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# Certain invariant sets of stochastic flows generated by stochastic differential equations

By

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## Introduction

First we consider the diffusion process $\{x_t\}_{t\geq 0}$ determined by the following stochastic differential equation;

$$(1) \qquad dx_t = \sum_{k=1}^{r} V_k(x_t) \circ dw^k(t) + V_0(x_t)dt\,, \qquad x_0 = x$$

where $V_0$, $V_1$, ..., $V_r$ are smooth vector fields on $\mathbf{R}^d$ and $w = (w^1, \dots, w^r)$ is an $r$-dimensional Wiener process. It seems to be a very natural and primitive problem to search for a subset E of $\mathbf{R}^d$ as small as possible satisfying that $x_t \in \mathrm{E}$ for all $t$ almost surely. In this paper, we will give an answer to it.

Let $\theta_{0,t}$, ..., $\theta_{r,t}$ be the flows generated respectively by $V_0$, ..., $V_r$. Let $\mathrm{E}_x$ be the set of the points $\theta_{k_n,t_n} \circ \cdots \circ \theta_{k_1,t_1}(x)$ where $n$ is a positive integer, $k_1$, ..., $k_n \in \{0, \dots, r\}$ and $t_1$, ..., $t_n \in (-\infty, +\infty)$. Since Sussmann showed in [3] that $\mathrm{E}_x$ becomes a $\mathrm{C}^\infty$-submanifold of $\mathbf{R}^d$, we know at least that $x_t$ stays in $\mathrm{E}_x$ for a while. On the other hand, $x_t$ does not exit from $\overline{\mathrm{E}}_x$, the closure of $\mathrm{E}_x$ in $\mathbf{R}^d$, as a trivial consequence of Stroock and Varadhan's result [2] considering the support of the probability law of $\{x_t\}_{t\geq 0}$ in the path space. In this paper, we will show that $x_t$ does not exit from $\mathrm{E}_x$ itself up to the explosion time. Indeed, we will show under Lipschitz condition that the stochastic flow generated by the equation (1) preserves all submanifolds of the type $\mathrm{E}_x$.

## 1. Statement of the result

Let $V_0$, $V_1$, ..., $V_r$ be fixed smooth vector fields on $\mathbf{R}^d$ and we assume that $V_0$, ..., $V_r$ and $V_0 + \sum_{k=1}^{r} \partial V_k \cdot V_k$ satisfy the global Lipschitz condition where $\partial V_k$ denotes the matrix $\left(\dfrac{\partial V_k^i}{\partial x^j}\right)$. For any absolutely continuous function $h = (h_0, h_1, \dots, h_r)$ on $(-\infty, +\infty)$, we consider the ordinary differential equation

$$(2) \qquad \dot{\varphi}_t = \sum_{k=0}^{r} V_k(\varphi_t)\dot{h}^k(t)\,, \qquad -\infty < t < +\infty$$

and put $\varphi_t \equiv \varphi_{s,t}(x, h)$ where $\varphi_t$ is the solution of (2) satisfying $\varphi_s = x$. Then, for any fixed $s$, $t$ and $h$, $\varphi_{s,t}(h) \equiv \varphi_{s,t}(\cdot, h)$ is a diffeomorphism on $\mathbf{R}^d$.

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Let $\mathbf{S}$ be the set of all polygonal functions $h=(h^0,\dots,h^r)$ which move parallel to one of the coordinate axes everytime. Then

$$\text{(3)} \qquad \mathrm{E}_x=\{\varphi_{0,1}(x,h): h\in\mathbf{S}\}=\{\varphi_{s,t}(x,h): h\in\mathbf{S},\ -\infty<s,t<+\infty\}$$

where $\mathrm{E}_x$ is defined in Introduction. $\mathrm{E}_x$ is the smallest set which contains $x$ and is invariant with respect to each of the flows generated respectively by $V_0$, $\dots$, $V_r$. Especially, we note that $y\in\mathrm{E}_x$ implies $\mathrm{E}_x=\mathrm{E}_y$.

Let $w(t)=(w^1(t),\dots,w^r(t))$, $-\infty<t<+\infty$ be an $r$-dimensional Wiener process such that $w(0)=0$. Let $\varphi_{s,t}(x)$, $-\infty<s$, $t<+\infty$, $x\in\mathbf{R}^d$ be a stochastic flow of diffeomorphisms generated by the stochastic differential equation (1), i.e. it is an $\mathbf{R}^d$-valued ramdom field satisfying with probability 1 that

$$\text{(4)} \qquad \varphi_{s,t}(x) \text{ is continuous with respect to } (s,t,x)\,,$$

$$\text{(5)} \qquad \varphi_{s,t} \text{ is a diffeomorphism on } \mathbf{R}^d \text{ for any } s,\ t\in(-\infty,+\infty)\,,$$

$$\text{(6)} \qquad \varphi_{t,u}\circ\varphi_{s,t}=\varphi_{s,u} \quad \text{for any } s,\ t,\ u\,,$$

and moreover, for any fixed $s$ and $x$, $\varphi_t\equiv\varphi_{s,t}(x)$ satisfies that

$$\text{(7)} \qquad \varphi_t=x+\sum_{k=1}^r\int_s^t V_k(\varphi_u)\circ dw^k(u)+\int_s^t V_0(\varphi_u)du \qquad \text{a.s.}$$

where, when $s>t$, we interpret the above stochastic integrals as backward integrals. See [1] for the existence of the stochastic flow.

Our main result is as follows:

**Theorem.** *It holds with probability* 1 *that, for any* $s$, $t$ *and* $x$, $\mathrm{E}_x$ *is an invariant set of* $\varphi_{s,t}$.

The following corollary is an immediate consequence of it.

**Corollary.** *Let* $\{x_t\}_{t\geq 0}$ *be the solution of* (1). *Then* $x_t\in\mathrm{E}_x$ *for all* $t\in[0,+\infty)$ *almost surely.*

**Remark.** We can easily extend above corollary to the case where the Lipschitz condition is not satisfied. In this case, the solutions of the equations (1) and (2) may explose. But we shall define $\mathrm{E}_x$ analogously considering only no-explosing solutions $\varphi_{s,t}(x,h)$ in (3). Then we can deduce from the above corollary by truncation argument that the diffusion process $x_t$ stays in $\mathrm{E}_x$ up to the explosion time.

## 2. Proof of Theorem

This section is devoted to prove Theorem. First we study the form of the set $\mathrm{E}_x$.

**Lemma 1.** *For any* $x\in\mathbf{R}^d$, *there exist a local coordinate* $f=(f^1,\dots,f^d)$ *on some neighborhood* U *of* $x$ *and some integer* $e$, $0\leq e\leq d$ *such that*

(i) $f^1(x) = \cdots = f^d(x) = 0$,
(ii) $\mathrm{M} \equiv \{\xi \in U: f^{e+1}(\xi) = \cdots = f^d(\xi) = 0\}$ *is contained in* $\mathrm{E}_x$,
(iii) $V_0, \ldots, V_r$ *are tangential to* M *at any point on* M.

*Proof.* Fix any $x \in \mathbf{R}^d$. Let $(\tilde{f}^1, \ldots, \tilde{f}^d)$ be a local coordinate on a neighborhood $\tilde{\mathrm{U}}$ of some point $\tilde{x}$ of $\mathrm{E}_x$ such that $\tilde{f}^1(\tilde{x}) = \cdots = \tilde{f}^d(\tilde{x}) = 0$ and $\tilde{\mathrm{M}} \equiv \{\xi \in \tilde{\mathrm{U}}: \tilde{f}^{e+1}(\xi) = \cdots = \tilde{f}^d(\xi) = 0\}$ is contained in $\mathrm{E}_x$ for some $e$, $0 \le e \le d$. It is clear that there exists at least one such local coordinate because we permit $e$ to be 0. We can choose $\tilde{x}$ and $(\tilde{U}; \tilde{f}^1, \ldots, \tilde{f}^d)$ so that $e$, which is the dimension of $\tilde{\mathrm{M}}$, becomes maximum. Since $\tilde{x} \in \mathrm{E}_x$, there exists some $h \in \mathrm{S}$ such that $\varphi_{0,1}(x, h) = \tilde{x}$ by (3). We put $f^i = \tilde{f}^i \circ \varphi_{0,1}(h)$ and $\mathrm{U} = \varphi_{0,1}(h)^{-1}(\tilde{\mathrm{U}})$. Then they satisfy (i) and (ii) since $\mathrm{M} \equiv \{\xi \in \mathrm{U}: f^{e+1}(\xi) = \cdots = f^d(\xi) = 0\} = \varphi_{0,1}(h)^{-1}(\tilde{\mathrm{M}}) = \varphi_{1,0}(h)(\tilde{\mathrm{M}}) \subset \mathrm{E}_x$.

Next, we shall show (iii). Suppose that there exists a point $p \in \mathrm{M}$ such that $V_k(p)$ is not tangential to M for some $k$. Then we define a map $g$ from a sufficiently small neighborhood of 0 in $\mathbf{R}^{e+1}$ into $\mathbf{R}^d$ by

$$g(y^1, \ldots, y^e, t) = \theta_t(f^{-1}(f^1(p) + y^1, \ldots, f^e(p) + y^e, 0, \ldots, 0))$$

where $\{\theta_t: -\infty < t < +\infty\}$ denotes the flow generated by $V_k$. It is easy to see $g(0) = p$. Noting that $f^{-1}(y^1 + f^1(p), \ldots, y^e + f^e(p), 0, \ldots, 0)$ lies in $M$, we see that the image of $g$ is contained in $\mathrm{E}_x$. Let $\mathrm{T}_p(\mathrm{M})$ denote the tangent space to M at $p$. Since $\left(\frac{\partial g}{\partial y^1}(0), \ldots, \frac{\partial g}{\partial y^e}(0)\right)$ is a basis of $\mathrm{T}_p(\mathrm{M})$ and $\frac{\partial g}{\partial t}(0) = V_k(p)$ does not lie in $\mathrm{T}_p(\mathrm{M})$, we can see that $\frac{\partial g}{\partial y^1}(0), \ldots, \frac{\partial g}{\partial y^e}(0)$ and $\frac{\partial g}{\partial t}(0)$ are linearly independent so that $g$ is nondegenerate at 0. So we can find some local coordinate $f_1 = (f_1^1, \ldots, f_1^d)$ on some neighborhood $\mathrm{U}_1$ of $p$ such that

$$f_1(g(y^1, \ldots, y^e, t)) = (y^1, \ldots, y^e, t, 0, \ldots, 0)$$

by the rank theorem. It is clear that $f_1(p) = 0$. Since $\mathrm{M}_1 \equiv \{\xi \in \mathrm{U}_1: f_1^{e+2}(\xi) = \cdots = f_1^d(\xi) = 0\}$ is contained in the image of $g$, it is contained in $\mathrm{E}_x$. But this contradicts with the maximality of $e$. So $V_0, \ldots, V_r$ are tangential to M and this completes the proof. □

**Remark.** Lemma 1 is also covered by a result of Sussmann [3] that $\mathrm{E}_x$ becomes a $\mathrm{C}^\infty$-submanifold of $\mathbf{R}^d$ by a suitable differentiable structure.

We know from Lemma 1 that, for any fixed $x$ and $s$, it holds almost surely that $\varphi_{s,t}(x)$ stays in $\mathrm{E}_x$ if $t$ is close enough to $s$. Indeed, $\varphi_{s,t}(x)$ stays in M until it exit from U by (iii) in Lemma 1. The next lemma states that the exceptional set does not depend on $x$ and $s$.

**Lemma 2.** *It holds almost surely that, for any* $x$ *and* $s$, *there exists some positive number* $\varepsilon$ *such that* $|t - s| < \varepsilon$ *implies* $\varphi_{s,t}(x) \in \mathrm{E}_x$.

*Proof.* Let $h_n = h_n(w)$, $n = 1, 2, \dots$ be an polygonal approximation of the process $(t, w^1, \dots, w^r)$. Then $\varphi_{s,t}(x, h_n)$ converges to $\varphi_{s,t}(x)$ in probability in the topology of local uniform convergence with respect to $(s, t, x)$ (see [1]). By choosing a good subsequence, we may suppose that the above local uniform convergence holds almost surely. Particularly, it holds almost surely that, for arbitrary $x$ and $s$, $\varphi_{s,t}(x, h_n)$ converges to $\varphi_{s,t}(x)$ uniformly with respect to $t$ on any finite intervals. That implies the assertion of Lemma 2. Indeed, suppose that $\varphi_{s,t}(x, h_n)$ converges to $\varphi_{s,t}(x)$ uniformly with respect to $t$ on any finite intervals. Let U be the coordinate neighborhood of $x$ introduced in Lemma 1 and K be a compact neighborhood of $x$ contained in U. Then, since the sequence $\{\varphi_{s,t}(x, h_n)\}_{n=1,2,\dots}$ is equicontinuous with respect to $t$, there exists an $\varepsilon > 0$ such that $|t - s| < \varepsilon$ implies that $\varphi_{s,t}(x, h_n) \in \mathrm{K}$ for all $n$. Therefore, if $|t - s| < \varepsilon$, then $\varphi_{s,t}(x, h_n) \in \mathrm{M} \cap \mathrm{K}$ for all $n$ where M is defined in Lemma 1. Since $\mathrm{M} \cap \mathrm{K}$ is compact and $\varphi_{s,t}(x, h_n)$ converges to $\varphi_{s,t}(x)$, we see $\varphi_{s,t}(x) \in \mathrm{M} \cap \mathrm{K} \subset \mathrm{E}_x$ if $|t - s| < \varepsilon$. It completes the proof. □

Before deducing the assertion of Theorem, we shall prepare the following set-theoretical fact.

**Lemma 3.** *Let the map* $\psi: (-\infty, +\infty) \times (-\infty, +\infty) \times \mathbf{R}^d \to \mathbf{R}^d$ *satisfy the following conditions* (i) *and* (ii);

(i) $\psi_{s,s}(x) = x$ *and* $\psi_{t,u}(\psi_{s,t}(x)) = \psi_{s,u}(x)$ *for any* $s$, $t$, $u$ *and* $x$.

(ii) *For any* $s$ *and* $x$, *there exists an* $\varepsilon > 0$ *such that* $|t - s| < \varepsilon$ *implies* $\psi_{s,t}(x) \in \mathrm{E}_x$.

*Then* $\psi_{s,t}(x) \in \mathrm{E}_x$ *for any* $s$, $t$ *and* $x$.

*Proof.* Let us fix any $s$ and $x$. Put $\mathrm{A} \equiv \{t \in (-\infty, +\infty): \psi_{s,t}(x) \in \mathrm{E}_x\}$. It suffices to show $\mathrm{A} = (-\infty, +\infty)$.

First we shall show that A is an open set in $(-\infty, +\infty)$. Suppose $t \in \mathrm{A}$. By the assumption (ii), we can choose $\varepsilon > 0$ such that $|u - t| < \varepsilon$ implies $\psi_{t,u}(\psi_{s,t}(x)) \in \mathrm{E}_{\psi_{s,t}(x)}$. By the assumption (i), the left-hand side is equal to $\psi_{s,u}(x)$. Moreover $t \in \mathrm{A}$ means that $\psi_{s,t}(x) \in \mathrm{E}_x$ or equivalently $\mathrm{E}_{\psi_{s,t}(x)} = \mathrm{E}_x$. Therefore $|u - t| < \varepsilon$ implies $\psi_{s,u}(x) \in \mathrm{E}_x$ which means that $u \in \mathrm{A}$. That shows that A is an open set.

Next we shall show that $\mathrm{A}^c$ is an open set. Suppose $t \in \mathrm{A}^c$. By the same argument as above, there exists an $\varepsilon > 0$ such that $|u - t| < \varepsilon$ implies $\psi_{s,u}(x) \in \mathrm{E}_{\psi_{s,t}(x)}$. Since $\mathrm{E}_x \cap \mathrm{E}_{\psi_{s,t}(x)} = \phi$, $|u - t| < \varepsilon$ implies $u \in \mathrm{A}^c$. That shows that $\mathrm{A}^c$ is an open set.

Since $\psi_{s,s}(x) = x$, $s$ belongs to A so that A is non-empty. Therefore $\mathrm{A} = (-\infty, +\infty)$. The proof is completed. □

*Proof of Theorem.* The assertion of Theorem is clear noting that $\varphi_{s,t}(x)$ satisfies the assumption of Lemma 3 almost surely by (6) and Lemma 2. □

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## References

[1] H. Kunita, Stochastic flows and stochastic differential equations, Cambridge Univ. Press (1990).

[2] D. W. Stroock and S. R. S. Varadhan, On the support of diffusion process with applications to the strong maximum principle, Proc. Sixth Berkeley Symp. Math. Statist. Prob. III (1972), 333–359.

[3] H. Sussmann, Orbits of families of vector fields and integrabilities of systems with singularities, Bull. Amer. Math. Soc., **180** (1973), 197–199.