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On the asymptotic behavior of Gaussian sequences with stationary increments

By

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1. Introduction

Let $\{X(n);\, n = 1, 2, \ldots\}$ be a stochastic process with discrete time parameter and introduce the following definitions.

Definition 1.1. The function $g(n)$ $(n = 1, 2, \ldots)$ belongs to the upper-upper class of the process $X(n)$ $(g \in \mathrm{UUC}(X))$ if almost surely there exists $n_0 > 0$ such that for all $n \geq n_0$, $X(n) < g(n)$ holds.

Definition 1.2. The function $g(n)$ $(n = 1, 2, \ldots)$ belongs to the upper-lower class of the process $X(n)$ $(g \in \mathrm{ULC}(X))$ if almost surely there exists an infinite sequence $0 < n_1 < n_2 < \cdots \to +\infty$ such that for all k, $g(n_k) \leq X(n_k)$ holds.

Recently, the second author [9] has investigated the asymptotic behavior of the increments of a Wiener process $W(t)$ $(0 \leq t < \infty)$ using the above notion of UUC and ULC obviously modified for processes with continuous time parameter. In order to compare our results with his, we briefly summarize main parts of his results: Let $\tilde{a}_T$ $(0 \leq T < \infty)$ be a real function of T satisfying the conditions

(i) $0 < \tilde{a}_T \leq T$,

(ii) $\tilde{a}_T$ is nondecreasing,

(iii) $T - \tilde{a}_T$ is nondecreasing.

Denote, for $0 \leq T < \infty$

$$X_0(T) = \sup_{0 \leq s \leq \tilde{a}_T} \sup_{0 \leq t \leq T-s} |W(s+t) - W(t)| / \sqrt{\tilde{a}_T}$$

$$X_1(T) = \sup_{0 \leq s \leq \tilde{a}_T} \sup_{0 \leq t \leq T-s} (W(s+t) - W(t)) / \sqrt{\tilde{a}_T}$$

$$X_2(T) = \sup_{0 \leq s \leq \tilde{a}_T} \sup_{0 \leq t \leq T-\tilde{a}_T} |W(s+t) - W(t)| / \sqrt{\tilde{a}_T}$$

$$X_3(T) = \sup_{0 \leq s \leq \tilde{a}_T} \sup_{0 \leq t \leq T-\tilde{a}_T} (W(s+t) - W(t)) / \sqrt{\tilde{a}_T}$$

$$X_4(T) = \sup_{0 \leq t \leq T-\tilde{a}_T} |W(t+\tilde{a}_T) - W(t)| / \sqrt{\tilde{a}_T}$$

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$$X_5(T) = \sup_{0 \le t \le T - \tilde{a}_T} (W(t + \tilde{a}_T) - W(t))/\sqrt{\tilde{a}_T}.$$

For $k \in \mathbf{N}$ (a set of positive integers), set

$$T_k = e^k,\ a_k = \tilde{a}_{T_k},\ b_k = T_k - a_k,\ \gamma_k = \sqrt{\log(T_k/a_k) + \log\log T_k}$$

and

$$\delta_k = a_k/\gamma_k^2. \tag{1.1}$$

We shall make use of the usual notation $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

Theorem A. *If*

$$I_U(g) \equiv \sum_k^\infty \left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp\left(-\frac{1}{2}g^2(T_k)\right) < +\infty,$$

then

$$g \in \mathrm{UUC}(X_i), \qquad i = 0, 1, 2, 3, 4, 5.$$

Theorem B. *If*

$$I_L(g) \equiv \sum_k^\infty \left(\frac{b_{k-1}}{\delta_k} \vee 1\right)\left(\frac{a_k - a_{k-1}}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp\left(-\frac{1}{2}g^2(T_k)\right) = +\infty,$$

then

$$g \in \mathrm{ULC}(X_i), \qquad i = 0, 1, 2, 3, 4, 5.$$

In this paper, we obtain similar results for a class of stationary Gaussian sequences. In discrete parameter cases, our Corollary 2.1 which is presented as an example of our main theorems covers a part of Theorem 1 in Csörgo and Révész [4] and Theorem 3 in Deheuvels and Steinebach [5] (see Remarks 4 and 5 in this paper).

Let $\{\xi_j; j = 1, 2, \ldots\}$ be a centered stationary Gaussian sequence with $E\xi_1 = 0$ and $E\xi_1^2 = 1$. Define $S_n = \xi_1 + \cdots + \xi_n$, $\sigma^2(n) = ES_n^2$ and $r_n = E\xi_1\xi_{1+n}$, $n \ge 1$. Assume that $\sigma(n)$ can be extended to a continuous function $\sigma(t)$ of $t > 0$ which is increasing and regularly varying at infinity with index $0 < \alpha < 1$, given in the canonical form (cf. Feller [6] or Seneta [11]):

$$\sigma(t) = t^\alpha \exp\left(\int_1^t \frac{\varepsilon(y)}{y} dy\right), \tag{1.2}$$

where $\varepsilon(t) \to 0$ as $t \to \infty$. Let $\{\tilde{a}_n; n = 1, 2, \ldots\}$ be a sequence of positive integers satisfying the conditions

(1.3) $1 \le \tilde{a}_n \le n$,

(1.4) $\tilde{a}_n$ is nondecreasing,

(1.5) $n - \tilde{a}_n$ is nondecreasing.

Define discrete parameter processes $X_0, X_1, \dots, X_5$ by

$$X_0(n) = \sup_{1 \le \ell \le \tilde{a}_n} \sup_{1 \le m \le n-\ell} |S_{\ell+m} - S_m|/\sigma(\tilde{a}_n)$$

$$X_1(n) = \sup_{1 \le \ell \le \tilde{a}_n} \sup_{1 \le m \le n-\ell} (S_{\ell+m} - S_m)/\sigma(\tilde{a}_n)$$

$$X_2(n) = \sup_{1 \le \ell \le \tilde{a}_n} \sup_{1 \le m \le n-\tilde{a}_n} |S_{\ell+m} - S_m|/\sigma(\tilde{a}_n)$$

$$X_3(n) = \sup_{1 \le \ell \le \tilde{a}_n} \sup_{1 \le m \le n-\tilde{a}_n} (S_{\ell+m} - S_m)/\sigma(\tilde{a}_n)$$

$$X_4(n) = \sup_{1 \le m \le n-\tilde{a}_n} |S_{m+\tilde{a}_n} - S_m|/\sigma(\tilde{a}_n)$$

$$X_5(n) = \sup_{1 \le m \le n-\tilde{a}_n} (S_{m+\tilde{a}_n} - S_m)/\sigma(\tilde{a}_n),$$

where $\ell, m \in \mathbf{N}$. Clearly we have $X_0(n) \ge X_2(n) \ge X_3(n)$, $X_4(n) \ge X_5(n)$ and $X_0(n) \ge X_1(n) \ge X_3(n) \ge X_5(n)$. For $k \in \mathbf{N}$, set

$$n_k = [e^k],\ a_k = \tilde{a}_{n_k},\ b_k = n_k - a_k,\ \tilde{\gamma}_n = \{\log(n/\tilde{a}_n) + \log\log n\}^{1/2} \quad \text{and } \gamma_k = \tilde{\gamma}_{n_k},$$

where $[y]$ denotes the greatest integer not exceeding y. Define

$$\delta_k = \delta^{-1}(\sigma(a_k)/\gamma_k) \vee 1, \tag{1.6}$$

where $\sigma^{-1}(\cdot)$ is the inverse function of $\sigma(\cdot)$. This δ_k is defined a little bit differently from (1.1), because in the discrete parameter case, we need not consider intervals with length smaller than one. When $\delta_k > 1$, we have the same results as in the continuous cases. But in the case of $\delta_k = 1$, the situations are different (cf. Remark 2).

Remark 1. We are concerned only with the behavior of functions at infinity and therefore all statements about functions in this paper are supposed to hold only at some neighborhood of the infinity.

Our main theorems are presented in the next section 2. In section 3, we state preliminary lemmas necessary in the proofs of the main theorems.

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2. Results

Theorem 2.1. *If*

$$I_U(g) = \sum_k^{\infty}\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp\left(-\frac{1}{2}g^2(n_k)\right) < +\infty,$$

then for $0 < \alpha < 1$

$$g \in \mathrm{UUC}(X_i), \qquad i = 4, 5.$$

Theorem 2.2. *If* $I_U(g) < +\infty$, *then for* $1/2 < \alpha < 1$

$$g \in \mathrm{UUC}(X_i), \qquad i = 0, 1, 2, 3, 4, 5. \tag{2.1}$$

In the special case of $\sigma(t)=\sqrt{t}$ *in* (1.2) (*i.e. Brownian motion case*), (2.1) *also holds.*

Theorem 2.3. *Assume that* $\sigma^2(t)$ *is concave for* $t>0$,

(i) $\quad a_k \le c_1 a_{k-1} \qquad$ *for some* $c_1>1$

and

(ii) $$I_L(g)=\sum_k^{\infty}\left(\frac{b_{k-1}}{\delta_k}\vee 1\right)\left(\frac{a_k-a_{k-1}}{\delta_k}\vee 1\right)\gamma_k^{-1}\exp\left(-\frac{1}{2}g^2(n_k)\right)=+\infty.$$

Then for $0<\alpha\le 1/2$

$$g\in \mathrm{ULC}(X_i),\qquad i=0,1,2,3,4,5.$$

Remark 2. Suppose that $\sigma(t)=\sqrt{t}$. For a choice of $\tilde{a}_T$ such that δ_k in (1.1) satisfy $\delta_k\ge 1$, (then clearly these δ_k also coincide with those given by (1.6)), the criterion $I_U(g)<+\infty$, in Theorem A for the continuous parameter case is exactly the same as that in Theorem 2.1 or Theorem 2.2 for the discrete parameter case. However, in such a case of $\tilde{a}_T$ that δ_k in (1.1) satisfy $\delta_k<1$, (then clearly δ_k in (1.6) is equal to 1), the criterion in Theorem A for the continuous parameter case and in Theorem 2.1 or Theorem 2.2 for the discrete parameter case should differ. In fact choose $\tilde{a}_T=(\log T)^\beta$ with $0<\beta<1$. Then in Theorem A, $I_U(g)<+\infty$ for

$$g(T)=\sqrt{2\log T+(3-2\beta)\log_{(2)}T+2\log_{(3)}T+\cdots+(2+\varepsilon)\log_{(m)}T}$$

if and only if $\varepsilon>0$, where $\log_{(m)}T=\log(\log_{(m-1)}T)$, while in Theorem 2.1 or Theorem 2.2, $I_U(g)<+\infty$ for

$$g(n)=\sqrt{2\log n+\log_{(2)}n+2\log_{(3)}n+\cdots+(2+\varepsilon)\log_{(m)}n}$$

if and only if $\varepsilon>0$.

Remark 3. It may be interesting to ask whether Theorem 2.2 is true or not when $0<\alpha\le 1/2$ and also Theorem 2.3 is true or not when $1/2<\alpha<1$.

Throughout this paper we shall let c denote a positive constant which can be changed in lines if necessary.

Corollary 2.1. *Let the sequence* $\{\tilde{a}_n;\, n=1,2,\ldots\}$ *satisfy the following conditions*

(i) $\quad a_{k+1}\le c_1 a_k \qquad$ *for some* $c_1>1$

(ii) $\quad a_k-a_{k-1}\ge c_2(a_{k+1}-a_k) \qquad$ *for some* $c_2>0$.

Suppose $\sigma(t)=t^\alpha$ *for* $0<\alpha<1$. *For any real* ε *let*

$$d_k=\log(\gamma_k^{1/\alpha}\wedge a_k)+\log\left(\frac{a_{k+1}-a_k}{\delta_k}\vee 1\right)-\log\gamma_k+\varepsilon\log k \tag{2.2}$$

and

$$x_k = \sqrt{2}\gamma_k + \frac{d_k}{\sqrt{2}\gamma_k}.$$

Note that we can define monotone sequences $\tilde{d}_n$ *and* $\tilde{x}_n$ *such that* $d_k = \tilde{d}_{n_k}$ *and* $x_k = \tilde{x}_{n_k}$. *Then we have the following*: *If* $\varepsilon > 0$,

(2.3) $\quad \tilde{x} \in \mathrm{UUC}(X_i),\ i = 4, 5 \qquad$ *for* $0 < \alpha < 1$

(2.4) $\quad \tilde{x} \in \mathrm{UUC}(x_i),\ i = 0, 1, 2, 3, 4, 5 \qquad$ *for* $1/2 \le \alpha < 1$.

If $\varepsilon < 0$,

(2.5) $\quad \tilde{x} \in \mathrm{ULC}(X_i)\ ,\ i = 0, 1, 2, 3, 4, 5 \qquad$ *for* $0 < \alpha \le 1/2$.

Proof. Since $\sigma^{-1}(x) = x^{1/\alpha}$, it follows that

$$\delta_k = \sigma^{-1}(\sigma(a_k)/\gamma_k) \vee 1 = a_k \gamma_k^{-1/\alpha} \vee 1.$$

Let us prove (2.3) and (2.4). We split $I_U(\tilde{x})$ into two sums as follows:

$$\begin{aligned} I_U(\tilde{x}) &= \sum_{\delta_k = 1} \left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp\left(-\frac{1}{2}x_k^2\right) \\ &\quad + \sum_{\delta_k > 1} \left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp\left(-\frac{1}{2}x_k^2\right) \\ &= I^U + II^U, \text{ say.} \end{aligned}$$

As for I^U, $\delta_k = 1$ implies that $a_k \le \gamma_k^{1/\alpha}$. Hence

$$d_k = \log a_k + \log\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right) - \log \gamma_k + \varepsilon \log k,$$

$$\begin{aligned} \exp\left(-\frac{1}{2}x_k^2\right) &\le \exp\left(-\gamma_k^2 - d_k\right) \\ &= \left(\frac{n_k}{a_k}\right)^{-1} k^{-1} a_k^{-1} \left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)^{-1} \gamma_k k^{-\varepsilon} \end{aligned}$$

and by (i)

$$I^U \le c \sum_{\delta_k = 1} n_k \left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp\left(-\frac{1}{2}x_k^2\right) \le c \sum_{\delta_k = 1} k^{-1-\varepsilon} < +\infty.$$

As for II^U, $\delta_k > 1$ implies that $\delta_k = a_k \gamma_k^{-1/\alpha}$, $a_k > \gamma_k^{1/\alpha}$ and

$$\exp\left(-\frac{1}{2}{x_k}^2\right) \le \left(\frac{n_k}{a_k}\right)^{-1} k^{-1} \gamma_k^{-1/\alpha} \left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)^{-1} \gamma_k k^{-\varepsilon}.$$

Thus by (i)

$$II^U \le c \sum_{\delta_k > 1} \left(\frac{n_k}{a_k}\right) \gamma_k^{1/\alpha} \left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right) \gamma_k^{-1} \exp\left(-\frac{1}{2} x_k^2\right)$$

$$= c \sum_{\delta_k > 1} k^{-1-\varepsilon} < +\infty.$$

Therefore from Theorems 2.1 and 2.2 we obtain (2.3) and (2.4) respectively.

Next let us prove (2.5). By (ii),

$$I_L(\tilde{x}) = \sum_k^{\infty} \left(\frac{b_{k-1}}{\delta_k} \vee 1\right) \left(\frac{a_k - a_{k-1}}{\delta_k} \vee 1\right) \gamma_k^{-1} \exp\left(-\frac{1}{2} x_k^2\right)$$

$$\ge c \sum_{\delta_k = 1} \left(\frac{b_{k-1}}{\delta_k} \vee 1\right) \left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right) \gamma_k^{-1} \exp\left(-\frac{1}{2} x_k^2\right)$$

$$+ c \sum_{\delta_k > 1} \left(\frac{b_{k-1}}{\delta_k} \vee 1\right) \left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right) \gamma_k^{-1} \exp\left(-\frac{1}{2} x_k^2\right)$$

$$= I^L + II^L, \text{ say.}$$

Now $\delta_k = 1$ implies that $a_k \le \gamma_k^{1/\alpha} \le c k^{1/(2\alpha)}$ and

$$b_{k-1} = n_{k-1} - a_{k-1} \ge n_{k-1} - a_k \ge c n_k.$$

Also, we have

$$\exp\left(-\frac{1}{2} x_k^2\right) \ge c \exp(-\gamma_k^2 - d_k)$$

because $d_k = o(\gamma_k)$. Thus for any $\varepsilon < 0$

$$I^L \ge c \sum_{\delta_k = 1} n_k \left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right) \gamma_k^{-1} \left(\frac{n_n}{a_k}\right)^{-1} k^{-1} a_k^{-1} \left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)^{-1} \gamma_k k^{-\varepsilon}$$

$$= c \sum_{\delta_k = 1} k^{-1-\varepsilon}.$$

As for II^L, we split II^L onto two sums again:

$$II^L = \sum_{\substack{a_{k-1} \le \frac{1}{2} n_{k-1} \\ \delta_k > 1}} c \left(\frac{b_{k-1}}{\delta_k} \vee 1\right) \left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right) \gamma_k^{-1} \exp\left(-\frac{1}{2} x_k^2\right)$$

$$+ \sum_{\substack{a_{k-1} \le \frac{1}{2} n_{k-1} \\ \delta_k > 1}} c \left(\frac{b_{k-1}}{\delta_k} \vee 1\right) \left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right) \gamma_k^{-1} \exp\left(-\frac{1}{2} x_k^2\right)$$

$$\equiv II_1{}^L + II_2{}^L.$$

As for $II_1{}^L$, $a_{k-1} \le \frac{1}{2} n_{k-1}$ implies that $b_{k-1} = n_{k-1} - a_{k-1} \ge \frac{1}{2} n_{k-1} \ge c n_k$. Also $\delta_k > 1$ imply that $\delta_k = a_k \gamma_k^{-1/\alpha}$ and $a_k > \gamma_k^{1/\alpha}$. Hence

$$\exp\left(-\frac{1}{2}x_k^2\right) = \left(\frac{n_k}{a_k}\right)^{-1} k^{-1}\gamma_k^{-1/\alpha}\left(\frac{a_{k+1}-a_k}{\delta_k} \vee 1\right)^{-1} \gamma_k k^{-\varepsilon} \exp\left(-\frac{d_k^2}{4\gamma_k^2}\right)$$

and by (i)

$$\begin{aligned} II_1{}^L &\geq c \sum_{\substack{a_{k-1}\leq\frac{1}{2}n_{k-1}\\ \delta_k>1}} \left(\frac{n_k}{a_k}\gamma_k{}^{1/\alpha}\right)\left(\frac{a_{k+1}-a_k}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp\left(-\frac{1}{2}x_k^2\right) \\ &= c \sum_{\substack{a_{k-1}\leq\frac{1}{2}n_{k-1}\\ \delta_k>1}} k^{-1-\varepsilon} \exp\left(-\frac{d_k^2}{4\gamma_k^2}\right) \\ &\geq c \sum_{\substack{a_{k-1}\leq\frac{1}{2}n_{k-1}\\ \delta_k>1}} k^{-1-\varepsilon} \exp\left\{-\frac{\varepsilon^2(\log k)^2}{4\{\log(n_k/a_k)+\log k\}}\right\} \\ &\geq c \sum_{\substack{a_{k-1}\leq\frac{1}{2}n_{k-1}\\ \delta_k>1}} k^{-1-\varepsilon-\frac{\varepsilon^2}{4}}. \end{aligned}$$

As for $II_2{}^L$, $a_{k-1} > \frac{1}{2}n_{k-1}$ implies that

$$\frac{n_k}{a_k} \leq \frac{en_{k-1}}{a_{k-1}} < 2e \qquad \text{and } \gamma_k \leq c\sqrt{\log k}.$$

Using (i) again, we have

$$\begin{aligned} II_2{}^L \geq & \sum_{\substack{a_{k-1}\leq\frac{1}{2}n_{k-1}\\ \delta_k>1}} c\left(\frac{a_{k+1}-a_k}{\delta_k} \vee 1\right)\gamma_k^{-1}\left(\frac{n_k}{a_k}\right)^{-1} k^{-1}\gamma_k^{-1/\alpha} \\ & \cdot\left(\frac{a_{k+1}-a_k}{\delta_k} \vee 1\right)^{-1} \gamma_k k^{-\varepsilon} \exp\left(-\frac{d_k{}^2}{4\gamma_k{}^2}\right) \\ \geq c & \sum_{\substack{a_{k-1}\leq\frac{1}{2}n_{k-1}\\ \delta_k>1}} (2e)^{-1}k^{-1-\varepsilon}(\log k)^{-\frac{1}{2\alpha}} \exp\left\{-\frac{\varepsilon^2(\log k)^2}{4\{\log(n_k/a_k)+\log k\}}\right\} \\ \geq c & \sum_{\substack{a_{k-1}\leq\frac{1}{2}n_{k-1}\\ \delta_k>1}} (\log k)^{-\frac{1}{2\alpha}} k^{-1-\varepsilon-\frac{\varepsilon^2}{4}}. \end{aligned}$$

Combining the above estimates we have $II_1^L + II_2^L = +\infty$. Therefore (2.5) immediately follows from Theorem 2.3.

From Corollary 2.1, we can easily prove the following Corollaries 2.2 and 2.3.

Corollary 2.2. *Under the assumptions of Corollary* 2.1, *we have*

$$\limsup_{n\to\infty} \frac{2\tilde{\gamma}_n^2}{\tilde{d}_n}\left(\frac{X_i(n)}{\sqrt{2}\tilde{\gamma}_n} - 1\right) = 1, \qquad \text{a.s.} \tag{2.6}$$

where $i = 4, 5$ *for* $0 < \alpha < 1/2$, *or* $i = 0, 1, 2, 3, 4, 5$ *for* $\alpha = 1/2$.

Corollary 2.3. *Under the assumptions of Corollary* 2.2, *we have*

$$\limsup_{n\to\infty} \frac{X_i(n)}{\{2\{\log(n/\tilde{a}_n) + \log\log n\}\}^{1/2}} = 1, \quad \text{a.s.} \tag{2.7}$$

Further assume that the sequence $\{\tilde{a}_n;\, n = 1, 2, \dots\}$ *satisfies the condition*

$$\lim_{n\to\infty} \frac{\log\log n}{\{\log(n/\tilde{a}_n)\}^{1/2}} = 0.$$

Then we have

$$\limsup_{n\to\infty} \{X_i(n) - \{2\log(n/\tilde{a}_n)\}^{1/2}\} = 0, \quad \text{a.s.} \tag{2.8}$$

Remark 4. In the discrete parameter case, the above (2.7) yields (1) and (2) of Theorem 1 in Csörgo and Révész [4]. It may be interesting to compare the above (2.8) with Corollary 2.1 in Révész [10] and Theorem 2.1 in Choi [2].

Corollary 2.4. *Suppose that the assumptions of Corollary* 2.2 *are satisfied.*
Case 1. *Let*

(i) $\lim_{k\to\infty} \dfrac{\log a_k}{\log k} = \beta_1 < +\infty,$

(ii) $\lim_{k\to\infty} \dfrac{\log(a_{k+1} - a_k)}{\log k} = \beta_2.$

Then

$$\limsup_{n\to\infty} \frac{2\tilde{\gamma}_n^2}{\log\log n}\left(\frac{X_i(n)}{\sqrt{2}\,\tilde{\gamma}_n} - 1\right) \tag{2.9}$$

$$= \left(\frac{1}{2\alpha} \wedge \beta_1\right) + \left\{\left\{\beta_2 - \left(\beta_1 - \frac{1}{2\alpha}\right) \vee 0\right\} \vee 0\right\} - \frac{1}{2}, \quad \text{a.s.}$$

Case 2. *Let*

(iii) $\lim_{k\to\infty} \dfrac{\log a_k}{\log k} = +\infty,$

(iv) $\lim_{k\to\infty} \dfrac{\log\{(a_{k+1} - a_k)/a_k\}}{\log k} = -\beta_3 \quad (\beta_3 \geq 0),$

(v) $\lim_{k\to\infty} \dfrac{\log \gamma_k}{\log k} = \beta_4.$

Then

(2.10)
$$\limsup_{n\to\infty} \frac{2\tilde{\gamma}_n^2}{\log\log n}\left(\frac{X_i(n)}{\sqrt{2}\tilde{\gamma}_n} - 1\right)$$
$$= \frac{\beta_4}{\alpha} + \left\{\left(\frac{\beta_4}{\alpha} - \beta_3\right) \vee 0\right\} - \beta_4, \qquad \text{a.s.}$$

Proof. Let us first prove (2.9). By (i) we note that for large k we have approximately

$$a_k \sim k^{\beta_1} \qquad \text{and} \qquad \gamma_k \sim \sqrt{k}.$$

Hence

$$\lim_{k\to\infty} \frac{\log(\gamma_k{}^{1/\alpha} \wedge a_k)}{\log k} = \frac{1}{2\alpha} \wedge \beta_1.$$

Since $\delta_k = a_k \gamma_k^{-1/\alpha} \vee 1$, it follows that

$$\lim_{k\to\infty} \frac{\log \delta_k}{\log k} = \left(\beta_1 - \frac{1}{2\alpha}\right) \vee 0$$

and from (ii)

$$\lim_{k\to\infty} \frac{\log\{\{(a_{k+1} - a_k)/\delta_k\} \vee 1\}}{\log k}$$
$$= \lim_{k\to\infty} \frac{\{\log(a_{k+1} - a_k) - \log \delta_k\} \vee 0}{\log k}$$
$$= \left\{\beta_2 - \left(\beta_1 - \frac{1}{2\alpha}\right) \vee 0\right\} \vee 0.$$

Since

$$\lim_{k\to\infty} \frac{d_k}{\log k} = \left(\frac{1}{2\alpha} \wedge \beta_1\right) + \left\{\left\{\beta_2 - \left(\beta_1 - \frac{1}{2\alpha}\right) \vee 0\right\} \vee 0\right\} - \frac{1}{2},$$

(2.9) immediately follows from Corollary 2.2.

Next let us prove (2.10). (iii) implies that for M big enough there exists $k_0 \geq 1$ such that for all $k \geq k_0$, $a_k > k^M$ holds. In general there exists $c > 0$ such that $\gamma_k \leq c\sqrt{k}$. Thus $\gamma_k^{1/\alpha} < a_k$ and by (v)

$$\log(\gamma_k^{1/\alpha} \wedge a_k) = \frac{1}{\alpha} \log \gamma_k \sim \frac{\beta_4}{\alpha} \log k.$$

Since $\delta_k > 1$, it follows that

$$\log\left(\frac{a_{k+1}-a_k}{\delta_k}\vee 1\right)=\left\{\log\left(\frac{a_{k+1}-a_k}{a_k}\right)+\frac{1}{\alpha}\log\gamma_k\right\}\vee 0$$

and by (iv)

$$\lim_{k\to\infty}\frac{\log\{([a_{k+1}-a_k]/\delta_k)\vee 1\}}{\log k}=\left(\frac{\beta_4}{\alpha}-\beta_3\right)\vee 0.$$

These facts yield (2.10).

Corollary 2.5. *Suppose that the assumptions of Corollary* 2.4 *are satisfied. Further assume that*

$$\text{(vi)}\qquad \lim_{n\to\infty}\frac{\log\log n}{\log(n/\tilde{a}_n)}=0.$$

Let

$d_k^*=\log(\gamma_k^{1/\alpha}\wedge a_k)+\log\left(\frac{a_{k+1}-a_k}{\delta_k}\vee 1\right)-\log\gamma_k$ *and* $\tilde{d}_n^*$ *be a monotone sequence such that* $\tilde{d}_{n_k}^*=d_k^*$. *Then*

$$\text{(2.11)}\qquad \limsup_{n\to\infty}\frac{2\log(n/\tilde{a}_n)}{\log\log n}\left(\frac{X_i(n)}{\{2\log(n/\tilde{a}_n)\}^{1/2}}-1\right)$$
$$=1+\lim_{n\to\infty}\frac{\tilde{d}_n^*}{\log\log n},\qquad \text{a.s.}$$

Proof. From (vi) we have

$$\begin{aligned}\tilde{\gamma}_n&=\{\log(n/\tilde{a}_n)+\log\log n\}^{1/2}\\&=\{\log(n/\tilde{a}_n)\}^{1/2}\left\{1+\frac{\log\log n}{2\log(n/\tilde{a}_n)}+o\left(\frac{\log\log n}{\log(n/\tilde{a}_n)}\right)\right\}\\&=\{\log(n/\tilde{a}_n)\}^{1/2}+\frac{\log\log n}{2\{\log(n/\tilde{a}_n)\}^{1/2}}+o\left(\frac{\log\log n}{\{\log(n/\tilde{a}_n)\}^{1/2}}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\tilde{\gamma}_n}&=\frac{1}{\{\log(n/\tilde{a}_n)\}^{1/2}}\left(1+\frac{\log\log n}{\log(n/\tilde{a}_n)}\right)^{-1/2}\\&=\frac{1}{\{\log(n/\tilde{a}_n)\}^{1/2}}\left\{1-\frac{\log\log n}{2\log(n/\tilde{a}_n)}+o\left(\frac{\log\log n}{\log(n/\tilde{a}_n)}\right)\right\}.\end{aligned}$$

It follows from Corollary 2.1 that for $\varepsilon>0$

$$X_i(n) < \sqrt{2}\tilde{\gamma}_n + \frac{\tilde{d}_n}{\sqrt{2}\tilde{\gamma}_n}$$

$$= \{2\log(n/\tilde{a}_n)\}^{1/2} + \frac{\log\log n}{\{2\log(n/\tilde{a}_n)\}^{1/2}} + o\left(\frac{\log\log n}{\{\log(n/\tilde{a}_n)\}^{1/2}}\right)$$

$$+ \frac{\tilde{d}_n}{\{2\log(n/\tilde{a}_n)\}^{1/2}}\left\{1 - \frac{\log\log n}{2\log(n/\tilde{a}_n)} + o\left(\frac{\log\log n}{\log(n/\tilde{a}_n)}\right)\right\}$$

and

$$X_i(n) - \{2\log(n/\tilde{a}_n)\}^{1/2} < \frac{\log\log n + \tilde{d}_n}{\{2\log(n/\tilde{a}_n)\}^{1/2}} + o\left(\frac{\log\log n}{\{\log(n/\tilde{a}_n)\}^{1/2}}\right).$$

Thus

$$\frac{\{2\log(n/\tilde{a}_n)\}^{1/2}}{\log\log n}\{X_i(n) - \{2\log(n/\tilde{a}_n)\}^{1'2}\} < 1 + \frac{\tilde{d}_n}{\log\log n} + o(1)$$

and since $\varepsilon > 0$ is arbitrary we have

$$\limsup_{n\to\infty}\frac{2\log(n/\tilde{a}_n)}{\log\log n}\left(\frac{X_i(n)}{\{2\log(n/\tilde{a}_n)\}^{1/2}} - 1\right)$$

$$\leq 1 + \liminf_{k\to\infty}\frac{\tilde{d}^*_{n_k}}{\log\log n_k}, \qquad \text{a.s.}$$

On the contrary we can also obtain for $\varepsilon < 0$

$$\limsup_{n\to\infty}\frac{2\log(n/\tilde{a}_n)}{\log\log n}\left(\frac{X_i(n)}{\{2\log(n/\tilde{a}_n)\}^{1/2}} - 1\right)$$

$$\geq 1 + \liminf_{k\to\infty}\frac{\tilde{d}^*_{n_k}}{\log\log n_k}, \qquad \text{a.s.}$$

Remembering, as in the proof of Corollary 2.4,

$$\lim_{k\to\infty}\frac{d^*_k}{\log k} = \lim_{k\to\infty}\frac{\log(\gamma_k^{1/\alpha}\wedge a_k) + \log\{([a_{k+1} - a_k]/\delta_k)\vee 1\} - \log\gamma_k}{\log k}$$

$$= \begin{cases} \left(\dfrac{1}{2\alpha}\wedge\beta_1\right) + \left\{\left\{\beta_2 - \left(\beta_1 - \dfrac{1}{2\alpha}\right)\vee 0\right\}\vee 0\right\} - \dfrac{1}{2} & \text{in Case 1} \\ \dfrac{\beta_4}{\alpha} + \left\{\left(\dfrac{\beta_4}{\alpha} - \beta_3\right)\vee 0\right\} - \beta_4 & \text{in Case 2,} \end{cases}$$

(2.11) immediately follows.

Remark 5. From (2.11) of Corollary 2.5 we get the following sharp result:

$$\limsup_{n\to\infty} \frac{2\log(n/\tilde{a}_n)}{\log\log n}\left(\frac{X_i(n)}{\{2\log(n/\tilde{a}_n)\}^{1/2}} - 1\right)$$

$$= \begin{cases} \left(\frac{1}{2\alpha} \wedge \beta_1\right) + \left\{\left\{\beta_2 - \left(\beta_1 - \frac{1}{2\alpha}\right) \vee 0\right\} \vee 0\right\} + \frac{1}{2} & \text{in Case 1} \\ \frac{\beta_4}{\alpha} + \left\{\left(\frac{\beta_4}{\alpha} - \beta_3\right) \vee 0\right\} - \beta_4 + 1 & \text{in Case 2.} \end{cases}$$

This is a general form, under milder conditions, of the Theorem 3 (6.5) in Deheuvels and Steinebach [5], which is the above Case 2 in Remark 5 with $\alpha = 1/2$, $\beta_3 = 1$ and $\beta_4 = 1/2$.

3. Preliminary lemmas

We list several lemmas essential to prove our theorems.

Lemma 3.1. *For $t > 0$, let $\sigma(t)$ be a regularly varying function with index $0 < \alpha < 1$ at ∞, given in the form of*

$$\sigma(t) = t^\alpha \exp\left(\int_1^t \frac{\varepsilon(y)}{y} dy\right),$$

where $\varepsilon(t) \to 0$ as $t \to \infty$. Then we have the following: For any $0 < \eta < \alpha$ there exists $T = T(n) \geq 1$ such that

(3.1) $$\lambda^{\alpha+\eta} \leq \frac{\sigma(\lambda t)}{\sigma(t)} \leq \lambda^{\alpha-\eta} \qquad \textit{for } 0 < \lambda \leq 1 \textit{ and } \lambda t \geq T,$$

(3.1)′ $$\lambda^{\alpha-\eta} \leq \frac{\sigma(\lambda t)}{\sigma(t)} \leq \lambda^{\alpha+n} \qquad \textit{for } \lambda \geq 1 \textit{ and } t \geq T,$$

(3.2-a) $$\lambda^{1/(\alpha-\eta)} \leq \frac{\sigma^{-1}(\lambda t)}{\sigma^{-1}(t)} \leq \lambda^{1/(\alpha+\eta)} \qquad \textit{for } 0 < \lambda \leq 1 \textit{ and } \lambda^{1/(\alpha-\eta)}\sigma^{-1}(t) \geq T,$$

(3.2-b) *for some $K(\eta) \geq \kappa(\eta) > 0$ and $K'(\eta) \geq k'(\eta) > 0$,*

$$\kappa(\eta) t^{\alpha-\eta} \leq \sigma(t) \leq K(\eta) t^{\alpha+\eta} \qquad \textit{for } t \geq T \textit{ and}$$

$$\kappa'(\eta) t^{1/(\alpha+\eta)} \leq \sigma^{-1}(t) \leq K'(\eta) t^{1/(\alpha-\eta)} \qquad \textit{for } t \geq T,$$

(3.2-c) $$a_k \gamma_k^{-1/(\alpha-\eta)} \leq \delta_k \leq a_k \gamma_k^{-1/(\alpha+\eta)} \qquad \textit{for } a_k \gamma_k^{-1/(\alpha-\eta)} \geq T,$$

where $\delta_k = \sigma^{-1}(\sigma(a_k)/\gamma_k) \vee 1$.

Proof. As for (3.1), we have for $0 < \lambda \leq 1$

$$\lambda^\alpha \exp\left(-\left|\int_t^{\lambda t} \frac{\varepsilon(y)}{y} dy\right|\right) \leq \frac{\sigma(\lambda t)}{\sigma(t)} \leq \lambda^\alpha \exp\left(\left|\int_t^{\lambda t} \frac{\varepsilon(y)}{y} dy\right|\right)$$

and

$$\left|\int_t^{\lambda t} \frac{\varepsilon(y)}{y} dy\right| \leq \int_{\lambda t}^t \frac{|\varepsilon(y)|}{y} dy.$$

By the assumption, for any $\eta > 0$ we can choose $T(\eta)$ such that for all $y \geq T(\eta)$, $|\varepsilon(y)| < \eta$ holds. Thus if $\lambda t \geq T$

$$\left|\int_t^{\lambda t} \frac{\varepsilon(y)}{y} dy\right| \leq \eta \int_{\lambda t}^t \frac{1}{y} dy = \log \lambda^{-\eta} \quad \text{and} \quad \lambda^{\alpha+\eta} \leq \frac{\sigma(\lambda t)}{\sigma(t)} \leq \lambda^{\alpha-\eta}.$$

As for (3.1)′, noting that we have for $\lambda \geq 1$

$$\left|\int_t^{\lambda t} \frac{\varepsilon(y)}{y} dy\right| \leq \int_t^{\lambda t} \frac{|\varepsilon(y)|}{y} dy,$$

we can easily deduce (3.1)′ by the same fashion as above.

Next let us prove (3.2-a). We shall make use of the result (3.1). Suppose that $0 < \lambda \leq 1$ and $\lambda^{1/(\alpha-\eta)}\sigma^{-1}(t) \geq T$. Since $\lambda^{1/(\alpha+\eta)}\sigma^{-1}(t) \geq T$, we have

$$\frac{\sigma(\lambda^{1/(\alpha+\eta)}\sigma^{-1}(t))}{\sigma(\sigma^{-1}(t))} \geq (\lambda^{1/(\alpha+\eta)})^{\alpha+\eta} = \lambda,$$

that is,

$$\lambda^{1/(\alpha+\eta)}\sigma^{-1}(t) \geq \sigma^{-1}(\lambda t).$$

Therefore,

$$\lambda^{1/(\alpha+\eta)} \geq \sigma^{-1}(\lambda t)/\sigma^{-1}(t).$$

In the same way as above, we can also obtain

$$\lambda^{1/(\alpha-\eta)} \leq \sigma^{-1}(\lambda t)/\sigma^{-1}(t).$$

As for (3.2-b), we take $B > 0$ such that $|\varepsilon(y)| \leq B$ for all $y \geq 1$. By the assumption, for any $\eta > 0$ there exists $T(\eta) \geq 1$ such that for all $y \geq T(\eta)$, $|\varepsilon(y)| < \eta$ holds. Thus for any $t \geq T(\eta)$

$$\begin{aligned}\sigma(t) &= t^\alpha \exp\left(\int_1^{T(\eta)} \frac{\varepsilon(y)}{y} dy\right) \exp\left(\int_{T(\eta)}^t \frac{\varepsilon(y)}{y} dy\right) \\ &\leq t^\alpha \exp\left(\int_1^{T(\eta)} \frac{B}{y} dy\right) \exp\left(\int_{T(\eta)}^t \frac{\eta}{y} dy\right) \leq K(\eta) t^{\alpha+\eta},\end{aligned}$$

where $K(\eta) = T(\eta)^{B-\eta}$. In the same way we also have $\sigma(t) \geq \kappa(\eta) t^{\alpha-\eta}$ where $\kappa(\eta) = T(\eta)^{-(B-\eta)}$.

Finally we prove (3.2-c). In order to apply (3.2-a), set $\lambda = 1/\gamma_k$ and $t = \sigma(a_k)$. Then

$$\lambda^{1/(\alpha-\eta)}\sigma^{-1}(t) = a_k \gamma_k^{-1/(\alpha-\eta)} \geq T(\eta).$$

By (3.2-a),

$$(1/\gamma_k)^{1/(\alpha-\eta)} \leq \frac{\sigma^{-1}(\sigma(a_k)/\gamma_k)}{\sigma^{-1}(\sigma(a_k))} \leq (1/\gamma_k)^{1/(\alpha+\eta)},$$

that is,

$$a_k \gamma_k^{-1/(\alpha-\eta)} \leq \sigma^{-1}(\sigma(a_k)/\gamma_k) \leq a_k \gamma_k^{-1/(\alpha+\eta)}.$$

Since $a_k \gamma_k^{-1/(\alpha-\eta)} \geq T(\eta) \geq 1$, we have (3.2-c).

In the same way as in the proof of Lemma 1 in Kôno [9], we can easily obtain the following

Lemma 3.2. *It is enough to prove Theorems* 2.1–2.3 *for the function* g *which satisfies for* c *big enough*

(i) $\quad \frac{1}{2}\gamma_k \leq g(n_k) \leq c\gamma_k$

(ii) $\quad \lim_{k\to\infty}\left(\frac{b_{k-1}}{\delta_k} \vee 1\right)\left(\frac{a_k - a_{k-1}}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp\left(-\frac{1}{2}g^2(n_k)\right) = 0$

when $I_L(g) = +\infty$.

We also need the following Lemma 3.3 due to Kôno [8], Berman and Kôno [1]: Let $\{X(t);\, t\in T\}$ be a centered Gaussian process such that for all $t \in T$

$$0 < \underline{\sigma} \leq \sqrt{Ex^2(t)} \equiv \sigma(t) \leq \bar{\sigma} < +\infty.$$

Let $d_X^2(s, t) = E\{X(s) - X(t)\}^2$ and (T, d_X) be a compact pseudo-metric space and assume that sample paths of $X(t)$ are separable and measurable. Denote

$$\Phi(x) = P(X(t)/\sigma(t) \geq x).$$

For $\varepsilon > 0$ let $N_X(T, \varepsilon)$ be a minimal number of balls $B(t, \varepsilon) \equiv \{s \in T;\, d_X(s, t) \leq \varepsilon\}$ whose union covers T.

Lemma 3.3. *For* $\sqrt{2} \geq \varepsilon_j > 0$, $\lambda_{j+1} > 0$, $j = 0, 1, 2, \dots$, *and any* $x > 0$

$$P\{\sup_{t\in T} X(t) \geq \bar{\sigma}(x + \sum_{j=0}^{\infty} \varepsilon_j \lambda_{j+1})\}$$
$$\leq \left\{N_X(T, \underline{\sigma}\varepsilon_0) + 3\sum_{j=1}^{\infty} N_X(T, \underline{\sigma}\varepsilon_j) \exp\left(-\frac{1}{2}\lambda_j^2\right)\right\}\Phi(x).$$

Lemma 3.4. *Let* $\{B_n;\, n = 1, 2, \dots\}$ *be a sequence of events satisfying the conditions*

(i) $\quad \sum_{n=1}^{\infty} P(B_n) = +\infty$

(ii) *there exist* $c>0$ *and* $\varepsilon>0$ *such that the following hold: for each* n *there exists a finite subsequence* $n \le n_1 < n_2 < \cdots < n_{i(n)}$ *such that*

(ii-a) $\sum_{k=1}^{i(n)} P(B_n \cap B_{n_k}) \le c\,P(B_n)$

(ii-b) $P(B_n \cap B_m) \le B_m) \le (1+\varepsilon) P(B_n) P(B_m)$ *holds for* $m \ne n_k$ *and* $n < m$.

Then

$$P(B_n, \text{i.o.}) \ge 1/(1+\varepsilon).$$

Lemma 3.5. (Chung-Erdös-Siral [3]) *Let* (X, Y) *be a two dimensional random variable with the Gaussian distribution such that* $EX = EY = 0$, $EX^2 = EY^2 = 1$ *and* $EXY = \mathbf{r}$. *Then*

(3.3) $P(X \ge x, Y \ge y) \le c_\varepsilon \Phi(x)\Phi(y)$ *for any* $-1 < \mathbf{r} < \varepsilon/(xy)$ *and* $x, y > 0$, *where* $\lim_{\varepsilon \to 0} c_\varepsilon = 1$.

(3.4) *for some* $c > 0$, $P(X \ge x, Y \ge y) \le c \exp\{-(1-\mathbf{r})y^2/4\}\Phi(x)$ *for any* $y \ge x \ge 0$.

4. Proofs of Theorems

Throughout this section we assume that for a fixed $k \in \mathbf{N}$, x_k is in the range $\frac{1}{3}\gamma_k \le x_k \le c\gamma_k$ where c is big enough, according to Lemma 3.2. Denote for some $c>0$, $y_k = x_k + \frac{c}{x_k}$. The proof of Theorem 2.1 is mainly based on the following Lemma 4.1.

Lemma 4.1. *For* $0 < \alpha < 1$ *in* (1.2), *we have*

$$P\left\{\sup_{a_k \le \ell \le a_{k+1}} \sup_{1 \le m \le n_{k+1} - a_{k+1}} \frac{|S_{\ell+m} - S_m|}{\sigma(\ell)} \ge y_k\right\}$$
$$\le c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)\Phi(y_k).$$

Proof. Set $A_k = \{(\ell, m);\, a_k \le \ell \le a_{k+1},\, 1 \le m \le n_{k+1} - a_{k+1}\}$ and $X(\ell, m) = (S_{\ell+m} - S_m)/\sigma(\ell)$, $(\ell, m) \in A_k$. Then $EX^2(\ell, m) = 1 \equiv \bar\sigma \equiv \underline{\sigma}$, say, in Lemma 3.3. For $\ell \ne \ell'$ and $m \ne m'$ we have

$$\begin{aligned} d_X{}^2((\ell, m), (\ell', m')) &\equiv E\{X(\ell, m) - X(\ell', m')\}^2 \\ &= 2 - \frac{2E\{(S_{\ell+m} - S_m)(S_{\ell'+m'} - S_{m'})\}}{\sigma(\ell)\sigma(\ell')} \\ &= 2 - \frac{E\{(S_{\ell+m} - S_m)^2 + (S_{\ell'+m'} - S_{m'})^2 - (S_{\ell+m} - S_{\ell'+m'} + S_{m'} - S_m)^2\}}{\sigma(\ell)\sigma(\ell')} \end{aligned}$$

$$\leq \frac{E\{[(S_{\ell+m} - S_{\ell'+m'}) + (S_{m'} - S_m)]^2\}}{\sigma(\ell)\sigma(\ell')}$$

$$\leq \frac{2\{E[(S_{\ell+m} - S_{\ell'+m'})^2] + E[(S_{m'} - S_m)^2]\}}{\sigma(\ell)\sigma(\ell')} \leq \frac{4\sigma^2(|\ell - \ell'| + |m - m'|)}{\sigma^2(a_k)}.$$

In order to apply Lemma 3.3, let us define

$$\zeta(\varepsilon) = \sigma^{-1}(\varepsilon\sigma(a_k)), \tag{4.1}$$

and for $j = 0, 1, 2, \ldots$, let

$$\varepsilon_j = (j+1)^{-3}/\gamma_k,\ \lambda_j = j,\ x = x_k$$

and $\zeta_j = \zeta(\varepsilon_j)$. Clearly

$$1 \leq \zeta_j = \{\sigma^{-1}(\varepsilon_j\sigma(a_k))\} \vee 1 = \{\sigma^{-1}((j+1)^{-3}\sigma(a_k)/\gamma_k)\} \vee 1$$

and

$$\begin{aligned} \bar{\sigma}(x + \sum_{j=0}^{\infty} \varepsilon_j\lambda_{j+1}) &= x_k + \sum_{j=0}^{\infty} (j+1)^{-2}/\gamma_k \\ &\leq x_k + \frac{c}{x_k} \equiv y_k. \end{aligned} \tag{4.2}$$

Let us estimate a lower bound of ζ_j. First we consider the case where for some $0 < \eta < \alpha$, there exists $T(\eta) \geq 1$ such that

$$(j+1)^{-3/(\alpha-\eta)}(\sigma^{-1}(\sigma(a_k)/\gamma_k)) \geq T(\eta).$$

Set $\lambda = (j+1)^{-3}$ and $t = \sigma(a_k)/\gamma_k$ in Lemma 3.1 (3.2-a). Then $\lambda^{1/(\alpha-\eta)}\sigma^{-1}(t) \geq T(\eta)$ and from Lemma 3.1 (3.2-a)

$$\frac{\sigma^{-1}((j+1)^{-3}\sigma(a_k)/\gamma_k)}{\sigma^{-1}(\sigma(a_k)/\gamma_k)} \geq (j+1)^{-3/(\alpha-\eta)}.$$

Hence

$$\zeta_j \geq \{(j+1)^{-3/(\alpha-\eta)}\sigma^{-1}(\sigma(a_k)/\gamma_k)\} \vee 1. \tag{4.3}$$

Next considering the case where

$$(j+1)^{-3/(\alpha-\eta)}(\sigma^{-1}(\sigma(a_k)/\gamma_k) \geq T(\eta).$$

we have

$$\begin{aligned} &\delta_k \leq T(\eta)(j+1)^{-3/(\alpha-\eta)} \text{ and} \\ &\frac{b_{k+1}}{\zeta_j} \vee 1 \leq b_{k+1} \leq T(\eta)(j+1)^{3/(\alpha-\eta)}\left(\frac{b_{k+1}}{\delta_k} \vee 1\right). \end{aligned} \tag{4.4}$$

Thus

$$N_X(A_k, \varepsilon_j) \le c\,N(A_k, \zeta_j) \le C\left(\frac{n_{k+1} - a_{k+1}}{\zeta_j} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\zeta_j} \vee 1\right) \tag{4.5}$$
$$\le c(j+1)^{6/(\alpha-\eta)}\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)$$

(where $N(A, \varepsilon)$ is a minimal number of balls by a metric $d((\ell, m), (\ell', m')) = |\ell - \ell'| + |m - m'|$ whose union covers $A \subset Z^2$) and

$$N_X(A_k, \varepsilon_0) + 3\sum_{j=1}^{\infty} N_X(A_k, \varepsilon_j)\exp\left(-\frac{1}{2}\lambda_j^2\right)$$
$$\le c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)\left\{1 + 3\sum_{j=1}^{\infty}(j+1)^{6/(\alpha-\eta)}\exp\left(-\frac{1}{2}\lambda_j^2\right)\right\} \tag{4.6}$$
$$\le c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right).$$

We note that for some $c > 0$

$$\Phi(x_k) \le c\,\Phi(y_k). \tag{4.7}$$

Therefore from (4.2), (4.6), (4.7) and Lemma 3.3 we have

$$P\left\{\sup_{(\ell, m)\in A_k} \frac{|S_{\ell+m} - S_m|}{\sigma(\ell)} \ge y_k\right\} \le P\left\{\sup_{(\ell, m)\in A_k} \frac{|S_{\ell+m} - S_m|}{\sigma(\ell)} \ge \bar{\sigma}(x_k + \sum_{j=0}^{\infty}\varepsilon_j\lambda_{j+1})\right\}$$
$$\le \left\{N_X(A_k, \varepsilon_0) + 3\sum_{j=1}^{\infty} N_X(A_k, \varepsilon_j)\exp\left(-\frac{1}{2}\lambda_j^2\right)\right\}\Phi(x_k)$$
$$\le c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)\Phi(y_k).$$

For proving Theorem 2.2, we shall need the following Lemmas 4.2 and 4.3. For $k \in \mathbf{N}$ let α_k be such that

$$\alpha_k = 1 - \frac{\delta_k}{a_k} > 0 \text{ and they increase to } 1. \tag{4.8}$$

Lemma 4.2. *For* $1/2 < \alpha < 1$ *in* (1.2), *we have*

$$P\left\{\sup\nolimits_{1 \le \ell \le a_k} \sup\nolimits_{1 \le m \le n_{k+1} - \ell} \frac{|S_{\ell+m} - S_m|}{\sigma(a_k)} \ge y_k\right\} \tag{4.9}$$
$$\le c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)\Phi(y_k).$$

In particular if $\sigma(t) = \sqrt{t}$ *in* (1.2), *then* (4.9) *also holds.*

Proof. Set $A_k^{(1)} = \{(\ell, m);\ 1 \le \ell \le a_k,\ 1 \le m \le n_{k+1} - \ell\}$. Set $q_0 = \min\{q;$ non-negative integer such that $s_{k,q} \equiv a_k\alpha_k^q \ge 1\}$. For $q \le q_0$. Set

$$\Lambda_q^k = \{(\ell, m);\ s_{k,q+1} \le \ell \le s_{k,q},\ 1 \le m \le n_{k+1} - \ell\}$$

and

$$X(\ell, m) = \frac{S_{\ell+m} - S_m}{\sigma(a_k)}, \qquad (\ell, m) \in \Lambda_q^k.$$

Now

$$\underline{\sigma} \equiv \frac{\sigma(s_{k,q+1})}{\sigma(a_k)} \le \sqrt{EX^2(\ell, m)} \le \frac{\sigma(S_{k,q})}{\sigma(a_k)} \equiv \bar{\sigma}$$

and for $\ell \ne \ell'$, $m \ne m'$ we have

$$d_X{}^2((\ell, m), (\ell', m')) \le \frac{4}{\sigma^2(a_k)} \sigma^2(|\ell - \ell'| + |m - m'|).$$

In order to apply Lemma 3.3, let us define

$$\sigma^{-1}(\underline{\sigma}\varepsilon\sigma(a_k)) \vee 1 \equiv \zeta(\varepsilon), \tag{4.10}$$

and for $j = 0, 1, 2, \ldots$ set

$$\varepsilon_j = \frac{\sigma(a_k)(j+1)^{-3}}{\sigma(s_{k,q+1})\gamma_k}, \quad \lambda_j = j, \quad x = \frac{\sigma(a_k)x_k}{\sigma(s_{k,q})}$$

and $\zeta_j = \zeta(\varepsilon_j)$. Clearly

$$1 \le \zeta_j = \sigma^{-1}(\underline{\sigma}\varepsilon_j\sigma(a_k)) \vee 1 = \sigma^{-1}((j+1)^{-3}\sigma(a_k)/\gamma_k) \vee 1,$$

$$\bar{\sigma}x = x_k \quad \text{and} \quad \bar{\sigma}\sum_{j=0}^{\infty} \varepsilon_j \lambda_{j+1} \le \frac{\sigma(s_{k,q})}{\sigma(s_{k,q+1})}\left(\frac{c}{\gamma_k}\right).$$

Using Lemma 3.1 (3.1), we can easily obtain $\sigma(s_{k,q})/\sigma(s_{k,q+1}) \le c$ for large k. Thus we have

$$\bar{\sigma}(x + \sum_{j=0}^{\infty} \varepsilon_j \lambda_{j+1}) \le x_k + \frac{c}{x_k} \equiv y_k. \tag{4.11}$$

We note that $s_{k,q} - s_{k,q+1} \le \delta_k$. Thus we have for $j = 0, 1, 2, \ldots$

$$\begin{aligned} N_X(\Lambda_q^k, \underline{\sigma}\varepsilon_j) &\le c\,N(\Lambda_q^k, \zeta_j) \le c\left(\frac{n_{k+1}}{\zeta_j} \vee 1\right)\left(\frac{s_{k,q} - s_{k,q+1}}{\zeta_j} \vee 1\right) \\ &\le c(j+1)^{6/(\alpha-\eta)}\left(\frac{n_{k+1}}{\delta_k} \vee 1\right)\left(\frac{s_{k,q} - s_{k,q+1}}{\delta_k} \vee 1\right) \\ &\le c(j+1)^{6/(\alpha-\eta)}\left(\frac{n_{k+1}}{\delta_k} \vee 1\right) \\ &\le c(j+1)^{6/(\alpha-\eta)}\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right). \end{aligned} \tag{4.12}$$

The last inequality is verified as below: If $a_{k+1} \leq \frac{1}{2}n_{k+1}$, then $b_{k+1} \geq \frac{1}{2}n_{k+1}$ and

$$\frac{n_{k+1}}{\delta_k} \vee 1 \leq c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right) \leq c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1}-a_k}{\delta_k} \vee 1\right).$$

On the other hand if $a_{k+1} > \frac{1}{2}n_{k+1}$, then $a_{k+1} - a_k \geq \left(\frac{1}{2} - \frac{1}{e}\right)n_{k+1}$ and

$$\frac{n_{k+1}}{\delta_k} \vee 1 \leq c\left(\frac{a_{k+1}-a_k}{\delta_k} \vee 1\right) \leq c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1}-a_k}{\delta_k} \vee 1\right).$$

Now (4.12) yields

$$\begin{aligned}(4.13)\qquad & N_X(\Lambda_q^k, \underline{\sigma}\varepsilon_0) + 3\sum_{j=1}^{\infty} N_X(\Lambda_q^k, \underline{\sigma}\varepsilon_j)\exp\left(-\frac{1}{2}\lambda_j^2\right) \\ & \leq c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1}-a_k}{\delta_k} \vee 1\right).\end{aligned}$$

Again from Lemma 3.1 (3.1) we have for $0 < \eta < \alpha$ and large k

$$(4.14)\qquad \frac{\sigma(s_{k,q})}{\sigma(a_k)} \leq \alpha_k^{q(\alpha-\eta)} \leq 1, \quad \frac{\sigma^2(a_k)}{\sigma^2(s_{k,q})} \geq \alpha_k^{-2q(\alpha-\eta)}$$

and

$$\frac{\sigma^2(a_k)}{\sigma^2(s_{k,0})} = 1.$$

It follows from (4.14) that

$$\begin{aligned}& \sum_{q=0}^{q_0} \Phi\left(\frac{\sigma(a_k)x_k}{\sigma(s_{k,q})}\right) \leq c\frac{1}{x_k}\sum_{q=0}^{q_0} \exp\left(-\frac{\sigma^2(a_k)x_k^2}{2\sigma^2(s_{k,q})}\right) \\ (4.15)\qquad & = c\frac{1}{x_k}\exp\left(-\frac{1}{2}x_k^2\right)\left(1 + \sum_{q=1}^{q_0}\exp\left\{-\frac{1}{2}\left(\frac{\sigma^2(a_k)}{\sigma^2(s_{k,q})} - 1\right)x_k^2\right\}\right) \\ & \leq c\frac{1}{x_k}\exp\left(-\frac{1}{2}x_k^2\right)\left(1 + \sum_{q=1}^{q_0}\exp\left\{-\frac{1}{2}[\alpha_k^{-2q(\alpha-\eta)} - 1]x_k^2\right\}\right).\end{aligned}$$

We claim that

$$(4.16)\qquad \alpha_k^{-2(\alpha-\eta)q} - 1 \geq \frac{q\delta_k}{a_k}.$$

Indeed, take η such that $\alpha > \alpha - \eta \geq 1/2$. Setting $\beta_k = \alpha_k^{2(\alpha-\eta)}$, we have $0 < \beta_k \leq \alpha_k < 1$ and by (4.8)

$$\begin{aligned}\alpha_k^{-2(\alpha-\eta)q} - 1 &= (\beta_k^{-1} - 1)(\beta_k^{-q+1} + \cdots + 1) \geq (\beta_k^{-1} - 1)q \\ &\geq (1 - \beta_k)q \geq (1 - \alpha_k)q = q\delta_k/a_k.\end{aligned}$$

Now we shall prove

$$\frac{\delta_k}{a_k} x_k^2 \geq c \qquad \text{for some } c > 0. \tag{4.17}$$

Suppose that for any $0 < \eta < \alpha$ there exists $T = T(\eta) \geq 1$ such that

$$a_k < T \gamma_k^{1/(\alpha-\eta)}.$$

Taking η such that $\alpha > \alpha - \eta \geq 1/2$, it follows that

$$\frac{\delta_k}{a_k} x_k^2 \geq \frac{x_k^2}{T \gamma_k^{1/(\alpha-\eta)}} \geq \frac{1}{9T} \gamma_k^{2 - \frac{1}{\alpha-\eta}} \geq c.$$

In the case that $a_k \geq T \gamma_k^{1/(\alpha-\eta)}$, (4.17) immediately follows from Lemma 3.1 (3.2-c). Therefore the relations (4.16) and (4.17) yield

$$(\alpha_k^{-2(\alpha-\eta)q} - 1) x_k^2 \geq q \left\{ \frac{\delta_k}{a_k} x_k^2 \right\} \geq cq$$

and

$$\sum_{q=1}^{q_0} \exp\left\{ -\frac{1}{2} [\alpha_k^{-2q(\alpha-\eta)} - 1] x_k^2 \right\} < +\infty. \tag{4.18}$$

In the sequel the upper bound of (4.15) reaches to

$$\sum_{q=0}^{q_0} \Phi\left(\frac{\sigma(a_k) x_k}{\sigma(s_{k,q})} \right) \leq c \frac{1}{x_k} \exp\left(-\frac{1}{2} x_k^2 \right) \leq c \Phi(x_k) \leq c \Phi(y_k). \tag{4.19}$$

It is easy to check that (4.19) also remains true when $\sigma(t) = \sqrt{t}$ in (1.2). From (4.11), (4.13), (4.19) and Lemma 3.3 we have

$$\begin{aligned}
& P\left\{ \sup_{(\ell,m) \in A_k^{(1)}} \frac{|S_{\ell+m} - S_m|}{\sigma(a_k)} \geq y_k \right\} \\
& \leq \sum_{q=0}^{q_0} P\left\{ \sup_{(\ell,m) \in A_q^k} \frac{|S_{\ell+m} - S_m|}{\sigma(a_k)} \geq \bar{\sigma}(x + \sum_{j=0}^{\infty} \varepsilon_j \lambda_{j+1}) \right\} \\
& \leq \sum_{q=0}^{q_0} \left\{ N_X(A_q^k, \underline{\sigma} \varepsilon_0) + 3 \sum_{j=1}^{\infty} N_X(A_q^k, \underline{\sigma} \varepsilon_j) \exp\left(-\frac{1}{2} \lambda_j^2 \right) \right\} \Phi(x) \\
& \leq c \left(\frac{b_{k+1}}{\delta_k} \vee 1 \right) \left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1 \right) \sum_{q=0}^{q_0} \Phi\left(\frac{\sigma(a_k) x_k}{\sigma(s_{k,q})} \right) \\
& \leq c \left(\frac{b_{k+1}}{\delta_k} \vee 1 \right) \left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1 \right) \Phi(y_k).
\end{aligned}$$

Lemma 4.3. *For* $1/2 < \alpha < 1$ *in* (1.2), *we have*

$$P_k \equiv P\left\{\sup_{n_k \le n \le n_{k+1}} \sup_{\substack{a_k \le \ell \le \tilde{a}_n \\ 1 \le m \le n-\ell}} \frac{|S_{\ell+m} - S_m|}{\sigma(\tilde{a}_n)} \ge y_k\right\} \tag{4.20}$$

$$\le c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)\Phi(y_k).$$

In particular if $\sigma(t) = \sqrt{t}$ *in* (1.2), *then* (4.20) *also holds.*

Proof. (i) *The case of* $a_{k+1} \le \frac{3}{4} n_{k+1}$.

Set $A_k^{(2)} = \{(\ell, m); a_k \le \ell \le a_{k+1}, 1 \le m \le n_{k+1} - \ell\}$ and $X(\ell, m) = (S_{\ell+m} - S_m)/\sigma(\ell)$, $(\ell, m) \in A_k^{(2)}$. By the same way as in the proof of Lemma 4.1, we can easily deduce that

$$P_k \le P\left\{\sup_{(\ell, m) \in A_k^{(2)}} \frac{|S_{\ell+m} - S_m|}{\sigma(\ell)} \ge y_k\right\}$$

$$\le c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)\Phi(y_k). \tag{4.21}$$

Note that (4.21) holds for $0 < \alpha < 1$.

(ii) *The case of* $a_{k+1} \ge \frac{3}{4} n_{k+1}$.

Using our conditions, we get the following:

$$a_{k+1} - a_k \ge \frac{3}{4} n_{k+1} - n_k = \left(\frac{3}{4}e - 1\right) n_k \ge a_k \tag{4.22}$$

$$a_k \ge n_k - (n_{k+1} - a_{k+1}) \ge \left(1 - \frac{e}{4}\right) n_k \tag{4.23}$$

$$\Delta_k \equiv n_{k+1} - n_k \le 4(e-1)\, a_k \tag{4.24}$$

$$a_{k+1} - a_k \le (4e - 1)\, a_k \ (\text{hence } a_{k+1} \le 4ea_k). \tag{4.25}$$

Set $i_k = [(a_{k+1} - a_k)/\delta_k]$ and for $i = 1, 2, \ldots, i_k$,

$$n_{k,i} = n_k + i\Delta_k[\delta_k/(a_{k+1} - a_k)] \text{ and } a_{k,i} = \tilde{a}_{n_{k,i}}.$$

It follows from (1.5), (4.22) and (4.24) that

$$a_{k,i+1} - a_{k,i} \le n_{k,i+1} - n_{k,i} = \Delta_k[\delta_k/(a_{k+1} - a_k)] \tag{4.26}$$

$$\le \Delta_k \delta_k / a_k \le 4(e-1)\, \delta_k.$$

Setting $\Lambda_i^k = \{(\ell, m); a_k \le \ell \le a_{k,i}, 1 \le m \le n_{k,i} - \ell\}$, we have

$$P_k \le \sum_{i=1}^{i_k - 1} P\left\{\sup_{(\ell, m) \in \Lambda_{i+1}^k - \Lambda_i^k} \frac{|S_{\ell+m} - S_m|}{\sigma(a_{k,i} \vee \ell)} \ge y_k\right\}. \tag{4.27}$$

Let

$$\Lambda_{i,1}^k = \{(\ell, m);\, a_{k,i} < \ell \le a_{k,i+1},\ 1 \le m \le n_{k,i+1} - \ell\}$$

and

$$\Lambda_{i,2}^k = \{(\ell, m);\, a_k \le \ell \le a_{k,i},\ n_{k,i} \le \ell + m \le n_{k,i+1}\}.$$

Obviously we have

$$\Lambda_{i+1}^k - \Lambda_i^k \subset \Lambda_{i,1}^k \cup \Lambda_{i,2}^k. \tag{4.28}$$

STEP 1. *For $0 < \alpha < 1$ in (1.2), we have*

$$P\left\{ \sup_{(\ell, m) \in \Lambda_{i,1}^k} \frac{|S_{\ell+m} - S_m|}{\sigma(\ell)} \ge y_k \right\} \le c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\Phi(y_k).$$

Proof. Setting $X(\ell, m) = (S_{\ell+m} - S_m)/\sigma(\ell)$, $(\ell, m) \in \Lambda_{i.1}^k$, we have $EX^2(\ell, m) = 1 \equiv \bar{\sigma} \equiv \underline{\sigma}$ and for $\ell \ne \ell'$, $m \ne m'$,

$$d_X{}^2((\ell, m), (\ell', m')) \le \frac{4}{\sigma^2(a_{k,i})}\sigma^2(|\ell - \ell'| + |m - m'|).$$

Let us define $\zeta(\varepsilon) = \{\sigma^{-1}(\varepsilon\sigma(a_{k,i}))\} \vee 1$, and for $j = 0, 1, 2, \ldots$, let $\varepsilon_j = (j+1)^{-3}/\gamma_k$, $\lambda_j = j$, $x = x_k$ and $\zeta_j = \zeta(\varepsilon_j)$. Clearly

$$\bar{\sigma}(x + \sum_{j=0}^{\infty} \varepsilon_j \lambda_{j+1}) \le x_k + \frac{c}{x_k} \equiv y_k.$$

Estimating a lower bound of ζ_j as in (4.3), (4.4) and (4.5), we have

$$\zeta_j \ge \{(j+1)^{-3/(\alpha-\eta)}\sigma^{-1}(\sigma(a_{k,i})/\gamma_k)\} \vee 1$$

and from (4.26)

$$\begin{aligned}
N_X(\Lambda_{i,1}^k, \varepsilon_j) &\le c N(\Lambda_{i,1}^k, \zeta_j) \\
&\le c\left(\frac{n_{k,i+1} - a_{k,i}}{\zeta_j} \vee 1\right)\left(\frac{a_{k,i+1} - a_{k,i}}{\zeta_j} \vee 1\right) \\
&\le c(j+1)^{6/(\alpha-\eta)}\left(\frac{n_{k,i+1} - a_{k,i}}{\delta_k} \vee 1\right)\left(\frac{a_{k,i+1} - a_{k,i}}{\delta_k} \vee 1\right) \\
&\le c(j+1)^{6/(\alpha-\eta)}\left(\frac{n_{k,i+1} - a_{k,i}}{\delta_k} \vee 1\right) \\
&= c(j+1)^{6/(\alpha-\eta)}\left\{\left(\frac{n_{k,i} - a_{k,i}}{\delta_k} + \frac{n_{k,i+1} - n_{k,i}}{\delta_k}\right) \vee 1\right\} \\
&\le c(j+1)^{6/(\alpha-\eta)}\left(\frac{n_{k,i} - a_{k,i}}{\delta_k} \vee 1\right) \\
&\le c(j+1)^{6/(\alpha-\eta)}\left(\frac{b_{+1}}{\delta_k} \vee 1\right).
\end{aligned}$$

Therefore

$$N_X(\Lambda_{i,1}^k, \varepsilon_0) + 3\sum_{j=1}^{\infty} N_X(\Lambda_{i,1}^k, \varepsilon_j)\exp\left(-\frac{1}{2}\lambda_j^2\right) \le c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)$$

and from Lemma 3.3

$$P\left\{\sup_{(\ell, m)\in\Lambda_{i,1}^k} \frac{|S_{\ell+m} - S_m|}{\sigma(\ell)} \ge y_k\right\} \le c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\Phi(y_k).$$

STEP 2. *For $1/2 < \alpha < 1$ in* (1.2), *we have*

$$P\left\{\sup_{(\ell, m)\in\Lambda_{i,2}^k} \frac{|S_{\ell+m} - S_m|}{\sigma(a_{k,i})} \ge y_k\right\} \le c\Phi(y_k), \tag{4.29}$$

where $\Lambda_{i,2}^k = \{(\ell, m);\ a_k \le \ell \le a_{k,i},\ n_{k,i} \le \ell + m \le n_{k,i+1}\}$. *In particular if* $\sigma(t) = \sqrt{t}$ *in* (1.2), *then* (4.29) *also holds.*

Proof. Let q_0 be the minimam nonnegative integer q such that

$$s_{k,i}^{(q)} \equiv a_{k,i} - \frac{q(ea_{k+1} - a_k)}{a_k}\delta_k \ge a_k.$$

For $q \le q_0$, set $\Lambda_{i,2}^{k,q} = \{(\ell, m);\ s_{k,i}^{(q+1)} \le \ell \le s_{k,i}^{(q)},\ n_{k,i} \le \ell + m \le n_{k,i+1}\}$. Note that $s_{k,i}^{(0)} = a_{k,i}$ and for $q = 1, 2, \ldots, q_0$.

$$a_k \le s_{k,i}^{(q)} \le a_{k,i}\left(1 - \frac{q\delta_k}{a_k}\right) \le a_{k,i}\left(1 - \frac{\delta_k}{a_k}\right)^q = a_{k,i}\alpha_k{}^q. \tag{4.30}$$

Thus

$$P\left\{\sup_{(\ell, m)\in\Lambda_{i,2}^k} \frac{|S_{\ell+m} - S_m|}{\sigma(a_{k,i})} \ge y_k\right\} \le \sum_{q=0}^{q_0} P\left\{\sup_{(\ell, m)\in\Lambda_{i,2}^{k,n}} \frac{|S_{\ell+m} - S_m|}{\sigma(a_{k,i})} \ge y_k\right\}. \tag{4.31}$$

Setting $X(\ell, m) = (S_{\ell+m} - S_m)/\sigma(a_{k,i})$, $(\ell, m) \in \Lambda_{i,2}^{k,q}$, we have

$$\underline{\sigma} \equiv \frac{\sigma(s_{k,i}^{(q+1)})}{\sigma(a_{k,i})} \le \sqrt{EX^2(\ell, m)} \le \frac{\sigma(s_{k,i}^{(q)})}{\sigma(a_{k,i})} \equiv \bar{\sigma}$$

and

$$d_X{}^2((\ell, m), (\ell', m')) \le \frac{4}{\sigma^2(a_{k,i})}\sigma^2(|\ell - \ell'| + |m - m'|).$$

To apply Lemma 3.3, we set $\sigma(|\ell - \ell'| + |m - m'|)/\sigma(a_{k,i}) \le \underline{\sigma}\varepsilon$. Then

$$1 \le |\ell - \ell'| + |m - m'| = \sigma^{-1}(\underline{\sigma}\varepsilon\sigma(a_{k,i})) \vee 1 \equiv \zeta.$$

For $j = 0, 1, 2, \ldots$, let

$$\varepsilon_j = \frac{\sigma(a_k)(j+1)^{-3}}{\sigma(s_{k,i}^{(q+1)})y_k}, \quad \lambda_j = j, \quad 1 \le \zeta_j = \sigma^{-1}(\underline{\sigma}\varepsilon_j\sigma(a_{k,i})) \vee 1$$

and

$$x=\frac{\sigma(a_{k,i})}{\sigma(s_{k,i}^{(q)})}x_k.$$

Obviously

$$\bar{\sigma}x=x_k \quad \text{and} \quad \bar{\sigma}\sum_{j=0}^{\infty}\varepsilon_j\lambda_{j+1}\le\frac{\sigma(s_{k,i}^{(q)})}{\sigma(s_{k,i}^{(q+1)})}\left(\frac{c}{\gamma_k}\right). \tag{4.32}$$

Using (4.25) and Lemma 3.1 ((3.1)′), we can easily obtain for some $c>0$

$$\frac{\sigma(s_{k,i}^{(q)})}{\sigma(s_{k,i}^{(q+1)})}\le c. \tag{4.33}$$

It follows from (4.32) and (4.33) that

$$\bar{\sigma}(x+\sum_{j=0}^{\infty}\varepsilon_j\lambda_{j+1})\le y_k.$$

Note that

$$s_{k,i}^{(q)}-s_{k,i}^{(q+1)}\le(4e^2-1)\delta_k.$$

Thus from (4.26) we have

$$\begin{aligned}N_x(\Lambda_{i,2}^{k,q},\underline{\sigma}\varepsilon_j)&\le c\,N(\Lambda_{i,2}^{k,q},\zeta_j)\\&\le c(j+1)^{6/(\alpha-\eta)}\left(\frac{n_{k,i+1}-n_{k,i}}{\delta_k}\vee 1\right)\left(\frac{s_{k,i}^{(q)}-s_{k,i}^{(q+1)}}{\delta_k}\vee 1\right)\\&\le c(j+1)^{6/(\alpha-\eta)}\end{aligned}$$

and

$$N_X(\Lambda_{i,2}^{k,q},\underline{\sigma}\varepsilon_0)+3\sum_{j=1}^{\infty}N_X(\Lambda_{i,2}^{k,q},\underline{\sigma}\varepsilon_j)\exp\left(-\frac{1}{2}\lambda_j{}^2\right)\le c.$$

Applying Lemma 3.3, we have

$$\begin{aligned}\sum_{q=0}^{q_0}P\left\{\sup_{(\ell,m)\in\Lambda_{i,2}^{k,q}}\frac{|S_{\ell+m}-S_m|}{\sigma(a_{k,i})}\ge y_k\right\}&\le c\sum_{q=0}^{q_0}\Phi\left(\frac{\sigma(a_{k,i})}{\sigma(s_{k,i}^{(q)})}x_k\right)\\&\le c\frac{1}{x_k}\exp\left(-\frac{1}{2}x_k^2\right)\left\{1+\sum_{q=0}^{q_0}\exp\left[-\frac{1}{2}\left(\frac{\sigma^2(a_{k,i})}{\sigma(s_{k,i}^{(q)})}-1\right)x_k^2\right]\right\}.\end{aligned} \tag{4.34}$$

Using (4.8), (4.30) and Lemma 3.1 (3.1), we obtain

$$\begin{aligned}\frac{\sigma^2(a_{k,i})}{\sigma^2(s_{k,i}^{(q)})}&\ge\frac{\sigma^2(a_{k,i})}{\sigma^2(a_{k,i}\{1-q\delta_k/a_k\})}\ge\left\{1-\frac{q\delta_k}{a_k}\right\}^{-2(\alpha-\eta)}\\&\ge\left\{1-\frac{\delta_k}{a_k}\right\}^{-2q(\alpha-\eta)}=\alpha_k^{-2q(\alpha-\eta)}.\end{aligned}$$

Thus from (4.18)

$$\sum_{q=1}^{q_0} \exp\left\{-\frac{1}{2}\left(\frac{\sigma^2(a_{k,i})}{\sigma^2(s_{k,i}^{(q)})}-1\right)x_k{}^2\right\} < +\infty \tag{4.35}$$

and from (4.31) and (4.34) we have

$$P\left\{\sup_{(\ell,m)\in\Lambda_{i,2}^k} \frac{|S_{\ell+m}-S_m|}{\sigma(a_{k,i})} \ge y_k\right\} \le c\frac{1}{x_k}\exp\left(-\frac{1}{2}x_k^2\right) \le c\,\Phi(y_k).$$

We can easily check that (4.35) also holds when $\sigma(t)=\sqrt{t}$ in (1.2). In the sequel in the case of $a_{k+1}\ge \frac{3}{4}n_{k+1}$, it follows from (4.27), (4.28) and Steps 1 and 2 that

$$\begin{aligned} P_k &\le \sum_{i=1}^{i_k-1} P\left\{\sup_{(\ell,m)\in\Lambda_{i,1}^k\cup\Lambda_{i,2}^k} \frac{|S_{\ell+m}-S_m|}{\sigma(a_{k,i}\vee \ell)} \ge y_k\right\} \\ &\le \sum_{i=1}^{i_k-1}\left(P\left\{\sup_{(\ell,m)\in\Lambda_{i,1}^k} \frac{|S_{\ell+m}-S_m|}{\sigma(\ell)} \ge y_k\right\} + P\left\{\sup_{(\ell,m)\in\Lambda_{i,2}^k} \frac{|S_{\ell+m}-S_m|}{\sigma(a_{k,i})} \ge y_k\right\}\right) \\ &\le c\,i_k\left(\frac{b_{k+1}}{\delta_k}\vee 1\right)\Phi(y_k) \\ &\le c\left(\frac{b_{k+1}}{\delta_k}\vee 1\right)\left(\frac{a_{k+1}-a_k}{\delta_k}\vee 1\right)\Phi(y_k). \end{aligned}$$

Combining Lemmas 4.2 and 4.3, we have

Lemma 4.4. *For* $1/2<\alpha<1$ *in* (1.2), *we have*

$$P\left\{\sup_{n_k\le n\le n_{k+1}} \sup_{\substack{1\le \ell\le \tilde{a}_n \\ 1\le m\le n-\ell}} \frac{|S_{\ell+m}-S_m|}{\sigma(\tilde{a}_n)} \ge y_k\right\}$$

$$\le c\left(\frac{b_{k+1}}{\delta_k}\vee 1\right)\left(\frac{a_{k+1}-a_k}{\delta_k}\vee 1\right)\Phi(y_k). \tag{4.36}$$

In particular if $\sigma(t)=\sqrt{t}$ *in* (1.2), *then* (4.36) *also holds.*

Proof of Theorem 2.1. For g in Lemma 3.2, there exist k_0 and $c>0$ such that $g(n_k)\ge c$ for all $k\ge k_0$. Set $x_k = g(n_k)-\dfrac{2c}{g(n_k)}$. Then

$$y_k \equiv x_k + \frac{c}{x_k} \le g(n_k) \tag{4.37}$$

and by Lemma 3.2 (i)

$$\Phi(y_k) \le c\,\Phi(g(n_k)) \le c\gamma_k^{-1}\exp\left\{-\frac{1}{2}g^2(n_k)\right\}. \tag{4.38}$$

Applying Lemma 4.1, we have for $0 < \alpha < 1$

$$P\left\{\sup_{a_k \le \ell \le a_{k+1}} \sup_{1 \le m \le n_{k+1} - a_{k+1}} \frac{|S_{\ell+m} - S_m|}{\sigma(\ell)} \ge g(n_k)\right\}$$

$$\le c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp\left\{-\frac{1}{2}g^2(n_k)\right\}.$$

Since $I_U(g) < +\infty$, it follows from the first Borel-Cantelli lemma that almost surely there exists $k_1 \ge 1$ such that

$$\sup_{a_k \le \ell \le a_{k+1}} \sup_{1 \le m \le n_{k+1} - a_{k+1}} \frac{|S_{\ell+m} - S_m|}{\sigma(\ell)} < g(n_k)$$

for all $k \ge k_1$. Clearly we have

$$\begin{aligned} X_4(n) &= \sup_{1 \le m \le n - \tilde{a}_n} \frac{|S_{m+\tilde{a}_n} - S_m|}{\sigma(\tilde{a}_n)} \\ &\le \sup_{n_k \le n \le n_{k+1}} \sup_{1 \le m \le n - \tilde{a}_n} \frac{|S_{m+\tilde{a}_n} - S_m|}{\sigma(\tilde{a}_n)} \\ &\le \sup_{a_k \le \ell \le a_{k+1}} \sup_{1 \le m \le n_{k+1} - a_{k+1}} \frac{|S_{\ell+m} - S_m|}{\sigma(\ell)} \\ &< g(n_k) \le g(n), \qquad \text{a.s.} \end{aligned}$$

This implies $g \in \mathrm{UUC}(X_i)$, $i = 4, 5$.

Proof of Theorem 2.2. From (4.37), (4.38) and Lemma 4.4, we have for $1/2 < \alpha < 1$ ($\alpha = 1/2$ when $\sigma(\mathrm{t}) = t^\alpha$)

$$P\left\{\sup_{n_k \le n \le n_{k+1}} \sup_{\substack{1 \le \ell \le \tilde{a}_n \\ 1 \le m \le n - \ell}} \frac{|S_{\ell+m} - S|}{\sigma(\tilde{a}_n)} \ge g(n_k)\right\}$$

$$\le c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{a_{k+1} - a_k}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp\left\{-\frac{1}{2}g^2(n_k)\right\}.$$

Since $I_U(g) < +\infty$, it follows that almost surely there exists $k_1 \ge 1$ such that

$$\sup_{n_k \le n \le n_{k+1}} \sup_{\substack{1 \le \ell \le \tilde{a}_n \\ 1 \le m \le n - \ell}} \frac{|S_{\ell+m} - S_m|}{\sigma(\tilde{a}_n)} < g(n_k)$$

for all $k \ge k_1$. Thus we have

$$\begin{aligned} X_0(n) = \sup_{\substack{1 \le \ell \le \tilde{a}_n \\ 1 \le m \le n - \ell}} \frac{|S_{\ell+m} - S_m|}{\sigma(\tilde{a}_n)} &\le \sup_{n_k \le q \le n_{k+1}} \sup_{\substack{1 \le \ell \le \tilde{a}_q \\ 1 \le m \le n - \ell}} \frac{|S_{\ell+m} - S_m|}{\sigma(\tilde{a}_q)} \\ &< g(n_k) \le g(n), \qquad \text{a.s.} \end{aligned}$$

This implies $g \in \mathrm{UUC}(X_i)$, $i = 0, 1, 2, 3, 4, 5$.

Proof of Theorem 2.3. Define i_k for $k = 2, 3, \ldots,$ by

$$i_k = \left[\frac{a_k - a_{k-1}}{\delta_k M_1} \right] \vee 1$$

where $M_1 \geq 1$ is chosen later big enough and $a_{k,i}$ for $i = 1, 2, \ldots, i_k$, by

$$a_{k,i} \equiv a_{k-1} + (i-1)\delta_k M_1.$$

For $k = 2, 3, \ldots,$ define

$$q_k = \begin{cases} \max\{p \geq 0;\ n_{k-1} - a_{k-1} > 2(n_p - a_p)\} \\ -1 \qquad \text{if } \{\quad\} = \phi, \end{cases}$$

$$d_{k,1} = \begin{cases} n_{q_k} - a_{q_k} & \text{if } q_k \geq 0 \\ 0 & \text{if } q_k = -1 \end{cases}$$

and $d_{1,1} = 0$. Note that for $k = 2, 3, \ldots,$

$$n_{k-1} - a_{k-1} - d_{k,1} > \frac{1}{2}(n_{k-1} - a_{k-1})$$

and $d_{k,1} \leq d_{k',1}$ for $k \leq k'$. Note also $q_k \leq k - 2$. Set

$$j_k = \left[\frac{n_{k-1} - a_{k-1} - d_{k,1}}{\delta_k M_1} \right] \vee 1,$$

$$d_{k,j} = d_{k,1} + (j-1)\delta_k M_1,\ j = 1, \ldots, j_k$$

and

$$B_{i,j,k} = \{S_{d_{k,j} + a_{k,i}} - S_{d_{k,j}} \geq \sigma(a_{k,i}) x_k\}$$

where $x_k = g(n_k)$ and g was referred in Lemma 3.2. Let

$$B_k = \bigcup_{1 \leq i \leq i_k} \bigcup_{1 \leq j \leq j_k} B_{i,j,k}.$$

In order to show that $g \in \mathrm{UUC}(X_5)$, it is enough to prove that for any $\varepsilon > 0$,

$$P(B_k, \text{ i.o.}) \geq 1 - \varepsilon. \tag{4.39}$$

We will prove (4.39) on the basis of Lemma 3.4.

STEP 1. *For any* $\varepsilon > 0$ *we can find* M_1 *and* M_2 *big enough such that*

$$\begin{aligned} P(B_k) &\geq (1 - \varepsilon) \sum_{i,j} P(B_{i,j,k}) \\ &\geq c\left(\frac{b_{k-1}}{\delta_k} \vee 1\right)\left(\frac{a_k - a_{k-1}}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp\left(-\frac{1}{2}x_k^2\right) \end{aligned}$$

for all $k \geq M_2$. *This gives, in particular,* $\sum_k P(B_k) = +\infty$ *by the assumption* (ii) *of Theorem* 2.3.

Proof. It is obvious that

$$P(B_k) \geq \sum_{i,j} P(B_{i,j,k}) - \sum_{(i,j) \neq (i',j')} P(B_{i,j,k} \cap B_{i',j',k}). \tag{4.40}$$

Set

$$X_{i,j,k} = \frac{S_{d_{k,j}+a_{k,i}} - S_{d_{k,j}}}{\sigma(a_{k,i})} \text{ and } \mathbf{r}_{i,j}^{i',j'} \equiv E(X_{i,j,k} X_{i',j',k}).$$

Obviously we have

$$\mathbf{r}_{i,j}^{i',j'} = \frac{R}{2\sigma(a_{k,i})\sigma(a_{k,i'})},$$

where

$$\begin{aligned} R = {} & \sigma^2(|d_{k,j} + a_{k,i} - d_{k,j'}|) + \sigma^2(|d_{k,j'} + a_{k,i'} - d_{k,j}|) \\ & - \sigma^2(|d_{k,j} + a_{k,i} - d_{k,j'} - a_{k,i'}|) - \sigma^2(|d_{k,j} - d_{k,j'}|). \end{aligned}$$

Denote the distance $|p_m - p_n|$ between p_m and p_n by r_{mn}, $m \neq n = 1, 2, 3, 4$, where

$$p_1 = d_{k,j},\ p_2 = d_{k,j} + a_{k,i},\ p_3 = d_{k,j'},\ p_4 = d_{k,j'} + a_{k,i'}.$$

Then, $R = \sigma^2(r_{23}) + \sigma^2(r_{14}) - \sigma^2(r_{24}) - \sigma^2(r_{13})$, $r_{12} = a_{k,i}$ and $r_{34} = a_{k,i'}$. Let

$$I_{i,j,k} = \{m;\ d_{k,j} \leq m \leq d_{k,j} + a_{k,i}\}$$

be a set of integers. Now we estimate the second term in the right-hand side of (4.40). We divide the summands of this second term into three cases.

Case 1. Let $I_{i,j,k} \cap I_{i',j',k} = \phi$. Then the concavity of σ^2 yields

$$R = \{\sigma^2(r_{14}) - \sigma^2(r_{24})\} - \{\sigma^2(r_{13}) - \sigma^2(r_{23})\} \leq 0.$$

It follows, in combination of Lemma 3.5 (3.3), Lemma 3.2 and Theorem 2.3 (ii), that

$$\begin{aligned} \sum_{i',j'}^{(1)} P(B_{i,j,k} \cap B_{i',j',k}) &\leq \sum_{i',j'}^{(1)} P(B_{i,j,k}) P(B_{i',j',k}) \\ &= P(B_{i,j,k})\, i'_k j'_k\, \Phi(x_k) \\ &\leq c\, P(B_{i,j,k}) \left(\frac{a_k - a_{k-1}}{\delta_k} \vee 1 \right) \left(\frac{b_{k-1}}{\delta_k} \vee 1 \right) \gamma_k^{-1} \exp\left(-\frac{1}{2} x_k^2 \right) \\ &\leq \frac{\varepsilon}{2} P(B_{i,j,k}) \end{aligned} \tag{4.41}$$

for all $k \geq M_2$ if M_2 is big enough. Here $\sum_{i',j'}^{(1)}$ denotes the sum over (i', j')

satisfying $I_{i,j,k} \cap I_{i',j',k} = \phi$.

Case 2. Let $I_{i,j,k} \cap I_{i',j',k} \neq \phi$ and

$$r_{14} \vee r_{23} = \max_{1 \le m \neq n \le 4} r_{mn}.$$

Now we see that

$$1 - \mathbf{r}_{i,j}^{i',j'} = \frac{2\sigma(r_{12})\sigma(r_{34}) - R}{2\sigma(r_{12})\sigma(r_{34})}$$
$$= \frac{\sigma^2(r_{24}) + \sigma^2(r_{13}) - \{\sigma(r_{12}) - \sigma(r_{34})\}^2 + R'}{2\sigma(r_{12})\sigma(r_{34})}$$

where $R' = \sigma^2(r_{12}) + \sigma^2(r_{34}) - \sigma^2(r_{23}) - \sigma^2(r_{14})$. Here we claim that $R' \ge 0$. In fact,

$$R' = \sigma^2(r_{12} \wedge r_{34}) + \sigma^2(r_{12} \vee r_{34}) - \sigma^2(r_{14}) - \sigma^2(r_{23})$$

and in case of $r_{14} = \max r_{mn}$ we have, from the concavity of σ^2,

$$\sigma^2(r_{14}) - \sigma^2(r_{12} \vee r_{34}) \le \sigma^2(r_{14} - (r_{12} \vee r_{34}) + r_{23}) - \sigma^2(r_{23}).$$
$$= \sigma^2(r_{12} \wedge r_{34}) - \sigma^2(r_{23}).$$

On the other hand if $r_{23} = \max r_{mn}$, then

$$\sigma^2(r_{23}) - \sigma^2(r_{12} \wedge r_{34}) \le \sigma^2(r_{23} - (r_{12} \wedge r_{34}) + r_{14}) - \sigma^2(r_{14})$$
$$= \sigma^2(r_{12} \vee r_{34}) - \sigma^2(r_{14}).$$

These imply $R' \ge 0$. In consequence we obtain

$$1 - \mathbf{r}_{i,j}^{i',j'} \ge \frac{\sigma^2(r_{24}) + \sigma^2(r_{13}) - \{\sigma(r_{12} - \sigma(r_{34})\}^2}{2\sigma(r_{12})\sigma(r_{34})}. \tag{4.42}$$

Further it follows that for some $0 < c < 1$

$$\{\sigma(r_{12}) - \sigma(r_{34})\}^2 \le c^2\sigma^2(r_{13} \vee r_{24}). \tag{4.43}$$

Indeed, by the condition (i) of Theorem 2.3

$$\frac{\sigma(|r_{12} - r_{34}|)}{\sigma(r_{12} \wedge r_{34})} \le \frac{\sigma(a_k - a_{k-1})}{\sigma(a_{k-1})}$$

is bounded and hence there exists c_1 with $0 < c_1 < 1$ such that

$$\left(1 + \frac{\sigma^2(|r_{12} - r_{34}|)}{\sigma^2(r_{12} \wedge r_{34})}\right)^{1/2} - 1 \le c_1 \frac{\sigma(|r_{12} - r_{34}|)}{\sigma(r_{12} \wedge r_{34})}.$$

So, this and the concavity of σ^2 yield

$$\{\sigma(r_{12}) - \sigma(r_{34})\}^2 = \sigma^2(r_{12} \wedge r_{34})\left(\frac{\sigma(r_{12} \vee r_{34})}{\sigma(r_{12} \wedge r_{34})} - 1\right)^2$$

$$= \sigma^2(r_{12} \wedge r_{34})\left(\left\{\frac{\sigma^2(r_{12} \vee r_{34}) - \sigma^2(r_{12} \wedge r_{34})}{\sigma^2(r_{12} \wedge r_{34})} + 1\right\}^{1/2} - 1\right)^2$$

$$\leq \sigma^2(r_{12} \wedge r_{34})\left(\left\{\frac{\sigma^2(r_{12} \vee r_{34} - r_{12} \wedge r_{34})}{\sigma^2(r_{12} \wedge r_{34})} + 1\right\}^{1/2} - 1\right)^2$$

$$= \sigma^2(r_{12} \wedge r_{34})\left(\left\{\frac{\sigma^2(|r_{12} - r_{34}|)}{\sigma^2(r_{12} \wedge r_{34})} + 1\right\}^{1/2} - 1\right)^2$$

$$\leq c_1^2\sigma^2(|r_{12} - r_{34}|) \leq c_1^2\sigma^2(r_{13} \vee r_{24}).$$

This proves (4.43). It follows from (4.42), (4.43) and Lemma 3.1 ((3.1)′ and (3.2-b)) that for some c_2 with $0 < c_2 < 1$

$$1 - \mathbf{r}_{i,j}^{i',j'} \geq \frac{\sigma^2(r_{24}) + \sigma^2(r_{13}) - c_2^2\sigma^2(r_{13} \vee r_{24})}{2\sigma(r_{12})\sigma(r_{34})}$$

$$\geq \frac{(1 - c_2^2)\sigma^2(r_{13} \vee r_{24})}{2\sigma^2(a_k)} \geq c\frac{\sigma^2((|i - i'| + |j - j'|)\delta_k M_1)}{\sigma^2(a_k)} \tag{4.44}$$

$$\geq c\{(|i - i'| + |j - j'|)M_1\}^{2(\alpha-\eta)}\{\sigma(\sigma^{-1}(\sigma(a_k)/\gamma_k))/\sigma(a_k)\}^2$$

$$\geq c\{(|i - i'| + |j - j'|)M_1\}^{2(\alpha-\eta)}x_k^{-2}.$$

Case 3. Let $I_{i,j,k} \cap I_{i',j',k} \neq \phi$ and

$$r_{12} \vee r_{34} = \max_{1 \leq m \neq n \leq 4} r_{mn}.$$

First consider the case $r_{12} = \max r_{mn}$. Again the concavity of σ^2 yields

$$\mathbf{r}_{i,j}^{i',j'} \leq \frac{\sigma^2(r_{14}) - \sigma^2(r_{13}) + \sigma^2(r_{23}) - \sigma^2(r_{24})}{2\sigma(r_{34})\sigma(r_{14} \vee r_{23})}$$

$$\leq \frac{\sigma^2(r_{14} \vee r_{23}) - \sigma^2(r_{13} \vee r_{24}) + \sigma^2(r_{34})}{2\sigma(r_{34})\sigma(r_{14} \vee r_{23})}$$

$$= 1 - \frac{\sigma^2(r_{13} \vee r_{24}) - \{\sigma(r_{14} \vee r_{23}) - \sigma(r_{34})\}^2}{2\sigma(r_{34})\sigma(r_{14} \vee r_{23})},$$

that is,

$$1 - \mathbf{r}_{i,j}^{i',j'} \geq \frac{\sigma^2(r_{13} \vee r_{24}) - \{\sigma(r_{14} \vee r_{23}) - \sigma(r_{34})\}^2}{2\sigma(r_{34})\sigma(r_{14} \vee r_{23})}.$$

Furthermore if follows that for some $0 < c_3 < 1$

$$\{\sigma(r_{14} \vee r_{23}) - \sigma(r_{34})\}^2$$

$$\leq \sigma^2(r_{34})\left(\left\{1 + \frac{\sigma^2(r_{14} \vee r_{23} - r_{34})}{\sigma^2(r_{34})}\right\}^{1/2} - 1\right)^2$$

$$= \sigma^2(r_{34})\left(\left\{1 + \frac{\sigma^2(r_{13} \vee r_{24})}{\sigma^2(r_{34})}\right\}^{1/2} - 1\right)^2$$

$$\leq c_3^2 \sigma^2(r_{13} \vee r_{24}).$$

Thus we have

$$1 - \mathbf{r}_{i,j}^{i',j'} \geq \frac{(1 - c_3^2)\sigma^2(r_{13} \vee r_{24})}{2\sigma(r_{34})\sigma(r_{12})} \geq c\frac{\sigma^2((|i - i'| + |j - j'|)\delta_k M_1)}{\sigma^2(a_k)}$$

$$\geq c\{(|i - i'| + |j - j'|)M_1\}^{2(\alpha - \eta)} x_k^{-2}. \tag{4.45}$$

By the same way as above we can easily show that (4.45) also holds when $r_{34} = \max r_{mn}$. Now applying Lemma 3.5 (3.4), it follows from (4.44) and (4.45) that

$$\sum_{i',j'}^{(2)} P(B_{i,j,k} \cap B_{i',j',k})$$

$$\leq c \sum_{i',j'}^{(2)} \exp\left\{-\frac{1}{4}(1 - \mathbf{r}_{i,j}^{i',j'})x_k^2\right\} P(B_{i,j,k})$$

$$\leq c\, P(B_{i,j,k}) \sum_{i',j'}^{(2)} \exp\{-c\{(|i - i'| + |j - j'|)M_1\}^{2(\alpha - \eta)}\} \tag{4.46}$$

$$\leq c\, P(\mathrm{B}_{i,j,k}) \sum_{m,n} \exp\{-c\{(m + n)M_1\}^{2(\alpha - \eta)}\}$$

$$\leq \frac{\varepsilon}{2} P(B_{i,j,k})$$

if M_1 is chosen big enough. Here $\sum_{i',j'}^{(2)}$ denotes the sum over (i', j') satisfying $I_{i,j,k} \cap I_{i',j',k} \neq \phi$. Combining the inequalities (4.41) and (4.46), we have

$$\sum_{i',j'} P(B_{i,j,k} \cap B_{i',j',k}) \leq \varepsilon P(B_{i,j,k}).$$

This and (4.40) complete the proof of Step 1.

In the following, we always assume that $k' \geq k$. Define

$$\mathbf{r}_{i,j,k}^{i',j',k'} = E(X_{i,j,k} X_{i',j',k'}).$$

Let $p_1 = d_{k,j}$, $p_2 = d_{k,j} + a_{k,i}$, $p_3 = d_{k',j'}$ and $p_4 = d_{k',j'} + a_{k',i'}$ be points on a positive half line. For $m \neq n = 1, 2, 3, 4$, we define a distance $|p_m - p_n|$ between p_m and p_n by r_{mn}. Then we have

$$\mathbf{r}_{i,j,k}^{i',j',k'} = \frac{\sigma^2(r_{23}) + \sigma^2(r_{14}) - \sigma^2(r_{24}) - \sigma^2(r_{13})}{2\sigma(a_{k,i})\sigma(a_{k',i'})}.$$

STEP 2. Let $d_{k',1} \geq n_k$. Then by the concavity of σ^2 we get

$$\mathbf{r}_{i,j,k}^{i',j',k'} \leq 0.$$

It follows from Lemma 3.5 (3.3) and Step 1 that for any $\varepsilon > 0$, there exists $M = M_\varepsilon$ big enough such that for all $k \geq M$,

$$\begin{aligned} P(B_k \cap B_{k'}) &\leq \sum_{i,j} \sum_{i',j'} P(B_{i,j,k} \cap B_{i',j',k'}) \\ &\leq \sum_{i,j} \sum_{i',j'} P(B_{i,j,k}) P(B_{i',j',k'}) \\ &\leq (1+\varepsilon) P(B_k) P(B_{k'}). \end{aligned}$$

STEP 3. *Let $d_{k',1} < n_k$, $q_{k'} \leq k' - 3$ and $k + c_2 \log k \leq k'$ for some $c_2 > 0$. Then for any $\varepsilon > 0$, we can find $M = M_\varepsilon$ such that for all $k \geq M$,*

$$P(B_k \cap B_{k'}) \leq (1+\varepsilon) P(B_k) P(B_{k'}).$$

Proof. Since $q_{k'} \leq k' - 3$, we have

$$\begin{aligned} n_{k'-2} &\geq n_{q_{k'}+1} \geq n_{q_{k'}+1} - a_{q_{k'}+1} \geq n_{q_{k'}} - a_{q_{k'}} \\ &\geq \frac{1}{2}(n_{k'-1} - a_{k'-1}) \end{aligned}$$

and

$$a_{k'-1} \geq n_{k'-1} - 2n_{k'-2} = \left(1 - \frac{2}{e}\right) n_{k'-1}. \tag{4.47}$$

By the concavity of σ^2, we obtain

$$\mathbf{r}_{i,j,k}^{i',j',k'} \leq \frac{\sigma(a_{k,i})}{\sigma(a_{k',i'})} \leq \frac{\sigma(a_k)}{\sigma(a_{k'} - 1)}. \tag{4.48}$$

Note that (4.47) yields

$$a_{k'-1} \geq (e-2)e^{-2} n_k, \text{ and } \gamma_{k'}^2 \leq c \log k'. \tag{4.49}$$

Thus applying Lemma 3.1 ((3.1)′), we have

$$\begin{aligned} |\mathbf{r}_{i,j,k}^{i',j',k'}| x_k x_{k'} &\leq c(\log k') \frac{\sigma(n_k)}{\sigma((e-2)e^{-2} n_{k'})} \\ &\leq c(\log k') \exp(-(k'-k)(\alpha - \eta)) \\ &\leq c\, k^{-c_2(\alpha-\eta)} \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty \end{aligned}$$

and that for any $\varepsilon' > 0$ there exists $k_{\varepsilon'}$ such that for all $k \geq k_{\varepsilon'}$

$$|\mathbf{r}_{i,j,k}^{i',j',k'}| \leq \frac{\varepsilon'}{x_k x_{k'}}.$$

Now applying Lemma 3.5 (3.3) and Step 1, we have

$$\begin{aligned} P(B_k \cap B_{k'}) &\le \sum_{i,j} \sum_{i',j'} P(B_{i,j,k} \cap B_{i',j',k'}) \\ &\le c_{\varepsilon'} \sum_{i,j} \sum_{i',j'} P(B_{i,j,k}) P(B_{i',j',k'}) \\ &\le c_{\varepsilon'} (1-\varepsilon')^{-2} P(B_k) P(B_{k'}) \\ &\le (1+\varepsilon) P(B_k) P(B_{k'}). \end{aligned}$$

Here we remark that $\lim_{\varepsilon' \downarrow 0} c_{\varepsilon'} = 1$.

STEP 4. *Let $d_{k',1} < n_k$, $q_{k'} \le k' - 3$ and $k + c_1 \le k' \le k + c_2 \log k$ for c_1 big enough. Then denoting by $\sum_{k'}$ the sum over such k', we have*

$$\sum_{k'} P(B_k \cap B_{k'}) \le c\ P(B_k).$$

Proof. By combination of (4.47), (4.48) and Lemma 3.1 ((3.1)′), we have

$$\begin{aligned} |\mathbf{r}_{i,j,k}^{i',j',k'}| &\le \frac{\sigma(a_k)}{\sigma(a_{k'-1})} \le \frac{\sigma(n_k)}{\sigma((e-2)e^{-2} n_{k'})} \\ &\le \frac{\sigma(n_k)}{\sigma(e^{-2+c_1}(e-2) n_k)} \le \{e^{-2+c_1}(e-2)\}^{-(\alpha-\eta)} \\ &\le \delta < 1, \text{ say.} \end{aligned}$$

It follows from Lemma 3.5 (3.4) that

$$\begin{aligned} P(B_{i,j,k} \cap B_{i',j',k'}) &\le c \exp\left\{-\frac{1}{4}(1 - \mathbf{r}_{i,j,k}^{i',j',k'}) x_k^2\right\} \Phi(x_k) \\ &\le c \exp\left(-\frac{1}{4}(1-\delta) x_k^2\right) \Phi(x_k). \end{aligned}$$

Using (4.47) and (4.49), we obtain

$$j_{k'} \le c \frac{n_{k'}}{\delta_{k'}} \le c \frac{n_{k'}}{a_{k'}} \gamma_{k'}{}^{1/(\alpha-\eta)} \le c (\log k')^{\frac{1}{2(\alpha-\eta)}}$$

and

$$i_{k'} \le c(\gamma_{k'})^{1/(\alpha-\eta)} \le c(\log k')^{\frac{1}{2(\alpha-\eta)}}.$$

Therefore

$$\begin{aligned} \sum_{k'} P(B_k \cap B_{k'}) &\le \sum_{i,j} \sum_{i',j',k'} P(B_{i,j,k} \cap B_{i',j',k'}) \\ &\le c \sum_{i,j} \sum_{i',j',k'} \exp(-c\ x_{k'}{}^2) \Phi(x_k) \end{aligned}$$

$$\le c(1-\varepsilon)^{-1} P(B_k) \sum_{i',j',k'} \exp(-c\, x_{k'}^2)$$

$$\le c(1-\varepsilon)^{-1} P(B_k)(\log k)^{\frac{1}{\alpha-\eta}} e^{-c(\log k)}$$

$$\le c\, P(B_k).$$

STEP 5. *Let* $d_{k',1} < n_k$, $q_{k'} = k'-2$ *and* $k + c_2 \log k \le k'$ *for some* $c_2 > 0$. *Then for any* $\varepsilon > 0$ *there exists* $M = M_\varepsilon$ *such that for all* $k \ge M$,

$$P(B_k \cap B_{k'}) \le (1+\varepsilon) P(B_k)\, P(B_{k'}).$$

Proof. By the assumption,

$$d_{k',1} = n_{k'-2} - a_{k'-2} < n_k < \frac{1}{e} n_{k'-2},$$

and

$$\left(1 - \frac{1}{e}\right) n_{k'-2} < a_{k'-2} \le a_{k'-1}. \tag{4.50}$$

Making use of (4.48) and (4.50), we obtain

$$|\mathbf{r}_{i,j,k}^{i',j',k'}| \le \frac{\sigma(a_k)}{\sigma(a_{k'-1})} \le \frac{\sigma(n_k)}{\sigma((e-1)e^{-3} n_{k'})} \le c\, e^{-(k'-k)(\alpha-\eta)}. \tag{4.51}$$

By (4.50)

$$\gamma_{k'}^2 = \log\left(\frac{n_{k'}}{a_{k'}} k'\right) \le c \log k' \tag{4.52}$$

and hence

$$|\mathbf{r}_{i,j,k}^{i',j',k'}| x_k x_{k'} \le c\, k^{-c_2(\alpha-\eta)} \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty.$$

This implies that for any $\varepsilon' > 0$ there exists $k_{\varepsilon'}$ such that for all $k \ge k_{\varepsilon'}$

$$|\mathbf{r}_{i,j,k}^{i',j',k'}| < \frac{\varepsilon'}{x_k x_{k'}}.$$

As in the proof of Step 3 we can obtain the desired result.

STEP 6. *Let* $d_{k',1} < n_k$, $q_{k'} = k'-2$ *and* $k + c_1 \le k' \le k + c_2 \log k$ *for* c_1 *big enough. Then*

$$\sum_{k'} P(B_k \cap B_{k'}) \le c\, P(B_k).$$

Proof. From (4.51) and the assumption we have

$$1 - \mathbf{r}_{i,j,k}^{i',j',k'} \ge 1 - c\, e^{-(k'-k)(\alpha-\eta)}$$

$$\geq 1 - c e^{-c_1(\alpha-\eta)} \geq \delta > 0, \text{ say.}$$

By (4.52) we see that $\gamma_{k'}{}^2 \leq c \log k'$. This, in combination with (4.50), yields

$$j_{k'} \leq c \frac{n_{k'}}{\delta_{k'}} \leq c(\log k')^{\frac{1}{2(\alpha-\eta)}}$$

and

$$i_{k'} \leq c(\log k')^{\frac{1}{2(\alpha-\eta)}}.$$

Therefore,

$$\begin{aligned}
\sum_{k'} P(B_k \cap B_{k'}) &\leq \sum_{i,j} \sum_{i',j',k'} P(B_{i,j,k} \cap B_{i',j',k'}) \\
&\leq c \sum_{i,j} \sum_{i',j',k'} \exp\left(-\frac{\delta}{4} x_k^2\right) \Phi(x_k) \\
&\leq c(1-\varepsilon)^{-1} P(B_k) \sum_{k'} i_{k'} j_{k'} \exp\left(-\frac{\delta}{4} x_k^2\right) \\
&\leq c(1-\varepsilon)^{-1} P(B_k) \sum_{k'} (\log k')^{1/(\alpha-\eta)} e^{-c(\log k')} \\
&\leq c P(B_k).
\end{aligned}$$

STEP 7. Combining the above Steps $1 \sim 6$ together, we see that all conditions of Lemma 3.4 are satisfied for the sequence $\{B_k\}$: the set $\{k_1, k_2, \ldots, k_{i(k)}\}$ is given by

$$\{k' ; d_{k',1} < n_k, \ k + c_1 \leq k' \leq k + c_2 \log k\}.$$

Thus we obtain (4.39).

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