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Characters of cuspidal unramified series for central simple algebras of prime degree

By

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Introduction

Let A be a central simple algebra of dimension n^2 over a non-archimedean local field F and L be a maximal unramified extension of F in A. Recall that any compact (mod centei) Cartan subgroup of $A^\times$ is isomorphic to $E^\times$ for some extension E/F of degree n. Gerardin [G2] constructed the 'cuspidal unramified series', which is the set of irreducible supercuspidal representations of $A^\times$ parametrized by regular quasi-characters of $L^\times$.

The aim of this paper is to get the character formula for the unramified cuspidal series on regular elements in a compact modulo center Cartan subgroup $E^\times$ of $A^\times$ when $[A: F]=l^2$, l a prime. Since the case $l=2$ is well-known, we assume l is an odd prime. We note that, when l is a prime, A is isomorphic to the division algebra of dimension l^2 over F or the algebra of $l\times l$ matrices over F. Our main results are Corollary 1.2.2, Theorem 1.2.7 and Theorem 2.2.3.

Let D_n be a division algebra of dimension n^2 over F. Deligne-Kazhdan-Vigneras [BDKV] and Rogawski [R] proved the abstract matching theorem: there is a bijection between irreducible representations of $D_n^\times$ and essentially square-integrable representations of $\mathrm{GL}_n(F)$ which preserves the characters up to $(-1)^{n-1}$. (cf. Theorem 2.2.1 in this paper). In the tame case n is prime to the residual characteristic of F, Moy [M] proved that there is a bijection between the same sets as above using the concrete construction of the representations given by Howe [H2]. In general, the relation of these two bijections is unknown. (See [Sa], [M]). We show that, if n is an odd prime, these two bijections coincide on the cuspidal unramified series. (See Theorem 2.2.3).

Howe and Corwin [CH], [Co] have considered characters of irreducible representations of $D_n^\times$ in the tame case. Their result ([Co], Theorem 1) is very interesting, but it is complicated and not practical in a sense. We treat only the special case, but our result is very simple and gives a complete knowledge of the representation π_θ. (See (1.1.2) and (1.1.6) for the definition of π_θ).

In section 1, we treat the division algebra case. Subsection 1.1 is devoted to review the construction of an irreducible representation π_θ of the multiplicative group of a division algebra D of dimension l^2 over F from a regular quasi-character θ of $L^\times$ according to [G2]. We note that in this case π_θ is monomial i.e. induced from a one-dimensional representation. Subsection 1.2 is the main part of this paper. We com-

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pute the character formula of π_θ. More precisely, we give the decomposition of π_θ as $E^\times$-module, where E/F is a separable extension of degree l in D. Theorem 1.2.1 and Corollary 1.2.2 are the main results of this section. To prove Theorem 1.2.1, we proceed as follows. Since $\pi_\theta=\mathrm{ind}_H^{D^\times}\rho_\theta$ (cf. 1.1.5), we get:

$$\pi_\theta|_{L^\times}=\bigoplus_{a\in L^\times\backslash D^\times/H}\mathrm{Ind}_{aHa^{-1}\cap L^\times}^{L^\times}\rho_\theta^a. \tag{1.2.5}$$

by Mackey decomposition. We determine a complete system of representatives of the double coset $L^\times\backslash D^\times/H$ and divide the representatives into the sets $K_{\mu,i}$ with the following good property:

(1) $aHa^{-1}\cap L^\times$ does not depend on the choice of $a\in K_{\mu,i}$,

(2) the number of elements in each fiber of the map:

$$K_{\mu,i}\ni a\longmapsto\rho_\theta^a\rho_\theta^{-1}\in(L^\times\cap aHa^{-1})^\wedge \quad \text{is constant on } K_{\mu,i}.$$

(See Lemma 1.2.11-Lemma 1.2.16). Then we can easily prove Theorem 1.2.1.

Section 2 is devoted to the case $A^\times=\mathrm{GL}_l(F)$. As in section 1, we first review the construction of an irreducible supercuspidal representation Π_θ of $\mathrm{GL}_l(F)$ from a regular quasi-character θ of $L^\times$ according to [G2]. In subsection 2.2, we show the correspondence $\pi_\theta\leftrightarrow\Pi_\theta$ coincides with the Deligne-Kazhdan abstract matching. It amounts to get the character formula of Π_θ. Theorem 2.2.3 is the main theorem of this section. For the proof, we use the result of Kutzko [K], which reduce the computation of the character of Π_θ to the computation of the character of the 'very cuspidal' representation of $F^\times\mathrm{GL}_l(\mathcal{O}_F^\times)$ whose compact-induction to $\mathrm{GL}_l(F)$ is Π_θ and compute the character only on the set of 'very cuspidal' elements. Then by virtue of Theorem 1.2.1, we get Theorem 2.2.3.

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Notation Let F be a non-archimedean local field. We denote by $\mathcal{O}_F$, P_F, ϖ_F, k_F and v_F the maximal order of F, the maximal ideal of $\mathcal{O}_F$, a prime element of P_F, the residue field of F and the valuation of F normalized by $v_F(\varpi_F)=1$. We set q be the number of elements in k_F. Let A be a central simple algebra over F. Its reduced norm is denoted by $N_{A/F}$ and its reduced trace by $\mathrm{tr}_{A/F}$. Hereafter we fix an additive character ψ of F whose conductor is P_F i.e. ψ is trivial on P_F and not trivial on $\mathcal{O}_F$. For an irreducible admissible representation π of $A^\times$, the conductoral exponent of π is defined to be the integer $f(\pi)$ such that the local constant $\varepsilon(s,\pi,\psi)$ of Godement-Jacquet [GJ] is of the form $aq^{-s(f(\pi)-n)}$ where $n^2=[A:F]$. We call π *minimal* if

$$f(\pi)=\min_\eta f(\pi\otimes(\eta\circ N_{A/F}))$$

where η runs through the quasi-characters of $F^\times$. For a quasi-character η of $F^\times$, $\eta\circ N_{A/F}$ is denoted simply by η when there is no risk of confusion. Let G be a totally disconnected, locally compact group. We denote by $\hat{G}$ the set of (equivalence classes of) irreducible admissible representations of G. For a closed subgroup H of G and a representation ρ of H, we denote by $\mathrm{Ind}_H^G\rho$ (resp. $\mathrm{ind}_H^G\rho$) the induced representation (resp. compactly induced representation) of ρ to G. For a representation π of G, we denote by $\pi|_H$ the restriction of π to H.

1. Non-split (division algebra) case

1.1 Construction of the representation. Let D be a division algebra of degree l (dimension l^2) over F with l an odd prime. We denote by $\mathcal{O}_D$, P_D, ϖ_D and v_D the maximal order of D, the maximal ideal of $\mathcal{O}_D$, a prime element of P_D and the valuation of D normalized by $v_D(\varpi_D)=1$.

Let L be an unramified extension of F of degree l. L can be embedded into D and, up to conjugacy, the embedding is unique.

Definition 1.1.1. Let θ be a quasi-character of $L^\times$.

(1) θ is called regular if all its conjugates by the action of $\mathrm{Gal}(L/F)$ are distinct. We denote by $\hat{L}^\times_{reg}$ the set of regular quasi-characters of $L^\times$.

(2) Let $f(\theta)=\min\{n\,|\,\mathrm{Ker}\theta\supset 1+P_L^n\}$. θ is called generic if either

(a) $f(\theta)=1$ and θ is not written in the form $\eta\circ N_{L/F}$ where η is a quasi-character of $F^\times$ or

(b) $f(\theta)>1$ and $k_F(\varpi^{f(\theta)-1}\gamma_\theta)=k_L$ where $\gamma_\theta\in P_L^{1-f(\theta)}-P_L^{2-f(\theta)}$ such that

$$\theta(1+x)=\psi(\mathrm{tr}_{L/F}(\gamma_\theta x)) \qquad \text{for} \quad x\in P_L^{f(\theta)-1}.$$

We note that any regular quasi-character of $L^\times$ is written in the form $(\eta\circ N_{L/F})\otimes\theta$ where η is a quasi-character of $F^\times$ and θ is a generic quasi-character of $L^\times$.

We construct an irreducible representation π_θ from $\theta\in\hat{L}^\times_{reg}$ according to [G2]. At first we treat the case θ is generic. If $f(\theta)=1$, then θ itself can be regarded as a quasi-character of $F^\times\mathcal{O}_D^\times$ since $F^\times\mathcal{O}_D^\times/1+P_D\cong L^\times/1+P_L$. Therefore we set

$$\pi_\theta=\mathrm{Ind}_{F^\times\mathcal{O}_D^\times}^{D^\times}\theta\,. \tag{1.1.2}$$

Then π_θ is an irreducible representation of $D^\times$ with $f(\pi_\theta)=l$. If $f(\theta)=m+1>1$, then there exists an element $\gamma_\theta\in P_L^{-m}-(F\cap P_L^{-m})+P_L^{1-m}$ such that

$$\theta(1+x)=\psi(\mathrm{tr}_{L/F}(\gamma_\theta x)) \qquad \text{for} \quad x\in P_L^{[(m+2)/2]} \tag{1.1.3}$$

where [ ] is the greatest integer function. (Recall that the conductor of ψ is P_F.) Let $\psi_{\gamma_\theta}(1+x)=\psi(\mathrm{tr}_{D/F}(\gamma_\theta x))$ for $x\in P_D^{[(ml+2)/2]}$. Then ψ_{γ_θ} is a quasi-character of $1+P_D^{[(ml+2)/2]}$. Set $H=L^\times(1+P_D^{[(ml+2)/2]})\subset D^\times$ and define a quasi-character ρ_θ of H by

$$\rho_\theta(h\cdot g)=\theta(h)\psi_{\gamma_\theta}(g) \qquad \text{for} \quad h\in L^\times,\quad g\in 1+P_D^{[(ml+2)/2]}. \tag{1.1.4}$$

We set

$$\pi_\theta=\mathrm{Ind}_H^{D^\times}\rho_\theta\,. \tag{1.1.5}$$

Then π_θ is an irreducible minimal representation of $D^\times$ with $f(\pi_\theta)=l(m+1)$. (cf. [H1], IV).

For a regular quasi-character θ written in the form $\theta=(\eta\circ N_{L/F})\otimes\theta'$ where η is a quasi-character of $F^\times$ and θ' is a non-trivial generic quasi-character of $L^\times$, we set

(1.1.6) $$\pi_\theta=\pi_{\theta'}\otimes\eta\,.$$

Now we get a correspondence $\theta\in\hat{L}^\times_{reg}\mapsto\pi_\theta\in\hat{D}^\times$. The following result is known about this correspondence. (cf. [G2], [H1]).

Proposition 1.1.7. *With the above notations, for any regular quasi-character θ of $L^\times$, π_θ is an irreducible representation of $D^\times$ such that:*

(a) *the representations π_θ and $\pi_{\theta'}$ associated to two regular quasi-characters θ and θ' are equivalent if and only if θ and θ' are conjugate under* $\mathrm{Gal}(L/F)$;

(b) *the central quasi-character of π_θ is the restriction of θ to $F^\times$;*

(c) *for any quasi-character η of $F^\times$, the twisted representation $\pi_\theta\otimes\eta$ is equivalent to $\pi_{\theta\otimes\eta\circ N_{L/F}}$;*

(d) *the contragredient representation of π_θ is equivalent to $\pi_{\theta^{-1}}$;*

(e) *the L-function of π_θ is* 1;

(f) *the ε-factor of π_θ is $\varepsilon(\pi_\theta,\psi)=\varepsilon(\theta,\psi\circ\mathrm{tr}_{L/F})$; in particular $f(\pi_\theta)=l\cdot f(\theta)$;*

(g) $\{\pi_\theta\mid\theta\in\hat{L}^\times_{reg}\}=\{\pi\in\hat{D}^\times\mid f(\pi)\equiv 0 \pmod{l}\}$.

1.2 Character formula. In this subsection we compute the character of π_θ. More precisely, for a separable extension E/F of degree l in D/F, we give the decomposition of π_θ as $E^\times$ module. First we treat the case E is unramified. We can assume $E=L$ because E is conjugate to L in D. We need some notations to state the main theorem of this section. Let $U_0=L^\times$, $U_i=F^\times(1+P_L^i)$ $(i\geqq 1)$, $U_i^*=U_i-U_{i+1}$ and $X_i=\bigoplus_{\chi\in(L^\times/U_i)^\wedge}\chi$. We set $\Gamma=\mathrm{Gal}(L/F)$ and denote by χ_{π_θ} the character of π_θ.

Theorem 1.2.1. *Let θ be a generic quasi-character of $L^\times$ with $f(\theta)=m+1$ and π_θ as in* (1.1.2) *and* (1.1.5).

(1) (*Decomposition of π_θ as $L^\times$-module*)

$$\pi_\theta|_{L^\times}=(\bigoplus_{\sigma\in\Gamma}\theta\circ\sigma)\otimes\Big(X_0+(q-1)\frac{q^{l(l-1)/2}-1}{q^l-1}\sum_{a=1}^{m}q^{(l-1)(l-2)(a-1)/2}X_a\Big).$$

(2) (*Character formula of π_θ on $L^\times$*)

$$\chi_{\pi_\theta}(x)=\begin{cases} q^{l(l-1)j/2}\Big(\sum_{\sigma\in\Gamma}\theta(x^\sigma)\Big) & \text{if } x\in U_j^*\ (0\leqq j<m)\\ q^{l(l-1)m/2}\Big(\sum_{\sigma\in\Gamma}\theta(x^\sigma)\Big) & \text{if } x\in U_m\,.\end{cases}$$

Corollary 1.2.2. *Let θ be a regular quasi-character of $L^\times$ with* $\min_\eta f(\theta\otimes(\eta\circ N_{L/F}))=m+1$ *and π_θ as in* (1.1.6).

(1) (*Decomposition of π_θ as $L^\times$-module*)

$$\pi_\theta|_{L^\times}=(\bigoplus_{\sigma\in\Gamma}\theta\circ\sigma)\otimes\Big(X_0+(q-1)\frac{q^{l(l-1)/2}-1}{q^l-1}\sum_{a=1}^{m}q^{(l-1)(l-2)(a-1)/2}X_a\Big).$$

(2) (*Character formula of π_θ on $L^\times$*)

$$\chi_{\pi_\theta}(x)=\begin{cases} q^{l(l-1)j/2}\Big(\sum_{\sigma\in\Gamma}\theta(x^\sigma)\Big) & \text{if} \quad x\in U_j^* \ (0\leqq j<m) \\ q^{l(l-1)m/2}\Big(\sum_{\sigma\in\Gamma}\theta(x^\sigma)\Big) & \text{if} \quad x\in U_m\,. \end{cases}$$

Proof of Corollary 1.2.2. This follows immediately from Proposition 1.1.7 (c) and Theorem 1.2.1.

We need several steps to prove Theorem 1.2.1. Let us start with the structure of D. By Skolem-Noether theorem, there exists a prime element $\xi\in\mathcal{O}_D$ such that

$$\xi^{-1}x\xi=x^\sigma \qquad \text{for any} \quad x\in L\,,$$

where σ is a generator of $\mathrm{Gal}(L/F)$. We set $\varpi=\xi^l$. Then it follows that ϖ is a prime element of $\mathcal{O}_F$ and

$$\begin{aligned} D &= L\oplus\xi L\oplus\cdots\oplus\xi^{l-1}L \\ \mathcal{O}_D &= \mathcal{O}_L\oplus\xi\mathcal{O}_L\oplus\cdots\oplus\xi^{l-1}\mathcal{O}_L \\ P_D &= P_L\oplus\xi\mathcal{O}_L\oplus\cdots\oplus\xi^{l-1}\mathcal{O}_L \\ &\ \ \vdots \\ P_D^{l-1} &= P_L\oplus\xi P_L\oplus\cdots\oplus\xi^{l-1}\mathcal{O}_L\,. \end{aligned} \tag{1.2.3}$$

Let θ be a generic quasi-character of $L^\times$ with $f(\theta)=m+1$. If $f(\theta)=1$, then $\pi_\theta=\mathrm{Ind}_{F^\times\mathcal{O}_D^\times}^{D^\times}\theta$. Since $\{1, \xi, \xi^2, \cdots, \xi^{l-1}\}$ is a complete system of representatives of $D^\times/F^\times\mathcal{O}_D^\times$, we get $\chi_{\pi_\theta}=\sum_{\sigma\in\Gamma}(\theta\circ\sigma)$. We assume $f(\theta)=m+1>1$. We recall that $\pi_\theta=\mathrm{Ind}_H^{D^\times}\rho_\theta$, where $H=L^\times(1+P_D^{[(ml+2)/2]})$. (See (1.1.4) for the definition of ρ_θ). It follows from (1.2.3) that

$$H=F^\times(\mathcal{O}_L^\times+\xi P_L^{[(m+1)/2]}+\cdots+\xi^{(l-1)/2}P_L^{[(m+1)/2]}+\xi^{(l+1)/2}P_L^{[m/2]}+\cdots+\xi^{l-1}P_L^{[m/2]})\,. \tag{1.2.4}$$

By Mackey decomposition [Se],

$$\pi_\theta|_{L^\times}=\bigoplus_{a\in L^\times\backslash D^\times/H}\mathrm{Ind}_{aHa^{-1}\cap L^\times}^{L^\times}\rho_\theta^a\,, \tag{1.2.5}$$

where $\rho_\theta^a(x)=\rho_\theta(a^{-1}xa)$ for $x\in aHa^{-1}\cap L^\times$.

At first, we shall investigate $L^\times\backslash D^\times/H$. We have only to consider $L^\times\backslash F^\times\mathcal{O}_D^\times/H$ because

$$L^\times\backslash D^\times/H=\bigcup_{i=0}^{l-1}\xi^i(L^\times\backslash F^\times\mathcal{O}_D^\times/H) \qquad \text{(disjoint union)}. \tag{1.2.6}$$

For convenience, we often use the following notation:

$$n(i)=\begin{cases} \left[\dfrac{m+1}{2}\right] & \left(1\leqq i\leqq\dfrac{l-1}{2}\right) \\ \left[\dfrac{m}{2}\right] & \left(\dfrac{l+1}{2}\leqq i\leqq l-1\right). \end{cases} \tag{1.2.7}$$

Lemma 1.2.8. *Let $a=1+\sum_{i=1}^{l-1}\xi^i\alpha_i$ and $b=1+\sum_{i=1}^{l-1}\xi^i\beta_i$ (α_i, $\beta_i\in\mathcal{O}_L$). Then $aH=bH$ if and only if $\alpha_i-\beta_i\in P_L^{n(i)}$ for $1\leqq i\leqq l-1$.*

Proof. By (1.2.4), $aH=bH$ implies that there exist $\gamma_0\in\mathcal{O}_L^\times$ and $\gamma_1,\cdots,\gamma_{l-1}\in P_L^{n(i)}$ such that $b=a(\sum_{i=0}^{l-1}\xi^i\gamma_i)$. Since $\mathcal{O}_D=\mathcal{O}_L\oplus\xi\mathcal{O}_L\oplus\cdots\oplus\xi^{l-1}\mathcal{O}_L$ and $\xi^{-1}x\xi=x^\sigma$ for $x\in L$, we obtain:

$$1=\gamma_0+\varpi\sum_{j=1}^{l-1}\gamma_j\alpha_{l-j}^{\sigma^j}$$

$$(*)\qquad \beta_i-\alpha_i=(\gamma_0-1)+\gamma_i+\sum_{j=1}^{i-1}\gamma_j\alpha_{i-j}^{\sigma^j}$$

$$+\varpi\sum_{j=i+1}^{l-1}\gamma_j\alpha_{l+i-j}^{\sigma^j}\ (1\leqq i\leqq l-1).$$

Therefore we have $\gamma_0\in 1+P_L^{[m/2]+1}$ and $\beta_i-\alpha_i\in P_L^{n(i)}$ $(1\leqq i\leqq l-1)$.

Conversely we assume $\beta_i-\alpha_i\in P_L^{n(i)}$ $(1\leqq i\leqq l-1)$. By putting $\gamma_0-1=-\varpi\sum_{j=1}^{l-1}\gamma_j\alpha_{l-j}^{\sigma^j}$ into $(*)$, we get

$$\beta_i-\alpha_i=(1-\varpi\alpha_{l-i}^{\sigma^i})\gamma_i+\sum_{j=1}^{i-1}\gamma_i(\alpha_{i-j}^{\sigma^j}-\varpi\alpha_{l-j}^{\sigma^j})+\varpi\sum_{j=i+1}^{l-1}\gamma_j(\alpha_{l+i-j}^{\sigma^j}-\alpha_{l-j}^{\sigma^j})\quad(1\leqq i\leqq l-1).$$

Thus it follows that

$$v_L(\gamma_i)\geqq\min\left(\left[\frac{m+1}{2}\right],v_L(\gamma_1),\cdots,v_L(\gamma_{i-1}),v_L(\gamma_{i+1})+1,\cdots,v_L(\gamma_{l-1})+1\right)$$

$$\text{for}\quad 1\leqq i\leqq\frac{l-1}{2},$$

$$v_L(\gamma_i)\geqq\min\left(\left[\frac{m}{2}\right],v_L(\gamma_1),\cdots,v_L(\gamma_{i-1}),v_L(\gamma_{i+1})+1,\cdots,v_L(\gamma_{l-1})+1\right)$$

$$\text{for}\quad \frac{l+1}{2}\leqq i\leqq l-1.$$

Hence our lemma follows from the simple fact that there is no solution to the system of inequations:

$$x_i\geqq\min(x_1,\cdots,x_{i-1},x_{i+1}+1,\cdots,x_{l-1}+1)\quad(1\leqq i\leqq l-1).$$

Lemma 1.2.9. *We put*

$$M=\{(\alpha^\sigma\alpha^{-1},\alpha^{\sigma^2}\alpha^{-1},\cdots,\alpha^{\sigma^{l-1}}\alpha^{-1})\,|\,\alpha\in L^\times\}\subset\mathcal{O}_L^{(1)}\times\cdots\times\mathcal{O}_L^{(1)}=(\mathcal{O}_L^{(1)})^{l-1},$$

where $\mathcal{O}_L^{(1)}=\mathrm{Ker}N_{L/F}$. *Then the map* $(\alpha_i)\in(\mathcal{O}_L)^{l-1}\mapsto 1+\sum_{i=1}^{l-1}\xi^i\alpha_i\in\mathcal{O}_L^\times$ *induces a bijection from* $M\backslash(\mathcal{O}_L)^{l-1}/(P_L^{[(m+1)/2]})^{(l-1)/2}\times(P_L^{[m/2]})^{(l-1)/2}$ *to* $L^\times\backslash F^\times\mathcal{O}_D^\times/H$.

Proof. For $\alpha\in L^\times$ and $\beta_1,\cdots,\beta_{l-1}\in\mathcal{O}_L$,

$$\alpha\left(1+\sum_{i=1}^{l-1}\xi^i\beta_i\right)H=\left(1+\sum_{i=1}^{l-1}\xi^i\alpha^{\sigma^i}\alpha^{-1}\beta_i\right)H.$$

Therefore our lemma is obtained from Lemma 1.2.8.

In order to prove Theorem 1.2.1, we need more information about $L^\times\backslash F^\times\mathcal{O}_D^\times/H$. We prepare some notations.

For $1\leqq i\leqq l-1$ and $0\leqq\mu<n(i)$, we set

$$I_{\mu,i}=\begin{cases} M\backslash(\mathcal{O}_L)^{i-1}\times\mathcal{O}_L^\times\times(\mathcal{O}_L)^{l-i-1}/(P_L^{[(m+1)/2]-\mu-1})^{i-1}\times(1+P_L^{[(m+1)/2]-\mu}) \\ \qquad\times(P_L^{[(m+1)/2]-\mu})^{((l-1)/2)-i}\times(P_L^{[m/2]-\mu})^{(l-1)/2} \quad \text{for } 1\leqq i\leqq\dfrac{l-1}{2}, \\ M\backslash(\mathcal{O}_L)^{i-1}\times\mathcal{O}_L^\times\times(\mathcal{O}_L)^{l-i-1}/(P_L^{[(m+1)/2]-\mu-1})^{(l-1)/2}\times(P_L^{[m/2]-\mu-1})^{i-((l+1)/2)} \\ \qquad\times(1+P_L^{[m/2]-\mu})\times(P_L^{[m/2]-\mu})^{l-1-i} \quad \text{for } \dfrac{l+1}{2}\leqq i\leqq l-1, \end{cases}$$

$$J_{\mu,i}=\begin{cases} (\mathcal{O}_L/P_L^{[(m+1)/2]-\mu-1})^{i-1}\times(\mathcal{O}_F^\times/1+P_F^{[(m+1)/2]-\mu})\times(\mathcal{O}_L/P_L^{[(m+1)/2]-\mu})^{((l-1)/2)-i} \\ \qquad\times(\mathcal{O}_L/P_L^{[m/2]-\mu})^{(l-1)/2} \quad \text{for } 1\leqq i\leqq\dfrac{l-1}{2}, \\ (\mathcal{O}_L/P_L^{[(m+1)/2]-\mu-1})^{(l-1)/2}\times(\mathcal{O}_L/P_L^{[m/2]-\mu-1})^{i-((l+1)/2)}\times(\mathcal{O}_F^\times/1+P_F^{[m/2]-\mu}) \\ \qquad\times(\mathcal{O}_L/P_L^{[m/2]-\mu})^{l-1-i} \quad \text{for } \dfrac{l+1}{2}\leqq i\leqq l-1, \end{cases}$$

and

$$K_{\mu,i}=\left\{1+\varpi^\mu\Big(\sum_{j=1}^{i-1}\varpi\xi^j\beta_j+\sum_{j=i}^{l-1}\xi^j\beta_j\Big)\,\Big|\,(\beta_1,\cdots,\beta_{l-1})\in I_{\mu,i}\right\}.$$

We define $\varphi_i\colon(\mathcal{O}_L)^{i-1}\times\mathcal{O}_L^\times\times(\mathcal{O}_L)^{l-i-1}\longrightarrow(\mathcal{O}_L)^{i-1}\times\mathcal{O}_F^\times\times(\mathcal{O}_L)^{l-i-1}$ as follows:

(1.2.10) $\qquad \varphi_i(\alpha_1,\cdots,\alpha_{l-1})=(\beta_1,\cdots,\beta_{l-1}),\qquad \beta_j=\alpha_j\alpha_i^{q^{-i}}\alpha_i^{q^{-2i}}\cdots\alpha_i^{q^{-ki}},$

where k is determined by $0\leqq k<l$ and $-ki\equiv j \pmod l$. (In particular $\beta_i=N_{L/F}\alpha_i$).

Lemma 1.2.11. (1) *A complete system of representatives of the double coset $L^\times\backslash F^\times\mathcal{O}_D^\times/H$ is given by* $\bigcup_{\substack{1\leqq i\leqq l-1\\ 0\leqq\mu<n(i)}}K_{\mu,i}\cup\{1\}$.

(2) *The map φ_i induces a bijection from $I_{\mu,i}$ to $J_{\mu,i}$.*

Proof. Part one follows immediately from Lemma 1.2.9. For part two, it suffices to see that φ_1 induces a bijection from $I_{0,1}$ to $J_{0,1}$. If $\beta_1,\gamma_1\in\mathcal{O}_L^\times$ and $\beta_2,\cdots,\beta_{l-1},\gamma_2,\cdots,\gamma_{l-1}\in\mathcal{O}_L$ satisfy $(\gamma_1,\cdots,\gamma_{l-1})\in M(\beta_1,\cdots,\beta_{l-1})((1+P_L^{[(m+1)/2]})\times(P_L^{[(m+1)/2]})^{(l-3)/2}\times(P_L^{[m/2]})^{(l-1)/2})$, then there exist $\alpha\in\mathcal{O}_L^\times$ and $y_i\in P_L^{n(i)}$ $(1\leqq i\leqq l-1)$ such that

$$\gamma_1=\alpha^\sigma\alpha^{-1}\beta_1(1+y_1),$$

$$\gamma_i=\alpha^{\sigma^i}\alpha^{-1}(\beta_i+y_i)\quad(2\leqq i\leqq l-1).$$

This implies:

$$N_{L/F}(\beta_1)\equiv N_{L/F}(\gamma_1)\bmod 1+P_L^{[(m+1)/2]}\quad\text{(multiplicative equivalence)},$$

$$\gamma_i\gamma_1^{q^{-1}}\cdots\gamma_1^{q^i}\equiv\beta_i\beta_1^{q^{-1}}\cdots\beta_1^{q^i}\bmod P_L^{n(i)}\qquad\text{for } 2\leqq i\leqq l-1.$$

Therefore φ_1 induces a well-defined map from $I_{0,1}$ to $J_{0,1}$. The induced map's bijectivity follows from the bijectivity of the map $\mathcal{O}_L^{(1)}\backslash\mathcal{O}_L^\times/1+P_L^j\xrightarrow{N_{L/F}}\mathcal{O}_F^\times/1+P_F^j$.

Next we consider the term $aHa^{-1}\cap L^\times$ in (1.2.5).

Lemma 1.2.12. *If $a\in K_{\mu,i}$, then $aHa^{-1}\cap L^\times=F^\times(1+P_L^{n(i)-\mu})$.*

Proof. Since $F^\times \subset aHa^{-1} \cap L^\times$, we have only to see $aHa^{-1} \cap \mathcal{O}_L^\times = \mathcal{O}_F^\times(1+P_L^{n(i)-\mu})$. If $\alpha \in aHa^{-1} \cap \mathcal{O}_L^\times$, then there exist $\gamma_0 \in \mathcal{O}_L^\times$ and $\gamma_i \in P_L^{n(i)-\mu}$ $(1 \leqq i \leqq l-1)$ such that $\alpha a = a\sum_{i=0}^{l-1} \xi^i \gamma_i$. Put $a = 1+\sum_{j=1}^{l-1} \xi^j \beta_j$. Then we have

$$\gamma_0 = \alpha - \varpi \sum_{j=1}^{l-1} \gamma_j \beta_{l-j}^{\sigma^j},$$

$$(\alpha^{\sigma^{-i}} - \gamma_0)\beta_i = \gamma_i + \sum_{j=1}^{i} \beta_{i-j}^{\sigma^j} \gamma_j + \varpi \sum_{j=i+1}^{l-1} \beta_{l+i-j}^{\sigma^j} \gamma_j \qquad (1 \leqq i \leqq l-1).$$

By replacing γ_0 by $\alpha - \varpi \sum_{j=1}^{l-1} \gamma_j \beta_{l-j}^{\sigma^j}$, we get

$$(\alpha^{\sigma^{-i}} - \alpha)\beta_i \in P_L^{n(i)} \qquad (1 \leqq i \leqq l-1).$$

Therefore $\alpha \in \mathcal{O}_F^\times(1+P_L^{n(i)-\mu})$ and $aHa^{-1} \cap \mathcal{O}_L^\times \subset \mathcal{O}_F^\times(1+P_L^{n(i)-\mu})$. As for $aHa^{-1} \cap \mathcal{O}_L^\times \supset \mathcal{O}_F^\times(1+P_L^{n(i)-\mu})$, we can prove it by the same argument in the proof of Lemma 1.2.8.

Our next task is to compute ρ_θ^a for $a \in L^\times \backslash D^\times / H$. The above lemma tells us that $\rho_\theta^a \in (F^\times(1+P_L^{n(i)-\mu}))^\wedge$ if $a \in K_{\mu,i}$. If $a' = \xi^j a$, then $a'Ha'^{-1} \cap L^\times = aHa^{-1} \cap L^\times$ and $\rho_\theta^{a'} = \rho_\theta^a \circ \sigma^j$. Therefore it suffices to consider ρ_θ^a for $a \in L^\times \backslash F^\times \mathcal{O}_D^\times / H$.

Lemma 1.2.13. *Let $c \in F^\times$, $y \in P_L^{n(i)-\mu}$ and $a = 1+\varpi^\mu(\varpi \sum_{j=1}^{i-1} \xi^j \alpha_j + \sum_{j=i}^{l-1} \xi^j \alpha_j) \in K_{\mu,i}$. Then*

$$(\rho_\theta^a \rho_\theta^{-1})(c(1+y)) = \psi\Big(\mathrm{tr}_{L/F}\, \varpi^{\mu+1}\Big(\varpi \sum_{j=1}^{i-1} (\gamma_\theta^{\sigma^{-j}} f_{l-j}(a) \alpha_j^{\sigma^{-j}} - \gamma_\theta (f_{l-j}(a))^{\sigma^j} \alpha_j)$$

$$+ \sum_{j=i}^{l-1} (\gamma_\theta^{\sigma^{-j}} f_{l-j}(a) \alpha_j^{\sigma^{-j}} - \gamma_\theta (f_{l-j}(a))^{\sigma^j} \alpha_j)\Big) y\Big),$$

where $f_j(a) \in L$ is defined by $a^{-1} = \sum_{j=0}^{l-1} \xi^j f_j(a)$.

Proof. Since $(\rho_\theta^a \rho_\theta^{-1})$ is trivial on $F^\times$, we can assume $c=1$. Put $g = 1+x$, then

$$\begin{aligned} a^{-1} g a g^{-1} &= (1+a-1)^{-1} g (1+a-1) g^{-1} \\ &= (1+a-1)^{-1}(1+g(a-1)g^{-1}) \\ &= 1+a^{-1}(g(a-1)g^{-1}-(a-1)) \\ &= 1+a^{-1}\varpi^\mu\Big(\varpi \sum_{j=1}^{i-1} \xi^j \alpha_j (g^{\sigma^j} g^{-1}-1) + \sum_{j=i}^{l-1} \xi^j \alpha_j (g^{\sigma^j} g^{-1}-1)\Big). \end{aligned}$$

Since $\varpi^\mu(\varpi \sum_{j=1}^{i-1} \xi^j \alpha_j + \sum_{j=i}^{l-1} \xi^j \alpha_j) \in P_D^{[(ml+2)/2]}$, $\rho_\theta(1+x) = \psi(\mathrm{tr}_{D/F} \gamma_\theta x)$ $(x \in P_D^{[(ml+2)/2]})$ and $\mathrm{tr}_{D/F} \gamma_\theta \xi^j L = 0$ $(1 \leqq j \leqq l-1)$,

$$\begin{aligned} (\rho_\theta^a \rho_\theta^{-1})(g) &= \rho_\theta(a^{-1} g a g^{-1}) \\ &= \psi\Big(\mathrm{tr}_{D/F} \gamma_\theta a^{-1} \varpi^\mu \Big(\varpi \sum_{j=1}^{i-1} \xi^j \alpha_j (g^{\sigma^j} g^{-1}-1) \\ &\qquad + \sum_{j=i}^{l-1} \xi^j \alpha_j (g^{\sigma^j} g^{-1}-1)\Big)\Big) \end{aligned}$$

$$=\psi\Big(\mathrm{tr}_{L/F}\gamma_\theta a^{-1}\varpi^{\mu+1}\Big(\varpi\sum_{j=1}^{i-1}(f_{l-j}(a))^{\sigma^j}\alpha_j(g^{\sigma^j}g^{-1}-1)$$

$$+\sum_{j=i}^{l-1}(f_{l-j}(a))^{\sigma^j}\alpha_j(g^{\sigma^j}g^{-1}-1)\Big)\Big).$$

In the last term of the above equations, $\gamma_\theta\in P_L^{-m}$, $f_{l-j}(a)\in P_L^{\mu}$ and $g^{\sigma^j}g^{-1}-1\equiv y^{\sigma^j}-y$ mod $P_L^{2(n(i)-\mu)}$. Therefore

$$(\rho_\theta^\sigma\rho_\theta^{-1})(g)=\psi\Big(\mathrm{tr}_{L/F}\gamma_\theta a^{-1}\varpi^{\mu+1}\Big(\varpi\sum_{j=1}^{i-1}(f_{l-j}(a))^{\sigma^j}\alpha_j(g^{\sigma^j}g^{-1}-1)$$

$$+\sum_{j=i}^{l-1}(f_{l-j}(a))^{\sigma^j}\alpha_j(g^{\sigma^j}g^{-1}-1)\Big)\Big).$$

(We note ψ is trivial on P_L). Hence our lemma follows from the following property:

$$\mathrm{tr}_{L/F}uv^{\sigma^j}=\mathrm{tr}_{L/F}u^{\sigma^{-j}}v \qquad \text{for any} \quad u, v\in L\,.$$

We prepare the next lemma for the purpose of writing $f_k(a)$ by $(\alpha_j)_{1\le j\le l-1}$.

Lemma 1.2.14. *For $a=\sum_{j=0}^{l-1}\xi^j\alpha_j$ $(\alpha_j\in L)$, put*

$$\Lambda(a)=(\varpi^{[1+((j-i)/l)]}\alpha_{i-j\ \mathrm{mod}\ l})_{0\le i,j\le l-1}$$

$$=\begin{pmatrix}\alpha_0 & \varpi\alpha_{l-1}^{\sigma} & \cdots & \varpi\alpha_1^{\sigma^{l-1}}\\ \alpha_1 & \alpha_0^{\sigma} & \ddots & \vdots\\ \vdots & \ddots & \ddots & \varpi\alpha_{l-1}^{\sigma^{l-1}}\\ \alpha_{l-1} & \cdots & \alpha_1^{\sigma^{l-2}} & \alpha_0^{\sigma^{l-1}}\end{pmatrix}\in M_l(L)\,,$$

and

$$\Lambda_k(a)=(-1)^k\begin{vmatrix}\alpha_1 & \cdots & \varpi\alpha_{l-k+1}^{\sigma^{k-1}} & \varpi\alpha_{l-k-1}^{\sigma^{k+1}} & \cdots & \varpi\alpha_2^{\sigma^{l-1}}\\ \vdots & & \vdots & \vdots & & \vdots\\ \alpha_k & \vdots & \alpha_1^{\sigma^{k-1}} & \varpi\alpha_{l-1}^{\sigma^{k+1}} & \vdots & \varpi\alpha_{k+1}^{\sigma^{l-1}}\\ \vdots & & \vdots & \vdots & & \vdots\\ \alpha_{l-1} & \cdots & \alpha_{l-k+2}^{\sigma^{k-1}} & \alpha_{l-k}^{\sigma^{k+1}} & \cdots & \alpha_0^{\sigma^{l-1}}\end{vmatrix}\in L^\times$$

i.e. $\Lambda_k(a)$ is the $(1, k+1)$-cofactor of $\Lambda(\alpha)$. Then

$$a^{-1}=\sum_{j=0}^{l-1}\xi^j\frac{\Lambda_j(a)}{|\Lambda(a)|}\,,$$

where $|\Lambda(a)|$ is the determinant of $\Lambda(a)$.

Proof. By the map Λ: $D\to M_l(L)$, we can embed D into $M_l(L)$. Then our lemma follows from the basic matrix theory.

We define L-valued functions $R_{\mu,i}$ on $\mathcal{O}_L^{i-1}\otimes\mathcal{O}_L^\times\times\mathcal{O}_L^{l-i-1}$ by:

$$R_{\mu,i}(\beta_1,\cdots,\beta_{l-1})=\varpi^{\mu+2}\sum_{j=1}^{i-1}(\gamma_\theta^{\sigma^j}f_{l-j}(a)\alpha_j^{\sigma^j}-\gamma_\theta(f_{l-j}(a))^{\sigma^j}\alpha_j)$$

$$+\varpi^{\mu+1}\sum_{j=i}^{l-1}(\gamma_\theta^{\sigma^j}f_{l-j}(a)\alpha_j^{\sigma^j}-\gamma_\theta(f_{l-j}(a))^{\sigma^j}\alpha_j)\,,$$

where $\varphi_i(\alpha_1, \cdots, \alpha_{l-1})=(\beta_1, \cdots, \beta_{l-1})$ and $a=1+\varpi^{\mu}(\varpi\sum_{j=1}^{i-1}\xi^j\alpha_k+\sum_{j=i}^{l-1}\xi^j\alpha_k)$. (As for the definition of φ_i and $f_j(a)$, see 1.2.10 and Lemma 1.2.12 respectively). It is easily seen that $R_{\mu,i}$ is well-defined. In fact, we can show by virtue of Lemma 1.2.14 that $R_{\mu,i}(\beta_1, \cdots, \beta_{l-1})$ is a rational function of $\{\beta_j^{q^k}\}_{1\leq j,k\leq l-1}$. We fix $\beta_j\,(1\leqq j\leqq l-1)$ for all j but $l-i$ and define a function $\tilde{R}_{\mu,i}$ on $\mathcal{O}_L$ by:

$$\tilde{R}_{\mu,i}(x)=R_{\mu,i}(\beta_1, \cdots, \beta_{l-i-1}, x, \beta_{l-i+1}, \cdots, \beta_{l-1}).$$

The next lemma is the key point in this proof of Theorem 1.2.1.

Lemma 1.2.15. *Let* $L^{(0)}=\{x\in L \mid \mathrm{tr}_{L/F}x=0\}$. *Then* $\tilde{R}_{\mu,i}$ *has the following property:*

(1) $\tilde{R}_{\mu,i}$ *induces a surjection from* $\mathcal{O}_L/P_L^{[m/2]-\mu}$ *to* $P_L^{2\mu+1-m}\cap L^{(0)}/P_L^{\mu+1-[(m+1)/2]}\cap L^{(0)}$ *and each fiber of the induced map has* $q^{[m/2]-\mu}$ *elements if* $1\leqq i\leqq(l-1)/2$,

(2) $\tilde{R}_{\mu,i}$ *induces a surjection from* $\mathcal{O}_L/P_L^{[(m+1)/2]-\mu-1}$ *to* $P_L^{2\mu+2-m}\cap L^{(0)}/P_L^{\mu+1-[m/2]}\cap L^{(0)}$ *and each fiber of the induced map has* $q^{[(m+1)/2]-\mu-1}$ *elements if* $(l+1)/2\leqq i\leqq l-1$.

Proof. We assume $1\leqq i\leqq(l-1)/2$. By virtue of Lemma 1.2.14 and Lemma 1.2.15, we can show

$$\tilde{R}_{\mu,i}(x)\equiv ax-(ax)^{\sigma^i}+b \qquad \mathrm{mod}\ P_L^{v_L(x)+2\mu+1-m},$$

where $a=\varpi^{2\mu+1}(\gamma_\theta^{\sigma^{-i}}-\gamma_\theta)\in P_L^{2\mu+1-m}-P_L^{2\mu+2-m}$ and b is a constant in $P_L^{2\mu+1-m}$. Therefore we can get our lemma by induction on $[m/2]-\mu$ since $\tilde{R}_{\mu,i}(x)$ mod $P_L^{\mu+1-[(m+1)/2]}$ is a polynomial of $\{x, x^{\sigma}, \cdots, x^{\sigma^{l-1}}\}$ whose coefficients belong to $P_L^{2\mu+1-m}$. The case $(l+1)/2\leqq i\leqq l-1$ is proved by the same way.

Summing up the above lemmas, we have the following result.

Lemma 1.2.16. (1) *If* $1\leqq i\leqq(l-1)/2$,

$$\begin{aligned} K_{\mu,i} &\longrightarrow (F^{\times}(1+P_L^{[(m+1)/2]-\mu}))^{\wedge} \\ a &\longmapsto \rho_\theta^a\rho_\theta^{-1} \end{aligned}$$

is a surjection to $(F^{\times}(1+P_L^{[(m+1)/2]-\mu})/F^{\times}(1+P_L^{m-2\mu}))^{\wedge}$ *and each fiber of the map has* $(q-1)q^{((l-1)(l-2)(m-2\mu)/2)-l(i-1)-1}$ *elements.*

(2) *If* $(l+1)/2\leqq i\leqq l-1$,

$$\begin{aligned} K_{\mu,i} &\longrightarrow (F^{\times}(1+P_L^{[m/2]-\mu}))^{\wedge} \\ a &\longmapsto \rho_\theta^a\rho_\theta^{-1} \end{aligned}$$

is a surjection to $(F^{\times}(1+P_L^{[m/2]-\mu})/F^{\times}(1+P_L^{m-2\mu-1}))^{\wedge}$ *and each fiber of the map has* $(q-1)q^{((l-1)(l-2)(m-2\mu-1)/2)-l(i-((l+1)/2))-1}$ *elements.*

Proof. Let $1\leqq s<t\leqq 2t$, $b\in P_L^s\cap L^{(0)}$, $c\in F^{\times}$ and $y\in P_L^{1-t}$. Then the map $b\mapsto\hat{b}=(c(1+y)\mapsto\psi(\mathrm{tr}_{L/F}\,by))$ induces an isomorphism between $P_L^s\cap L^{(0)}/P_L^t\cap L^{(0)}$ and $(F^{\times}(1+P_L^{1-t})/F^{\times}(1+P_L^{1-s}))^{\wedge}$ since the conductor of ψ is P_L and L/F is unramified. Hence our lemma holds by virtue of Lemma 1.2.15 and Lemma 1.2.12.

Proof of Theorem 1.2.1. By Lemma 1.2.16,

$$\bigoplus_{a\in K_{\mu,i}} \mathrm{Ind}_{aHa^{-1}\cap L^\times}^{L^\times}\rho_\theta^a=\theta\otimes\begin{cases}(q-1)q^{((l-1)(l-2)(m-2\mu)/2)-l(i-1)-1}X_{m-2\mu} & \text{if } 1\leqq i\leqq \dfrac{l-1}{2},\\ (q-1)q^{((l-1)(l-2)(m-2\mu-1)/2)-l(i-((l+1)/2))-1}X_{m-2\mu-1} & \text{if } \dfrac{l+1}{2}\leqq i\leqq l-1,\end{cases}$$

where $X_j=\bigoplus_{\chi\in(L^\times/F^\times(1+P_L^j))^\wedge}\chi$. Thus by Lemma 1.2.11 and (1.2.5), we have:

$$\pi_\theta|_{L^\times}=(\bigoplus_{\sigma\in\Gamma}\theta\circ\sigma)\otimes\Big(X_0+(q-1)\frac{q^{l(l-1)/2}-1}{q^l-1}\sum_{a=1}^{m}q^{(l-1)(l-2)(a-1)/2}X_a\Big).$$

The rest of Theorem 1.2.1 follows immediately from the above formula.

Next we consider the case $E\not\cong L$. Then E is a totally ramified extension of F of degree l. This case is very easy.

Theorem 1.2.17. *Let θ be a regular quasi-character of $L^\times$ with $\min_\eta f(\theta\otimes(\eta\circ N_{L/F}))=m+1$ and π_θ as in (1.1.6).*

(1) (*Decomposition of π_θ as $E^\times$-module*)

$$\pi_\theta|_{E^\times}=\theta\otimes q^{(l-1)(l-2)m/2}\bigoplus_{\chi\in(E^\times/F^\times(1+P_E^{lm+1}))^\wedge}\chi$$

(2) (*Character formula of π_θ on $E^\times$*)

$$\chi_{\pi_\theta}(x)=\begin{cases}0 & \text{if } x\notin F^\times(1+P_E^{lm+1})\\ \theta(c)lq^{l(l-1)m/2} & \text{if } x=c(1+y)\in F^\times(1+P_E^{lm+1}).\end{cases}$$

Proof. It suffices to say that $\chi_{\pi_\theta}(x)=0$ if $[(lm+2)/2]\leqq v_E(x-1)<lm$. (We note that $F^\times(1+P_E^{lm})=F^\times(1+P_E^{lm+1})$). Set $r=v_E(x-1)$. From the definition of π_θ,

$$\begin{aligned}\chi_{\pi_\theta}(x)&=\sum_{g\in D^\times/H}\rho_\theta(g^{-1}xg)\\ &=\frac{1}{q^{l(lm+1-r-[(lm+1-r)/2])}}\sum_{g\in D^\times/H}\ \sum_{k\in P_D^{[(lm+1-r)/2]}/P_D^{lm+1-r}}\rho_\theta((1+k)^{-1}g^{-1}xg(1+k)).\end{aligned}$$

Set $g^{-1}xg=1+h$. By virtue of $(1+k)^{-1}(1+h)(1+k)\equiv 1+hk-kh \bmod P_D^{lm+1}$, $\rho_\theta((1+k)^{-1}(1+h)(1+k))=\psi(\mathrm{tr}_{D/F}(\gamma_\theta h-h\gamma_\theta)k)$. Since $h\in P_D^r$ and $h\notin P_L^r+P_D^{r+1}$, the map $k\mapsto\psi(\mathrm{tr}_{D/F}(\gamma_\theta h-h\gamma_\theta)k)$ is a non-trivial character of $P_D^{[(lm+1-r)/2]}/P_D^{lm+1-r}$. (cf. 6.7 [Ca]). Therefore $\chi_{\pi_\theta}(x)=0$.

2. Split (matrix algebra) case

2.1 Construction of the representation. In this section, we treat the case $A=M_l(F)$ with l an odd prime. We set $G=\mathrm{GL}_l(F)$, $K=F^\times\mathrm{GL}_l(\mathcal{O}_F)$, $K_0=\mathrm{GL}_l(\mathcal{O}_F)$, $A_i=P_F^iM_l(\mathcal{O}_F)$ and $K_i=1+A_i$ $(i\geqq 1)$. Let L be an unramified extension of F of degree l, then L can be embedded into $M_l(F)$ and, up to conjugacy, the embedding is unique. As in the division algebra case, we construct an irreducible representation Π_θ from

$\theta \in \hat{L}^{\times}_{reg}$ according to [G]. At first we treat the case θ is generic. (cf. Definition 1.1.1) If $f(\theta)=1$, then there is an irreducible representation κ'_{θ} of K_0 which is trivial on K_1 and such that its tensor product with the pull-back of the Steinberg representation of $K_0/K_1 \cong \mathrm{GL}_l(k_F)$ is the representaion induced by the one-dimensional representation $tx \mapsto \theta(t)$, $t \in \mathcal{O}_L^{\times}$, $x \in K_1$, of the subgroup $\mathcal{O}_L^{\times} K_1$. We denote by κ_{θ} the representation $tx \mapsto \theta(t)\kappa'_{\theta}(x)$, $t \in F^{\times}$, $x \in K_0$, of K and set

$$\Pi_{\theta}=\mathrm{ind}_K^G \kappa_{\theta}\,.$$

Then Π_{θ} is an irreducible supercuspidal representation with $f(\Pi_{\theta})=l$. We assume $f(\theta)=m+1>1$. Let $\psi_{\gamma_{\theta}}(1+x)=\psi(\mathrm{tr}(\gamma_{\theta}x))$ for $x \in A_{[(m+2)/2]}$. Then $\psi_{\gamma_{\theta}}$ is a quasi-character of $K_{[(m+2)/2]}$. (See 1.1.3 for the definition of γ_{θ}.) Set $H=L^{\times}(1+A_{[(m+2)/2]}) \subset K$ and define a quasi-character ρ_{θ} of H by

$$\rho_{\theta}(h \cdot g)=\theta(h)\psi_{\gamma_{\theta}}(g) \qquad \text{for} \quad h \in L^{\times}, \quad g \in K_{[(m+2)/2]}\,. \tag{2.1.1}$$

We consider two cases according to the parity of $m+1$.

Case $m+1$ even. Set

$$\kappa_{\theta}=\mathrm{Ind}_H^K \rho_{\theta} \quad \text{and} \quad \Pi_{\theta}=\mathrm{ind}_K^G \kappa_{\theta}\,. \tag{2.1.2}$$

Then κ_{θ} is an irreducible very cuspidal representation of K and Π_{θ} is an irreducible supercuspidal representation of G with $f(\Pi_{\theta})=l(m+1)$.

Case $m+1$ odd. Set $H'=L^{\times}K_{m/2}$. Then there is an irreducible component ρ'_{θ} of the induced representation $\mathrm{Ind}_H^{H'} \rho_{\theta}$ which is characterized by:

$$\chi_{\rho'_{\theta}}(x)=\theta(x) \qquad \text{for} \quad x \in H-F^{\times}(1+P_L)K_{(m+2)/2}\,. \tag{2.1.3}$$

(See Lemma 3.5.36 in [M].) We set

$$\kappa_{\theta}=\mathrm{Ind}_{H'}^K \rho_{\theta} \quad \textit{and} \quad \Pi_{\theta}=\mathrm{ind}_K^G \kappa_{\theta}\,. \tag{2.1.4}$$

Then κ_{θ} is an irreducible very cuspidal representation of K and Π_{θ} is an irreducible supercuspidal representation of G with $f(\Pi_{\theta})=l(m+1)$.

For a regular quasi-character θ written in the form $\theta=(\eta \circ N_{L/F}) \otimes \theta'$ where η is a quasi-character of $F^{\times}$ and θ' is a non-trivial generic quasi-character of $L^{\times}$, we set

$$\Pi_{\theta}=\Pi_{\theta'} \otimes \eta\,. \tag{2.1.5}$$

Now we get a correspondence $\theta \in \hat{L}^{\times}_{reg} \mapsto \Pi_{\theta} \in \hat{G}$. As in the division algebra case, the following proposition is known about this correspondence. (cf. [G2], [H2], [Ca], [M]).

Proposition 2.1.6. *With the above notations, for any regular quasi-character θ of $L^{\times}$, Π_{θ} is an irreducible supercuspidal representation of* $\mathrm{GL}_l(F)$ *such that:*

(a) *the representations Π_{θ} and $\Pi_{\theta'}$ associated to two regular quasi-characters θ and θ' are equivalent if and only if θ and θ' are conjugate under* $\mathrm{Gal}(L/F)$;

(b) *the central quasi-character of Π_{θ} is the restriction of θ to $F^{\times}$;*

(c) *for any quasi-character η of $F^{\times}$, the twisted representation $\Pi_{\theta} \otimes \eta$ is equivalent to $\Pi_{\theta \otimes \eta \circ N_{L/F}}$;*

(d) *the contragredient representation of Π_{θ} is equivalent to $\Pi_{\theta^{-1}}$;*

(e) *the L-function of* Π_θ *is* 1;

(f) *the* ε*-factor of* Π_θ *is* $\varepsilon(\Pi_\theta, \psi)=\varepsilon(\theta, \psi\circ \mathrm{tr}_{L/F})$; *in particular* $f(\Pi_\theta)=l\cdot f(\theta)$;

(g) $\{\Pi_\theta \mid \theta\in\hat{L}^\times_{reg}\}=\{\Pi\in\hat{G}\mid \Pi$ *is supercuspidal and* $f(\Pi)\equiv 0 \pmod{l}\}$.

We set $\hat{D}^\times_{unr}=\{\pi_\theta\mid\theta\in\hat{L}^\times_{reg}\}$ and $\hat{G}^0_{unr}=\{\Pi_\theta\mid\theta\in\hat{L}^\times_{reg}\}$. Then by Proposition 1.1.7. (g) and Proposition 2.1.6 (g),

$$\hat{D}^\times_{unr}=\{\pi\in\hat{D}^\times\mid f(\pi)\equiv 0 \pmod{l}\}$$

$$\hat{G}^0_{unr}=\{\Pi\in\hat{G}\mid \Pi \text{ is supercuspidal and } f(\Pi)\equiv 0 \pmod{l}\}.$$

By Proposition 1.1.7 and Proposition 2.1.6, we get the correspondence between $\hat{D}^\times_{unr}$ and $\hat{G}^0_{unr}$.

Proposition 2.1.7. *The correspondence* $\hat{D}^\times_{unr}\ni\pi_\theta\leftrightarrow\Pi_\theta\in\hat{G}^0_{unr}$ *is a bijection and preserves* ε*-factors, central quasi-characters and conductors.*

Remark. Our correspondence is a special case of Howe bijection, which is a bijective correspondence between the irreducible representations of the multiplicative group of a division algebra of degree n over F and essentially square-integrable representations of $\mathrm{Gl}_n(F)$ (n is prime to the residual characteristic of F) via admissible characters.

2.2 Character formula. We shall compute the character χ_{Π_θ} on the set of elliptic regular elements of G. At first we recall the following important result by Deligne-Kazhdan-Vigneras [BDKV] and Rogawski [R].

Theorem 2.2.1. *Let* $D^\times_n$ *be the multiplicative group of a division algebra of degree* n *over* F *and* E *be a separable extension of degree* n *over* F. ($E^\times$ *can be imbedded as a compact (mod center) Cartan subgroup in both* $D^\times_n$ *and* $\mathrm{GL}_n(F)$). *There is a bijection* $\pi\leftrightarrow\Pi$ *between irreducible representations of* $D^\times_n$ *and essentially square-integrable representations of* $\mathrm{GL}_n(F)$ *with the following properties*:

(1) *if* x *is a regular element in a compact (mod center) Cartan subgroup* $E^\times$, *then*

$$\chi_\pi(x)=(-1)^{n-1}\chi_\Pi(x),$$

(2) $\varepsilon(\pi_\theta, \psi)=(-1)^{n-1}\varepsilon(\Pi_\theta, \psi)$.

By the above theorem, we get:

Corollary 2.2.2. *There is a bijection between* $\hat{D}^\times_{unr}$ *and* $\hat{G}^0_{unr}$ *which preserves characters,* ε*-factors, central quasi-characters and conductors.*

We shall show two correspondences in Proposition 2.1.7 and Corollary 2.2.2 coincide.

Theorem 2.2.3. (*Character formula of* Π_θ) *Let* θ *be a regular quasi-character of* $L^\times$ *with* $\min_\eta f(\theta\otimes(\eta\circ N_{L/F}))=m+1$ *and* Π_θ *as in* (2.1.5). *If* x *is a regular element in a compact (mod center) Cartan subgroup* $E^\times$, *then*

$$\chi_{\pi_\theta}(x)=\chi_{\Pi_\theta}(x),$$

i.e. if $E\cong L$, *then*

$$\chi_{\Pi_\theta}(x)=\begin{cases} q^{l(l-1)j/2}\Big(\sum_{\sigma\in\Gamma}\theta(x^\sigma)\Big) & \text{if} \quad x\in U_j^* \ (0\leqq j<m) \\ q^{l(l-1)m/2}\Big(\sum_{\sigma\in\Gamma}\theta(x^\sigma)\Big) & \text{if} \quad x\in U_m, \end{cases}$$

where $U_j=F^\times(1+P_E^j)$ *and* $U_j^*=U_j-U_{j+1}$ *and if* $E\not\cong L$, *then*

$$\chi_{\Pi_\theta}(x)=\begin{cases} 0 & \text{if} \quad x\notin F^\times(1+P_E^{lm+1}) \\ \theta(c)lq^{m(l-1)} & \text{if} \quad x=c(1+y)\in F^\times(1+P_E^{lm+1}). \end{cases}$$

Remark. (a) When $l=3$, this theorem follows from the matching of ε-factors between $\pi_\theta\otimes\eta$ and $\Pi_\theta\otimes\eta$ for $\eta\in F^\times$ by virtue of the converse theorem [JPS].

(b) When $x\in E\cong L$ and $j<f(\theta)/3$, Theorem 5 in [G1] tells us

$$\chi_{\Pi_\theta}(x)=\chi_{\pi_\theta}(x)=q^{l(l-1)j/2}\Big(\sum_{\sigma\in\Gamma}\theta(x^\sigma)\Big) \qquad \text{for} \quad x\in U_j^*.$$

Proof of Theorem 2.2.3. By Proposition 2.1.6 (c), we can assume θ is a generic character with $f(\theta)=m+1$. By Corollary 1.2.2 and Proposition 2.1.7, there exists a generic character θ' such that $\chi_{\Pi_\theta}(x)=\chi_{\pi_{\theta'}}(x)$ if x is a regular element in $L^\times$. We shall show θ is conjugate to θ' under $\mathrm{Gal}(L/F)$. Let $L_{reg}^\times$ be the set of regular elements in $L^\times$. Then $L_{reg}^\times=L^\times-F^\times$. It is rather difficult to calculate $\chi_{\Pi_\theta}(x)$ for all $x\in L_{reg}^\times$, but it is easy if x is a 'very cuspidal' element in $L^\times$. (For the definition of the term 'very cuspidal', see [C]). Let $L_{vc}^\times$ be the set of very cuspidal elements in L. In our case, $L_{vc}^\times=L^\times-F^\times(1+P_L)=U_0^*$.

Proposition 2.2.4. *Let* $\Gamma=\mathrm{Gal}(L/F)$. *If* x *belongs to* $L_{vc}^\times$, *then*

$$\chi_{\Pi_\theta}(x)=\sum_{\sigma\in\Gamma}\theta(x^\sigma)$$

i.e.

$$\chi_{\pi_{\theta'}}(x)=\chi_{\pi_\theta}(x) \qquad \textit{for} \quad x\in U_0^*.$$

We pıove this proposition afterward. By the above proposition, we get:

$$\sum_{\sigma\in\Gamma}(\theta\circ\sigma)=\sum_{\sigma\in\Gamma}(\theta'\circ\sigma) \quad \text{on} \quad U_0^*. \tag{2.2.5}$$

We prepare the following simple lemma.

Lemma 2.2.6. *Let* $A_1\neq\{1\}$, A_2 *be finite abelian groups and* $A=A_1\times A_2$. *We assume* $\{\eta_1,\cdots,\eta_r\}$ *and* $\{\eta_1',\cdots\eta_r'\}\subset\hat{A}$ *satisfy the following conditions*:

(a) *either* η_i *and* η_i' *are trivial on* A_1 *for all* i *or non-trivial on* A_1 *for all* i,

(b) *for* $a\in(A_1-\{1\})\times A_2$,

$$\sum_{i=1}^r\eta_i(a)=\sum_{i=1}^r\eta_i'(a).$$

Then $\{\eta_1,\cdots,\eta_r\}=\{\eta_1',\cdots,\eta_r'\}$.

Proof. For $\eta\in\hat{A}$,

$$\sum_{g\in A_1-\{1\}}\eta(ga)=\begin{cases}(|A_1|-1)\eta(a) & \eta \text{ is trivial on } A_1,\\ \eta(a) & \text{otherwise}.\end{cases}$$

Therefore we get from (a) and (b), $\sum_{i=1}^{r}\eta_i(a)=\sum_{i=1}^{r}\eta'_i(a)$ for $a\in A$. Hence our lemma.

Since $\theta|_{F^\times}=\theta'|_{F^\times}$, we may assume θ and θ' is trivial on $F^\times$. By virtue of $f(\theta)=f(\theta')=m+1$, we can regard θ and θ' as quasi-characters of $L^\times/F^\times(1+P_L^{m+1})$. Moreover $L^\times/F^\times(1+P_L^{m+1})$ can be regarded as the direct product of $F^\times(1+P_L)/F^\times(1+P_L^{m+1})$ and $L^\times/F^\times(1+P_L)$ since the order of $F^\times(1+P_L)/F^\times(1+P_L^{m+1})$ is prime to the order of $L^\times/F^\times(1+P_L)$. Therefore we can apply the above lemma to $\{\theta\circ\sigma\}_{\sigma\in\Gamma}$ and $\{\theta'\circ\sigma\}_{\sigma\in\Gamma}$ by virtue of (2.2.5). Thus $\theta'=\theta\circ\sigma$ for some $\sigma\in\mathrm{Gal}(L/F)$.

The rest of our work is to prove Proposition 2.2.4.

Proof of Proposition 2.2.4. We recall $\Pi_\theta=\mathrm{ind}_K^G\kappa_\theta$. By Proposition 6.11 in [K],

$$\chi_{\Pi_\theta}(x)=\chi_{\kappa_\theta}(x)\qquad\text{for}\quad x\in L_{vc}^\times.$$

Thus we have only to consider χ_{κ_θ}. We start with the case $f(\theta)=1$. In this case, $\kappa_\theta|_K$ is a pull-back of an irreducible cuspidal reprentation of $\mathrm{GL}_l(k_F)$, whose character formula is well-known. For example, Theorem 7.12 in [Sp] tells us $\chi_{\kappa_\theta}(x)=\sum_{\sigma\in\Gamma}\theta(x^\sigma)$. We consider the case $f(\theta)>1$. If $f(\theta)$ is even, it follows from (2.1.2) that:

$$\chi_{\kappa_\theta}(t)=\sum_{g\in K/H}\dot{\rho}_\theta(g^{-1}tg)\qquad\text{for}\quad t\in L_{reg}^\times,$$

where $\dot{\rho}_\theta$ is defined by

$$\dot{\rho}_\theta=\begin{cases}\rho_\theta(x) & \text{if}\quad x\in H\\ 0 & \text{otherwise}.\end{cases}$$

If $f(\theta)$ is odd, it follows from (2.1.4) that:

$$\chi_{\kappa_\theta}(t)=\sum_{g\in K/H'}\dot{\chi}_{\rho'_\theta}(g^{-1}tg)\qquad\text{for}\quad t\in L_{reg}^\times,$$

where $\dot{\chi}_{\rho'_\theta}$ is defined by

$$\dot{\chi}_{\rho'_\theta}=\begin{cases}\chi_{\rho'_\theta}(x) & \text{if}\quad x\in H\\ 0 & \text{otherwise}.\end{cases}$$

We note that

(2.2.7) $$\chi_{\rho'_\theta}(t)=\rho_\theta(t)=\theta(t)\qquad\text{for}\quad t\in L_{vc}^\times.$$

(See (2.1.1) and (2.1.3)).

By Skolem-Noether theorem, there exists an element $\xi\in K_0$ such that:

$$\xi^{-1}x\xi=x^\sigma\qquad\text{for any}\quad x\in L,$$

where σ is a generator of $\mathrm{Gal}(L/F)$ and

$$M_l(\mathcal{O}_D)=\mathcal{O}_L\oplus\xi\mathcal{O}_L\oplus\cdots\oplus\xi^{l-1}\mathcal{O}_L.$$

Our prposition follows immediately from (2.2.7) and the following lemma.

Lemma 2.2.8. *Let $g \in K$ and $t \in L_{vc}^{\times}$. If $g^{-1}tgt^{-1} \in L^{\times}K_i$, then $g \in \xi^j L^{\times} K_i$ for some j.*

Proof. Let $g^{-1}tgt^{-1}=u(1+v)$ for $u \in \mathcal{O}_L^{\times}$ and $v \in A_i$. Let $g=\sum_{j=0}^{l-1}\xi^j \alpha_j$ where $\alpha_j \in \mathcal{O}_L$. Since $tgt^{-1} \equiv gu \mod A_i$, we get $(t^{\sigma^j}t^{-1}-u)\alpha_j \equiv 0 \mod P_L^i$. If there exist j_1 and j_2 such that $j_1 \neq j_2$ and $t^{\sigma^{j_1}}t^{-1} \equiv t^{\sigma^{j_2}}t^{-1} \mod P_L$, then $t \in F^{\times}(1+P_L)$. Since $t \in L_{vc}^{\times}=L^{\times}-F^{\times}(1+P_L)$, we get our lemma.

Remark. Proposition 2.2.4 follows directly from Theorem 5 in [G1].

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