n} \xrightarrow{\simeq} U(m+n)/U(m) \times U(n).$$

The KO^i-groups of $M_{m,n}$ are studied in [3]. The free parts of them are determined, but the torsion parts are partially known ([3], [4]). Here we compute them for arbitrary m and n, using only the Atiyah-Hirzebruch spectral sequence.

Main Theorem. *Let $k = [\frac{m}{2}]$, $l = [\frac{n}{2}]$, $a = (m, n)$ and $b = (k, l)$. The KO^i-groups of $M_{m,n}$ are as follows:*

i	$m = 2k+1, n = 2l+1$		all other cases
	$k+l =$ even	$k+l =$ odd	
0	$\frac{a}{2}Z \oplus bZ_2$	$\frac{a}{2}Z$	$\frac{a+b}{2}Z$
−1	bZ_2	bZ_2	bZ_2
−2	$\frac{a}{2}Z \oplus bZ_2$	$\frac{a}{2}Z \oplus bZ_2$	$\frac{a-b}{2}Z \oplus bZ_2$
−3	0	bZ_2	0
−4	$\frac{a}{2}Z$	$\frac{a}{2}Z \oplus bZ_2$	$\frac{a+b}{2}Z$
−5	0	0	0
−6	$\frac{a}{2}Z$	$\frac{a}{2}Z$	$\frac{a-b}{2}Z$
−7	bZ_2	0	0

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From this theorem, we have many corollaries about the relations to the complex K-theory of $M_{m,n}$. (See [3] and [2, Theorem 2].) For example,

Corollary. *If m or n is even, the complexification*:

$$c: KO(M_{m,n}) \longrightarrow K(M_{m,n})$$

is a monomorphism.

§ 1. The Atiyah-Hirzebruch spectral sequence

Recall that the coefficient ring of the real K-theory KO is

$$KO^* = \mathrm{Z}[\eta, \alpha, \beta, \beta^{-1}]/(2\eta, \eta^3, \alpha^2 - 4\beta),$$

with $deg\,\eta = -1$, $deg\,\alpha = -4$, $deg\,\beta = -8$.

Consider the Atiyah-Hirzebruch spectral sequence

$$E_r^{*,*} \Longrightarrow KO^*(X), \ E_2^{*,*} \simeq H^*(X\,;KO^*).$$

It is well known that the first differential d_2 is given as follows [1]:

$$(1.1) \qquad d_2^{p,*} = \begin{cases} Sq^2\pi_2 & (\text{if } p \equiv 0 \ (8)) \\ Sq^2 & (\text{if } p \equiv -1 \ (8)) \\ 0 & (\text{otherwise}), \end{cases}$$

where $\pi_2: H^*(X\,;\mathrm{Z}) \to H^*(X\,;\mathrm{Z}_2)$ is modulo 2 reduction.

Here we detect the next possible non trivial differentials.

Proposition 1. *Let X be a CW complex with cells only in even dimensions, and $E_r^{*,*}$ be its Atiyah-Hirzebruch spectral sequence of KO-theory. We have*

$$(1.2) \qquad E_3^{*,-1} \simeq H(H^*(X\,;\mathrm{Z}_2);Sq^2).$$

Suppose there are non trivial differentials d_r ($r \geq 3$). The first one is given by

$$d_r: E_r^{*,0} \longrightarrow E_r^{*,1-r}, \quad r \equiv 2 \ (8),$$

with $x \in E_r^{*,0}$ such that $\eta x \neq 0$ and $\eta d_r x \neq 0$.

Proof. As $H^*(X\,;\mathrm{Z})$ is torsion free, π_2 is epimorphic and we have (1.2). Using the facts that $E_3^{*,q}$ is a torsion free group for $q \equiv 0, -4$ (8), a torsion group for $q \equiv -1, -2$ (8), and all elements in $E_3^{*,even}$ have even total degrees. We see the candidates of the first non trivial differential d_r (≥ 3) are:

(i) $d_r: E_r^{*,-4} \longrightarrow E_r^{*,q}$ $\quad (q \equiv -1 \ (8))$,
(ii) $d_r: E_r^{*,-2} \longrightarrow E_r^{*,q}$ $\quad (q \equiv -1 \ (8))$,
(iii) $d_r: E_r^{*,-1} \longrightarrow E_r^{*,q}$ $\quad (q \equiv -2 \ (8))$,
(iv) $d_r: E_r^{*,0} \longrightarrow E_r^{*,q}$ $\quad (q \equiv -1 \ (8))$.

When $q \equiv -1$ (8), $\eta: E_3^{*,q} \to E_3^{*,q-1}$ is monomorphic. Thus, if $d_r x \in E_r^{*,q}$ and $d_r x \neq 0$, then $\eta d_r x = d_r(\eta x) \neq 0$, hence $\eta x \neq 0$. This makes the cases (i) and

(ii) impossible.

Consider the case (iii). If there is $y \in E_r^{*,-1}$, such that $d_r y \in E_r^{*,q}$ ($q \equiv -2$) and $d_r y \neq 0$, then there is x, such that $y = \eta x$, since $\eta\colon E_3^{*,0} \to E_3^{*,-1}$ is epimorphic. Moreover, we have $d_r x \neq 0$, because $d_r y = d_r(\eta x) = \eta d_r x \neq 0$, and $\eta\colon E_r^{*,q} \to E_r^{*,q-1}$ ($q \equiv -1$) is monomorphic. Consequently, as $x \in E_r^{*,0}$ and $d_r x \in E_r^{*,q}$ ($q \equiv -1$), we can reduce the case (iii) to (iv).

In the case (iv), again considering the monomorphism $\eta\colon E_r^{*,q} \to E_r^{*,q-1}$ ($q \equiv -1$), we have $\eta d_r x \neq 0$, and hence $\eta x \neq 0$.

§2. Computation of $E_3^{*,-1}$

For an arbitrary ring K,

$$H^*(M_{m,n}; K) \simeq K[a_1,\dots,a_m, b_1,\dots,b_n]/(c_1,\dots,c_{m+n}),$$

where a_i and b_i are the images of the Chern classes by maps which arise from the fibre sequence:

$$U(m+n)/U(m) \times U(n) \longrightarrow BU(m) \times BU(n) \longrightarrow BU(m+n),$$

and $c_i = \sum_j a_{i-j} b_j$.

Let $A = H^*(M_{m,n}; Z_2)$ and $d = Sq^2$, then (A, d) is a differential algebra. We compute the homology group $H(A)$.

Proposition 2. *Let B be the algebra*

$$Z_2[a_2{}^2,\dots,a_{2k}{}^2, b_2{}^2,\dots,b_{2l}{}^2]/(c_2{}^2,\dots,c_{2k+2l}{}^2).$$

Then we have the following isomorphisms.

(i) *If* $(m, n) = (2k+1, 2l)$, *then* $H(A) \simeq B$.

(ii) *If* $(m, n) = (2k, 2l)$, *then* $H(A) \simeq B$.

(iii) *If* $(m, n) = (2k+1, 2l+1)$, *then* $H(A) \simeq B \oplus B\ \langle a_{2k+1} b_{2l}\rangle$.

Proof. Let $R = Z_2[a_1,\dots,a_m, b_1,\dots,b_n]$, and $c_i = \sum_j a_{i-j} b_j$. The differentials d of A are given by:

(2.1) $$dx_{2j} = x_{2j+1} + x_1 x_{2j}, \quad dx_{2j+1} = x_1 x_{2j+1},$$

for $x_i = a_i$, b_i, or c_i. We construct inductively R_i by the following short exact sequences:

$$R_1 = R/(c_1).$$

(2.2) $$0 \longrightarrow R_{2j-1} \xrightarrow{\cdot c_{2j+1}} R_{2j-1} \xrightarrow{\pi} R_{2j} \longrightarrow 0.$$

(2.3) $$0 \longrightarrow R_{2j} \xrightarrow{\cdot c_{2j}} R_{2j} \xrightarrow{\pi} R_{2j+1} \longrightarrow 0,$$

for $2j+1 \leq m+n$. The multiplications by c_{2j+1} and by c_{2j} commute with d, thus R_i's are differential modules. We show the following lemma.

Lemma 3. *When* $2j+1 \leq m+n$,

(2.4) $$H(\cdot c_{2j+1}) = 0.$$

(2.5) $$H(\cdot\ c_{2j}) \text{ is monomorphic.}$$

(2.6) $$H(R_{2j}) \simeq H(R_{2j-1}) \oplus H(R_{2j-1})\ \langle c_{2j}\rangle.$$

(2.7) $$H(R_{2j+1}) \simeq \begin{cases} Z_2[a_2{}^2,\dots,a_{2k}{}^2,\ b_2{}^2,\dots,b_{2l}{}^2,\ u]/(c_2{}^2,\dots,c_{2j}{}^2,\ u^2 - a_{2k}{}^2 b_{2l}{}^2) & (\textit{if } (m,n) = (2k, 2l)) \\ Z_2[a_2{}^2,\dots,a_{2k}{}^2,\ b_2{}^2,\dots,b_{2l}{}^2]/(c_2{}^2,\dots,c_{2j}{}^2) & (\textit{otherwise}). \end{cases}$$

Proof of Lemma. We demonstrate the lemma by induction on j. Let $\bar{x}_{2j+1} = x_1 x_{2j+1}$, for $x_i = a_i$, b_i and c_i, respectively.

In R_1, $\bar{c}_{2j+1} = c_{2j+1}$, and (2.1) implies:

$$d\ x_{2j} = \bar{x}_{2j+1},\quad d\ \bar{x}_{2j+1} = 0,$$

for $x_i = a_i$, b_i, and c_i. By easy calculations we have

$$H(R_1) \simeq \begin{cases} Z_2[a_2{}^2,\dots,a_{2k}{}^2,\ b_2{}^2,\dots,b_{2l}{}^2,\ u]/(u^2 - a_{2k}{}^2 b_{2l}{}^2) & (\textit{if } (m,n) = (2k, 2l)) \\ Z_2[a_2{}^2,\dots,a_{2k}{}^2,\ b_2{}^2,\dots,b_{2l}{}^2] & (\textit{otherwise}), \end{cases}$$

This is (2.7) for $j = 0$.

If $d(x) = 0$, then $xc_{2j+1} = d(xc_{2j})$. So (2.4) follows. Consider the long exact sequence derived from homology of (2.2):

$$\cdots \longrightarrow H(R_{2j-1}) \xrightarrow{H(\cdot c_{2j+1})} H(R_{2j-1}) \xrightarrow{H(\pi)} H(R_{2j}) \xrightarrow{\delta} \cdots.$$

As $H(\cdot c_{2j+1}) = 0$, $\delta(c_{2j}) = 1$ and the maps are $H(R_{2j-1})$-module homomorphism, we can conclude (2.6). Consider the long exact sequence derived from homology of (2.3):

(2.8) $$\cdots \longrightarrow H(R_{2j}) \xrightarrow{H(\cdot c_{2j})} H(R_{2j}) \xrightarrow{H(\pi)} H(R_{2j+1}) \xrightarrow{\delta} \cdots.$$

In order to obtain (2.5), i.e., $H(\cdot c_{2j})$ is monomorphic, it suffices to show that $H(\cdot c_{2j}{}^2)$: $H(R_{2j-1}) \to H(R_{2j-1})$ is monomorphic, since the form of $H(R_{2j})$ is given by (2.6). As (2.7) for $H(R_{2j-1})$ is supposed inductively, this is done by using the following fact (Here we rewrite $a_{2i}{}^2$, $b_{2i}{}^2$ and $c_{2i}{}^2$, as α_i, β_i and γ_i, respectively.):

Let

$$S = Z_2[\alpha_1,\dots,\alpha_k,\ \beta_1,\dots,\beta_l]$$

or

$$S = Z_2[\alpha_1,\dots,\alpha_k,\ \beta_1,\dots,\beta_l,\ u]/(u^2 - \alpha_l \beta_k)$$

and $\gamma_i = \sum_j \alpha_{i-j}\beta_j$ *then* $\gamma_1, \gamma_2,\dots,\gamma_{l+k}$ *is regular sequence of S.*

Now, (2.8) splits into the short exact sequence, and

$$H(R_{2j+1}) \simeq \operatorname{Coker} H(\cdot c_{2j}{}^2) \simeq H(R_{2j-1})/(c_{2j}{}^2).$$

Thus we have (2.7) for $H(R_{2j+1})$. This completes the induction.

We continue the proof of Proposition 2. When (m, n) = (2k + 1, 2l), $A \simeq R_{2k+2l+1}$, and hence $H(A) \simeq H(R_{2k+2l+1})$. We get (i). When m + n = even, A is obtained by the next exact sequence:

$$(2.9) \qquad 0 \longrightarrow R_{m+n-1} \xrightarrow{\cdot c_{m+n}} R_{m+n-1} \longrightarrow A \longrightarrow 0.$$

Consider the long exact sequence derived from (2.9). In the case (m, n) = (2k, 2l), $H(\cdot c_{m+n}) = H(\cdot u)$ is monomorphic, since $H(\cdot c_{m+n}{}^2) = H(\cdot \gamma_{m+n})$ is so. Thus we have $H(A) \simeq H(R_{2k+2l-1})/(u)$, and (ii). If (m, n) = (2k + 1, 2l + 1), we have the short exact sequence:

$$0 \longrightarrow H(R_{2k+2l-1}) \longrightarrow H(A) \xrightarrow{\delta} H(R_{2k+2l-1}) \longrightarrow 0,$$

as $H(\cdot c_{m+n}) = 0$. It is easy to check $\delta(a_{2k+1}b_{2l}) = 1$. This implies (iii).

§3. Proof of Main Theorem

Proposition 4. *The Atiyah-Hirzebruch spectral sequence* $E_r^{*,*}$ *for* $KO^*(M_{m,n})$ *collapses for* $r \geq 3$.

Proof. Consider the maps induced by canonical inclusions $U(n) \to Sp(n)$ and $Sp(n) \to U(2n)$:

$$q : M_{m,n} = U(m+n)/U(m) \times U(n) \longrightarrow Sp(m+n)/Sp(m) \times Sp(n)$$
$$c' : Sp(k+l)/Sp(k) \times Sp(l) \longrightarrow U(2k+2l)/U(2k) \times U(2l)$$

It is well known that

$$H^*(Sp(m+n)/Sp(m) \times Sp(n); Z_2) = Z_2[q_1, q_2,\dots,q_m, r_1, r_2,\dots,r_n]/(s_1, s_2,\dots,s_{m+n}),$$

with $deg\ q_i = deg\ r_i = deg\ s_i = 4i$, and

$$q^* q_i = a_i^2,$$
$$c'^* a_i = \begin{cases} q_{i/2} & (\text{if } i = \text{even}) \\ 0 & (\text{if } i = \text{odd}). \end{cases}$$

Similarly r_i corresponds to b_i, and s_i to c_i, under q^* and c'^* respectively. First we consider the case (m, n) = (2k, 2l), that is, $M_{m,n} = U(2k+2l)/U(2k) \times U(2l)$. The Atiyah-Hirzebruch spectral sequence for $KO^*(Sp(m+n)/Sp(m) \times Sp(n))$ collapses, by degree reason. Consider the maps between the Atiyah-Hirzebruch spectral sequences:

$$E_3^{*,q}(q)\colon E_3^{*,q}(Sp(m+n)/Sp(m)\times Sp(n)) \longrightarrow E_3^{*,q}(M_{m,n}),$$
$$E_3^{*,q}(c')\colon E_3^{*,q}(M_{m,n}) \longrightarrow E_3^{*,q}(Sp(k+l)/Sp(k)\times Sp(l)).$$

If $q\equiv -1$ (8), the elements of $E_3^{*,q}(M_{m,n})$ are in the image of $E_3^{*,q}(q)$, and $E_3^{*,q}(c')$ is an monomorphism by Proposition 2 (ii). Hence the triviality of $E_r^{*,q}(Sp(m+n)/Sp(m)\times Sp(n))$ implies $E_r^{*,q}(M_{m,n})\simeq E_3^{*,q}(M_{m,n})(r\geq 3)$. Therefore the non trivial candidates of sources or targets of d_r are in $E_r^{*,q}$, with $q\equiv 0, -2, -4$ (8). So we conclude that $d_r=0$ for $r\geq 3$, since q's concentrate in even degrees.

Next we consider the case (m, n) = (2k, 21 + 1), that is, $M_{m,n}=U(2k+2l+1)/U(2k)\times U(2l+1)$.

Let

$$U(2k+2l)/U(2k)\times U(2l) \xrightarrow{\ i\ } M_{m,n} \xrightarrow{\ j\ } U(2k+2l+2)/U(2k)\times U(2l+2)$$

be the inclusions. By Proposition 2 (i), we know that $E_3^{*,q}(j)$ is epimorphic and $E_3^{*,q}(i)$ is monomorphic for $q\equiv -1$ (8). Thus, because of the triviality of the spectral sequences of the both sides, the non trivial elements of $E_3^{*,q}(M_{m,n})$, $q\equiv -1$, survive permanently. By same arguments as above, we have the theorem for (even, odd)-case.

Finally, we consider the case (m, n) = (2k + 1, 21 + 1), that is, $M_{m,n}=U(2k+2l+2)/U(2k+1)\times U(2l+1)$.

Let

$$U(2k+2l+1)/U(2k+1)\times U(2l) \xrightarrow{\ i\ } M_{m,n} \xrightarrow{\ j\ } U(2k+2l+3)/U(2k+2)\times U(2l+1)$$

be the inclusions. By Proposition 2 (iii), we have

$$E_3^{*,-1}\simeq B\oplus B\ \langle a_{2k+1}b_{2l}\rangle,$$

where $B=Z_2[a_1{}^2,\dots,a_{2k}{}^2, b_1{}^2,\dots,b_{2l}{}^2]/(c_1{}^2,\dots,c_{2k+2l}{}^2)\ \langle\eta\rangle$. Moreover it is clear that $E_3^{*,-1}(i)$ is monomorphic on B, $E_3^{*,-1}(j)$ is surjective onto B and $\mathrm{Ker}(E_3^{*,-1}(i))\simeq B\ \langle a_{2k+1}b_{2l}\rangle$. Therefore B survives permanently, and we can exclude B from this spectral sequence.

Suppose there are non trivial differentials. By Proposition 1, we can conclude that the first non trivial differential ($r\geq 3$) is

$$d_r\colon E_r^{p-r,0}\longrightarrow E_r^{p,q},\quad a_{2k+1}b_{2l}\longmapsto d_r(a_{2k+1}b_{2l}),$$

with $r=1-q$, $q\equiv -1$ (8). Because $p=4k+4l+3-q\equiv 0$ (4), the target is not in $B\ \langle a_{2k+1}b_{2l}\rangle$ but in B, which is already excluded from this spectral sequence. This contradiction affirms the theorem for the case (m, n) = (odd, odd).

Proof of Main Theorem. The rank of the free part of $KO^i(M_{m,n})$ is already given in [3], and

$$\text{Torsion part of } KO^{2i}(M_{m,n}) \simeq KO^{2i+1}(M_{m,n}) \simeq s(Z_2),$$

(See [3, Lemma 2.1].), where s is the dimension of $\bigoplus_{p\equiv 2i+2(8)} E_\infty^{p,-1}$. The theorem follows from Proposition 2 and Proposition 4.

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