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The main scalar of two-dimensional Finsler spaces with special metric

By

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Some geometrical meanings of the main scalar of two-dimensional Finsler space have been given in a previous paper [2]. The purpose of the present paper is to consider the main scalar of Finsler spaces with some special metric in relation to the indicatrix.

The main scalar vanishes identically if and only if the space is Riemannian and, in consequence, the value of the scalar may be regarded as expressing the *degree of Finslerian slippage from a Riemannian space.* It is only reasonable to think that a Finsler space with non-zero constant main scalar may be of some natural shape in a sense, because the degree of slippage is not equal to zero but stable all over the space.

After the classification theorem of two-dimensional Berwald spaces with 1-form metric is shown in §1, we shall be concerned with Finsler spaces with non-zero constant main scalar in §2. It is, however, a real surprise to observe that the indicatrix of such a space is all not an usual oval but quite abnormal curve.

Conversely we deal with a Finsler metric defined by a quite normal indicatrix in §4. Then we meet with a little strange circumstances that the main scalar has the upper limit $3/\sqrt{2}$. The final section is devoted to Wrona's metric, the indicatrix of which may be regarded as a limit of that treated in §4, and the number $3/\sqrt{2}$ appears again.

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§1. Berwald spaces with one-form metric

We shall be concerned with an n-dimensional Finsler space $F^n=(M^n, L)$ with 1-form metric $L=L(a^\alpha)$ ([5], [6]), where $a^\alpha=a^\alpha_i(x)y^i$, $\alpha=1, \cdots, n$, are n linearly independent differential 1-forms on a differentiable n-manifold M^n and $L(a^\alpha)$ is a (1)p-homogeneous function [3] of n variables a^α. We denote by $C1=(\Gamma_j{}^i{}_k, \Gamma_0{}^i{}_j, C_j{}^i{}_k)$ the 1-form Cartan connection [4]: The linear connection $(\Gamma_j{}^i{}_k(x))$ is the 1-form linear connection [5] defined by $\Gamma_j{}^i{}_k=a^i_\alpha\partial_k a^\alpha_j$ where the matrix (a^i_α) is the inverse of (a^α_i), and $C_j{}^i{}_k$ is the well-known C-tensor. The Finsler connection $C1$ has the $(h)h$-torsion $T: T_j{}^i{}_k=\Gamma_j{}^i{}_k-\Gamma_k{}^i{}_j$. The T vanishes if and only if all the a^α are gradient vector fields.

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Let $C\Gamma=(F_j{}^i{}_k, F_0{}^i{}_j, C_j{}^i{}_k)$ be the Cartan connection of the F^n. Then the difference tensor $D_j{}^i{}_k=F_j{}^i{}_k-\Gamma_j{}^i{}_k$ of $C\Gamma$ from $C1$ [5] is given by

$$D_{ijk}=-A_{ijk}+C_i{}^r{}_jA_{rk}+C_j{}^r{}_kA_{ri}-C_i{}^r{}_kA_{rj}, \tag{1.1}$$

where $D_{ijk}=g_{jr}D_i{}^r{}_k$, $T_{ijk}=g_{jr}T_i{}^r{}_k$ and

$$2A_{ijk}=T_{ijk}-T_{jki}+T_{kij}, \qquad A_{ij}=A_{0ij}-C_i{}^r{}_jA_{0r0}.$$

Now we shall deal with the Berwald connection $B\Gamma=(G_j{}^i{}_k, G^i{}_j, 0)$ of the F^n. Since the nonlinear connection $(G^i{}_j)=(G_0{}^i{}_j)$ coincides with $(F_0{}^i{}_j)$ of $C\Gamma$, we have $G^i{}_j=\Gamma_0{}^i{}_j+D_0{}^i{}_j$ and $2G^i=G^i{}_0=\Gamma_0{}^i{}_0+D_0{}^i{}_0$. From (1.1) we get $D_{0jk}=-A_{jk}$ and $D_{0j0}=T_{j00}$. Therefore we have

$$2G^i=\Gamma_0{}^i{}_0+T^i_{\cdot 00} \qquad (T^i_{\cdot jk}=g^{ir}T_{rjk}), \tag{1.2}$$

from which we have

$$G^i{}_j=\Gamma_0{}^i{}_j-(T_0{}^i{}_j-\dot\partial_jT^i_{\cdot 00})/2, \tag{1.3}$$
$$G_j{}^i{}_k=\Gamma_j{}^i{}_k-(T_j{}^i{}_k-\dot\partial_k\dot\partial_jT^i_{\cdot 00})/2.$$

The equation (1.2) leads us to

Proposition 1. *A geodesic of a Finsler space with* 1*-form metric is given by the differential equations*

$$d^2x^i/ds^2+\{\Gamma_j{}^i{}_k(x)+T^i_{\cdot jk}(x, dx/ds)\}(dx^j/ds)(dx^k/ds)=0.$$

According to Berwald's original definition, a Finsler space is called a Berwald space (affinely connected space), if the Berwald connection $B\Gamma$ is linear, that is, $G_j{}^i{}_k$ are functions of position x^i alone. Then (1.3) leads us to

Theorem 1. *A Finsler space with* 1*-form metric is a Berwald space, if and only if* $\dot\partial_k\dot\partial_jT^i_{\cdot 00}$ *are functions of position* x^i *alone, that is,* $T^i_{\cdot 00}$ *are quadratic forms of* y^j.

In a previous paper [5] we showed the condition for the F^n to be a Berwald space, that is, $C_j{}^i{}_{k|h}=0$ in $C\Gamma$. Here we shall show another proof of the condition.

First we have in $C\Gamma$

$$C_{hij|k}=\partial_kC_{hij}-\dot\partial_rC_{hij}(\Gamma_0{}^r{}_k+D_0{}^r{}_k)-\mathfrak{S}_{(hij)}\{C_{rij}(\Gamma_h{}^r{}_k+D_h{}^r{}_k)\},$$

where $\mathfrak{S}_{(hij)}$ denotes the cyclic permutation of h, i, j and summation. Since $C_{hij;k}=0$ in $C1$ [5, Proposition 3], we have

$$C_{hij|k}=-\dot\partial_rC_{hij}D_0{}^r{}_k-\mathfrak{S}_{(hij)}\{C_{rij}D_h{}^r{}_k\}.$$

According to the definition of the v-covariant derivative $C_{hij}|_r$ in $C\Gamma$ [3], we have

$$C_{hij|k}=-C_{hij}|_rD_0{}^r{}_k-\mathfrak{S}_{(hij)}\{C_{rij}(D_h{}^r{}_k+C_h{}^r{}_sD_0{}^s{}_k)\}.$$

Therefore (1.1) and the identity $C_{hij}|_r=C_{ijr}|_h$ lead us to

$$C_{hij|k}=C_i{}^r{}_j|_hA_{rk}+\mathfrak{S}_{(hij)}\{C_i{}^r{}_j(A_{hrk}+C_h{}^s{}_kA_{sr}-C_r{}^s{}_kA_{sh})\}. \tag{1.4}$$

Transvecting (1.4) by y^k we get

$$C_{hij|0}=C_i{}^r{}_j|_h A_{r0}+\mathfrak{S}_{(hij)}\{C_i{}^r{}_j A_{hr0}\}. \tag{1.5}$$

Consequently we have

Theorem 2. *A Finsler space with* 1*-form metric is*
(1) *a Berwald space if and only if we have*

$$C_i{}^r{}_j|_h A_{rk}+\mathfrak{S}_{(hij)}\{C_i{}^r{}_j(A_{hrk}+C_h{}^s{}_k A_{sr}-C_r{}^s{}_k A_{sh})\}=0,$$

(2) *a Landsberg space if and only if we have*

$$C_i{}^r{}_j|_h A_{r0}+\mathfrak{S}_{(hij)}\{C_i{}^r{}_j A_{hr0}\}=0.$$

In the remainder of this section we shall restrict our discussion to a two-dimensional Finsler space F^2 with 1-form metric. In the paper [5] we have already Theorem 4: If the F^2 is a Landsberg space, then it is a Berwald space. Here we shall investigate this fact in detail.

For this purpose we shall refer to the Berwald frame (l, m) [3] of the F^2. The equation [3, (28.2)] gives

$$\dot\partial_j l^i=m^i m_j/L, \qquad \dot\partial_j l_i=m_i m_j/L, \tag{1.6}$$
$$\dot\partial_j m^i=-(l^i+Im^i)m_j/L, \qquad \dot\partial_j m_i=-(l_i-Im_i)m_j/L,$$

where the I is the main scalar. Then we get

$$\dot\partial_k(l_i m_j-l_j m_i)=I(l_i m_j-l_j m_i)m_k/L. \tag{1.7}$$

We first deal with the $(h)h$-torsion tensor $T_j{}^i{}_k$ of $C\Gamma$. From [3, Proposition 28.2] it follows that $T_j{}^i{}_k$ can be written in the form

$$T_j{}^i{}_k=(T_1 l^i+T_2 m^i)(l_j m_k-l_k m_j), \tag{1.8}$$

with two scalars T_1 and T_2. It should be remarked that the components $T_j{}^i{}_k$ are functions of position x^i alone. It then follows from (1.6) and (1.7) that $\dot\partial_h T_j{}^i{}_k=0$ is given in terms of T_1 and T_2 by

$$\dot\partial_h T_1+(IT_1-T_2)m_h/L=0, \qquad \dot\partial_h T_2+T_1 m_h/L=0. \tag{1.9}$$

Next, from (1.8) we have

$$A_{ijk}=(l_i m_j-l_j m_i)(T_1 l_k+T_2 m_k),$$
$$A_{ij}=LT_1 m_i l_j+L(T_2-IT_1)m_i m_j.$$

From $LC_{ijk}=Im_i m_j m_k$ we have

$$C_j{}^r{}_k A_{rh}=Im_j m_k\{T_1 l_h+(T_2-IT_1)m_h\},$$
$$A_{hrk}+C_h{}^s{}_k A_{sr}-C_r{}^s{}_k A_{sh}=(l_h m_r-l_r m_h)\{T_1 l_k+(T_2-IT_1)m_k\}.$$

Further [3, (28.19)] gives

$$C_i{}^r{}_j|_h=[(I_{;2}m^r-Il^r)m_im_jm_h-Im^r\mathfrak{S}_{(hij)}\{l_hm_im_j\}]/L^2,$$

$$C_i{}^r{}_j|_hA_{rk}=[I_{;2}m_hm_im_j-I\mathfrak{S}_{(hij)}\{l_hm_im_j\}]\{T_1l_k+(T_2-IT_1)m_k\}/L.$$

Consequently we obtain from (1.4) and (1.5)

$$C_{hij|k}=I_{;2}m_hm_im_j\{T_1l_k+(T_2-IT_1)m_k\}/L, \tag{1.10}$$

$$C_{hij|0}=I_{;2}T_1m_hm_im_j. \tag{1.11}$$

Then the equation (1.11) shows that the F^2 is a Landsberg space if and only if (1) $T_1=0$ or (2) $I_{;2}=0$. In the former case we have also $T_2=0$ from (1.9), so that $T_j{}^i{}_k$ vanishes identically. In the latter case we have $I=I(x)$ because of $I_{;1}=0$ obviously. Since the main scalar I of the F^2 is a function of variables a^α from [5, Proposition 5] or (2.1), we get $\dot{\partial}_iI=0=(\partial I/\partial a^\alpha)a^\alpha_i$ and $\partial I/\partial a^\alpha=0$, so the I becomes constant. In any case we have $C_{hij|k}=0$ from (1.10), that is, the F^2 is a Berwald space.

Summarizing the above we have

Theorem 3. *A two-dimensional Finsler space with 1-form metric is a Landsberg space, if and only if it is a Berwald space. The Berwald spaces are divided into two classes as follows:*

(1) *It is a T-Minkowski space, i.e.,* $T_j{}^i{}_k=0$.

(2) *Its main scalar is constant.*

§2. The indicatrix of Finsler spaces with constant main scalar

The main scalar I of a two-dimensional Finsler space with 1-form metric was studied in the paper [5] and given by the equations

$$I^2=4F^4(F_{\alpha\alpha\alpha})^2/\{2FF_{\alpha\alpha}-(F_\alpha)^2\}^3, \qquad \alpha=1 \text{ or } 2, \tag{2.1}$$

where $F=L^2/2$ and the subscript α stands for partial derivative by a^α.

We shall show more direct proof of (2.1) than that in the paper [5]. First we have

$$g_{ij}=F_{\alpha\beta}a^\alpha_ia^\beta_j, \quad g^{ij}=F^{\alpha\beta}a^i_\alpha a^j_\beta, \quad C_{ijk}=F_{\alpha\beta\gamma}a^\alpha_ia^\beta_ja^\gamma_k/2,$$

where $(F^{\alpha\beta})$ is the inverse matrix of $(F_{\alpha\beta})$. Then we have $C_i=C_{ijk}g^{jk}=E_\alpha a^\alpha_i/2$ where $E_\alpha=F_{\alpha\beta\gamma}F^{\beta\gamma}$. From $LC_i=Im_i$ we get $I^2=2FC_iC_jg^{ij}=FE_\alpha E_\beta F^{\alpha\beta}/2$. If we put $f=F_{11}F_{22}-(F_{12})^2$, then we have $E_\alpha=(F_{\alpha11}F_{22}-2F_{\alpha12}F_{12}+F_{\alpha22}F_{11})/f$. From the homogeneity we have

$$F_{\alpha\beta1}a^1+F_{\alpha\beta2}a^2=0, \quad F_{\alpha1}a^1+F_{\alpha2}a^2=F_\alpha, \quad F_1a^1+F_2a^2=2F.$$

These identities lead us easily to $E_1=2FF_{111}/f(a^2)^2$ and $E_2=\{2FF_{111}/f(a^2)^2\}(-a^1/a^2)$. Then we have

$$\begin{aligned}I^2&=F\{(E_1)^2F_{22}-2E_1E_2F_{12}+(E_2)^2F_{11}\}/2f\\&=\{2F^3(F_{111})^2/f^2(a^2)^4\}\{F_{11}(a^1)^2+2F_{12}a^1a^2+F_{22}(a^2)^2\}\\&=4F^4(F_{111})^2/f^3(a^2)^6.\end{aligned}$$

Finally the equations $F_{\alpha1}a^1+F_{\alpha2}a^2=F_\alpha$, $\alpha=1, 2$, give $a^2=(F_{11}F_2-F_{12}F_1)/f$, and the equations $F_{11}a^1+F_{12}a^2=F_1$ and $F_1a^1+F_2a^2=2F$ give $a^2=\{2FF_{11}-(F_1)^2\}/(F_{11}F_2-F_{12}F_1)$. Then we get $(a^2)^2=\{2FF_{11}-(F_1)^2\}/f$ and (2.1) in case of $\alpha=1$. Similarly we get (2.1) in case of $\alpha=2$.

Now Theorem 3 picks up our interest in two-dimensional Finsler spaces with constant main scalar. The fundamental metric functions of such spaces were found by Berwald in 1927 [3, Theorem 28.4] by making use of the Landsberg angle. His result shows that the metrics are all 1-form metrics. On account of this recognition, in the previous paper [5], those metrics were again derived by straightforward integration of the differential equations (2.1) with constant I and, in consequence, we got

(2.2₁) $I^2<4:\ 2F=(\alpha^2+\beta^2)\exp\{2J\,\mathrm{Arctan}(\beta/\alpha)\},\quad J=I/\sqrt{4-I^2},$

(2.2₂) $I^2=4:\ 2F=\alpha^2\exp(I\beta/\alpha),$

(2.2₃) $I^2>4:\ 2F=\alpha^{1-J}\beta^{1+J},\quad J=I/\sqrt{I^2-4},$

where α and β are arbitrary, linearly independent 1-forms.

From the definition $LC_{ijk}=Im_im_jm_k$ of the main scalar I of a two-dimensional Finsler space F^2 it follows that the I vanishes identically if and only if F^2 is a Riemannian space, and hence it may be regarded as the *degree of Finslerian slippage from a Riemannian space.* Certain geometrical meanings of the I are shown in the paper [2]. For instance, if we denote by $R(\theta)$ the radius vector of the osculating indicatrix I_y as a function of the Landsberg angle θ, we have $\lim_{\theta\to0}(R-1)/\theta^3=I_0/3$ where I_0 is the value of the I at the osculating point $y(\theta=0)$ of I_y with the indicatrix. Then, can we naturally imagine that the indicatrix of F^2 with constant main scalar has some normal shape?

Then we shall describe the indicatrices of spaces with the metric given by (2.2). Since α and β are arbitrary, linearly independent linear forms in y^1 and y^2, we may describe them in (x, y)-plane as putting $\alpha=x$ and $\beta=y$. It is remarked that the I may be supposed to be positive.

We first deal with the metric (2.2₁). The indicatrix C_1 is given by the equation $(x^2+y^2)e^{2Ju}=1$ where $u=\mathrm{Arctan}(y/x)$. Referring to the polar coordinates (ρ, ϕ) defined by $x=\rho\cos\phi$, $y=\rho\sin\phi$, the principal value u has the value $u=\phi$ $(0\leqq\phi<\pi/2)$, $u=\phi-\pi$ $(\pi/2<\phi<3\pi/2)$, $u=\phi-2\pi$ $(3\pi/2<\phi\leqq2\pi)$. Thus, in the polar coordinates is C_1 given by

$$C_1:\ \begin{array}{ll}\rho=e^{-J\phi}\ (0\leqq\phi<\pi/2), & \rho=e^{-J(\phi-\pi)}\ (\pi/2<\phi<3\pi/2),\\ \rho=e^{-J(\phi-2\pi)}\ (3\pi/2<\phi\leqq2\pi). & \end{array}$$

The indicatrix C_1 is described in Fig. 1, where $OA=OC=1$, $OB_1=\lim_{\phi\to\pi/2-0}\rho=e^{-J\pi/2}$, $OB_2=\lim_{\phi\to\pi/2+0}\rho=e^{J\pi/2}$, $OD_1=OB_1$ and $OD_2=OB_2$ Then we observe that $OB_1\cdot OB_2=1$ and $\lim_{I\to0}OB_1=\lim_{I\to0}OB_2=1$. Therefore this C_1 may be regarded as the deformation of the unit circle O-$ABCD$ $(I=0)$ by separation of B into B_1 and B_2 and of D into D_1 and D_2 as the I increases from zero.

In Fig. 1 the main scalar I is to be taken as $I=0.3$, and Fig. 2 shows C_1 for $I=1$. Consequently the indicatrix has two escarpments at the y-axis and their height increases with the I.

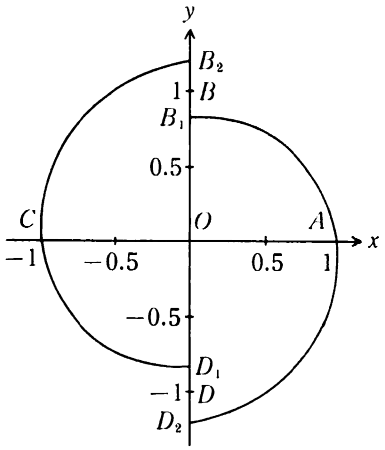

Fig. 1

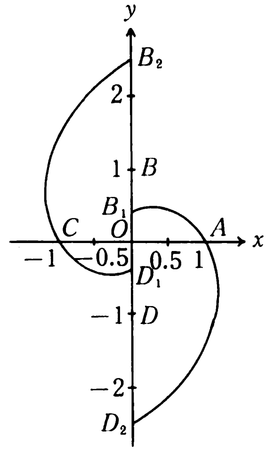

Fig. 2

Secondly we deal with the metric (2.2_2). The indicatrix C_2 is given by the equation $x^2e^{2y/x}=1$, that is, $y=-x\log|x|$. Fig. 3 shows this C_2, where $OA=OC=1$ and C_2 is tangent to the y-axis at the origin O. It is observed that C_2 is just the limit of C_1 of Fig. 2 as the I tends to 2 infinitly: Both B_1 and D_1 converge to O while B_2 and D_2 diverge to the points at infinity.

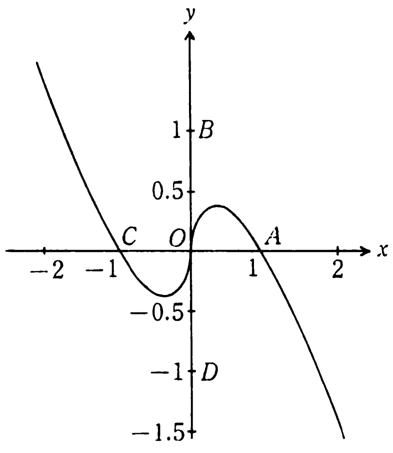

Fig. 3

Finally we consider the indicatrix C_3 of the metric (2.2_3). The equation of C_3 is simply written as the form $y^{p+2}=x^p$ $(p=J-1)$. When the I increases from 2 to $+\infty$, p decreases from $+\infty$ to 0. This C_3 is, however, quite complicated. For instance, C_3 for $p=1$ is in the first and third quadrates alone, while C_3 for $p=2$ consists of two parabolas going through all the quadrates.

Consequently the indicatrix of space with constant main scalar is not an usual oval in either case and it should be concluded that spaces with constant main scalar are very strange spaces, contrary to our conjecture.

§3. The main scalar of 2-dimensional Finsler space with (α, β)-metric

We shall consider a two-dimensional Finsler space F^2 with (α, β)-metric. α is a Riemannian metric $(\alpha^2=a_{ij}(x)y^iy^j)$ and β is a differential 1-form $(\beta=b_i(x)y^i)$. The fundamental function L of the F^2 is a positively homogeneous function in (α, β) of degree one.

The main scalar I of the F^2 has been studied in the paper [1], but we referred to the isothermal coordinates with respect to the α for convenience sake. Here we shall give it in arbitrary coordinate system.

Let g and a be the determinants $g=\det(g_{ij})$ and $a=\det(a_{ij})$ respectively. Then we have equations in §30 of [3] as follows:

$$g=aT, \qquad T=p(p+p_0b^2+p_{-1}\beta)+(p_0p_{-2}-p_{-1}^2)(\alpha^2b^2-\beta^2), \tag{3.1}$$

$$\begin{aligned} &p=F_1/\alpha, \qquad p_0=F_{22}, \qquad p_{-1}=F_{12}/\alpha, \\ &p_{-2}=(F_{11}-F_1/\alpha)/\alpha^2, \qquad b^2=a^{ij}b_ib_j, \end{aligned} \tag{3.2}$$

where the subscripts 1 and 2 of F stand for $\partial/\partial\alpha$ and $\partial/\partial\beta$ respectively. The T in (3.1) may be written in the form

$$T=2FF_1/\alpha^3+\{F_{11}F_{22}-(F_{12})^2\}(b^2-\beta^2/\alpha^2). \tag{3.3}$$

It is noted here that by virtue of $F_{11}\alpha+F_{12}\beta=F_1$ and $F_{21}\alpha+F_{22}\beta=F_2$ the term $F_{11}F_{22}-(F_{12})^2$ in (3.3) may be written also as

$$F_{11}F_{22}-(F_{12})^2=\{2FF_{11}-(F_1)^2\}/\beta^2=\{2FF_{22}-(F_2)^2\}/\alpha^2. \tag{3.4}$$

Now we get $\dot{\partial}_ig=a\dot{\partial}_iT=a(T_1Y_i/\alpha+T_2b_i)=aT_2p_i$, where $T_1=\partial T/\partial\alpha$, $T_2=\partial T/\partial\beta$ and $p_i=b_i-(\beta/\alpha^2)Y_i$, $Y_i=a_{ij}y^j$. Referring to the Berwald frame (l, m), we have $p_il^i=0$, so there is a scalar p satisfying $p_i=pm_i$. The p can be found on account of $(m_1, m_2)=\sqrt{g}(-l^2, l^1)$ [3, §28]:

$$b_1-(\beta/\alpha^2)Y_1=-py^2\sqrt{g}/L, \qquad b_2-(\beta/\alpha^2)Y_2=py^1\sqrt{g}/L,$$

from which we get $p=L(Y_1b_2-Y_2b_1)/\alpha^2\sqrt{g}$. Then the well-known equation [3, (24.1)]: $C_i=\dot{\partial}_ig/2g$ leads us to $C_i=(T_2p/2T)m_i$, that is, $I=LT_2p/2T$. Consequently we have

$$I=FT_2(Y_1b_2-Y_2b_1)/\alpha^2\sqrt{T^3}\sqrt{a}. \tag{3.5}$$

Example. (1) The main scalar of the *Randers metric* $L=\alpha+\beta$ is given by

$$I=3(Y_1b_2-Y_2b_1)/2\sqrt{a\alpha(\alpha+\beta)}. \tag{3.6}$$

(2) The (α, β)-metric $L=\alpha^2/\beta$ is called the *Kropina metric* and $L=\alpha^{m+1}/\beta^m$ $(m\neq 0, -1)$ is called the *generalized m-Kropina metric*. For the generalized one we have the main scalar as

$$\begin{aligned} I=&\{-m(m+1)(2m+1)b^2\alpha^2+2m(m^2-1)\beta^2\}(Y_1b_2-Y_2b_1)/ \\ &\{m(m+1)b^2\alpha^2-(m^2-1)\beta^2\}^{3/2}\sqrt{a}. \end{aligned} \tag{3.7}$$

§ 4. The main scalar of the circular-Randers space

In reverse to the discussion in § 2 we shall be concerned with a two-dimensional Minkowski space with the metric defined by one of the most simple indicatrix and study the behavior of its main scalar.

In an euclidean (x, y)-plane the indicatrix at the origin O is taken as a circle A with the radius a, the center A of which has the coordinates $(-b, 0)$, $(b<a)$. This circle A is given by the equation $(x+b)^2+y^2=a^2$ (Fig. 4) and [3, Example 16.4] shows that the fundamental function L is given by $(x/L+b)^2+(y/L)^2=a^2$, that is, $L=bx/c^2+\sqrt{(ax/c)^2+y^2}/c$ where $c=\sqrt{a^2-b^2}$. Therefore, if we put

$$\alpha^2=\{(ax/c)^2+y^2\}/c^2, \qquad \beta=bx/c^2, \tag{4.1}$$

the metric is a Randers metric $L=\alpha+\beta$ [3, Example 16.1]. We shall call this metric the *circular-Randers metric*.

Then we can apply the equation (3.6) to this metric and obtain

$$I=3by/2ac\sqrt{\alpha(\alpha+\beta)}, \tag{4.2}$$

where the minus sign is omitted.

Now we put $b=a\cos\varepsilon$, $c=a\sin\varepsilon$, and, for convenience sake, we refer to the coordinates (ρ, ϕ) defined by $ax/c=\rho\cos\phi$, $y=\rho\sin\phi$. Then we have

$$I=(3/2)\cos\varepsilon\cdot f(\phi), \qquad f(\phi)=\sin\phi/\sqrt{1+\cos\varepsilon\cos\phi}. \tag{4.3}$$

The graph of this function $I(\phi)$ is shown in Fig. 5.

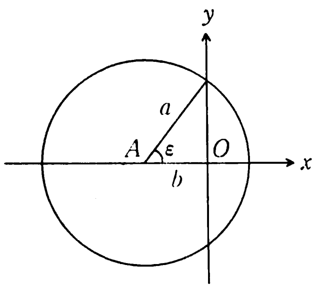

Fig. 4

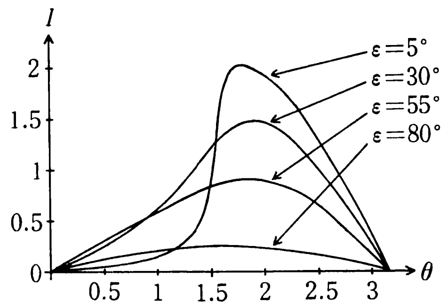

Fig. 5

We shall consider the maximum $I(\varepsilon)_0$ of the $I(\phi)$. The value $\phi=\phi_0$ giving the maximum is easily found as

$$\tan\phi_0=-\sqrt{2\sin\varepsilon/(1-\sin\varepsilon)}. \tag{4.4}$$

We now refer to the usual polar coordinates (r, θ) defined by $x=r\cos\theta$, $y=r\sin\theta$. Then we have the relation $\tan\theta=(a/c)\tan\phi$ and the $\theta=\theta_0$ corresponding to $\phi=\phi_0$ is given by

$$\tan(\theta_0-\pi/2)=\sqrt{\sin\varepsilon(1-\sin\varepsilon)/2}. \tag{4.5}$$

We get the maximum value $I(\varepsilon)_0$ as follows:

(4.6) $$I(\varepsilon)_0=(3/\sqrt{2})\sqrt{1-\sin\varepsilon}\ .$$

Consequently the maximum I_0 of the main scalar of the circular-Randers metrics does not reach $3/\sqrt{2}$.

§ 5. The main scalar of the Finsler spaces with Wrona's metric

We shall consider the $\lim_{b\to a}$ (or $\lim_{\varepsilon\to 0}$) of the circular-Randers metric. In this case the indicatrix at a point O (Fig. 4) becomes a circle through O and the metric may be regarded as that of the tangent Minkowski space of a two-dimensional Finsler space with the metric which was introduced by Wrona [7] in an euclidean plane.

Wrona's metric is defined as follows [3, Example 16.3]: Let O be a fixed point of the plane (Fig. 6). The indicatrix at a point P consists of two circles which are tangent at P to the straight line OP and have the diameter equal to the length OP. Then Wrona's norm of a segment PQ is defined as PQ/OH, where the straight line OH is perpendicular to PQ. From $PR=OH$ it is seen that Wrona's norm of PR is equal to the unit, so that R is on the indicatrix at P.

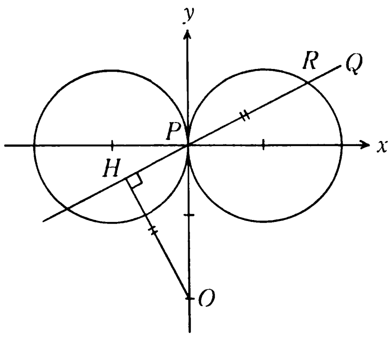

Fig. 6

We refer to the orthonormal coordinates (x, y) such that P is the origin and the two circles are given by $(x\pm a)^2+y^2=a^2$. Then the fundamental function L is given by $(x/L\pm a)^2+(y/L)^2=a^2$, that is,

(5.1) $$L=(x^2+y^2)/2ax\ ,$$

where we treat the circle of the right side only.

Therefore Wrona's metric is a kind of Kropina metric $L=\alpha^2/\beta$ where $\alpha^2=x^2+y^2$ and $\beta=2ax$. Then (3.7) gives its main scalar I as

(5.2) $$I=3y/\sqrt{2(x^2+y^2)}\ .$$

In the usual polar coordinates (r, θ) we have $I=(3/\sqrt{2})\sin\theta$ and, in consequence, the upper limit of this I is again equal to $3/\sqrt{2}$.

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