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Refinement in terms of capacities of certain limit theorems on an abstract Wiener space

By

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Introduction

The notion of (p, r)-capacities was introduced by P. Malliavin [13] on an abstract Wiener space (B, H, μ). A set of (p, r)-capacity 0 is always a set of μ-measure 0 and therefore it is important and useful to sharpen statements or properties holding μ-almost everywhere on B to those holding *quasi everywhere on* B, i.e., everywhere except on a set of capacity 0. This kind of sharpening was studied by, e.g., Fukushima [5] and Takeda [17] for the familiar almost sure sample properties of Wiener processes. Also the existence of quasi continuous modifications and their applications for a class of Wiener functions were discussed by Malliavin [13], Sugita [16], and Airault-Malliavin [1], among others.

The first aim of this paper is to give a similar refinement of limit theorems for a class of independent random variables defined on an abstract Wiener space (B, H, μ). A main result is Theorem 2.2 in Section 2, where a criterion of the Kolmogorov type for the almost everywhere convergence or divergence is sharpened to the criterion for the quasi everywhere convergence or divergence in the case of sums of a class of independent Wiener polynomial functions. As a by-product of a law of large numbers obtained as a corollary to Theorem 2.2, we can show that the Cameron-Martin subspace is slim (cf. Feyel-de La Pradelle [3]).

For a given positive generalized Wiener function Φ on (B, H, μ) with $(\Phi, 1) = 1$, Sugita [16] constructed a Borel probability measure $\nu = \nu_\Phi$ on B. In Section 3, we study limit theorems (central limit theorems, criterions for almost sure convergence and divergence) for the same class of independent polynomial Wiener functions as in Theorem 2.2 *with respect to the probability space* (B, ν). An interesting point in these theorems is that these random variables are no more independent on the space (B, ν).

In Section 4, we collect some supplementary results obtained in the course of study in Sections 2 and 3, which are of independent interest. In particular, we obtain an estimate for the tail capacities of the norm and a refinement of the Itô-Nisio theorem.

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§ 1 Preliminaries

Here we collect some of fundamental notions and facts in the Malliavin caluculus which will be needed in the subsequent sections.

An *abstract Wiener space* (B, H, μ) is a triplet (B, H, μ) where B is a real separable Banach space with the norm $\|\cdot\|_B$, H is a real separable Hilbert space (called the *Cameron-Martin subspace*) with the inner product $\langle\cdot,\cdot\rangle_H$ which is contained in B densely so that the injection $j: H \to B$ is continuous, μ is the standard Gaussian measure on B, that is, the Borel probability measure on B satisfying

$$\int_B e^{i_{B^*}\langle f, x\rangle_B}\, d\mu(x) = e^{-\frac{1}{2}|f|_H^2}$$

for every $f \in B^* \subset H^*$ and H^* is identified with H by the Riesz theorem.

In the following, we assume that $\dim H = \infty$. Then we have always

$$\mu(H) = 0. \tag{1.1}$$

It is well-known that for every $h \in H$, the linear function on H:

$$h' \in H \longrightarrow \langle h, h'\rangle_H \in \mathbf{R}$$

can be extended to a μ-measurable function. This extension, called the *stochastic linear function*, is denoted by (h, x). The family $\{(h, \cdot), h \in H\}$ forms a mean 0 Gaussian system on the probability space (B, μ) with the covariance coinciding with the H-inner product.

As usual, a μ-measurable function on B is called a *Wiener function* and two Wiener functions with values in the same space are identified whenever they coincide with each other μ-almost everywhere (μ-a.e.). For $1 \le p < \infty$, we denote by L_p the usual L_p-space of Wiener functions and more generally, by $L_p(E)$ the L_p-space of E-valued Wiener functions where, generally in this paper, we denote by E a real separable Hilbert space. A real Wiener function $F(x)$ on B is called a *polynomial function* if it can be expressed in the form

$$F(x) = p((h_1, x), \ldots, (h_n, x))$$

where $p(t_1, \ldots, t_n)$ is a real polynomial and $h_1, \ldots, h_n \in H$. If $\mathscr{P}$ is the totality of polynomial functions, then $\mathscr{P} \subset L_p$ densely for every $1 \le p < \infty$. Let $H_n(t)$ be the Hermite polynomial

$$H_n(t) = \frac{(-1)^n}{n!} e^{\frac{t^2}{2}} \frac{d^n}{dt^n} e^{-\frac{t^2}{2}}, \qquad n = 0, 1, 2, \ldots. \tag{1.2}$$

Fix an orthonormal base (ONB) $\{h_i\}_{i=1}^{\infty}$ in H and, for each $\alpha = (\alpha_i)_{i=1}^{\infty}$, $\alpha_i \in \mathbf{Z}_+$, such that $|\alpha| = \Sigma_{i=1}^{\infty} \alpha_i < \infty$, define $H_\alpha \in \mathscr{P}$ by

$$(1.3) \qquad H_\alpha(x) = \prod_{i=1}^{\infty} \sqrt{\alpha_i!}\, H_{\alpha_i}((h_i, x)).$$

Since $H_0(t) \equiv 1$ and $\alpha_i = 0$ except for finitely many i, (1.3) actually defines a polynomial function. It is well-known that $\{H_\alpha; |\alpha| < \infty\}$ forms an ONB in L_2 and for each $n = 0, 1, 2, \ldots \{H_\alpha; |\alpha| = n\}$ forms an ONB in the subspace $\mathscr{C}_n$ (the subspace of n-th homogeneous Wiener chaos) so that $L_2 = \Sigma_{n=0}^{\infty} \oplus \mathscr{C}_n$.

Now we review very rapidly the Sobolev spaces formed of Wiener functions and generalized Wiener functions, cf. e.g. [8], [9], [15] and [19]. For every $s \in \mathbf{R}$ and $1 < p < \infty$, the Sobolev space $\mathbf{D}_{p,s}$ is defined roughly by

$$(1.4) \qquad \mathbf{D}_{p,s} = (I - L)^{-\frac{s}{2}}(L_p)$$

with the norm

$$(1.5) \qquad \| F \|_{p,s} = \| (I - L)^{\frac{s}{2}} F \|_{L^p}.$$

Here L is the Ornstein-Uhlenbeck operator

$$(1.6) \qquad L = \sum_{n=0}^{\infty} (-n) J_n$$

where J_n is the orthogonal projection to the subspace $\mathscr{C}_n$ so that

$$(1.7) \qquad (I - L)^{\frac{s}{2}} = \sum_{n=0}^{\infty} (1 + n)^{\frac{s}{2}} J_n.$$

More precisely, we first define the norm $\|\cdot\|_{p,s}$ for $F \in \mathscr{P}$ by (1.5) and define $\mathbf{D}_{p,s}$ to be the completion of $\mathscr{P}$ by the norm. Since the family of norms on $\mathscr{P}$ satisfies

(1.8) (monotonicity) $\|F\|_{p,s} \le \|F\|_{p',s'}$ if $p \le p'$ and $s \le s'$

(1.9) (compatibility) $\{F_n\} \subset \mathscr{P}$ is such that $\|F_n\|_{p,s} \to 0$ and $\|F_n - F_m\|_{p',s'} \to 0$ as $n, m \to \infty$, then $\|F_n\|_{p',s'} \to 0$ as $n \to \infty$,

(1.10) (duality) if $\dfrac{1}{p} + \dfrac{1}{q} = 1$,

$$\| F \|_{p,s} = \sup\left\{ \int_B F G d\mu;\ G \in \mathscr{P},\ \| G \|_{q,-s} \le 1 \right\}, \qquad \forall F \in \mathscr{P},$$

we have immediately that

$$(1.11) \qquad \mathbf{D}_{p,0} = L_p$$

(1.12) $\mathbf{D}_{p',s'} \subset \mathbf{D}_{p,s}$ if $p \le p'$ and $s \le s'$ and the inclusion is continuous,

$$(1.13) \qquad \mathbf{D}'_{p,s} = \mathbf{D}_{q,-s}.$$

We set

(1.14) $$\mathbf{D}_\infty = \bigcap_{p>1}^{\infty} \bigcap_{s>0} \mathbf{D}_{p,s} \quad \text{and}$$

(1.15) $$\mathbf{D}_{-\infty} = \bigcup_{p>1}^{\infty} \bigcup_{s>0} \mathbf{D}_{p,-s}.$$

$\mathbf{D}_\infty$ is a Fréchet space, called the *space of test Wiener functions*, and $\mathbf{D}_{-\infty}$ is the dual of $\mathbf{D}_\infty$, called the *space of generalized Wiener functions.* If E is a real separable Hilbert space, we can extend the definition of polynomial functions and Sobolev spaces to the case of E-valued Wiener functions in an obvious way. The spaces are denoted by $\mathscr{P}(E)$, $\mathbf{D}_{p,s}(E)$, $\mathbf{D}_\infty(E)$ and $\mathbf{D}_{-\infty}(E)$ and norms by $\| \ \|_{p,s;E}$. The Ornstein-Uhlenbeck operator L, defined by (1.6) on $\mathscr{P}$, can be extended to an operator L: $\mathbf{D}_{-\infty} \to \mathbf{D}_{-\infty}$ which is continuous $\mathbf{D}_{p,s+2} \to \mathbf{D}_{p,s}$ for every s and p. The Fréchet derivative D is defined, first for $F \in \mathscr{P}$, by

$$\langle DF(x), h\rangle_H = \lim_{\varepsilon \to 0} \frac{F(x+\varepsilon h) - F(x)}{\varepsilon}, \qquad h \in H$$

and $DF \in \mathscr{P}(H)$. Then D can be extended to an operator D: $\mathbf{D}_{-\infty} \to \mathbf{D}_{-\infty}(H)$ which is continuous $\mathbf{D}_{p,s+1} \to \mathbf{D}_{p,s}(H)$ for every s and p. Therefore, the dual operator D^* of D is defined: D^*: $\mathbf{D}_{-\infty}(H) \to \mathbf{D}_{-\infty}$ which is continuous $\mathbf{D}_{p,s+1}(H) \to \mathbf{D}_{p,s}$. Furthermore, it holds that $L = -D^*D$. The following equivalence of norms is due to P. A. Meyer: for every $1 < p < \infty$, $s \in \mathbf{R}$ and $k = 0, 1, 2, \ldots$, there exist positive constants $c_{p,s,k} < C_{p,s,k}$ such that

(1.16) $$c_{p,s,k} \| D^k F \|_{p,s;\underbrace{H \otimes \ldots \otimes H}_{k \text{ times}}} \le \| F \|_{p,s+k} \le C_{p,s,k} \sum_{l=0}^{k} \| D^l F \|_{p,s;\underbrace{H \otimes \ldots \otimes H}_{l \text{ times}}}, \qquad F \in \mathbf{D}_\infty.$$

From now on, *we let p and r denote some constants* $1 < p < \infty$ *and* $r > 0$ *and q denote the dual index to p*: $\frac{1}{p} + \frac{1}{q} = 1$. We now define the (p, r)-*capacity* for subsets of B. First, for any open set O, we set

(1.17) $$C_{p,r}(O) = \inf\{ \| F \|_{p,r}^p;\ F \in \mathbf{D}_{p,r},\ F \ge 1\ \mu\text{-a.e. on } O\}$$

and for an arbitrary subset A, define

(1.18) $$C_{p,r}(A) = \inf\{C_{p,r}(O);\ A \subset O,\ O: \text{open}\}.$$

These (p, r)-capacities possess the following properties:

(1.19) $$\bar{\mu}(A) \le C_{p,r}(A), \qquad \bar{\mu} \text{ being the outer measure for } \mu,$$

(1.20) $$C_{p,r}(A_1) \le C_{p,r}(A_2) \qquad \text{if } A_1 \subset A_2$$

(1.21) $$C_{p,r}\Big(\bigcup_{n=1}^{\infty} A_n\Big) \le \sum_{n=1}^{\infty} C_{p,r}(A_n)$$

(1.22) $$C_{p,r}(A_n) \uparrow C_{p,r}(A) \qquad \text{if } A_n \uparrow A.$$

From these properties, we have the following:

(1.23) (Borel-Cantelli lemma for capacities)

$$\text{If } \sum_{n=1}^{\infty} C_{p,r}(A_n) < \infty, \text{ then } C_{p,r}(\overline{\lim_{n\to\infty}} A_n) = 0$$

(1.24) (Fatou's lemma for capacities)

$$C_{p,r}(\underline{\lim}_{n\to\infty} A_n) \leq \underline{\lim}_{n\to\infty} C_{p,r}(A_n).$$

Following Malliavin, we say that a subset A of B is *slim* if $C_{p,r}(A) = 0$ for all $1 < p < \infty$ and $r > 0$. Also any property holding everywhere except on a set of (p, r)-capacity 0 (a slim set) is said to hold (p, r)-*quasi everywhere* (resp. *quasi everywhere*), and we write simply (p, r)-*q.e.* (resp. *q.e.*)

A function F defined on B with values in a metric space is called (p, r)-quasi continuous if, for every $\varepsilon > 0$ there exists an open set $O \subset B$ with $C_{p,r}(O) < \varepsilon$ and the restriction $F \restriction B \backslash O$ is continuous. A Wiener function F with values in a topological space is said to possess a (p, r)-*quasi continuous modification* $F^{\sim}$, if among the equivalence class of F coinciding each other μ-*a.e.*, there exists a (p, r)-quasi continuous function $F^{\sim}$. Two (p, r)-quasi continuous modifications, when they exist, must coincide (p, r)-*q.e.*

It is well-known ([6], [16]) that every $F \in \mathbf{D}_{p,r}$ possesses a (p, r)-quasi continuous modification $F^{\sim}$. Furthermore, if $F \geq 0$ μ-*a.e.*, then $F^{\sim} \geq 0$ (p, r)-*q.e.* Also the following Tchebyshev-type inequality holds:

(1.25) $$C_{p,r}(F^{\sim} \geq \lambda) \leq \frac{1}{\lambda^p} \| F \|_{p,r}^p, \qquad \lambda > 0.$$

A generalized Wiener function $\Phi \in \mathbf{D}_{q,-r}$ is said to be *positive* and written as $\Phi \geq 0$ if $(\Phi, F) \geq 0$ for every $F \in \mathbf{D}_{p,r}$ such that $F \geq 0$, μ-*a.e.*, where $(\cdot, \cdot)$ is the natural coupling between $\mathbf{D}'_{p,r} = \mathbf{D}_{q,-r}$ and $\mathbf{D}_{p,r}$. Sugita [16] showed that to every $\Phi \in \mathbf{D}_{q,-r}$, $\Phi \geq 0$, there corresponds the unique finite Borel measure ν_{Φ} on B such that

i) $\bar{\nu}_{\Phi}(A) = 0$ if $C_{p,r}(A) = 0$

ii) if $F \in \mathbf{D}_{p,r}$, then $F^{\sim}$ is ν_{Φ}-integrable and

iii) $(\Phi, F) = \int_B F^{\sim} \, d\nu_{\Phi}$.

i) can be sharpened in the form of the inequality

(1.26) $$\bar{\nu}_{\Phi}(A) \leq \| \Phi \|_{q,-r} C_{p,r}(A)^{\frac{1}{p}}.$$

Sugita also showed that a Borel set A is slim if and only if $\nu_{\Phi}(A) = 0$ for every positive generalized Wiener function Φ.

§2. Quasi everywhere convergence and divergence

Let (B, H, μ) be an abstract Wiener space. Suppose we are given an orthonormal system (ONS) $\{h_n\}_{n=1}^{\infty}$ of H and a sequence of positive integers $\{p_n\}_{n=1}^{\infty}$. Define a sequence $\{\xi_n\}_{n=1}^{\infty}$ of polynomial functions by

$$\xi_n(x) = \sqrt{p_n!}\, H_{p_n}((h_n, x)), \qquad n = 1, 2, \ldots \tag{2.1}$$

where $H_n(t)$ is the Hermite polynomial defined by (1.2). Since (h_n, x), $n = 1, 2, \ldots$ are standard Gauss-distributed i.i.d. variables, $\xi_n(x)$ are independent random variables with mean 0 and variance 1. Also, by taking the quasi continuous modification, *we may and do assume that* $\xi_n(x)$ *is* (p, r)*-quasi continuous on* B *for every* $1 < p < \infty$ *and* $r > 0$ (in such a case, we say simply quasi continuous).

First we state the following general lemma valid for independent, mean 0 and p-th integrable random variables X_n, $n = 1, 2, \ldots$ defined on any probability space.

Lemma 2.1. *For every* $p \geq 2$, *there exists a positive constant* c_p *such that, for every real sequence* a_n, $n = 1, 2, \ldots,$

$$E(\sup_{n \geq 1} | \sum_{j=1}^{n} a_j X_j |^p) \leq c_p (\sum_{n=1}^{\infty} a_n^2)^{\frac{p}{2}} \sup_{n \geq 1} E | X_n |^p. \tag{2.2}$$

Proof. Set for each n, $f_n = \Sigma_{j=1}^{n} a_j X_j$ and $\mathscr{F}_n = \sigma(\{X_j\}_{j=1}^{n}) :=$ the σ-algebra generated by $X_j, j = 1, \ldots, n$. Then $\{f_n\}_{n=1}^{\infty}$ is a martingale with respect to $\{\mathscr{F}_n\}_{n=1}^{\infty}$. Setting $f^* = \sup_{n \geq 1} |f_n|$, it follows from Burkholder's inequality ([7]) that

$$E|f^*|^p \; c_p E(\sum_{n=1}^{\infty} |f_n - f_{n-1}|^2)^{\frac{p}{2}} = c_p E | \sum_{n=1}^{\infty} a_n^2 X_n^2 |^{\frac{p}{2}},$$

where c_p is a positive constant depending only on p. Since $\frac{p}{2} \geq 1$, it follows from the triangle inequality that

$$E | \sum_{n=1}^{\infty} a_n^2 X_n^2 |^{\frac{p}{2}} \leq (\sum_{n=1}^{\infty} a_n^2 (E|X_n|^p)^{\frac{2}{p}})^{\frac{p}{2}} \leq (\sum_{n=1}^{\infty} a_n^2)^{\frac{p}{2}} \sup_{n \geq 1} E|X_n|^p.$$

This completes the proof of Lemma 2.1.

Theorem 2.2. *Let* $\{\xi_n\}_{n=1}^{\infty}$ *be the sequence of random variables defined by* (2.1) *and let* $\{a_n\}_{n=1}^{\infty}$ *be a real sequence. Suppose that*

$$\sigma = \sup_n p_n < \infty. \tag{2.3}$$

Then the following dichotomy holds for the convergence or divergence of the series $\Sigma_{n=1}^{\infty} a_n \xi_n$:

(1) *If* $\Sigma_{n=1}^{\infty} a_n^2 < \infty$, *then the series* $\Sigma_{n=1}^{\infty} a_n \xi_n$ *converges quasi everywhere.*
(2) *If* $\Sigma_{n=1}^{\infty} a_n^2 = \infty$, *then for any real sequence* $\{b_n\}_{n=1}^{\infty}$, *the series* $\Sigma_{n=1}^{\infty} (a_n \xi_n - b_n)$ *diverges quasi everywhere.*

Proof. First we prove (1). For this, set

$$\alpha_{n,\varepsilon} = C_{p,r}(\sup_{m \geq n} |\sum_{j=n}^{m} a_j \xi_j| > \varepsilon), \qquad \varepsilon > 0, \ n \in \mathbf{N}.$$

Since $\xi_j \in C_{p_j}$ and the operator $(I - L)^{-\frac{r}{2}} = \Gamma\left(\frac{r}{2}\right)^{-1} \int_0^\infty t^{\frac{r}{2}-1} e^{-t(I-L)} dt$ is positivity preserving, we have

$$\begin{aligned} \sup_{m \geq n} |\sum_{j=n}^{m} a_j \xi_j| &= \sup_{m \geq n} |(I - L)^{-\frac{r}{2}} \sum_{j=n}^{m} a_j (I - L)^{\frac{r}{2}} \xi_j| \\ &\leq (I - L)^{-\frac{r}{2}} \sup_{m \geq n} |\sum_{j=n}^{m} a_j (1 + p_j)^{\frac{r}{2}} \xi_j|, \qquad \mu\text{-}a.e. \end{aligned}$$

By taking a quasi continuous modification of the right-hand side, this estimation can be sharpened as

$$\sup_{m \geq n} |\sum_{j=n}^{m} a_j \xi_j| \leq \{(I - L)^{-\frac{r}{2}} \sup_{m \geq n} |\sum_{j=n}^{m} a_j (1 + p_j)^{\frac{r}{2}} \xi_j|\}^{\sim}, \qquad q.e.$$

Hence, by the Tchebyshev inequality (1.25), $\alpha_{n,\varepsilon}$ can be estimated as

$$\begin{aligned} \alpha_{n,\varepsilon} &\leq \frac{1}{\varepsilon^p} \|(I - L)^{-\frac{r}{2}} \sup_{m \geq n} |\sum_{j=n}^{m} a_j (1 + p_j)^{\frac{r}{2}} \xi_j| \|_{p,r}^p \\ &= \frac{1}{\varepsilon^p} \int_B \sup_{m \geq n} |\sum_{j=n}^{m} a_j (1 + p_j)^{\frac{r}{2}} \xi_j|^p \, d\mu. \end{aligned}$$

Applying Lemma 2.1, we can estimate the above further as

$$\begin{aligned} \alpha_{n,\varepsilon} &\leq \frac{c_p}{\varepsilon^p} (\sum_{j=n}^{\infty} a_j^2 (1 + p_j)^r)^{\frac{p}{2}} \sup_{m \geq n} \int_B |\xi_m(x)|^p \, d\mu(x) \\ &\leq \frac{c_p}{\varepsilon^p} (1 + \sigma)^{\frac{pr}{2}} (\sum_{j=n}^{\infty} a_j^2)^{\frac{p}{2}} \max_{1 \leq m \leq \sigma} \left(\int_{-\infty}^{\infty} |\sqrt{m!}\, H_m(t)|^p \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \right) \\ &= A_{p,r,\varepsilon,\sigma} (\sum_{j=n}^{\infty} a_j^2)^{\frac{p}{2}}, \end{aligned}$$

where we set

$$A_{p,r,\varepsilon,\sigma} := \frac{c_p}{\varepsilon^p} (1 + \sigma)^{\frac{pr}{2}} \max_{1 \leq m \leq \sigma} \left(\int_{-\infty}^{\infty} |\sqrt{m!}\, H_m(t)|^p \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \right).$$

Combining this estimate with Fatou's lemma for capacities, we can now conclude that if $\sum_{n=1}^{\infty} a_n^2 < \infty$, the set $G_\varepsilon = \varlimsup_{n\to\infty} \{\sup_{m\geq n} |\sum_{j=n}^{m} a_j\xi_j| > \varepsilon\}$ has $C_{p,r}$-capacity 0 for every $\varepsilon > 0$ and $1 < p < \infty$, $r > 0$; that is, G_ε is a slim set for every $\varepsilon > 0$. Then setting $G = \bigcup_{k\geq 1} G_{\frac{1}{k}}$, G is also slim. It is clear that $\{\sum_{j=1}^{n} a_j\xi_j\}_{n=1}^{\infty}$ is a Cauchy sequence outside G and hence $\sum_{j=1}^{\infty} a_j\xi_j$ converges outside G. This completes the proof of (1).

Next we show the assertion (2). Given $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} a_n^2 = \infty$, it is sufficient to show that

$$C_{p,r}(\bigcap_{l\geq 1} \bigcup_{N\geq 1} \bigcap_{n\geq N} \bigcap_{m\geq 0} G_{l,m,n}) = 0 \tag{2.5}$$

for every $1 < p < \infty$ and $r > 0$, where

$$G_{l,m,n} = \left\{x;\, |\sum_{j=n}^{n+m} (a_j\xi_j(x) - b_j)| < \frac{1}{l}\right\}, \qquad l, m, n \in \mathbf{N}.$$

The left-hand side in (2.5) is dominated by

$$C_{p,r}(\bigcup_{N\geq 1} \bigcap_{n\geq N} \bigcap_{m\geq 0} G_{1,m,n}) = \lim_{N\to\infty} C_{p,r}(\bigcap_{n\geq N} \bigcap_{m\geq 0} G_{1,m,n})$$

and hence it is enough to show that

$$\lim_{N\to\infty} C_{p,r}(\bigcap_{n\geq N} \bigcap_{m\geq 0} G_{1,m,n}) = 0. \tag{2.6}$$

Take two sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ of non-negative integers such that

$$n_1 \leq n_1 + m_1 < n_2 \leq n_2 + m_2 < \cdots. \tag{2.7}$$

Then, if we choose k_0, $K \in \mathbf{N}$ such that $N < k_0 < K$,

$$C_{p,r}(\bigcap_{n\geq N} \bigcap_{m\geq 0} G_{1,m,n}) \leq C_{p,r}(\bigcap_{k=k_0}^{K} G_{1,m_k,n_k}). \tag{2.8}$$

Choose a C^∞-function $u(t)$ on $\mathbf{R}$ such that $0 \leq u(t) \leq 1$ and

$$\begin{cases} u(t) = 1 & \text{if } |t| \leq 1 \\ u(t) = 0 & \text{if } |t| \geq 2. \end{cases} \tag{2.9}$$

Set

$$\Delta_k(x) = \sum_{j\in J_k} (a_j\xi_j(x) - b_j), \qquad x \in B. \tag{2.10}$$

where

$$J_k = \{j;\ n_k \le j \le n_k + m_k\} \tag{2.11}$$

Since

$$\prod_{k=k_0}^{K} u(\Delta_k(x)) = 1 \qquad \text{if } x \in \bigcap_{k=k_0}^{K} G_{1,m_k,n_k},$$

we have

$$C_{p,r}\Big(\bigcap_{k=k_0}^{N} G_{1,m_k,n_k}\Big) \le \Big\| \prod_{k=k_0}^{K} u(\Delta_k) \Big\|_{p,r}^{p}. \tag{2.12}$$

If $r = 1$, then by the equivalence of norms,

$$\Big\| \prod_{k=k_0}^{K} u(\Delta_k) \Big\|_{p,r}^{p} \le A_1\Big(\Big\| \prod_{k=k_0}^{K} u(\Delta_k) \Big\|_{p}^{p} + \Big\| D \prod_{k=k_0}^{K} u(\Delta_k) \Big\|_{p}^{p}\Big) \tag{2.13}$$

where $\|\cdot\|_p$ is the L_p-norm and here and in the following, $A_1, A_2, \dots$ are positive constants depending only on p and σ in (2.3). By the chain rule for the operator D (cf. [9]) and the fact that $H_n'(t) = H_{n-1}(t)$, we have

$$D \prod_{k=k_0}^{K} u(\Delta_k) = \sum_{k=k_0}^{K} u'(\Delta_k)^2 \sum_{j \in J_k} a_j \sqrt{p_j!}\, H_{p_j - 1}((h_j, x))\, h_j \prod_{\substack{l=k_0 \\ l \ne k}} u(\Delta_l).$$

Since h_j are orthonormal in H,

$$\begin{aligned} \Big| D \prod_{k=k_0}^{K} u(\Delta_k) \Big|_H^2 &= \sum_{k=k_0}^{K} u'(\Delta_k)^2 \sum_{j \in J_k} a_j^2 p_j!\, H_{p_j-1}((h_j, x))^2 \prod_{\substack{l=k_0 \\ l \ne k}}^{K} u(\Delta_l)^2 \\ &\le \|u'\|_\infty^2 \sum_{k=k_0}^{K} \sum_{j \in J_k} a_j^2 p_j!\, H_{p_j-1}((h_j, x))^2 \prod_{l=k_0}^{K} \chi(\Delta_l) \end{aligned}$$

where $\chi(t)$ is the indicator function of the interval $[-2, 2]$. From this and the Schwarz inequality, we deduce that

$$\begin{aligned} \Big\| D \prod_{k=k_0}^{K} u(\Delta_k) \Big\|_p^p &\le \|u'\|_\infty^2 \int_B \Big(\sum_{k=k_0}^{K} \sum_{j \in J_k} a_j^2 p_j!\, H_{p_j-1}((h_j, x))^2\Big)^{\frac{p}{2}} \prod_{l=k_0}^{K} \chi(\Delta_l)\, d\mu \\ &\le \|u'\|_\infty^2\, I_1^{\frac{1}{2}} I_2^{\frac{1}{2}} \end{aligned} \tag{2.14}$$

where

$$I_1 = \int_B \Big(\sum_{k=k_0}^{K} \sum_{j \in J_k} a_j^2 p_j!\, H_{p_j-1}((h_j, x))^2\Big)^p d\mu$$

and

$$I_2 = \int_B \prod_{l=k_0}^{K} \chi(\Delta_l)\, d\mu.$$

We first estimate I_1. Since $p_j \leq \sigma$,

$$I_1 \leq \sigma^p \int_B |\sum_{k=k_0}^{K} \sum_{j \in J_k} a_j^2 [(p_j - 1)! H_{p_j - 1}((h_j, x))^2 - 1] + \sum_{k=k_0}^{K} \sum_{j \in J_k} a_j^2 |^p d\mu(x)$$
$$\leq 2^p \sigma^p \left\{ \int_B |\sum_{k=k_0}^{K} \sum_{j \in J_k} a_j^2 [(p_j - 1)! H_{p_j - 1}((h_j, x))^2 - 1]|^p d\mu(x) + |\sum_{k=k_0}^{K} \sum_{j \in J_k} a_j^2 |^p \right\}.$$

Applying Lemma 2.1 to centered random variables $\{X_j := (p_j - 1)! H_{p_j - 1}((h_j, x))^2 - 1\}$, $j \in J := \bigcup_{k=k_0}^{K} J_k$, we obtain

$$I_1 \leq 2^p \sigma^p \left\{ c_p (\sum_{j \in J} a_j^4)^{\frac{p}{2}} \max_{j \in J} \int_B |X_j|^p d\mu + (\sum_{j \in J} a_j^2)^p \right\} \tag{2.15}$$
$$\leq A_2 (\sum_{j \in J} a_j^2)^p.$$

From (2.12), (2.13), (2.14) and (2.15), we can conclude that

$$C_{p,1}(\bigcap_{k=k_0}^{K} G_{1,m_k,n_k}) \leq A_3 (1 + |\sum_{k=k_0}^{K} \sum_{j \in J_k} a_j^2 |^{\frac{p}{2}}) I_2^{\frac{1}{2}}. \tag{2.16}$$

In order to estimate I_2, we consider the following three possible cases:

(i) $$\lim_{n \to \infty} a_n = 0$$

(ii) $$0 < \overline{\lim_{n \to \infty}} |a_n| < \infty$$

(iii) $$\overline{\lim_{n \to \infty}} |a_n| = \infty.$$

First, we assume (i). Then, since $\sum_{j=1}^{\infty} a_j^2 = \infty$, we can choose the above $\{m_k\}$ and $\{n_k\}$ so that

$$\max_{j \geq n_k} |a_j|^2 < \frac{1}{2^k} \tag{2.17}$$

$$\sum_{j \in J_k \setminus \{(n_k + m_k)\}} a_j^2 < 1 \leq \sum_{j \in J_k} a_j^2 . \tag{2.18}$$

Then

$$\sum_{k=k_0}^{K} \sum_{j \in J_k} a_j^2 \leq \sum_{k=k_0}^{K} \left(1 + \frac{1}{2^k}\right) \leq K + 1.$$

We introduce the following notations:

$$A_k = \sqrt{\sum_{j \in J_k} a_j^2}, \qquad B_k = \sum_{j \in J_k} b_j,$$

$$\sigma_{k,j} = \frac{a_j}{A_k} \qquad \text{and } \eta_k = \sum_{j \in J_k} \sigma_{k,j} \xi_j.$$

Then we have

$$\Delta_k (= \sum_{j \in J_k} (a_j \xi_j - b_j)) = A_k \eta_k - B_k.$$

We claim that $\eta_k \xrightarrow{(d)} N(0, 1)$ as $k \to \infty$. Indeed, since η_k is a sum of independent random variables with mean 0 and variance 1, it is enough to verify the Lindeberg condition:

$$\lim_{k \to \infty} \sum_{j \in J_k} \int_{\{|\sigma_{k,j}\xi_j| > \varepsilon\}} |\sigma_{k,j}\xi_j|^2 \, d\mu = 0, \qquad \forall \varepsilon > 0. \tag{2.19}$$

It is clear that there exists a polynomial $P(t)$ such that

$$\max_{1 \le n \le \sigma} |\sqrt{n!}\, H_n(t)| \le P(t), \qquad \forall t \in \mathbf{R}.$$

Hence,

$$|\xi_j(x)| \le P((h_j, x)).$$

By (2.17) and (2.18), we have

$$\sigma_{k,j} = \frac{a_j}{A_k} \le \frac{1}{2^{\frac{k}{2}}}$$

and consequently

$$\begin{aligned}\sum_{j \in J_k} \int_{\{|\sigma_{k,j}\xi_j| > \varepsilon\}} |\sigma_{k,j}\xi_j|^2 \, d\mu &\le \sum_{j \in J_k} \sigma_{k,j}^2 \int_{\{2^{-\frac{k}{2}} P(t) > \varepsilon\}} P((h_j, x))^2 \, d\mu(x) \\ &= \int_{\{P(t) > 2^{\frac{k}{2}}\varepsilon\}} P(t)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \, dt.\end{aligned}$$

This proves (2.19). Now

$$I_2 = \prod_{k=k_0}^{K} \mu(|\Delta_k| \le 2) = \prod_{k=k_0}^{K} \mu(|A_k \eta_k - B_k| \le 2)$$

and $1 \le A_k^2 \le 1 + \dfrac{1}{2^k} \le 2$. Hence

$$\mu(|A_k \eta_k - B_k| \le 2) \le \mu(|\eta_k| \le 5) \qquad \text{if } |B_k| \le 3$$

and

$$\mu(|A_k\eta_k - B_k| \leq 2) \leq \mu\left(|\eta_k| \geq \frac{1}{\sqrt{2}}\right) \qquad \text{if } |B_k| \geq 3.$$

From these estimates and the fact that $\eta_k \xrightarrow{(d)} N(0, 1)$ as $k \to \infty$, we can find $k_0 > N$ and $0 < \alpha < 1$ such that

$$\mu(|A_k\eta_k - B_k| \leq 2) \leq \alpha \qquad \text{for all } k \geq k_0$$

and consequently $I_2 \leq \alpha^{K-k_0}$. This estimate, combined with (2.16), yields the following:

$$C_{p,1}\left(\bigcap_{k=k_0}^{K} G_{1,m_k,n_k}\right) \leq A_4 K^{\frac{p}{2}} \alpha^{\frac{k}{2}}$$

and by (2.8), we can conclude (2.6) if $r = 1$. The proof in the case of $r = 2, 3, \ldots$ can be given in the same way and this completes the proof of (2.6) in the case of (i).

In the case of (ii), there exists $\varepsilon > 0$ such that $|a_n| \geq \varepsilon$ for infinitely many n and we choose the above $\{n_k\}$ and $\{m_k\}$ so that $|a_{n_k}| \geq \varepsilon$ and $m_k = 0$. Let $\beta = \sup_n |a_n| < \infty$. Then

$$\sum_{k=k_0}^{K} \sum_{j \in J_k} a_j^2 = \sum_{k=k_0}^{K} a_{n_k}^2 \leq \beta^2 K.$$

Also

$$\mu(|\Delta_k| \leq 2) = \mu(|a_{n_k}\xi_{n_k} - b_{n_k}| \leq 2) \leq \mu\left(|\xi_{n_k}| \leq \frac{5}{\varepsilon}\right) \qquad \text{if } |b_{n_k}| \leq 3$$

and, if $|b_{n_k}| > 3$,

$$\mu(|\Delta_k| \leq 2) \leq \mu(\xi_{n_k} > 0) \qquad \text{or } \leq \mu(\xi_{n_k} < 0)$$

according as $a_{n_k} b_{n_k} > 0$ or $a_{n_k} b_{n_k} < 0$. Now we can easily find $0 < \alpha < 1$ such that

$$\mu(|\Delta_k| \leq 2) < \alpha \qquad \text{for all } k$$

because the number of different laws of $\{\xi_{n_k}\}_{k=1}^{\infty}$ is at most σ. Then (2.6) can be concluded in the same way as in the case of (ii). The proof for the case (iii) will be given in the next section.

Corollary 2.3 (Law of large numbers). *Let* $\{\xi_n\}_{n=1}^{\infty}$ *be defined by* (2.1) *and suppose that* $\sigma = \sup p_n < \infty$. *Then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \xi_j = 0 \qquad \textit{quasi everywhere.}$$

Proof. This is an immediate consequence of the previous theorem and Kronecker's lemma; let $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $\lim_{n \to \infty} b_n = \infty$ and $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If $\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ converges, then

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{j=1}^{n} a_j = 0.$$

Corollary 2.4. *H is slim.*

Proof. Choose an ONS $\{h_n\}_{n=1}^{\infty}$ in H such that $h_n \in B^*$. By Corollary 2.3, it holds that $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{1}{\sqrt{2}}\{(h_n, x)^2 - 1\} = 0$ *q.e.* and hence that $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}(h_j, x)^2 = 1$ *q.e.* But, for every $h \in H$, we have

$$\lim_{n\to\infty}\sum_{j=1}^{n}(h_j, h)^2 = \lim_{n\to\infty}\sum_{j=1}^{n}\langle h_j, h\rangle_h^2 \le |h|_H^2$$

and hence $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}(h_j, h)^2 = 0$. From this the assertion immediately follows.

§3 Central limit theorems under ν and convergence in $L^k(\nu)$

Let Φ be a positive generalized Wiener function belonging to $\mathbf{D}_{q,-r}$ such that $(\Phi, 1) = 1$ and $\nu = \nu_\Phi$ be the corresponding probability measure as explained in §1. Since every set of (p, r)-capacity 0 is a set of ν-measure 0, we can state Theorem 2.2 as a theorem concerning ν-almost everywhere convergence. We can also obtain other limit theorems for the same partial sums with respect to the probability ν. We start by giving the following approximation lemma:

Lemma 3.1. *For a given Φ as above and an orthonormal base (ONB) $\{l_n\}_{n=1}^{\infty}$ of H, there exists a sequence $\{\Phi_m\}_{m=1}^{\infty}$ of Wiener functions satisfying the following conditions:*

(1) $\Phi_m \in \mathbf{D}_{q,\infty}$, $\Phi_m \ge 0$ *and* $(\Phi_m, 1) = 1$

(2) $\lim_{m\to\infty}\|\Phi - \Phi_m\|_{q,-r} = 0$

(3) *for each m, there exist $N_m \in \mathbf{N}$ and $\varphi: \mathbf{R}^{N_m} \to \mathbf{R}$ such that*

$$\Phi_m(x) = \varphi_m((l_1, x), \ldots, (l_{N_m}, x)).$$

Proof. First we define Ψ_m by $\Psi_m = e^{\frac{1}{m}L}\Phi$. Then, we have $\Psi_m \ge 0$, $(\Psi_m, 1) = 1$ and $\lim_{m\to\infty}\|\Phi - \Psi_m\|_{q,-r} = 0$ ([16]). Let $\mathscr{B}_k$ be the σ-algebra generated by the (l_j, x) $j = 1, \ldots, k$. The conditional expectation $E_{\mathscr{B}_k}$ can extend to a contraction operator from $\mathbf{D}_{p,r}$ onto itself for all $p > 1$ and $r \in \mathbf{R}$ ([13]). For each m, we define $\Psi_{m,k}$ by $\Psi_{m,k} = E_{\mathscr{B}_k}\Psi_m$, then we have

$$\Psi_{m,k} \in \mathbf{D}_{q,\infty},\ \Psi_{m,k} \ge 0,\ (\Psi_{m,k}, 1) = 1 \quad \text{and} \quad \lim_{k\to\infty}\|\Psi_m - \Psi_{m,k}\|_{q,-r} = 0$$

Moreover $\Psi_{m,k}$ can be expressed as

$$\Psi_{m,k} = \psi_{m,k}((l_1, x), \dots, (l_k, x))$$

where $\psi_{m,k}\colon \mathbf{R}^k \to \mathbf{R}$ is a smooth function. Consequently, the assertion of Lemma 3.1 is evident.

Proposition 3.2 (Central limit theorem). *If $f_n \in H$, $n = 1, 2, \dots$ are such that $|f_n|_H = 1$ and $f_n \to 0$ weakly as $n \to \infty$, then, on the probability space (B, ν), we have $(f_n, x) \xrightarrow{(d)} N(0, 1)$.*

Proof. It suffices to show that

$$\lim_{n\to\infty} \int_B e^{it(f_n,x)}\, d\nu(x) = e^{-\frac{t^2}{2}} \tag{3.1}$$

for all $t \in \mathbf{R}$. Let $\{l_m\}_{m=1}^{\infty}$ be an ONB of H. By Lemma 3.1, there exists a sequence $\{\Phi_m\}_{m=1}^{\infty}$ satisfying the following conditions:

(i) $$\lim_{m\to\infty} \|\Phi - \Phi_m\|_{q,-r} = 0$$

(ii) $$\Phi_m \geq 0,\ (\Phi_m, 1) = 1,\ \Phi_m \in \mathbf{D}_{q,\infty}$$

(iii) $$\Phi_m = \varphi_m((l_1, x), \dots, (l_{N_m}, x))$$

where $\varphi_m\colon \mathbf{R}^{N_m} \to \mathbf{R}$ is a smooth function such that $\int_{\mathbf{R}^{N_m}} |\varphi_m|^q\, d\gamma_{N_m} < \infty$ and γ_N stands for the N-dimensional standard Gaussian distribution. Then, we have

$$\begin{aligned}\left|\int_B e^{it(f_n,x)}\, d\nu(x) - e^{-\frac{t^2}{2}}\right| &= |\langle \Phi, e^{it(f_n,\cdot)}\rangle - e^{-\frac{t^2}{2}}| \\ &\leq |\langle \Phi - \Phi_m, e^{it(f_n,\cdot)}\rangle| + |\langle \Phi_m, e^{it(f_n,\cdot)}\rangle - e^{-\frac{t^2}{2}}| \\ &\leq \|\Phi - \Phi_m\|_{q,-r} \|e^{it(f_n,\cdot)}\|_{p,r} + |\langle \Phi_m, e^{it(f_n,\cdot)}\rangle - e^{-\frac{t^2}{2}}|\end{aligned}$$

and

$$\langle \Phi_m, e^{it(f_n,\cdot)}\rangle = \int_B e^{it(f_n,x)} \varphi_m((l_1, x), \dots, (l_{N_m}, x))\, d\mu(x).$$

It is easy to see that the right-hand side of the above identity is equal to

$$\exp\left[-\frac{t^2}{2}\Big(1 - \sum_{j=1}^{N_m} \langle f_n, l_j\rangle_H^2\Big)\right] \int_{\mathbf{R}^{N_m}} \exp\Big(it \sum_{j=1}^{N_m} x_j \langle f_n, l_j\rangle_H\Big) \varphi_m(x_1, \dots, x_{N_m})\, d\gamma_{N_m}$$

Since $f_n \to 0$ weakly, $\lim_{n\to\infty} \langle f_n, l_j\rangle_H = 0$ for each j. Since

$$\int_{\mathbf{R}^{N_m}} |\varphi_m|\, d\gamma_{N_m} \leq \left(\int_{\mathbf{R}^{N_m}} |\varphi_m|^q\, d\gamma_{N_m}\right)^{\frac{1}{q}} < \infty,$$

it follows by the dominate convergence theorem that $\lim_{n\to\infty}\langle \Phi_m, e^{it(f_n,\cdot)}\rangle = e^{-\frac{t^2}{2}}$ for each fixed m. Choose an integer $N > r$. By the properties of norms (1.8) and (1.16), we have

$$\begin{aligned}\|e^{it(f_n,\cdot)}\|_{p,r} &\le \|e^{it(f_n,\cdot)}\|_{p,N} \le A_{p,N}(\|e^{it(f_n,\cdot)}\|_p + \|D^N e^{it(f_n,\cdot)}\|_p)\\ &= A_{p,N}(1 + \|(it)^N e^{it(f_n,\cdot)}\|_p) = A_{p,N}(1 + |t|^N).\end{aligned}$$

Cosequently, we obtain

$$\left|\int_B e^{it(f_n,x)}\,dv(x) - e^{-\frac{t^2}{2}}\right| \le A_{p,N}(1 + |t|^N)\|\Phi - \Phi_m\|_{q,-r} + |\langle \Phi_m, e^{it(f_n,\cdot)}\rangle - e^{-\frac{t^2}{2}}|.$$

By letting $n \to \infty$ and then $m \to \infty$, (3.1) follows.

Corollary 3.3. *Let $\{h_n\}_{n=1}^{\infty}$ be an ONS of H and $\{a_n\}_{n=1}^{\infty}$ be a real sequence. Set $A_n = (\sum_{j=1}^{n} a_j^2)^{\frac{1}{2}}$. If $\lim_{n\to\infty} A_n = \infty$, then the distribution of $\frac{1}{A_n}\sum_{j=1}^{n} a_j(h_j, x) \xrightarrow{(d)} N(0, 1)$ on the probability space (B, v).*

Proof. Set $f_n = \frac{1}{A_n}\sum_{j=1}^{n} a_j h_j$ so that $|f_n|_H^2 = \frac{1}{A_n^2}\sum_{j=1}^{n} a_j^2 = 1$. By the previous proposition, it is sufficient to prove that f_n converges weakly to 0 as $n \to \infty$. If $N < n$ and $g \in H$, then we have

$$\langle f_n, g\rangle_H = \frac{1}{A_n}\sum_{j=1}^{n} a_j\langle h_j, g\rangle_H = \frac{1}{A_n}\sum_{j=1}^{N} a_j\langle h_j, g\rangle_H + \frac{1}{A_n}\sum_{j=N+1}^{n} a_j\langle h_j, g\rangle_H.$$

By Cauchy-Schwarz's inequality,

$$|\langle f_n, g\rangle_H| \le \frac{1}{A_n}\sum_{j=1}^{N} |a_j\langle h_j, g\rangle_H| + \frac{1}{A_n}\sqrt{\sum_{j=N+1}^{n} a_j^2}\sqrt{\sum_{j=N+1}^{n} \langle h_j, g\rangle_H^2}.$$

Since $\{h_n\}$ is an ONS,

$$\overline{\lim_{n\to\infty}}\,|\langle f_n, g\rangle_H| \le \sqrt{\sum_{j=N+1}^{\infty} |\langle h_j, g\rangle_H|^2}.$$

Letting $N \to \infty$, we obtain the desired result.

Proposition 3.4 (Central limit theorem). *Let $\{a_n\}_{n=1}^{\infty}$ be a real sequence and $\{\xi_n\}_{n=1}^{\infty}$ be the sequence of random variables defined by* (2.1). *Suppose that*

$$\beta = \sup_{n\ge 1}|a_n| < \infty \quad A_n = (\sum_{j=1}^{n} a_j^2)^{\frac{1}{2}} \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty.$$

Then, $\frac{1}{A_n}\sum_{j=1}^{n} a_j\xi_j \xrightarrow{(d)} N(0, 1)$ under the probability v.

Proof. The basic idea of the proof of this proposition is the same as that of Proposition 3.2. Without loss of generality we can assume that $\{h_j\}_{j=1}^{\infty}$ is an ONB. Let $\{\Phi_m\}_{m=1}^{\infty}$ be an approximate sequence of Φ in Lemma 3.1 with respect to this ONB. It is enough to show that

$$\lim_{n\to\infty}\int_B \exp\left(it\frac{1}{A_n}\sum_{j=1}^{n} a_j\xi_j\right)dv(x) = e^{-\frac{t^2}{2}}$$

for all $t\in\mathbf{R}$. We have

$$\left|\int_B e^{it\frac{1}{A_n}\Sigma_{j=1}^n a_j\xi_j}dv - e^{-\frac{t^2}{2}}\right| = |(\Phi, e^{it\frac{1}{A_n}\Sigma_{j=1}^n a_j\xi_j}) - e^{-\frac{t^2}{2}}|$$

$$\leq (\Phi - \Phi_m, e^{it\frac{1}{A_n}\Sigma_{j=1}^n a_j\xi_j})| + |(\Phi_m, e^{it\frac{1}{A_n}\Sigma_{j=1}^n a_j\xi_j}) - e^{-\frac{t^2}{2}}|$$

$$\leq \|\Phi - \Phi_m\|_{q,-r}\|e^{it\frac{1}{A_n}\Sigma_{j=1}^n a_j\xi_j}\|_{p,r} + |(\Phi_m, e^{it\frac{1}{A_n}\Sigma_{j=1}^n a_j\xi_j}) - e^{-\frac{t^2}{2}}|.$$

First we show that

$$\sup_{n\geq 1}\|e^{it\frac{1}{A_n}\Sigma_{j=1}^n a_j\xi_j}\|_{p,r} < \infty. \tag{3.2}$$

For this purpose, we state the following assertion which is a consequence of (1.16) and the chain rule; for each $p>1$ and $r>0$, there exist $p'>1$, $r'>0$, $k\in\mathbf{N}$ and $K_{p,r}>0$ such that

$$\|e^{iF}\|_{p,r} \leq K_{p,r}(1+\|F\|_{p',r'}^k) \qquad \text{for every } F\in\mathscr{P}. \tag{3.3}$$

Using (3.3) and Lemma 2.1, we obtain

$$\|e^{it\frac{1}{A_n}\Sigma_{j=1}^n a_j\xi_j}\|_{p,r}$$

$$\leq K_{p,r}\left(1+\left\|\frac{t}{A_n}\sum_{j=1}^n a_j\xi_j\right\|_{p',r'}^k\right) \leq K_{p,r}\left(1+\left\|\frac{t}{A_n}\sum_{j=1}^n a_j(1+p_j)^{\frac{r'}{2}}\xi_j\right\|_{p'}^k\right)$$

$$\leq K_{p,r}\left[1+c_p^{\frac{k}{p'}}\frac{|t|^k}{A_n^k}\left(\sum_{j=1}^n a_j^2(1+p_j)^{r'}\right)^{\frac{k}{2}}\max_{1\leq j\leq n}\left(\int_B|\xi_j|^{p'}d\mu\right)^{\frac{k}{p'}}\right]$$

$$\leq K'_{p,r}\left[1+c_p^{\frac{k}{p'}}\frac{|t|^k}{A_n^k}(1+\sigma)^{\frac{kr'}{2}}\left(\sum_{j=1}^n a_j^2\right)^{\frac{k}{2}}\right] = K'_{p,r}(1+c_p^{\frac{k}{p'}}|t|^k(1+\sigma)^{\frac{kr'}{2}}).$$

Hence (3.2) follows immediately.

Next we show that

$$(\Phi_m, e^{it\frac{1}{A_n}\Sigma_{j=1}^n a_j\xi_j}) \longrightarrow e^{-\frac{t^2}{2}} \qquad \text{as } n\longrightarrow\infty \text{ for each fixed } m. \tag{3.4}$$

If $n>N_m$, from the independence of the (h_j, x), $1\leq j\leq n$, it follows that

$$(\Phi_m, e^{it\frac{1}{A_n}\Sigma_{j=1}^n a_j\xi_j}) = \int_B e^{it\frac{1}{A_n}\Sigma_{j=1}^n a_j\xi_j}\varphi_m((h_1, x),\ldots,(h_{N_m}, x))\,d\mu$$

$$= (\Phi_m, e^{it\frac{1}{A_n}\Sigma_{j=1}^{N_m} a_j\xi_j})\int_B e^{it\frac{1}{A_n}\Sigma_{j=N_m+1}^n a_j\xi_j}\,d\mu.$$

By the facts: $|e^{it\frac{1}{A_n}\Sigma_{j=1}^{N_m} a_j\xi_j}\Phi_m| \le \Phi_m \in \mathbf{D}_{q,\infty} \subset L_q \subset L_1$, $A_n \to \infty$ as $n \to \infty$ and $(\Phi_m, 1) = 1$, we can apply the dominate convergence theorem to obtain

$$\lim_{n\to\infty}(\Phi_m, e^{it\frac{1}{A_n}\Sigma_{j=1}^{N_m} a_j\xi_j}) = 1. \tag{3.5}$$

Next we claim that

$$\lim_{n\to\infty}\int_B e^{it\frac{1}{A_n}\Sigma_{j=N_m+1}^n a_j\xi_j}\,d\mu = e^{-\frac{t^2}{2}}. \tag{3.6}$$

To see this, setting $A_{n,m} = (\sum_{j=N_m+1}^n a_j^2)^{\frac{1}{2}}$ it is sufficient to show that

$$\frac{1}{A_{n,m}}\sum_{j=N_m+1}^n a_j\xi_j \xrightarrow{(d)} N(0, 1) \qquad \text{as } n \longrightarrow \infty$$

because $\lim_{n\to\infty}\frac{A_{n,m}}{A_n} = \lim_{n\to\infty}\left\{1 - \frac{1}{A_n^2}(\sum_{j=1}^{N_m} a_j^2)\right\}^{\frac{1}{2}} = 1$. For each $n > N_m$, $\left\{\frac{a_j}{A_{n,m}}\xi_j\right\}_{j=N_m+1}^n$ is a family of independent random variables satisfying

$$\int_B \frac{a_j}{A_{n,m}}\xi_j\,d\mu = 0 \qquad \text{and} \qquad \sum_{j=N_m+1}^n \int_B \left|\frac{a_j}{A_{n,m}}\xi_j\right|^2 d\mu = 1.$$

Hence we have only to verify the Lindeberg condition to obtain (3.6). As before we choose a polynomial $P(t)$ such that $\max_{1\le j\le\sigma}|\sqrt{j!}\,H_j(t)| \le P(t)$. For $\varepsilon > 0$, we have

$$\sum_{j=N_m+1}^n \int_{\{|\frac{a_j}{A_{n,m}}\xi_j| > \varepsilon\}} \left|\frac{a_j}{A_{n,m}}\xi_j\right|^2 d\mu$$

$$\le \sum_{j=N_m+1}^n \frac{a_j^2}{A_{n,m}^2}\int_{\{\frac{\beta}{A_{n,m}}P((h_j,x)) > \varepsilon\}} P((h_j, x))\,d\mu(x)$$

$$= \int_{\{P(t) > \frac{\varepsilon A_{n,m}}{\beta}\}} P(t)\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,dt \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

From (3.2) and (3.4) we obtain the desired result.

Remark. As is easily seen from the above proof, we can replace the conditions on $\{a_n\}_{n=1}^{\infty}$ by a weaker one: $\lim_{n\to\infty} \frac{1}{A_n} \max_{1\le j\le n} |a_j| = 0$ where $A_n = (\sum_{j=1}^{n} a_j^2)^{\frac{1}{2}}$

We are now in a position to complete the proof of Theorem 2.2 in the previous section. For this, we need the following lemma:

Lemma 3.5. *Let $P(t)$ be a polynomial, $\alpha_1,\dots,\alpha_r$ be the real zeros of $P(t)$ and m_j be the multiplicity of α_j. Then there exist $\gamma > 0$ and $\delta > 0$ such that*

$$\{t; |P(t)| \le \lambda\} \subset \bigcup_{j=1}^{r} \{t; |t-\alpha_j| \le \gamma\lambda^{\frac{1}{m_j}}\} \qquad \textit{for all } \lambda \in (0, \delta].$$

Proof. Suppose that the assertion of the lemma does not hold. Then for each positive integer n, there exist λ_n and t_n such that

$$0 < \lambda_n \le \frac{1}{n}, \ |P(t_n)| \le \lambda_n, \ |t_n - \alpha_j| > n\lambda_n^{\frac{1}{m_j}} \qquad 1 \le \forall j \le r.$$

real number α. By the continuity of $P(t)$ and the fact that $|P(t_n)| \le \frac{1}{n}$, we have $P(\alpha) = 0$ and hence α coincides with one of $\alpha_1,\dots,\alpha_r$, say α_j. We observe that

$$\lim_{t\to\alpha_j} \frac{P(t)}{(t-\alpha_j)^{m_j}} = \frac{P^{(m_j)}(\alpha_j)}{m_j!} \neq 0.$$

Thus there exist $d > 0$ and an integer n_0 such that $\left|\frac{P(t_n)}{(t_n-\alpha_j)^{m_j}}\right| \ge d$ for all $n \ge n_0$. Consequently it follows that for all $n \ge n_0$

$$\lambda_n \ge |P(t_n)| \ge d|t_n - \alpha_n|^{m_j} \ge dn^{m_j}\lambda_n$$

and hence that $dn^{m_j} \le 1$ for all $n \ge n_0$. This is a contradiction.

Proof of Theorem 2.2 *in the case* (iii). By assumption we can choose a sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers such that

$$n_1 < n_2 < n_3 < \cdots, \ |a_{n_k}| > k, \ \lim_{k\to\infty} \frac{b_{n_k}}{a_{n_k}} \text{ exists (possibly } \infty \text{ or } -\infty).$$

Set $A = \varliminf_{k\to\infty} \{|a_{n_k}\xi_{n_k} - b_{n_k}| \le 1\}$. Then A is a Borel set and by Fatou's lemma it holds that $\nu(A) \le \varliminf_{k\to\infty} \nu(|a_{n_k}\xi_{n_k} - b_{n_k}| \le 1)$ for every $\nu = \nu_\Phi$, $\Phi \in \mathbf{D}_{-\infty}$, $\Phi \ge 0$. If we can show that

$$\varliminf_{k\to\infty} \nu(|a_{n_k}\xi_{n_k} - b_{n_k}| \le 1) = 0, \tag{3.7}$$

then by the fact mentioned in § 1, we can conclude that A is slim. On the other hand $\lim_{n\to\infty}(a_n\xi_n - b_n) \neq 0$ on A^c and hence $\sum_{n=1}^{\infty}(a_n\xi_n - b_n)$ diverges on A^c, therefore Theorem 2.2 is completely proved.

Let us show (3.7). Set $u_k = \dfrac{b_{n_k}}{a_{n_k}}$, then

$$\nu(|a_{n_k}\xi_{n_k} - b_{n_k}| \le 1) = \nu\left(|\xi_{n_k} - u_k| \le \frac{1}{|a_{n_k}|}\right) \le \nu\left(|\xi_{n_k} - u_k| \le \frac{1}{k}\right).$$

Here we devide the proof into two parts:

1) The case $-\infty < \lim_{k\to\infty} u_k < \infty$

Setting $u = \lim_{k\to\infty} u_k$, we obtain

$$\nu\left(|\xi_{n_k} - u_k| \le \frac{1}{k}\right) \le \nu(|\xi_{n_k} - u| \le \lambda_k)$$

where $\lambda_k = |u - u_k| + \dfrac{1}{k}$. Let $\{\alpha_j\}_{j=1}^{l}$ be the totality of real zeros of the polynomials $\sqrt{n!}\,H_n(t) - u$, $1 \le n \le \sigma$ and m_j be the maximum of the multiplicities corresponding to α_j. Then, by Lemma 3.5 there exist $\gamma > 0$ and k_0 such that

$$\{|\xi_{n_k} - u| < \lambda_k\} \subset \bigcup_{j=1}^{l} \{|(h_{n_k}, x) - \alpha_j| \le \gamma_k^{\frac{1}{m_k}}\}, \qquad \forall k > k_0.$$

Thus we obtain

$$\nu(|\xi_{n_k} - u| \le \lambda_k) \le \sum_{j=1}^{l} \nu(|(h_{n_k}, x) - \alpha_j| \le \gamma\lambda_k^{\frac{1}{m_j}}).$$

Let ε be a positive number. Then we have

$$\nu(|\xi_{n_k} - u| \le \lambda_k) \le \sum_{j=1}^{l} \nu(|(h_{n_k}, x) - \alpha_j| \le \varepsilon)$$

for all sufficiently large k. Note that $\{h_n\}_{n=1}^{\infty}$ converges weakly to 0, and hence by Proposition 3.2 $(h_n, x) \xrightarrow{(d)} N(0, 1)$ as $n \to \infty$ under the probability ν. Therefore we finally obtain

$$\varlimsup_{k\to\infty} \nu(|a_{n_k}\xi_{n_k} - b_{n_k}| \le 1) \le \sum_{j=1}^{l} \int_{\{|t-\alpha_j| \le \varepsilon\}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt.$$

Letting $\varepsilon \downarrow 0$, (3.7) follows immediately.

2) The case $\lim_{k\to\infty} u_k = -\infty$ or ∞

Since the proof for the case $\lim_{k\to\infty} u_k = -\infty$ is the same as for the case $\lim_{k\to\infty} u_k = \infty$, we prove (3.7) only for the latter case. Suppose that $\lim_{k\to\infty} u_k = \infty$, then for every $M > 0$, there exists k_0 such that $M < u_k - \frac{1}{k}$, $\forall k > k_0$. Hence it follows that

$$\nu\left(|\xi_{n_k} - u_k| \le \frac{1}{k} \right) \le \nu(M < \xi_{n_k}) \qquad \forall k > k_0.$$

A moment's reflection shows that there exists a real valued function ϕ defined on an interval (M_0, ∞) such that

$$\lim_{M\to\infty} \phi(M) = \infty, \qquad \bigcup_{j=1}^{\sigma} \{t; \sqrt{j!}\, H_j(t) > M\} \subset \{t; |t| > \phi(M)\}.$$

Thus, if $M > M_0$ and $k > k_0$, we obtain $\nu(M < \xi_{n_k}) \le \nu(|(h_{n_k}, x)| > \phi(M))$. Letting $k \to \infty$, it follows from Proposition 3.2 that

$$\varlimsup_{k\to\infty} \nu(|a_{n_k}\xi_{n_k} - b_{n_k}| \le 1) \le \int_{\{|t| > \phi(M)\}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$$

Letting $M \to \infty$, we obtain (3.7).

Theorem 3.6. *Given an ONS $\{h_n\}_{n=1}^{\infty}$ of H and a sequence $\{p_n\}_{n=1}^{\infty}$ of positive integers, we define ξ_n by* (2.1). *Let $\{a_n\}_{n=1}^{\infty}$ be a real sequence, and suppose that $\sigma = \sup_{n\ge1} p_n < \infty$. Then the following statements (i) $\sim$ (iv) are equivalent:*

(i) $\sum_{n=1}^{\infty} a_n^2 < \infty$.

(ii) *The series $\sum_{n=1}^{\infty} a_n\xi_n$ converges ν-almost everywhere.*

(iii) *For all positive k, the series $\sum_{n=1}^{\infty} a_n\xi_n$ converges in $L^k(\nu)$.*

(iv) *For some positive k, the $\sum_{j=1}^{n} a_j\xi_j$ are bounded in $L^k(\nu)$.*

If $\sum_{n=1}^{\infty} a_n^2 = \infty$, then for any real sequence $\{b_n\}_{n=1}^{\infty}$, $\sum_{n=1}^{\infty} (a_n\xi_n - b_n)$ diverges ν-almost everywhere.

Proof. The equivalence (i)$\Leftrightarrow$(ii) is immediate by Theorem 2.2, and the implication (iii)$\Rightarrow$(iv) is trivial. So we prove the implications (i)$\Rightarrow$(iii) and (iv)$\Rightarrow$(i).

To see that (i) implies (iii), it is enough to notice that for each $k > 0$ there exists a positive constant A_k such that

$$\| \sum_{j=n}^{m} a_j\xi_j \|_{L^k(\nu)} \le A_k \left(\sum_{j=n}^{m} a_j^2 \right)^{\frac{1}{2}}$$

for any positive integers m, n with $m \geq n$, which is shown as follows: choose an even number, say $2N$, such that $2N > k$. Then we have

$$\| \sum_{j=n}^{m} a_j \xi_j \|_{L^k(\nu)} \leq \| \sum_{j=n}^{m} a_j \xi_j \|_{L^{2N}(\nu)} \leq \| \Phi \|_{q,-r}^{\frac{1}{2N}} \| (\sum_{j=n}^{m} a_j \xi_j)^{2N} \|_{p,r}^{\frac{1}{2N}}$$

It is known that for all $k \in \mathbf{N}$ and all $p, q, r > 1$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ it holds that $\| F \otimes G \|_{r,k} \leq \| F \|_{p,k} \| G \|_{q,k}$, $\forall F, \forall G \in \mathscr{P}$ ([9], [15]). Iterating this property several times, we obtain

$$\| (\sum_{j=n}^{m} a_j \xi_j)^{2N} \|_{p,r}^{\frac{1}{2N}} \leq A'_N \| \sum_{j=n}^{m} a_j \xi_j \|_{2Np,r}$$

where A'_N is a constant depending only on N. Applying Lemma 2.1 we have

$$\begin{aligned} \| \sum_{j=n}^{m} a_j \xi_j \|_{2Np,r} &= \| \sum_{j=n}^{m} a_j (1 + p_j)^{\frac{r}{2}} \xi_j \|_{2Np} \\ &\leq c_{2Np}^{\frac{1}{2Np}} (\sum_{j=n}^{m} a_j^2 (1 + p_j)^r)^{\frac{1}{2}} \max_{n \leq j \leq m} \left(\int_B |\xi_j|^{2Np} d\mu \right)^{\frac{1}{2Np}} \\ &\leq c_{2Np}^{\frac{1}{2Np}} (1 + \sigma)^{\frac{r}{2}} \max_{1 \leq j \leq \sigma} \left(\int_{-\infty}^{\infty} |\sqrt{j!}\, H_j(t)|^{2Np} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \right)^{\frac{1}{2Np}} (\sum_{j=n}^{m} a_j^2)^{\frac{1}{2}}. \end{aligned}$$

The above argument shows that we can take A_N as

$$A_N = A'_N c_{2Np}^{\frac{1}{2Np}} (1 + \sigma)^{\frac{r}{2}} \max_{1 \leq j \leq \sigma} \left(\int_{-\infty}^{\infty} |\sqrt{j!}\, H_j(t)|^{2Np} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \right)^{\frac{1}{2Np}}.$$

Next we prove the implication (iv) ⇒ (i). Suppose that (iv) holds and set $M = \sup_{n \geq 1} \| \sum_{j=1}^{n} a_j \xi_j \|_{L^k(\nu)}$. First we show that $\sup_{n \geq 1} |a_n| < \infty$ so that Proposition 3.4 is applicable. By the fact that $\| \cdot \|_{L^k(\nu)}$ is a norm or a quasi norm according as $k > 1$ or $0 < k < 1$ respectively, we have

$$\| a_n \xi_n \|_{L^k(\nu)} \leq 2 d_k M. \tag{3.8}$$

where d_k is a positive constant.

On the other hand, we have

$$\| a_n \xi_n \|_{L^k(\nu)} \geq |a_n|\, \nu(\xi_n \geq 1)^{\frac{1}{k}}. \tag{3.9}$$

It is easily seen that there exists β such that $\inf_{1 \leq j \leq \sigma} \inf_{t \geq \beta} \sqrt{j!}\, H_j(t) \geq 1$. Therefore we have

$$\nu(\xi_n \geq 1) \geq \nu((h_n, x) \geq \beta). \tag{3.10}$$

Since $(h_n, x) \xrightarrow{(d)} N(0, 1)$ by Proposition 3.2, we have

$$\lim_{n\to\infty} \nu((h_n, x) \geq \beta) = \int_\beta^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt > 0.$$

Thus there exist a positive constnt α_1 and a positive integer N_1 such that

$$\nu((h_n, x) \geq \beta) \geq \alpha_1, \qquad \forall n \geq N_1. \tag{3.11}$$

Combining (3.8), (3.9), (3.10) and (3.11) shows $|a_n| \leq \alpha_1^{-\frac{1}{k}} 2d_k M$ which proves our assertion.

Now we return to the proof of (i). Set $A_n = (\sum_{j=1}^n a_j^2)^{\frac{1}{2}}$ and we observe as before

$$M \geq \| \sum_{j=1}^n a_j \xi_j \|_{L^k(\nu)} = A_n \| \frac{1}{A_n} \sum_{j=1}^n a_j \xi_j \|_{L^k(\nu)} \geq A_n \nu\left(\frac{1}{A_n} \sum_{j=1}^n a_j \xi_j \geq 1 \right)^{\frac{1}{k}}.$$

Now suppose, on the contrary, that $\lim_{n\to\infty} A_n = \infty$. Then by Proposition 3.2 we have

$$\lim_{n\to\infty} \nu\left(\frac{1}{A_n} \sum_{j=1}^n a_j \xi_j \geq 1 \right) = \int_1^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt > 0$$

which allows us to take $\alpha_2 > 0$ and an integer N_2 such that

$$\nu\left(\frac{1}{A_n} \sum_{j=1}^n a_j \xi_j \geq 1 \right) \geq \alpha_2, \qquad \forall n \geq N_2$$

and hence it holds that $A_n \leq \alpha_2^{-\frac{1}{k}} M$, $\forall n \geq N_2$, this contradicts our assumption. Therefore (i) is proved.

Corollary 3.7. *Let $\{X_n\}_{n=1}^\infty$ be a Gaussian system on a probability space $(\Omega, \mathscr{B}, P)$ such that $E(X_n) = 0$, $E(X_n X_m) = \delta_{nm} - c_n c_m$ where $\{c_n\}_{n=1}^\infty$ is a real sequence such that $\sum_{n=1}^\infty c_n^2 \leq 1$. Let $p_n \in \mathbf{N}$, and set $\xi_n = \sqrt{p_n!}\, H_{p_n}(X_n)$. Suppose $\sup_{n\geq 1} p_n < \infty$. Then the assertions of Proposition* 3.4 *and Theorem* 3.6 *remain valid by replacing ν by P.*

Proof. Choose an ONB $\{h_n\}_{n=1}^\infty$ of H. We define f by $f = \sum_{n=0}^\infty c_n h_n$ where $c_0 = (1 - \sum_{n=1}^\infty c_n^2)^{\frac{1}{2}}$. Then we have $\langle f, h_n \rangle_H = c_n$, $|f|_H = 1$. Let ν be the probability measure corresponding to the positive generalized Wiener function $\delta_0((f, x))$ ([9], [15]). Then the sequence $\{(h_n, x)\}_{n=1}^\infty$ of random variables on the probability space (B, ν) forms a Gaussian system such that

$$\int_B (h_n, x)\, d\nu(x) = 0$$

$$\int_B (h_n, x)(h_m, x)\, d\nu(x) = \delta_{nm} - \langle f, h_n\rangle_H \langle f, h_m\rangle_H = \delta_{nm} - c_n c_m$$

Hence $\{X_n(\omega)\}_{n=1}^{\infty} \overset{(d)}{\sim} \{(h_n, x)\}_{n=1}^{\infty}$. Therefore we have $\{\sqrt{p_n!}\, H_{p_n}(X_n(\omega))\}_{n=1}^{\infty} \overset{(d)}{\sim}$ $\{\sqrt{p_n!}\, H_{p_n}((h_n, x))\}_{n=1}^{\infty}$. Since the assertions of Proposition 3.4 and Theorem 3.6 are concerned only with the law under ν of the random sequence $\{\xi_n\}_{n=1}^{\infty}$ defined by (2.1), our assertion is clear.

§4 Other results

In this section, we present some results obtained in the course of the study of the previous sections. The asymptotic behavior of the tail measure $\mu(\|x\|_B > \lambda)$ as $n \to \infty$ is known:

$$\lim_{\lambda \to \infty} \frac{1}{\lambda^2} \log \mu(\|x\|_B > \lambda) = -\frac{1}{2\gamma} \tag{4.1}$$

where γ is the positive constant defined in Theorem 4.1 given below ([2], [14]). From this fact Fernique's result follows in a strengthened form:

$$\int_B e^{\alpha \|x\|_B^2}\, d\mu(x) < \infty \qquad \text{if and only if } \alpha < \frac{1}{2\gamma}. \tag{4.2}$$

Our result is as follows:

Theorem 4.1. *Let* $p > 1$ *and* $\gamma > 0$. *Then*

$$-\frac{1}{2\gamma} = \lim_{\lambda \to \infty} \frac{1}{\lambda^2} \log C_{p,r}(\|x\|_B > \lambda) \tag{4.3}$$

where γ *is the norm of the injection* $j\colon H \to B$ *i.e.* $\gamma = \sup_{|h|_H \le 1} \|h\|_B$.

Proof. The basic idea of the proof is due to Takeda [17]. The point is that the following inequality holds:

$$\|x\|_B \le \left(\frac{\alpha + 1}{\alpha}\right)^{\frac{r}{2}} (V_{\alpha,r} \|\cdot\|_B)(x) \qquad \mu\text{-a.e. } x$$

where $V_{\alpha,r} = \alpha^{\frac{r}{2}}(\alpha I - L)^{-\frac{r}{2}}$. To see this, note that the norm $\|\cdot\|_B$ is expressed as

$\|x\|_B = \sup_{f\in\mathscr{C}} |(f, x)|$ where $\mathscr{C}$ is a countable subset of $\{f\in B^*;\ \|f\|_{B^*}\le 1\}$. From the positivity of $V_{\alpha,r}$ and the fact that $(f, x)\in\mathscr{C}_1$, it follows that

$$\begin{aligned}\|x\|_B &= \sup_{f\in\mathscr{C}}|(f,x)| = \sup_{f\in\mathscr{C}}|[(\alpha I - L)^{-\frac{r}{2}}(\alpha+1)^{\frac{r}{2}}(f,\cdot)](x)| \\ &= \left(\frac{\alpha+1}{\alpha}\right)^{\frac{r}{2}} \sup_{f\in\mathscr{C}}|[V_{\alpha,r}(f,\cdot)](x)| \\ &\le \left(\frac{\alpha+1}{\alpha}\right)^{\frac{r}{2}} [V_{\alpha,r}\sup_{f\in\mathscr{C}}|(f,\cdot)|](x) = \left(\frac{\alpha+1}{\alpha}\right)^{\frac{r}{2}} [V_{\alpha,r}\|\cdot\|_B](x).\end{aligned}$$

Recall that $V_{\alpha,r}$ has the integral expression mentioned in the proof of Theorem 2.2 (1) and $V_{\alpha,r}1 = 1$. Hence by Jensen's inequality, we obtain

$$\|x\|_B^2 \le \left(\frac{\alpha+1}{\alpha}\right)^r \{[V_{\alpha,r}\|\cdot\|_B](x)\}^2 \le \left(\frac{\alpha+1}{\alpha}\right)^r [V_{\alpha,r}\|\cdot\|_B^2](x).$$

Let c be a positive constant with $c < \dfrac{1}{2p\gamma}$. Then again by Jensen's inequality, we have

$$\exp\left[c\left(\frac{\alpha+1}{\alpha}\right)^{-r}\|x\|_B^2\right] \le \exp[(V_{\alpha,r}c\|\cdot\|_B^2)(x)] \le [V_{\alpha,r}e^{c\|\cdot\|_B^2}](x) \qquad \mu\text{-}a.e.\ x.$$

Hence we have

$$\exp\left[c\left(\frac{\alpha+1}{\alpha}\right)^{-r}\|x\|_B^2\right] \le [V_{\alpha,r}e^{c\|\cdot\|_B^2}]^{\sim}(x) \qquad q.e.\ x.$$

Thus it follows that

$$\begin{aligned}C_{p,r}(\|x\|_B > \lambda) &= C_{p,r}\left(\exp\left[c\left(\frac{\alpha+1}{\alpha}\right)^{-r}\|x\|_B^2\right] > \exp\left[c\left(\frac{\alpha+1}{\alpha}\right)^{-r}\lambda^2\right]\right) \\ &\le C_{p,r}\left([V_{\alpha,r}e^{c\|\cdot\|_B^2}]^{\sim}(x) > \exp\left[c\left(\frac{\alpha+1}{\alpha}\right)^{-r}\lambda^2\right]\right) \\ &\le \exp\left[-cp\left(\frac{\alpha+1}{\alpha}\right)^{-r}\lambda^2\right]\|V_{\alpha,r}e^{c\|\cdot\|_B^2}\|_{p,r}^p \\ &= \exp\left[-cp\left(\frac{\alpha+1}{\alpha}\right)^{-r}\lambda^2\right]\|(I-L)^{\frac{r}{2}}\alpha^{\frac{r}{2}}(\alpha I - L)^{-\frac{r}{2}}e^{c\|\cdot\|_B^2}\|_p^p.\end{aligned}$$

By Meyer-Shigekawa's multiplier theorem ([19]), the operator $(I-L)^{\frac{r}{2}}(\alpha I - L)^{-\frac{r}{2}}$ is bounded on $L^p(\mu)$. Thus

$$\alpha^{\frac{pr}{2}} \exp\left[-cp\left(\frac{\alpha+1}{\alpha}\right)^{-r}\lambda^2\right] \|(I-L)^{\frac{r}{2}}(\alpha I - L)^{-\frac{r}{2}} e^{c\|\cdot\|_B^2}\|_p^p$$
$$\leq A_{p,c,\alpha} \exp\left[-cp\left(\frac{\alpha+1}{\alpha}\right)^{-r}\lambda^2\right]$$

where

$$A_{p,c,\alpha} = \alpha^{\frac{pr}{2}} \|(I-L)^{\frac{r}{2}}(\alpha I - L)^{-\frac{r}{2}}\|_p^p \int_B e^{pc\|x\|_B^2} d\mu(x).$$

Note that by (4.2) $A_{p,c,\alpha} < \infty$. Consequently we obtain

$$\frac{1}{\lambda^2}\log C_{p,r}(\|x\|_B > \lambda) \leq \frac{1}{\lambda^2}\log A_{c,p,\alpha} - cp\left(\frac{\alpha+1}{\alpha}\right)^{-r}.$$

Letting $\lambda \to \infty$ first, and $\alpha \to \infty$, $c \to \frac{1}{2p\gamma}$, we have

$$\overline{\lim_{\lambda\to\infty}} \frac{1}{\lambda^2}\log C_{p,r}(\|x\|_B > \lambda) \leq -\frac{1}{2\gamma}$$

Next we show the rest. By the property (1.19) we have $\mu(\|x\|_B > \lambda) \leq C_{p,r}(\|x\|_B > \lambda)$ and hence

$$\frac{1}{\lambda^2}\log \mu(\|x\|_B > \lambda) \leq \frac{1}{\lambda^2}\log C_{p,r}(\|x\|_B > \lambda).$$

From (4.1) it follows that

$$-\frac{1}{2\gamma} \leq \underline{\lim_{\lambda\to\infty}} \frac{1}{\lambda^2}\log C_{p,r}(\|x\|_B > \lambda).$$

This completes the proof of the theorem.

From the above theorem the following corollary follows immediately (the same result was obtained by Sugita [16] in a different way).

Corollary 4.2. *Let ν be the probability measure corresponding to a positive generalized Wiener function $\Phi \in \mathbf{D}_{q,-r}$ with $(\Phi, 1) = 1$. Then* $\overline{\lim_{\lambda\to\infty}} \frac{1}{\lambda^2}\log \nu(\|x\|_B \geq \lambda) \leq -\frac{1}{2p\gamma}$.

Our next theorem is a refinement of what is called Itô-Nisio's theorem: For any ONB $\{e_n\}_{n=1}^\infty$ of H it holds that

$$\lim_{n\to\infty} \|x - x_n\|_B = 0 \qquad a.e.\ x \tag{4.4}$$

where x_n is defined by $x_n = \sum_{j=1}^{n} (e_j, x)e_j$. Our result is that we can reduce the exceptional set of Itô-Nisio's theorem to a slim set (this result was also obtained by Feyel-de La Pradelle [3]):

Theorem 4.3. *Let $\{e_n\}_{n=1}^{\infty}$ be an ONB of H and x_n be the quasi continuous mapping defined by $x_n = \sum_{j=1}^{n} (e_j, x)e_j$. Then it holds that $\lim_{n\to\infty} \|x - x_n\|_B = 0$ q.e. x.*

Remark. The proof of the theorem is based on Itô-Nisio's theorem and the fact that

$$\lim_{n\to\infty} \int_B \|x - x_n\|_B^p \, d\mu(x) = 0 \qquad \text{for every } p > 0. \tag{4.5}$$

Hence we cannot deduce Itô-Nisio's theorem as a corollary. For the proof of (4.4) and (4.5), see [12] and [18] p.290.

Proof. Given an $\varepsilon > 0$ we define $\alpha_{n,m}$ by

$$\alpha_{n,m} = C_{p,r}\Big(\max_{n+1\le k\le m} \|x_k - x_n\|_B > \varepsilon\Big)$$

for each pair of positive integers n, m with $n < m$. As in the proof of Theorem 4.1 we have

$$\max_{n+1\le k\le m} \|x_k - x_n\|_B \le 2^{\frac{r}{2}}[(I - L)^{-\frac{r}{2}} \max_{n+1\le k\le m} \|\cdot - \cdot\|_B]^{\sim}(x) \qquad q.e.\ x.$$

Thus by the Tshebyshev-type inequality we have

$$\alpha_{n,m} \le \left(\frac{2^{\frac{r}{2}}}{\varepsilon}\right)^p \int_B \max_{n+1\le k\le m} \|x_k - x_n\|_B^p \, d\mu(x).$$

Note that $\{\|x_k - x_n\|_B\}_{k=n+1}^{m}$ is a martingale, hence it follows from Doob's inequality that

$$\alpha_{n,m} \le \left(\frac{2^{\frac{r}{2}} p}{\varepsilon(p-1)}\right)^p \int_B \|x_m - x_n\|_B^p \, d\mu(x).$$

Since $\{\max_{n+1\le k\le m} \|x_k - x_n\| > \varepsilon\} \uparrow \{\sup_{n+1\le k} \|x_k - x_n\|_B > \varepsilon\}$ as $m \to \infty$, we have by (1.22)

$$\alpha_n = C_{p,r}\Big(\sup_{k\ge n+1} \|x_k - x_n\|_B > \varepsilon\Big) = \lim_{m\to\infty} \alpha_{n,m}.$$

On the other hand, it holds from (4.5) that

$$\lim_{m\to\infty}\int_B \|x_m - x_n\|_B^p \, d\mu(x) = \int_B \|x - x_n\|_B^p \, d\mu(x).$$

Consequently we obtain

$$\alpha_n \le \left(\frac{2^{\frac{r}{2}}p}{\varepsilon(p-1)}\right)^p \int_B \|x - x_n\|_B^p \, d\mu(x).$$

By (4.5) again, the right-hand side of the above inequality tends to 0 as $n \to \infty$. Therefore by Fatou's lemma we finally obtain

$$C_{p,r}(\varlimsup_{n\to\infty}\{\sup_{n+1\le k}\|x_k - x_n\|_B > \varepsilon\}) = 0$$

for all $\varepsilon > 0$. Set

$$G = \bigcup_{l=1}^{\infty} G_l \qquad \text{where } G_l = \varlimsup_{n\to\infty}\left\{\sup_{n+1\le k}\|x_k - x_n\|_B > \frac{1}{l}\right\}.$$

It is clear that G is slim and that $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence on G^c. Similary setting

$$G' = \bigcup_{l=1}^{\infty} G'_l \qquad \text{where } G'_l = \varlimsup_{n\to\infty}\left\{\|x - x_n\|_B > \frac{1}{l}\right\}.$$

we can prove that G' is slim and that we can extract a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ so that $\lim_{j\to\infty}\|x - x_{n_j}\|_B = 0$ on G'^c. This proves Theorem 4.3.

Corollary 4.4. *Let $\{\varphi_n\}_{n=1}^{\infty}$ be an ONB of $L^2(0, 1)$ and $\{\xi_n\}_{n=1}^{\infty}$ be a Gaussian system such that*

$$E(\xi_n) = 0 \qquad \textit{and } E(\xi_n\xi_m) = \delta_{nm} - c_n c_m$$

where $c_n = \int_0^1 \varphi_n(s)\,ds$. The stochastic process $\{X_n(t)\}_{0\le t\le 1}$ defined by

$$X_n(t, \omega) = \sum_{j=1}^{n} \xi_j(\omega)\int_0^t \varphi_j(s)\,ds$$

converges uniformly to a pinned Brownian motion for almost all ω.

Proof. It is enough to show that there exist a certain probability space $(\Omega, \mathscr{B}, P)$ and a Gaussian system $\{\eta_n\}_{n=1}^{\infty}$ on it such that

$$E(\eta_n) = 0 \qquad E(\eta_n\eta_m) = \delta_{nm} - c_n c_m \tag{4.6}$$

and that the stochastic process $\{Y_n(t)\}_{0\le t\le 1}$ defined by $Y_n(t) = \sum_{j=1}^{n}\eta_j\int_0^t \varphi_j(s)\,ds$ converges uniformly to a pinned Brownian motion.

Let (W_0^1, H, μ) be the one-dimentional Wiener space, that is, W_0^1 is the set of real continuous functions on [0, 1] vanishing at 0, H is the Cameron-Martin subspace of W_0^1, i.e., $H=\{h\colon[0,1]\to\mathbf{R}; h$ is absolutely continuous and $|h|_H^2=\int_0^1|h'(t)|^2\,dt<\infty\}$, and μ is the standard Wiener measure. W_0^1, equipped with the sup norm $\|w\|_\infty=\max_{0\le t\le 1}|w(t)|$, is a Banach space. Let ν be the probability measure corresponding to the positive generalized Wiener function $\delta_0(w(1))$. ν is nothing but the probability law of pinned Brownian motion ([8], [15]).

We take $(W_0^1, \mathscr{B}(W_0^1), \nu)$ as $(\Omega, \mathscr{B}, P)$ where $\mathscr{B}(W_0^1)$ is the Borel σ-algebra on W_0^1. Let $\{e_n\}_{n=1}^\infty$ be the ONB of H defined by $e_n(t)=\int_0^t\varphi_n(s)\,ds$, and we set

$$\eta_n(w)=(e_n, w)=\int_0^1\varphi_m(s)\,dw(s).$$

Then $\{\eta_n\}_{n=1}^\infty$ satisfies (4.6) and by Theorem 4.3 we have

$$\lim_{n\to\infty}\max_{0\le t\le 1}|w(t)-\sum_{j=1}^n(e_n, w)e_n(t)|=0 \qquad a.e.\ w.$$

Therefore Corollary 4.4 is shown.

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