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# Semi-stably polarized fiber spaces and Bogomolov-Gieseker's inequality

Dedicated to Professor Heisuke Hironaka on his sixtieth birthday

By

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## Introduction

Let $S$ be a non-singular projective surface defined over an algebraically closed field of characteristic zero, $H$ an ample line bundle on $S$ and $E$ a locally free sheaf on $S$ of rank 2. $E$ is said to be $H$-semi-stable if for any sub-sheaf $L$ of $E$ of rank one, the following inequality holds.

$$(c_1(L)\cdot H)\leqq\frac{(c_1(E)\cdot H)}{2}.$$

F.A. Bogomolov [2] and D. Gieseker [6] proved that if $E$ is $H$-semi-stable, then

$$(c_1^2(E))\leqq 4(c_2(E)).$$

Let $X$ be the associated projective bundle of $E$, $L$ the tautological line bundle of $X$ and $f: X\to S$ the natural projection. Then, $c_1(E)$ and $c_2(E)$ are determined cohomologically by the following equations:

$$\begin{cases}(L^2\cdot f^*(x))=(f^*(c_1(E))\cdot L\cdot f^*(x)) & \text{for all} \quad x\in H^2(S, \boldsymbol{Q})\\ (L^3-f^*(c_1(E))\cdot L^2+L\cdot f^*(c_2(E)))=0.\end{cases}$$

Furthermore, it is easy to see that the condition of the semi-stability is equivalent to that

$$\left(L-\frac{f^*(c_1(E))}{2}\cdot\Gamma\cdot f^*(H)\right)\geqq 0$$

for all $f$-dominant subvariety $\Gamma$ of $X$ (see Remark (2.20)). The above notions are easily translated into the case of a polarized fiber space, that is, a surjective morphism $f: X\to S$ of non-singular projective varieties and an $f$-ample line bunle $L$ on $X$ (see Section 2). With this notation, we shall prove a generalized inequality of Bogomolov-Gieseker's type in this paper (see Theorem (3.1) and Theorem (5.7)). Our Proof is based on Miyaoka's proof of Bogomolov-Gieseker's inequality [13]. First, we shall prove the generalized inequality of Bogomolov-Gieseker's type under a strong assumption (Theorem (3.1)). In this stage, we shall not restrict ourselves to the case of $\dim f=1$. Next, we shall prove Mumford-Mehta-Ramanathan's type theorem, that is,

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a theorem about restriction of semi-stable fibration to ample curves (Proposition (5.2) and Theorem (5.4)). Up to now, we cannot prove Mumford-Mehta-Ramanathan's type theorem without restriction of $\dim f=1$. Main reason for this is that the existence of Zariski decomposition on a higher dimensional variety or weaker theorem (cf. Lemma (5.5)) is not established. For example, Harder-Narasimhan filtration of vector bundle is closely related to Zariski decomposition (see [17, §4, Examples (II)]).

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## 1. Notation and Conventions

**(1.1)** Let $K$ be an algebraically closed field. We shall fix this field $K$ throughout this paper. Without any statements or comments, every algebraic scheme will be defined over $K$.

**(1.2)** Let $X$ be an algebraic scheme. We denote the free abelian group generated by $i$-dimensional sub-varieties of $X$ by $Z_i(X)$, whose element is called *i-cycle*. The group of $i$-cycles modulo rational equivalence on $X$ is denoted by $A_i(X)$. We set $Z_i(X)_{\boldsymbol{Q}}=Z_i(X)\otimes\boldsymbol{Q}$ and $A_i(X)_{\boldsymbol{Q}}=A_i(X)\otimes\boldsymbol{Q}$. For a zero-cycle $\alpha=\sum_P n_P P$, the degree of $\alpha$, denoted by $(\alpha)$, is defined by $(\alpha)=\sum_P n_P$. Let $f: X\to Y$ a morphism of algebraic schemes. The push-forward and pull-back of algebraic cycles are denoted by $f_*$ and $f^*$ respectively if they are defined. These symbols might be confused with symbols of direct images and pull-backs of sheaves. But we think that reader can distinguish them in their context. Details of push-forward, pull-back and intersection product of algebraic cycles will be found in Fulton's book [4].

**(1.3)** Let $X$ be a complete algebraic scheme over a field $k$, which is not necessarily algebraically closed. We set $\mathrm{Div}(X)=\{$Cartier divisors on $X\}$ and $\mathrm{Div}(X)_{\boldsymbol{Q}}=\mathrm{Div}(X)\otimes\boldsymbol{Q}$, whose element is called a *$\boldsymbol{Q}$-Cartier divisor*. If $X$ is locally factorial, then $\mathrm{Div}(X)=Z_{\dim X-1}(X)$ and $\mathrm{Pic}(X)=A_{\dim X-1}(X)$. Let $D$ be an element of $\mathrm{Div}(X)_{\boldsymbol{Q}}$. $D$ is said *nef* if $(D\cdot C)\geqq 0$ for all integral curve $C$ on $X$. $D$ is called *pseudo-effective* if there are effective $\boldsymbol{Q}$-Cartier divisors $D_n$ such that $D$ is the limit of the sequence $\{D_n\}$ in the vector space $(\mathrm{Div}(X)/\equiv)\otimes\boldsymbol{R}$, where $\equiv$ is the numerical equivalence.

**(1.4)** Let $f: X\to Y$ be a morphism of algebraic schemes. Let $\Gamma$ be a subvariety of $X$. We say that $\Gamma$ is *f-dominant* if the image of $\Gamma$ by $f$ is $Y$. Assume that $f$ is proper. Let $L$ be a Cartier divisor on $X$. We call $L$ is *f-nef* if for every complete irreducible curve $C$ on $X$ with $f(C)$ being a point, $(L\cdot C)\geqq 0$.

**(1.5)** We denote by $\boldsymbol{Z}_{>0}$ the set of positive integers. Let $S$ be a projective variety with a very ample divisor $H$. Let $s$ be a positive integer with $1\leqq s<\dim S$. For $m=(m_1,\cdots,m_s)\in(\boldsymbol{Z}_{>0})^s$, we set

$$S(H,m)=\{(x,D_1,\cdots,D_s)\in S\times|m_1H|\times\cdots\times|m_sH|\mid x\in D_i \text{ for all } 1\leqq i\leqq s.\}.$$

Let $q_m: S(H, m)\to|m_1H|\times\cdots\times|m_sH|$ be the natural projection, whose general fibers are $s$-times complete intersection varieties by elements of $|m_1H|, \cdots, |m_sH|$. The generic fiber of $q_m$ is denoted by $S(H, m)_\eta$.

**(1.6)** Let $r$ be a real number. We set $\lceil r\rceil=\min\{n\in\boldsymbol{Z}\,|\,r\leqq n\}$, which is called the *rounding-up* of $r$.

## 2. Chern classes and semi-stability for polarized fiber spaces

First, we introduce the notion of a polarized fiber space. Our terminology is slightly different from usual sense, that is, we do not assume that fibers are connected.

**Definition (2.1)** Let $X$ be an equi-dimensional algebraic scheme and $S$ a variety. A morphism $f: X\to S$ is called a *fiber space* if $f$ is proper and every irreducible component of $X$ is mapped surjectively to $S$. Let $L$ be a Cartier divisor on $X$. The pair $(X/S, L)$ is said a *pre-polarized fiber space.* Let $X_\eta$ be the generic fiber of $f$ and $r$ the relative dimension of $f$. We set $\deg(X/S, L)=(L^r)_{X_\eta}$, which is called a *degree of the pre-polarized fiber space* $(X/S, L)$. $\deg(X/S, L)$ is also determined by the formula $f_*(L^r)=\deg(X/S, L)\,[S]$. If $L$ is ample on $X_\eta$, $L$ is said a *weak polarization* of the fiber space $f: X\to S$. Moreover, if $L$ is $f$-ample, the pair $(X/S, L)$ is called a *polarized fiber space.*

For a weak polarized fiber space, we define Chern classes as elements of algebraic cycles with $\boldsymbol{Q}$-coefficients, which is an analogy of Chern classes of vector bundle (see Remark (2.4)).

**Definition (2.2)** Let $f: X\to S$ be a fiber space from an equi-dimensional algebraic scheme $X$ to a non-singular variety $S$ with a weak polarization $L$. Let $r$ be the relative dimension of $f$ and $d$ the dimension of $S$. First, we set $c_0(X/S, L)=1$ as an element of $A_d(S)_{\boldsymbol{Q}}$. Inductively, we define $c_k(X/S, L)\in A_{d-k}(S)_{\boldsymbol{Q}}(k=1, 2, \cdots, d)$ as follows.

$$c_1(X/S, L)=\frac{1}{\deg(X/S, L)}f_*(L^{r+1}),$$

$$c_2(X/S, L)=\frac{-1}{\deg(X/S, L)}(f_*(L^{r+2})-c_1(X/S, L)\cdot f_*(L^{r+1})),$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$c_k(X/S, L)=\frac{(-1)^{k-1}}{\deg(X/S, L)}\sum_{i=0}^{k-1}(-1)^i c_i(X/S, L)\cdot f_*(L^{r+k-i}),$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$c_d(X/S, L)=\frac{(-1)^{d-1}}{\deg(X/S, L)}\sum_{i=0}^{d-1}(-1)^i c_i(X/S, L)\cdot f_*(L^{r+d-i}).$$

$c_k(X/S, L)$ is said *$k$-th Chern class of the weak polarized fiber space* $(X/S, L)$.

**Remark (2.3)** By the definition of Chern classes, it is easy to see that

$$\deg(X/S,\ L)^k c_k(X/S,\ L) \in A_{d-k}(S).$$

**Remark** (2.4) Let $S$ be a $d$-dimensional non-singular variety and $E$ a locally free sheaf on $S$ of rank $r+1$. Let $X$ be the associated projective bundle of $E$, that is, $\mathrm{Proj}(\bigoplus_{i=0}^{\infty}\mathrm{Sym}^i(E))$, $L$ the tautological line bundle of $X$ and $f: X\to S$ the natural projection. Then, there is the identity

$$L^{r+1}-f^*(c_1(E))L^r+\cdots+(-1)^{r+1}f^*(c_{r+1}(E))=0.$$

Multiplying $L^{k-1}$, we have

$$L^{r+k}-f^*(c_1(E))L^{r+k-1}+\cdots+(-1)^{k-1}f^*(c_{k-1}(E))L^{r+1}+(-1)^k f^*(c_k(E))L^r+\cdots=0.$$

Thus, taking the push-forward of the above identity, we get

$$f_*(L^{r+k})-c_1(E)f_*(L^{r+k-1})+\cdots+(-1)^{k-1}c_{k-1}(E)f_*(L^{r+1})+(-1)^k c_k(E)=0.$$

This shows that $c_k(X/S,\ L)=c_k(E)$.

Cohomologically, Chern classes of a weak polarized fiber space are determined by equations in the following proposition.

**Proposition** (2.5) *With notation as in Definition* (2.2), *we assume that $X$ and $S$ are complete and the characteristic of the defining field is zero. We consider cohomological equations* ($\mathrm{Eq}_k$) ($k=1, 2, \cdots, d$) *with respect to $c_i\in H^{2i}(S, \boldsymbol{Q})$.*

$$(\mathrm{Eq}_k)\qquad \sum_{i=0}^{k}(-1)^i(f^*(c_i)\cdot L^{r+k-i}\cdot f^*(x))=0 \qquad \textit{for all}\quad x\in H^{2(d-k)}(S,\ \boldsymbol{Q})$$

*We set*

$$V=\left\{(c_0,\ \cdots,\ c_d)\in H^0(S,\ \boldsymbol{Q})\times\cdots\times H^{2d}(S,\ \boldsymbol{Q})\ \middle|\ \begin{array}{l}\{c_i\}_{i=0,\cdots,d}\ \textit{satisfies the above}\\ \textit{equations}\ (\mathrm{Eq}_k)\ (k=1,\ \cdots,\ d)\end{array}\right\}.$$

*Then,* $\dim_{\boldsymbol{Q}}V=1$ *and the cohomological class of* $(c_0(X/S,\ L),\ \cdots,\ c_d(X/S,\ L))$ *is an element of* $V$.

*Proof.* ($\mathrm{Eq}_k$) is equivalent to the equation:

$$(\mathrm{Eq}'_k)\qquad (-1)^{k-1}\frac{\sum_{i=0}^{k-1}(-1)^i(f^*(c_i)\cdot L^{r+k-i}\cdot f^*(x))}{\deg(X/S,\ L)}=(c_k\cdot x)\quad \text{for all } x\in H^{2(d-k)}(S,\ \boldsymbol{Q}).$$

Here, we assume that we have already got $c_0, c_1, \cdots, c_{k-1}$. Then,

$$(-1)^{k-1}\frac{\sum_{i=0}^{k-1}(-1)^i(f^*(c_i)\cdot L^{r+k-i}\cdot f^*(x))}{\deg(X/S,\ L)}$$

defines an element of $\mathrm{Hom}_{\boldsymbol{Q}}(H^{2(d-k)}(S,\ \boldsymbol{Q}),\ \boldsymbol{Q})$. Since the pairing

$$H^{2k}(S,\ \boldsymbol{Q})\times H^{2(d-k)}(S,\ \boldsymbol{Q})\longrightarrow \boldsymbol{Q}$$

is non-degenerate, we have the unique element $c_k\in H^{2k}(S,\ \boldsymbol{Q})$ satisfying ($\mathrm{Eq}'_k$), which proves the first assertion.

For $x\in H^{2(d-k)}(S,\ \boldsymbol{Q})$, using the projection formula of algebraic cycles (cf. [4, Proposition 8.3]),

$$(c_k(X/S,\ L)\cdot x)=\frac{(-1)^{k-1}}{\deg(X/S,\ L)}\Big(\Big(\sum_{i=0}^{k-1}(-1)^i c_i(X/S,\ L)\cdot f_*(L^{r+k-i})\Big)\cdot x\Big)$$
$$=\frac{(-1)^{k-1}}{\deg(X/S,\ L)}\Big(\Big(\sum_{i=0}^{k-1}(-1)^i f^*(c_i(X/S,\ L))\cdot L^{r+k-i}\Big)\cdot f^*(x)\Big),$$

which is nothing more than ($\mathrm{Eq}'_k$). □

**Remark** (2.6) In the case where the characteristic $p$ is positive, if we consider the étable cohomology over $\boldsymbol{Q}_l(l\neq p)$, we obtain the same results in Proposition (2.5).

When we change a polarization of a fiber space, the first Chern class of this changes as follows.

**Proposition** (2.7) *With notation as in Definition* (2.2), *let $D$ be a divisor on $S$ and $k$ a positive integer. Then,*

(1) $c_1(X/S,\ L+f^*(D))=c_1(X/S,\ L)+(r+1)D$.

(2) $c_1(X/S,\ kL)=kc_1(X/S,\ L)$.

*Proof.* The first assertion is easily derived from the following observation.

$$f_*((L+f^*(D))^{r+1})=f_*(L^{r+1}+(r+1)f^*(D)L^r+\cdots)$$
$$=f_*(L^{r+1})+(r+1)\deg(X/S,\ L)D\,.$$

The second is almost trivial because

$$f_*((kL)^{r+1})=k^{r+1}f_*(L^{r+1}),$$
$$\deg(X/S,\ kL)=k^r\deg(X/S,\ L)\,.$$ □

The following proposition is one of key steps to prove Theorem (5.4).

**Proposition** (2.8) *Let $f: X\to C$ be a fiber space of a non-singular projective variety $X$ to a non-singular projective curve $C$ with an $f$-ample divisor $L$ and the relative dimension $r$. Then, $L-(f^*(c_1(X/C,\ L))/r+1)$ is pseudo-effective.*

*Proof.* Set $\bar{L}=L-(f^*(c_1(X/C,\ L))/r+1)$. Let $F$ be a general fiber of $f$, $\varepsilon$ an arbitrary positive rational number and $d$ a positive integer such that $d(\bar{L}+\varepsilon F)$ is a divisor, which means that every coefficient of $d(\bar{L}+\varepsilon F)$ is an integer. Consider the Leray spectral sequence

$$E_2^{i,j}=H^i(C,\ R^jf_*\mathcal{O}_X(nd(\bar{L}+\varepsilon F)))\Longrightarrow H^{i+j}(X,\ \mathcal{O}_X(nd(\bar{L}+\varepsilon F)))\,.$$

Since $(\bar{L}+\varepsilon F)$ is $f$-ample, we have

$$R^jf_*\mathcal{O}_X(nd(\bar{L}+\varepsilon F)))=0 \qquad \text{for } j>0 \text{ and } n\gg 0\,.$$

Thus, we get

$$H^i(X,\ \mathcal{O}_X(nd(\bar{L}+\varepsilon F)))=0 \qquad \text{for } i>1 \text{ and } n\gg 0\,.$$

because $\dim C=1$. Hence,

$$h^0(X, \mathcal{O}_X(nd(\bar{L}+\varepsilon F)))\geqq \chi(X, \mathcal{O}_X(nd(\bar{L}+\varepsilon F)))>0$$

for sufficiently large integer $n$, because

$$\begin{aligned}((\bar{L}+\varepsilon F)^{r+1})&=(\bar{L}^{r+1})+\varepsilon(r+1)(\bar{L}^r\cdot F)\\&=(L^{r+1}-f^*(c_1(X/S, L))L^r)+\varepsilon(r+1)\deg(X/S, L)\\&=(f_*(L^{r+1})-c_1(X/S, L)f_*(L^r))+\varepsilon(r+1)\deg(X/S, L)\\&=\deg(X/S, L)(c_1(X/S, L)-c_1(X/S, L))+\varepsilon(r+1)\deg(X/S, L)\\&=\varepsilon(r+1)\deg(X/S, L)\\&>0.\end{aligned}$$

Therefore, $\bar{L}+\varepsilon F$ is linearly equivalent to an effective $\boldsymbol{Q}$-divisor. Thus, $\bar{L}=\lim_{\varepsilon\downarrow 0}(\bar{L}+\varepsilon F)$ is pseudo-effective. □

We shall show some examples of calculations of the first Chern classes. These calculations are based on the following lemmas.

**Lemma (2.9)** *Let $f: X\to S$ be a surjective, flat, projective and generically smooth morphism of non-singular varieties with $\dim f=1$ and $f_*\mathcal{O}_X=\mathcal{O}_S$. Then, we have*

$$f_*(K_{X/S}^2)=-12c_1(R^1f_*\mathcal{O}_X)-f_*(c_2(\Omega_{X/S}^1)).$$

*Moreover, if $R^1f_*\mathcal{O}_X$ is torsion free in codimension one, then*

$$-c_1(R^1f_*\mathcal{O}_X)=c_1(f_*\omega_{X/S}).$$

*Proof.* Let $d$ be the dimension of $S$. Applying Grothendieck's Riemann-Roch theorem (cf. [4, Theorem 15.2]) to $f$, we have

$$\begin{aligned}(2.9.1)\qquad f_*(\mathrm{td}(T_X)f^*(\mathrm{td}(T_S)^{-1}))&=f_*(\mathrm{td}(T_X))\cdot\mathrm{td}(T_S)^{-1}\\&=((\mathrm{ch}(f_*\mathcal{O}_X)-\mathrm{ch}(R^1f_*\mathcal{O}_X))\cdot\mathrm{td}(T_S))\cdot\mathrm{td}(T_S)^{-1}\\&=\mathrm{ch}(\mathcal{O}_S)-\mathrm{ch}(R^1f_*\mathcal{O}_X).\end{aligned}$$

Since $f$ is generically smooth, we obtain the exact sequence

$$0\longrightarrow f^*(\Omega_S^1)\longrightarrow\Omega_X^1\longrightarrow\Omega_{X/S}^1\longrightarrow 0,$$

which implies that

$$\mathrm{td}(T_X)f^*(\mathrm{td}(T_S)^{-1})=\mathrm{td}(\Omega_{X/S}^1)^{-1}.$$

An eary calculation shows that

$$\mathrm{td}(\Omega_{X/S}^1)^{-1}=1-\frac{1}{2}c_1(\Omega_{X/S}^1)+\frac{1}{12}(c_1(\Omega_{X/S}^1)^2+c_2(\Omega_{X/S}^1))+(\text{terms of higher codimension}).$$

Hence, considering the $A_{d-1}(S)_{\boldsymbol{Q}}$-part of (2.9.1), we get

$$f_*\left(\frac{1}{12}(c_1(\Omega^1_{X/S})^2+c_2(\Omega^1_{X/S}))\right)=-c_1(R^1f_*\mathcal{O}_X),$$

which asserts the first part of our lemma.

By Grothendieck's duality theorem (cf. [7]), we have

$$R^{\cdot}f_*\omega_{X/S}[1]=R^{\cdot}\mathcal{H}om_{\mathcal{O}_S}(R^{\cdot}f_*\mathcal{O}_X, \mathcal{O}_S),$$

which produces the spectral sequence

$$E_2^{i,j}=\mathcal{E}xt^i_{\mathcal{O}_S}(R^{-j}f_*\mathcal{O}_X, \mathcal{O}_S)\Longrightarrow E^{i+j}=R^{i+j+1}f_*\omega_{X/S}.$$

Since $E_2^{i,j}=0$ if $i<0$ or $j\neq 0, -1$, $E_\infty^{0,-1}=E_2^{0,-1}$. Hence, we obtain

$$\mathcal{H}om_{\mathcal{O}_S}(R^1f_*\mathcal{O}_X, \mathcal{O}_S)=f_*\omega_{X/S}.$$

Thus, we have the second assertion of the lemma. □

**Lemma** (2.10) *With notation as in Lemma* (2.9), *assume that there is a Zariski open set $U$ of $S$ such that* $\mathrm{codim}(S\backslash U)\geqq 2$ *and $f^{-1}(x)$ is a reduced curve with only normal crossings for every $x\in U$. Let $\Gamma$ be a reduced and irreducible divisor on $S$ and $R$ be the discrete valuation ring associated with $\Gamma$. We denote by $\sigma_\Gamma$ the number of singularities of the special fiber of $X_R$. We set $\Delta_f=\sum_\Gamma\sigma_\Gamma\Gamma$. Then, we have $f_*(c_2(\Omega^1_{X/S}))=\Delta_f$.*

*Proof.* Set $\tilde{\Delta}_U=\{x\in X_U \mid f$ is not smooth at $x\}$, which is a closed set of $X_U$. Let $\tilde{\Delta}$ be a closure of $\tilde{\Delta}_U$ in $X$. Shrinking $U$ slightly, we may assume that $\tilde{\Delta}_U$ is non-singular and $f_U(\tilde{\Delta}_U)_{\mathrm{red}}$ is also non-singular. Here we can keep the condition $\mathrm{codim}(S\backslash U)\geqq 2$. Then, we have the following exact sequence:

$$0\longrightarrow\Omega^1_{X_U/U}\longrightarrow\omega_{X_U/U}\longrightarrow\omega_{X_U/U}\otimes\mathcal{O}_{\tilde{\Delta}_U}\longrightarrow 0,$$

which shows that $c_2(\Omega_{X_U/U})=\tilde{\Delta}_U$. Therefore, we have $c_2(\Omega_{X/S})=\tilde{\Delta}+z$, where $z$ is $(d-1)$-cycle whose support is in $f^{-1}(S\backslash U)$. Thus, we get

$$f_*(c_2(\Omega^1_{X/S}))=f_*(\tilde{\Delta}+z)=f_*(\tilde{\Delta})=\Delta_f.$$ □

**Example** (2.11) Let $f: X\to S$ be a conic bundle over a non-singular projective variety $S$. Then, by Lemma (2.9) and Lemma (2.10), we get

$$c_1(X/S, -K_{X/S})=\frac{-\Delta_f}{2}.$$

**Example** (2.12) Let $f: X\to S$ be a fiber space of non-singular projective varieties with connected fibers and $F$ a general fiber of $f$. Assume that $\dim(F)=1$, the genus of $F$ is greater than or equal to 2, $f$ is generically smooth and that there is an open set $U$ of $S$ such that $\mathrm{codim}(S\backslash U)\geqq 2$ and $f^{-1}(x)$ is reduced and has only node singularities for every $x\in U$. Then, by Lemma (2.9) and Lemma (2.10), we have

$$c_1(X/S, K_{X/S})=\frac{12\det(f_*\omega_{X/S})-\Delta_f}{2(g(F)-1)}.$$

Next, we introduce the notion of semi-stability of polarized fiber spaces.

**Definition** (2.13) With notation as in Definition (2.2), assume that $X$ and $S$ are projective. Let $H$ be an ample divisor on $S$. We say the weak polarization $L$ of the fiber space $f: X \to S$ is *H-semi-stable* if for any $f$-dominant irreducible subvariety $\Gamma$ of $X$ with the relative dimension $s$, we have

$$\left(\left(L-\frac{f^*(c_1(X/S, L))}{r+1}\right)^{s+1}\cdot\Gamma\cdot f^*(H)^{d-1}\right)\geqq 0.$$

For simplicity, we say the weak polarized fiber space $(X/S, L)$ is $H$-semi-stable if the weak polarization $L$ is $H$-semi-stable.

The following proposition shows that the above semi-stability of polarized fiber spaces is one of analogies of the semi-stability of vector bundles (see Remark (2.20) for more details).

**Proposition** (2.14) *With notation as in Definition* (2.13), *the following are equivalent.*

(1) *The weak polarized fiber space* $(X/S, L)$ *is* $H$*-semi-stable.*

(2) *For any* $f$*-dominant irreducible subvariety* $\Gamma$ *of* $X$ *with the relative dimension* $s$,

$$\frac{(c_1(X/S, L)\cdot H^{d-1})}{r+1}\leqq\frac{(c_1(\Gamma/S, L)\cdot H^{d-1})}{s+1}.$$

*Proof.* Let $\Gamma$ be an $f$-dominant irreducible subvariety of $X$ with the relative dimension $s$. Using the projection formula of algebraic cycles, we get

$$\begin{aligned}
&\left(\left(L-\frac{f^*(c_1(X/S, L))}{r+1}\right)^{s+1}\cdot\Gamma\cdot f^*(H)^{d-1}\right)\\
&=\left(\left(L^{s+1}-\frac{s+1}{r+1}f^*(c_1(X/S, L))L^s+\cdots\right)\cdot\Gamma\cdot f^*(H)^{d-1}\right)\\
&=\left((L\,|\,_\Gamma)^{s+1}-\frac{s+1}{r+1}f^*(c_1(X/S, L))(L\,|\,_\Gamma)^s+\cdots)\cdot f^*(H)^{d-1}\right)\\
&=\left((f_*((L\,|\,_\Gamma)^{s+1})-\deg(\Gamma/S, L)\frac{s+1}{r+1}c_1(X/S, L))\cdot H^{d-1}\right)\\
&=(s+1)\deg(\Gamma/S, L)\left(\frac{(c_1(\Gamma/S, L)\cdot H^{d-1})}{s+1}-\frac{(c_1(X/S, L)\cdot H^{d-1})}{r+1}\right).
\end{aligned}$$

This proves our proposition. □

Later, we need the following proposition.

**Proposition** (2.15) *With notation as in Definition* (2.13), *if* $L$ *is* $f$*-nef and the weak polarization* $L$ *of* $f$ *is* $H$*-semi-stable, then we obtain that for any* $t$*-dimensional subvariety* $\Gamma$ *of* $X$,

$$\left(\left(L-\frac{f^*(c_1(X/S, L))}{r+1}\right)^{t-d+1}\cdot\Gamma\cdot f^*(H)^{d-1}\right)\geqq 0.$$

*Proof*. Clearly, we may assume that $H$ is very ample. If $\Gamma$ is $f$-dominant, the proposition is obvious by the definition of semi-stability. Set $\Delta=f(\Gamma)$. If $\operatorname{codim}\Delta\geqq 2$, $\Delta$ does not intersect with a general complete intersection curve by members of $|H|$. Hence, in this case,

$$\left(\left(L-\frac{f^*(c_1(X/S,\ L))}{r+1}\right)^{t-d+1}\cdot\Gamma\cdot f^*(H)^{d-1}\right)=0.$$

Next, we assume that $\operatorname{codim}\Delta=1$. Let $C$ be a general complete intersection curve by members of $|H|$. Then, $\Delta\cap C$ consists of points, say, $\{x_1, \cdots, x_r\}$, where $r=(\Delta\cdot C)$. Since

$$\left(L-\frac{f^*(c_1(X/S,\ L))}{r+1}\right)\bigg|_{f^{-1}(x_i)}\cong L|_{f^{-1}(x_i)}$$

is nef for each $i$, by [9, Chapter III, §2, Theorem 1] we have

$$\left(\left(L-\frac{f^*(c_1(X/S,\ L))}{r+1}\right)^{t-d+1}\cdot\Gamma\cdot f^*(H)^{d-1}\right)\geqq 0. \qquad \square$$

The following is a numerical characterization of semi-stability over a curve.

**Proposition** (**2.16**) *With notation as in Definition* (2.13), *assume that $S$ is a curve and $L$ is $f$-nef. Then, the following are equivalent.*

(1) *The weak polarized fiber space $(X/S,\ L)$ is $H$-semi-stable.*

(2) $L-\dfrac{f^*(c_1(X/S,\ L))}{r+1}$ *is nef.*

*Proof*. By Proposition (2.15), (1) implies (2). We assume (2). Let $\Gamma$ be an $f$-dominant irreducible subvariety of $X$ with the relative dimension $s$. Then, by [9, Chapter III, §2, Theorem 1], we have

$$\left(\left(L-\frac{f^*(c_1(X/S,\ L))}{r+1}\right)^{s+1}\cdot\Gamma\right)\geqq 0,$$

which proves this proposition. $\square$

The above proposition leads us to a stronger notion of semi-stability.

**Definition** (**2.17**) With notation as in Definition (2.2), assume that $X$ and $S$ are projective and that $\dim S\geqq 2$. We use the notation in (1.5). Let $H$ be a very ample divisor on $S$ and $A$ a nef Cartier divisor on $X$. We say the weak polarization $L$ of the fiber space $f: X\to S$ is *$A$-strongly $H$-semi-stable* if for any positive rational number $\varepsilon$, there is an element $m$ of $(\mathbf{Z}_{>0})^{d-1}$ such that

$$L-\frac{f^*(c_1(X/S,\ L))}{r+1}+\varepsilon A$$

is nef on the fiber space $X\times_S S(H,\ m)_\eta$ over the generic complete intersection curve $S(H,\ m)_\eta$. In the case where $A$ is numerically trivial, we say for simplicity that $L$

is *strongly H-semi-stable*. Moreover, for convenience, we say the weak polarized fiber space $(X/S, L)$ is $A$-strongly $H$-semistable if $L$ is $A$-strongly $H$-semi-stable.

First, we must show that the strong semi-stability is really stronger than the usual semi-stability.

**Proposition** (**2.18**) *With notation as in Definition* (2.17), *if the weak polarization $L$ of the fiber space $f: X \to S$ is $A$-strongly $H$-semi-stable, then for any $t$-dimensional subvariety $\Gamma$ of $X$,*

$$\left(\left(L-\frac{f^*(c_1(X/S, L))}{r+1}\right)^{t-d+1}\cdot\Gamma\cdot f^*(H)^{d-1}\right)\geqq 0,$$

*In particular, $L$ is $H$-semi-stable.*

*Proof*. First, we need the following lemma.

**Lemma** (**2.19**) *Let $X$ be a complete algebraic scheme over a field $k$ and $L$ a line bundle on $X$. Let $\bar{k}$ be the algebraic closure of $k$. Then, we have the following.*

(1) *$L$ is nef on $X$ if and only if $L_{\bar{k}}$ is nef on $X_{\bar{k}}$.*

(2) *$L$ is ample on $X$ if and only if $L_{\bar{k}}$ is ample on $X_{\bar{k}}$.*

*Proof*. Suppose $L$ is nef on $X$. Let $C$ be an irreducible curve on $X_{\bar{k}}$. Take a finite extension field $k'$ of $k$ such that $C$ is defined over $k'$. Let $\pi: X_{k'} \to X$ be the canonical morphism of $k$-scheme. Then, by [4, Proposition 2.5(c) and Example 6.2.9],

$$(L_{\bar{k}}\cdot C)=(\pi^*(L)\cdot C)=(L\cdot\pi_*(C))\geqq 0,$$

which shows 'only if' part of (1).

Assume $L_{\bar{k}}$ is nef on $C_{\bar{k}}$. Let $C$ be an irreducible curve on $X$. Then,

$$(L\cdot C)=(L_{\bar{k}}\cdot C_{\bar{k}})\geqq 0.$$

Thus we have 'if' part of (1).

Next, we assume that $L$ is ample on $X$. Take a positive integer $m$ such that $L^{\otimes m}$ is very ample. Then, it is obvious that $L_{\bar{k}}^{\otimes m}$ is also very ample on $X_{\bar{k}}$. This shows 'only if' part of (2).

Finally, we suppose that $L_{\bar{k}}$ is ample on $X_{\bar{k}}$. Let $F$ be a coherent sheaf on $X$. To show that $L$ is ample on $X$, it is sufficient to see that $H^i(X, L^{\otimes n}\otimes F)=0$ for sufficiently large $n$ and $i>0$ by the cohomological criterion of ampleness. Since $L_{\bar{k}}$ is ample on $X_{\bar{k}}$, for sufficiently large $n$ and $i>0$,

$$H^i(X_{\bar{k}}, L_{\bar{k}}^{\otimes n}\otimes_{\mathcal{O}_X}F)=0.$$

By the base change theorem, we have

$$H^i(X_{\bar{k}}, L_{\bar{k}}^{\otimes n}\otimes_{\mathcal{O}_X}F)=H^i(X, L^{\otimes n}\otimes F)\otimes_k\bar{k},$$

which shows 'if' part of (2). □

Let us start the proof of Proposition (2.18). Let $\varepsilon$ be a positive rational number.

Then, there is an element $m$ of $(\mathbf{Z}_{>0})^{d-1}$ such that

$$L-\frac{f^*(c_1(X/S,\ L))}{r+1}+\varepsilon A$$

is nef on $X\times_S S(H,\ m)_\eta$. Hence, by [9, Chapter III, §2, Theorem 1] and Lemma (2.19), we get

$$\left(\left(L-\frac{f^*(c_1(X/S,\ L))}{r+1}+\varepsilon A\right)^{t-d+1}\cdot\Gamma\right)_{X\times_S S(H,\ m)_\eta}\geqq 0.$$

Thus, we obtain

$$\left(\left(L-\frac{f^*(c_1(X/S,\ L))}{r+1}+\varepsilon A\right)^{t-d+1}\cdot\Gamma\cdot f^*(H)^{d-1}\right)\geqq 0,$$

which implies our proposition because $\varepsilon$ is arbitrary. □

**Remark** (2.20) Let $S$ be a $d$-dimensional non-singular projective variety, $H$ an ample divisor on $S$ and $E$ a vector bundle on $X$ of rank $r+1$. Let $X$ be the associated projective bundle, $L$ the tautological line bundle of $X$ and $f: X\to S$ the canonical projection. According to Mumford, $E$ is $H$-semi-stable if for any quotient $Q$ of $E$ with rank $s+1$,

$$\frac{(c_1(E)\cdot H^{d-1})}{r+1}\leqq\frac{(c_1(Q)\cdot H^{d-1})}{s+1}.$$

By Remark (2.4) and Proposition (2.14), it is easy to see that if $(X/S,\ L)$ is $H$-semi-stable in our sense, then $E$ is $H$-semi-stable in the sense of Mumford. Next we observe that Mumford's semistability of $E$ implies our semi-stability of $(X/S,\ L)$ if $S$ is defined over a field of characteristic zero. By [12, Theorem 1.6], for sufficiently large integer $m$, there is a complete intersection curve $C$ by $|mH|$ such that $E|_C$ is semi-stable. Hence by [13, Theorem 3.1],

$$L-\frac{f^*(c_1(E))}{r+1}\bigg|_{f^{-1}(C)}$$

is nef. Thus, $(X/S,\ L)$ is $H$-semi-stable. If the characteristic is positive, Mumford's semi-stability does not imply our semi-stability (see Remark (2.25)).

We give more examples.

**Example** (2.21) Let $f: X\to S$ be an elliptic fiber bundle of non-singular projective varieties, $L$ an $f$-ample divisor on $X$ and $H$ an ample divisor on $S$. We will show that $(X/S,\ L)$ is strongly $H$-semi-stable. For this purpose, we may assume that $\dim S=1$. In this case, there is an etale covering such that $\pi: S\times E\to X$, where $E$ is an elliptic curve. Since $L-(f^*(c_1(X/S,\ L))/2)$ is pseudo-effective by Proposition (2.8), $\pi^*(L-(f^*(c_1(X/S,\ L))/2))$ is also pseudo-effective. Hence, $\pi^*(L-(f^*(c_1(X/S,\ L))/2))$ is nef because the pseudo-effective cone is equal to the nef cone on $S\times E$. Hence, $L-(f^*(c_1(X/S,\ L))/2)$ is nef, which proves $(X/S,\ L)$ is $H$-semi-stable.

**Example** (2.22) Let $\Sigma_1$ be the blowing-up of $\boldsymbol{P}^2$ at a point of $\boldsymbol{P}^2$ and $f: \Sigma_1 \to \boldsymbol{P}^1$ the natural induced morphism. Let $\mu: S \to \Sigma_1$ be the blowing-up at general eight points of $\Sigma_1$. We set $g = f \cdot \mu$. Then, it is easy to see that $-K_S$ is nef and $g$-ample and that $((-K_S)^2)=0$. Hence, $(S/\boldsymbol{P}^1, -K_S)$ is $\mathcal{O}_{\boldsymbol{P}^1}(1)$-semi-stable. But $(\Sigma_1/\boldsymbol{P}^1, -K_{\Sigma_1})$ is not $\mathcal{O}_{\boldsymbol{P}^1}(1)$-semi-stable.

**Example** (2.23) Y. Fujimoto raised the following example. Here, we assume $\mathrm{char}(K) \neq 2$. Let $X$ be an Enriques surface, that is, a non-singular projective surface with $H^1(X, \mathcal{O}_X)=H^2(X, \mathcal{O}_X)=0$ and $\omega_X^{\otimes 2} \cong \mathcal{O}_X$. By [1, Theorem 17.5], there is a morphism $f: X \to \boldsymbol{P}^1$ such that the generic fiber of $f$ is elliptic. Let $F$ be a general fiber of $f$. Due to [1, Proposition 17.6], we have an irreducible curve $G$ such that $(G \cdot F) = 2$ and $(G^2) \leqq 0$. If $G$ is a (-2)-curve, $X$ is called special. In the moduli space of Enriques surfaces, general Enriques surfaces are not special. Considering the polarized fiber space $(X/\boldsymbol{P}^1, G)$, we get

(2.23.1) *$(X/\boldsymbol{P}^1, G)$ is semi-stable if and only if $X$ is not special.*

*Proof.* First, we assume that $X$ is special. Then,

$$\left(G - \frac{f^*(c_1(X/\boldsymbol{P}^1, G))}{2} \cdot G\right) = \left(G - \frac{(G^2)}{2(G \cdot F)} F \cdot G\right) = -1 .$$

Hence, $(X/\boldsymbol{P}^1, G)$ is not semi-stable. Next, we suppose that $X$ is not special. In this case, $(G^2)=0$, which implies $c_1(X/\boldsymbol{P}^1, G) \equiv 0$. Hence $(X/\boldsymbol{P}^1, G)$ is semi-stable because $G$ is nef. □

In the case where $X$ is not special, one can prove that the nef cone is equal to the pseudo-effective cone. To show this, it is sufficient to see that $(C^2) \geqq 0$ for any irreducible curve $C$ on $X$. By the adjunction formula, we have $(C^2) \geqq 0$ or $(C^2)=-2$. If $(C^2)=-2$, then $C$ is a smooth rational curve. By [3], $X$ has no smooth rational curve. Hence, we get $(C^2) \geqq 0$. Thus, for any polarization $L$, $(X/\boldsymbol{P}^1, L)$ is semi-stable because $L-(f^*(c_1(X/P^1, L))/2)$ is pseudo-effective by Proposition (2.8)

Next, we make a remark about the moduli of semi-stably polarized fiber spaces.

**Remark** (2.24) Here, we assume $\mathrm{char}(K)=0$. Let $C$ be a non-singular projective curve and $E$ a vector bundle on $C$ of rank $r+1$. Let $f: \boldsymbol{P}(E) \to C$ be the associated projective bundle and $F$ the generic fiber of $f$. We can easily see that

$$\chi(\mathcal{O}_F(n)) = \binom{n+r}{r}$$

$$\chi(\mathcal{O}_{\boldsymbol{P}(E)}(n)) = \deg(c_1(E))\binom{n+r}{r+1} + (1-g(C))\binom{n+r}{r}.$$

Hence, fixing polynomials $\chi(\mathcal{O}_F(n))$ and $\chi(\mathcal{O}_{\boldsymbol{P}(E)}(n))$ is equivalent to fixing invariants $g(C)$, $r$ and $\deg(c_1(E))$. We set polynomials

$$P(n)=\binom{n+r}{r} \quad \text{and} \quad Q(n)=d\binom{n+r}{r+1}+(1-g(C))\binom{n+r}{r}.$$

Then, the moduli space of semi-stable vector bundles $E$ over $C$ with $\deg(c_1(E))=d$ and $\mathrm{rank}(E)=r+1$ is equal to the moduli space of semi-stable fiber spaces $(\boldsymbol{P}(E)/C, \mathcal{O}_{\boldsymbol{P}(E)}(1))$ over $C$ with $\chi(\mathcal{O}_F(n))=P(n)$ and $\chi(\mathcal{O}_{\boldsymbol{P}(E)}(n))=Q(n)$.

Next, we consider more general cases. Let $f: X\to C$ be a fiber space of a non-singular projective variety $X$ to a non-singular projective curve $C$ with $\dim f=r$, $L$ an $f$-ample divisor on $X$ and $F$ a general fiber of $f$. Since

$$\deg(c_1(X/C, L))=\frac{(L^{r+1})}{\deg(X/C, L)},$$

$\chi(\mathcal{O}_F(nL))$ and $\chi(\mathcal{O}_X(nL))$ determine $\deg(c_1(X/C, L))$ and the relative dimension $r$ of $f$. Set

$$a=\max\left\{n\in\boldsymbol{Z}\,|\,n<\frac{\deg(c_1(X/C, L))}{r+1}\right\},$$

which depends only on $\deg(c_1(X/C, L))/r+1$. Then, it is easily checked that $L-aF$ is ample if $(X/C, L)$ is semi-stable and that

$$\chi(\mathcal{O}_X(n(L-aF)))=\chi(\mathcal{O}_X(nL))-an\chi(\mathcal{O}_F(nL)).$$

Let $P(n)$ and $Q(n)$ be polynomials. The above observation and Matsusaka's big theorem (cf. [10]) show us that the family of semi-stable fiber spaces $(X/C, L)$ over $C$ with $\chi(\mathcal{O}_F(nL))=P(n)$ and $\chi(\mathcal{O}_X(nL))=Q(n)$ is bounded. But we don't know at present the existence of the moduli space of this family.

Finally, we want to discuss about openness of a family of semi-stably polarized fiber spaces.

**Remark** (2.25) Let $S$ be a non-singular projective variety with an ample line bundle $H$, $T$ an algebraic scheme, $f: \mathcal{X}\to S\times T$ a morphism of algebraic schemes and $\mathcal{L}$ an $f$-ample line bundle on $\mathcal{X}$. Let $p: S\times T\to T$ be the natural projection. Assume that $p\cdot f: \mathcal{X}\to T$ is flat and that for every $t\in T$, $(\mathcal{X}_t/S\times t, \mathcal{L}_t)$ is a polarized fiber space. With this notation, there is a natural question whether the set of points $t\in T$ such that $(\mathcal{X}_t/S\times t, \mathcal{L}_t)$ is $H$-semi-stable is open. In the case of vector bundles, this was proved in [11] if the characteristic of the defining field is zero. (If the characteristic is positive, our definition of semi-stability is stronger than usual Mumford sense. Hence, we can not apply [11] to our question even if we consider the projective bundle arising from a vector bundle. For more details, see the example of this Remark.) But in our general situation, we know a little about this. First one is

(**2.25.1**) (**Valuative specialization**) *In the case where $T$ is a spectrum of a discrete valuation ring, if $(\mathcal{X}/S\times T, \mathcal{L})$ is not $H$-semi-stable at the generic point of $T$, then $(\mathcal{X}/S\times T, \mathcal{L})$ is not $H$-semi-stable at the special point of $T$.*

*Proof of Valuative specialization.* Let $\eta$ be the generic point and $o$ the special point of $T$. Since $(\mathcal{X}_\eta/S\times\eta, \mathcal{L}_\eta)$ is not $H$-semi-stable, there is an $f_\eta$-dominant subvariety $\Gamma$ such that

$$\left(\left(\mathcal{L}-\frac{f^*(c_1(\mathcal{X}/S\times T, \mathcal{L}))}{r+1}\right)^{t-d+1}\cdot\Gamma\cdot f^*(H)^{d-1}\right)_{\mathcal{X}_\eta}<0,$$

where $r=\dim f$, $d=\dim S$ and $t=\dim \Gamma$. Take a subvariety $\Gamma'$ of $\mathcal{X}$ such that $\Gamma'_\eta=\Gamma$. Since $\Gamma'$ is flat over $T$, we have

$$\left(\left(\mathcal{L}-\frac{f^*(c_1(\mathcal{X}/S\times T, \mathcal{L}))}{r+1}\right)^{t-d+1}\cdot\Gamma'_o\cdot f^*(H)^{d-1}\right)_{\mathcal{X}_0}<0,$$

which shows $(\mathcal{X}_0/S\times o, \mathcal{L}_0)$ is not $H$-semi-stable by Proposition (2.15). □

In the case where $\dim S=1$, we have the following.

(**2.25.2**) *Assume that* $\dim S=1$, $T$ *is irreducible and that* $(\mathcal{X}/S\times T, \mathcal{L})$ *is semi-stable at the generic point of* $T$. *Then, there are countably many dense Zariski open sets* $\{U_n\}_{n=1,2,\dots}$ *of* $T$ *such that if* $x\in\bigcap_{n=1}^\infty U_n$, *then* $(\mathcal{X}/S\times T, \mathcal{L})$ *is semi-stable at* $x$.

*Proof.* By the numerical characterization of semi-stability (Proposition (2.16)), it is sufficient to show the following.

(**2.25.3**) *Let* $f: X\to S$ *be a projective and surjective morphism of an algebraic scheme* $X$ *to an algebraic variety* $S$ *and* $L$ *a line bundle on* $X$. *If* $L$ *is nef on the generic fiber* $X_\eta$ *of* $f$, *there are countably many dense Zariski open sets* $\{U_n\}_{n=1,2,\dots}$ *such that if* $s\in\bigcap_{n=1}^\infty U_n$, $L$ *is nef on* $X_s$.

*Proof.* Let $A$ be an $f$-ample line bundle on $X$. Take a positive integer $n$. Since $L$ is nef on $X_\eta$, $nL+A$ is ample on $X_\eta$ by Lemma (2.19) and [9, Chapter IV, §2, Theorem 1]. Hence, there is a dense Zariski open set $U_n$ such that if $s\in U_n$, $nL+A$ is ample on $X_s$. Thus, if $s\in\bigcap_{n=1}^\infty U_n$, $nL+A$ is ample on $X_s$ for all $n$, which implies that $L$ is nef on $X_s$. □

Finally, we give a negative example to the above question in positive characteristic. First of all, we note the following fact, which is easily proved if we refer to [13, Proof of Theorem 3.1].

(**2.25.4**) *Let* $C$ *be a non-singnlar projective curve over an algebraically closed field of positive characteristic and* $E$ *a vector bundle on* $C$. *Let* $X$ *be the projective bundle of* $E$, $L$ *the tautological line bundle of* $X$ *and* $f: X\to C$ *the natural projection. Let* $F$ *be the Frobenius morphism over* $C$. *Then, the following are equivalent.*

(1) $(X/C, L)$ *is semi-stable in our sense.*

(2) $(F^n)^*(E)$ *is semi-stable in the sense of Mumford for all* $n\geqq 0$.

By [5], for each prime $p$ and integer $g>1$, there is a non-singular projective curve $C$ of genus $g$ in characteristic $p$ and a sequence of vector bundles $\{E_n\}_{n=1,2,\dots}$ so that

(1) $E_n$ is of rank 2 and of degree 0,
(2) $E_n$ is semi-stable in the sense of Mumford,
(3) $F^*(E_n)$ is isomorphic to $E_{n-1}$ and
(4) $F^*(E_1)$ is not semi-stable in the sense of Mumford.

We can take a quasi-projective scheme $T$ and a vector bundle $\mathcal{E}$ of rank 2 on $C\times T$ such that for all semi-stable vector bundle $E$ on $C$ with rank 2 and degree 0, there is a point $t\in T$ such that $E$ is isomorphic to $\mathcal{E}_t$. Let $\mathcal{X}$ be the projective bundle of $\mathcal{E}$, $\mathcal{L}$ the tautological line bundle of $\mathcal{X}$ and $f:\mathcal{X}\to C\times T$ the natural projection. We set

$$U_n=\{t\in T\,|\,(F^n)^*(\mathcal{E}_t)\text{ is semi-stable in the sense of Mumford.}\},$$

which is open by [11]. Clearly, $U_{n+1}\subseteqq U_n$. But the above sequence $\{E_n\}$ shows that $U_{n+1}\neq U_n$. Hence, $\bigcap_n U_n$ is not open. On the other hand, by (2.25.4), for $t\in T$, $(\mathcal{X}_t/C\times t,\ \mathcal{L}_t)$ is semi-stable in our sense if and only if $t\in\bigcap_n U_n$. Therefore, the openness of a family of semi-stably polarized fiber spaces does not hold in this example.

## 3. Inequality of Bogomolov-Gieseker's type

In this section, we shall prove a generalized Bogomolov-Gieseker's inequality for a fiber space with a strongly stable polarization.

**Theorem** (3.1) *Let $f: X\to S$ be a fiber space over a d-dimensional non-singular projective variety with* $\dim f=r$ *and* $\dim S\geqq 2$. *Let $L$ be an f-ample Cartier divisor on $X$, $A$ a nef Cartier divisor on $X$ and $H$ a very ample divisor on $S$. Assume the polarized fiber space $(X/S, L)$ is $A$-strongly $H$-semi-stable. Then, we have the following inequality.*

$$r(c_1^2(X/S,\ L)\cdot H^{d-2})\leqq 2(r+1)(c_2(X/S,\ L)\cdot H^{d-2}).$$

*Proof*. Replacing $L$ by a multiple of $L$, we may assume that $c_1(X/S,\ L)/r+1$ is represented by a divisor $M$ on $S$ using formulae in Proposition (2.7). Moreover, adding some ample divisor to $A$, we may assume that $A$ is ample and that for any positive rational number $\varepsilon$, there is an element $m=(m_1,\cdots,m_{d-1})$ of $(\boldsymbol{Z}_{>0})^{d-1}$ such that

$$L-\frac{f^*(c_1(X/S,\ L))}{r+1}+\varepsilon A$$

is ample on $X\times_S S(H,\ m)_\eta$ by Proposition (2.19) and [9, Chapter IV, § 2, Theorem 1]. We fix $\varepsilon$ and $m$ for a moment. Let $t$ be a positive integer such that $t\varepsilon$ is integer. Let $H_1,\cdots,H_{d-1}$ be general members of $|m_1H|,\cdots,|m_{d-1}H|$ respectively. Set $T=H_2\cap\cdots\cap H_{d-1}$ and $C=H_1\cap H_2\cdots\cap H_{d-1}$. Then,

$$\left(L-\frac{f^*(c_1(X/S,\ L))}{r+1}+\varepsilon A\right)\Big|_{f^{-1}(C)}$$

is ample. We set $\bar{L}=L-f^*(M)$, $W=f^{-1}(T)$ and $V=f^{-1}(C)$.

$$\begin{array}{ccccc} V & \subset & W & \subset & X \\ \downarrow & & \downarrow & & \downarrow \\ C & \subset & T & \subset & S \end{array}$$

**Claim** (3.1.1) *There are positive integers $T_1$ and $T_2$ such that*

$$H^0(W, \mathcal{O}_W(n(\bar{L}-T_1 f^*(H))))=0 \quad \textit{and} \quad H^2(W, \mathcal{O}_W(n(\bar{L}+T_2 f^*(H))))=0$$

*for all sufficiently large integer $n$.*

Since $((\bar{L}+\varepsilon A)^r \cdot f^*(H)^d)>0$, there is a positive integer $T_1$ such that

$$((\bar{L}-T_1 f^*(H))\cdot(\bar{L}+\varepsilon A)^r \cdot f^*(H)^{d-1})<0 .$$

Hence, $H^0(W, \mathcal{O}_W(n(\bar{L}-T_1 f^*(H))))=0$ by the same argument as in the proof of Proposition (2.18). Because of $f$-ampleness of $\bar{L}$, there is a positive integer $T_2$ such that $\bar{L}+T_2 f^*(H)$ is ample. Thus the second assertion is an immediate consequence of Serre's vanishing theorem.

**Claim** (3.1.2) $H^i(W, \mathcal{O}_W(n\bar{L}))=0$ *for $i>2$ and $n\gg 0$.*

Since $R^i f_* \mathcal{O}_W(n\bar{L})=0$ for $i>0$ and $n\gg 0$ by $f$-ampleness of $\bar{L}$, the Leary spactral sequence

$$E_2^{ij}=H^i(T, R^j f_* \mathcal{O}_W(n\bar{L})) \Longrightarrow H^{i+j}(W, \mathcal{O}_W(n\bar{L}))$$

is degenerate. Thus $H^i(W, \mathcal{O}_W(n\bar{L}))=H^i(T, f_*\mathcal{O}_W(n\bar{L}))$. Therefore we have our claim because $\dim T=2$.

**Claim** (3.1.3)

$$\frac{h^0(W, \mathcal{O}_W(nt\bar{L}))}{(nt)^{r+2}} \leqq \varepsilon m_2 \cdots m_{d-1}(T_1+1)\left(\frac{\sum_{i=1}^{r+1} \varepsilon^{i-1}\binom{r+1}{i}(\bar{L}^{r+1-i}\cdot A^i \cdot f^*(H)^{d-1})}{(r+1)!}+\frac{1}{(r+1)!}\right)$$

*for sufficiently large integer $n$.*

The short exact sequence

$$0 \longrightarrow \mathcal{O}_W(nt\bar{L}-(k+1)V) \longrightarrow \mathcal{O}_W(nt\bar{L}-kV) \longrightarrow \mathcal{O}_V(nt\bar{L}-kV) \longrightarrow 0$$

gives rise to a left-exact sequence

$$0 \longrightarrow H^0(W, \mathcal{O}_W(nt\bar{L}-(k+1)V)) \longrightarrow H^0(W, \mathcal{O}_W(nt\bar{L}-kV)) \longrightarrow H^0(V, \mathcal{O}_V(nt\bar{L}-kV)),$$

which provides us with an inequality

$$h^0(W, \mathcal{O}_W(nt\bar{L}-kV))-h^0(W, \mathcal{O}_W(nt\bar{L}-(k+1)V))\leqq h^0(V, \mathcal{O}_V(nt\bar{L}-kV)).$$

Taking the summation from $k=0$ to $k=\lceil ntT_1/m_1\rceil-1$ of the above inequality, we have

$$h^0(W, \mathcal{O}_W(nt\bar{L}))-h^0\left(W, \mathcal{O}_W\left(nt\bar{L}-\left\lceil\frac{ntT_1}{m_1}\right\rceil V\right)\right)\leqq \sum_{k=0}^{\lceil ntT_1/m_1\rceil-1} h^0(V, \mathcal{O}_V(nt\bar{L}-kV))$$

$$\leqq \left\lceil\frac{ntT_1}{m_1}\right\rceil h^0(V, \mathcal{O}_V(nt\bar{L})).$$

Since $\lceil ntT_1/m_1\rceil \geqq ntT_1/m_1$, by Claim (3.1.1) we obtain

$$h^0(W, \mathcal{O}_W(nt\bar{L})) \leqq \frac{nt(T_1+1)}{m_1} h^0(V, \mathcal{O}_V(nt(\bar{L}+\varepsilon A))) \tag{3.1.4}$$

for sufficiently large $n$. On one hand, since $\bar{L}+\varepsilon A$ is ample on $V$,

$$\begin{aligned} h^0(V, \mathcal{O}_V(nt(\bar{L}+\varepsilon A))) &= \chi(V, \mathcal{O}_V(nt(\bar{L}+\varepsilon A))) \\ &= \frac{((\bar{L}+\varepsilon A)^{r+1})_V}{(r+1)!}(nt)^{r+1} + (\text{lower terms}) \\ &= m_1 \cdots m_{d-1} \frac{((\bar{L}+\varepsilon A)^{r+1} \cdot f^*(H)^{d-1})}{(r+1)!}(nt)^{r+1} + (\text{lower terms}) \\ &\leqq m_1 \cdots m_{d-1} \frac{((\bar{L}+\varepsilon A)^{r+1} \cdot f^*(H)^{d-1}) + \varepsilon}{(r+1)!}(nt)^{r+1} \end{aligned}$$

for sufficiently large $n$. On the other hand, we have

$$\begin{aligned} (\bar{L}^{r+1} \cdot f^*(H)^{d-1}) &= ((L^{r+1} - L^r \cdot f^*(c_1(X/S, L)) + \cdots) \cdot f^*(H)^{d-1}) \\ &= ((f_*(L^{r+1}) - f_*(L^r) \cdot c_1(X/S, L)) \cdot H^{d-1}) \\ &= \deg(X/S, L)((c_1(X/S, L) - c_1(X/S, L)) \cdot H^{d-1}) \\ &= 0 . \end{aligned}$$

Hence, we get

$$\begin{aligned} & h^0(V, \mathcal{O}_V(nt(\bar{L}+\varepsilon A))) \\ & \leqq \varepsilon m_1 \cdots m_{d-1} \frac{\sum_{i=1}^{r+1} \varepsilon^{i-1} \binom{r+1}{i} (\bar{L}^{r+1-i} \cdot A^i \cdot f^*(H)^{d-1}) + 1}{(r+1)!}(nt)^{r+1} \end{aligned} \tag{3.1.5}$$

for sufficiently large $n$. Thus, by (3.1.4) and (3.1.5), we have Claim (3.1.3).

**Claim (3.1.6)**

$$\frac{h^2(W, \mathcal{O}_W(nt\bar{L}))}{(nt)^{r+2}} \leqq \varepsilon m_2 \cdots m_{d-1}(T_2+1)\left(\frac{\sum_{i=1}^{r+1} \varepsilon^{i-1} \binom{r+1}{i} (\bar{L}^{r+1-i} \cdot A^i \cdot f^*(H)^{d-1})}{(r+1)!} + \frac{1}{(r+1)!}\right)$$

*for sufficiently large integer $n$.*

The proof of Claim (3.1.6) is almost the same as that of Claim (3.1.3). But details are slightly different. By the same argument as in Claim (3.1.2), we see

$$H^i(V, \mathcal{O}_V(nt(\bar{L}+kV))) = 0 \quad i>1 \quad \text{and} \quad n \gg 0 .$$

Hence, the short exact sequence

$$0 \longrightarrow \mathcal{O}_W(nt\bar{L}+kV) \longrightarrow \mathcal{O}_W(nt\bar{L}+(k+1)V) \longrightarrow \mathcal{O}_V(nt\bar{L}+(k+1)V) \longrightarrow 0$$

gives rise to a right-exact sequence

$$H^1(V, \mathcal{O}_V(nt\bar{L}+(k+1)V)) \to H^2(W, \mathcal{O}_W(nt\bar{L}+kV)) \to H^2(W, \mathcal{O}_W(nt\bar{L}+(k+1)V)) \to 0 ,$$

which shows an inequality

$$h^2(W, \mathcal{O}_W(nt\bar{L}+kV))-h^2(W, \mathcal{O}_W(nt\bar{L}+(k+1)V))\leqq h^1(V, \mathcal{O}_V(nt\bar{L}+(k+1)V)).$$

Taking the summation from $k=0$ to $k=\lceil ntT_2/m_1\rceil-1$ of the above inequality, we have

$$h^2(W, \mathcal{O}_W(nt\bar{L}))-h^2\left(W, \mathcal{O}_W\left(nt\bar{L}+\left\lceil\frac{ntT_2}{m_1}\right\rceil V\right)\right)\leqq \sum_{k=0}^{\lceil ntT_2/m_1\rceil-1} h^1(V, \mathcal{O}_V(nt\bar{L}+(k+1)V)).$$

Since $\lceil ntT_2/m_1\rceil \geqq ntT_2/m_1$, Claim (3.1.1) implies $H^2(W, \mathcal{O}_W(nt\bar{L}+\lceil ntT_2/m_1\rceil V))=0$. Let $F$ be a general member of $|(k+1)V|_V|$. Clearly, $F$ is a sum of general fibers of $f|_V$. Hence, considering the exact sequence

$$0 \longrightarrow \mathcal{O}_V(nt\bar{L}) \longrightarrow \mathcal{O}_V(nt\bar{L}+(k+1)V) \longrightarrow \mathcal{O}_F(nt\bar{L}) \longrightarrow 0,$$

we have

$$h^1(V, \mathcal{O}_V(nt\bar{L}))\geqq h^1(V, \mathcal{O}_V(nt\bar{L}+(k+1)V)).$$

Thus we get

$$\begin{aligned} h^2(W, \mathcal{O}_W(nt\bar{L}))&\leqq\left\lceil\frac{ntT_2}{m_1}\right\rceil h^1(V, \mathcal{O}_V(nt\bar{L})) \\ &\leqq\frac{nt(T_2+1)}{m_1} h^1(V, \mathcal{O}_V(nt\bar{L})) \\ &=\frac{nt(T_2+1)}{m_1}(h^0(V, \mathcal{O}_V(nt\bar{L}))-\chi(V, \mathcal{O}_V(nt\bar{L}))) \\ &\leqq\frac{nt(T_2+1)}{m_1}(h^0(V, \mathcal{O}_V(nt(\bar{L}+\varepsilon A)))-\chi(V, \mathcal{O}_V(nt\bar{L}))) \end{aligned}$$

for sufficiently large integer $n$. Finally noting that the degree of $\chi(V, \mathcal{O}_V(nt\bar{L}))$ with respect to $n$ is $r$ at most, we conclude our Claim by (3.1.5).

Let us continue the proof of Theorem (3.1). By Claim (3.1.3) and Claim (3.1.6), we have

$$\begin{aligned} \frac{(\bar{L}^{r+2}\cdot f^*(H)^{d-2})}{(r+2)!}&=\frac{(\bar{L}^{r+2})_W}{m_2\cdots m_{d-1}\cdot(r+2)!} \\ &=\lim_{n\to\infty}\frac{\chi(X_m, \mathcal{O}_{X_m}(nt\bar{L}))}{m_2\cdots m_{d-1}\cdot(nt)^{r+2}} \\ &\leqq\limsup_{n\to\infty}\frac{h^0(W, \mathcal{O}_W(nt\bar{L}))+h^2(W, \mathcal{O}_W(nt\bar{L}))}{m_2\cdots m_{d-1}\cdot(nt)^{r+2}} \\ &\leqq\varepsilon(T_1+T_2+2)\left(\frac{\sum_{i=1}^{r+1}\varepsilon^{i-1}\binom{r+1}{i}(\bar{L}^{r+1-i}\cdot A^i\cdot f^*(H)^{d-1})}{(r+1)!}+\frac{1}{(r+1)!}\right). \end{aligned}$$

Since $\varepsilon$ is arbitrary, we get $(\bar{L}^{r+2}\cdot f^*(H)^{d-2})\leqq 0$. On the other hand, since

$$\frac{f_*(L^{r+2})}{\deg(X/S, L)}=c_1^2(X/S, L)-c_2(X/S, L),$$

we get

$$
\begin{aligned}
&(\bar{L}^{r+2}\cdot f^*(H)^{d-2})\\
&=\left(\left(L^{r+2}-\frac{r+2}{r+1}L^{r+1}f^*(c_1(X/S,\ L))+\frac{r+2}{2(r+1)}L^r f^*(c_1(X/S,\ L))^2+\cdots\right)\cdot f^*(H)^{d-2}\right)\\
&=\left(\left(f_*(L^{r+2})-\frac{r+2}{r+1}f_*(L^{r+1})c_1(X/S,\ L)+\frac{r+2}{2(r+1)}f_*(L^r)c_1^2(X/S,\ L)\right)\cdot H^{d-2}\right)\\
&=\deg(X/S,\ L)\left(\left(\frac{f_*(L^{r+2})}{\deg(X/S,\ L)}-\frac{r+2}{2(r+1)}c_1^2(X/S,\ L)\right)\cdot H^{d-2}\right)\\
&=\deg(X/S,\ L)\left(\frac{r}{2(r+1)}c_1^2(X/S,\ L)\cdot H^{d-2}-c_2(X/S,\ L)\cdot H^{d-2}\right).
\end{aligned}
$$

This completes the proof of Theorem (3.1). □

An immediate corollary of Theorem (3.1) for $\dim f=0$ is the following.

**Corollary** (3.2) *Let $f: S'\to S$ be a generically finite fiber space over a non-singular projective surface $S$. Let $L$ be an $f$-ample Cartier divisor on $S'$ and $H$ an ample divisor on $S$. Assume that $(S'/S,\ L)$ is $H$-semi-stable. Then, $(S'/S,\ L)$ is strongly $H$-semi-stable. In particular, $(c_2(S'/S,\ L))\geqq 0$.*

*Proof.* Let $S'_{\text{red}}=\sum S_i$ be the irreducible decomposition of the reduced structure $S'_{\text{red}}$ of $S'$ and $f_i: S_i\to S$ the induced morphism. By Bertini's theorem, for sufficiently large integer $m$, $|mH|$ contains an member $\Gamma$ such that $\Gamma_i=f_i^{-1}(\Gamma)$ are irreducible for all $i$. To see that $L-f^*(c_1(S'/S,\ L))$ is nef on $f^{-1}(\Gamma)$, it is sufficient to prove that

$$(L-f^*(c_1(S'/S,\ L))\cdot\Gamma_i)\geqq 0\,.$$

The definition of semi-stability implies that

$$(L-f^*(c_1(S'/S,\ L))\cdot\Gamma_i)=(L-f^*(c_1(S'/S,\ L))\cdot S_i\cdot f^*(mH))\geqq 0\,.$$

Hence, we have our corollary. □

The Hodge index theorem is derived from our inequality.

**Corollary** (3.3) (**Hodge index theorem**) *Let $S$ be a non-singular projective surface and* NS($S$) *the Neron-Severi group of $S$. Then, the signature of the intersection pairing on* NS$(S)\otimes\boldsymbol{Q}$ *is* $(1,\ \rho-1)$, *where $\rho$ is the rank of* NS($S$).

*Proof.* Since the intersection pairing on NS$(S)\otimes\boldsymbol{Q}$ is non-degenerate, the signature of this must have type $(a,\ b)$ with $a+b=\rho$. Hence, we can take a basis $x_1,\cdots,x_\rho$ NS$(S)\otimes\boldsymbol{Q}$ such that

(1) $x_1$ is ample,
(2) $(x_i\cdot x_j)=0$ for $i\neq j$,
(3) $(x_i^2)>0$ for $i=1,\cdots,a$ and
(4) $(x_j^2)<0$ for $j=a+1,\cdots,\rho$.

Therefore, our corollary is deduced directly from the following.

**Lemma (3.4)** *Let $S$ be a non-singular projective surface. Let $H$ be an ample divisor on $S$ and $D$ a divisor such that $(H\cdot D)=0$. Then, $(D^2)\leqq 0$.*

*Proof.* Let $S_1$ and $S_2$ be two copies of $S$. We set $S'=S_1\amalg S_2$. Let $f: S'\to S$ be the natural morphism. We define $L$ as follow.

$$L=\begin{cases} D & \text{on } S_1 \\ 0 & \text{on } S_2. \end{cases}$$

Then, it is easy to see that $c_1(S'/S, L)=(D/2)$ and $(c_2(S'/S, L))=-(D^2)/4$. Since

$$(L-f^*(c_1(S'/S, L))\cdot f^*(H))_{S_1}=(L-f^*(c_1(S'/S, L))\cdot f^*(H))_{S_2}=0,$$

$(S'/S, L)$ is $H$-semi-stable. Hence, by Corollary (3.2), we get

$$(c_2(S'/S, L))=\frac{-(D^2)}{4}\geqq 0. \qquad \square$$

## 4. Weil's lemma for flat fiber spaces

Let $f: X\to S$ be a flat fiber space of non-singular projective varieties with $f_*\mathcal{O}_X=\mathcal{O}_S$ and $H$ a very ample divisor on $S$. Let $s$ be a positive integer with $1\leqq s<\dim S$ and $m=(m_1, \cdots, m_s)$ an element of $(\boldsymbol{Z}_{>0})^s$. We use the notation in (1.5). Set $S_m=S(H, m)$, $(S_m)_\eta=S(H, m)_\eta$ and $T_m=|m_1H|\times\cdots\times|m_sH|$. Let $p_m: S_m\to S$ and $q_m: S_m\to T_m$ be the natural projections. Set $X_m=X\times_S S_m$ and $(X_m)_\eta=X\times_S(S_m)_\eta$. Furthermore, we set the induced morphisms as follows.

$$\begin{array}{ccccc}
X & \xleftarrow{\tilde{p}_m} & X_m & \xleftarrow{\tilde{\tau}_m} & (X_m)_\eta \\
\downarrow f & & \downarrow f_m & & \downarrow (f_m)_\eta \\
S & \xleftarrow{p_m} & S_m & \xleftarrow{\tau_m} & (S_m)_\eta \\
 & & \downarrow q_m & & \\
 & & T_m & &
\end{array}$$

$$g_m=q_m\cdot f_m, \qquad \lambda_m=p_m\cdot\tau_m, \qquad \tilde{\lambda}_m=\tilde{p}_m\cdot\tilde{\tau}_m.$$

It is easy to see that $S_m$ is non-singular and $p_m$ is smooth. With this notation, we will prove the following lemma in this section.

**Lemma (4.1) (Weil's lemma)** *If* $\dim S\geqq 2$, *$f$ is generically smooth and $m_i\geqq 3$ for all $1\leqq i\leqq s$, then the homomorphism*

$$\tilde{\lambda}_m^*: \mathrm{Pic}(X)\longrightarrow \mathrm{Pic}((X_m)_\eta)$$

*is isomorphic.*

*Proof.* First, we need:

**Sublemma (4.1.1)** *Assume $s=1$. If we set $A_m=\{x\in T_m \mid g_m^{-1}(x)$ is not integral$\}$, then* $\operatorname{codim} A_m\geqq 2$.

*Proof.* Let $E$ be a reduced effective divisor on $S$ such that $f$ is smooth over $S\setminus E$. Let $\Gamma$ be a reduced and irreducible divisor such that $\Gamma\nsubseteq E$. Let $f^{-1}(\Gamma)=\sum a_i E_i$ be the irreducible decomposition of $f^{-1}(\Gamma)$ and $\gamma$ the generic point of $\Gamma$. Flatness of $f$ shows that $f(E_i)=\Gamma$. Hence, $f^{-1}(\gamma)\cap E_i\neq\emptyset$ for all $i$. Thus, $f^{-1}(\Gamma)$ must be reduced and irreducible because $f^{-1}(\gamma)$ is smooth and connected. This proves that

$$A_m\subseteqq\{x\in T_m \mid q_m^{-1}(x) \text{ is not integral}\}\cup\{x\in T_m \mid q_m^{-1}(x)\subseteqq E\}$$

Since $m_1\geqq 3$, due to [12, 2.1.3 iii)], $\operatorname{codim}\{x\in T_m \mid q_m^{-1}(x) \text{ is not integral}\}\geqq 2$, which asserts our corollary. □

Next We need:

**Sublemma (4.1.2)** $\operatorname{Pic}(X_m)=\tilde{p}_m^*(\operatorname{Pic}(X))\oplus g_m^*(\operatorname{Pic}(T_m))$.

*Proof.* Let $L$ be a line bundle on $X_m$. Since $\tilde{p}_m^{-1}(x)$ is a product of hyperplanes in the projective spaces $|m_iH|$ for $x\in X$, there is a line bundle $M_1$ on $T_m$ such that $L\otimes g_m^*(M_1^{-1})$ is trivial for all fibers of $\tilde{p}_m$. Hence, $L\otimes g_m^*(M_1^{-1})\cong\tilde{p}_m^*(M_2)$ for some line bundle $M_2$ on $X$, which proves that $\operatorname{Pic}(X_m)$ is spanned by $\tilde{p}_m^*(\operatorname{Pic}(X))$ and $g_m^*(\operatorname{Pic}(T_m))$.

Next, we assume that $\mathcal{O}_{X_m}\cong\tilde{p}_m^*(M_1)\otimes g_m^*(M_2)$ for some $M_1\in\operatorname{Pic}(X)$ and $M_2\in\operatorname{Pic}(T_m)$. Then, $g_m^*(M_2)|_{\tilde{p}_m^{-1}(x)}$ is trivial for all $x\in X$. Hence, $M_2$ must be trivial by the same reason as above. Therefore, we have $M_1\cong\mathcal{O}_X$ because $(\tilde{p}_m)_*\mathcal{O}_{X_m}\cong\mathcal{O}_X$. □

Now let us start the proof of Lemma (4.1).

**(4.1.3) (The surjectivity of $\tilde{\lambda}_m^*$)** Let $L$ be a line bundle on $(X_m)_\eta$. Since any line bundle on $X_m$ can be represented by a divisor, there is a line bundle $L'$ on $X_m$ such that $\tilde{\tau}_m^*(L')=L$. By Sublemma (4.1.2), we have $M_1\in\operatorname{Pic}(X)$ and $M_2\in\operatorname{Pic}(T_m)$ such that $L'\cong\tilde{p}_m^*(M_1)\otimes g_m^*(M_2)$. Then,

$$\tilde{\lambda}_m^*(M_1)\cong\tilde{\tau}_m^*(L'\otimes g_m^*(M_2^{-1}))\cong\tilde{\tau}_m^*(L')\cong L$$

**(4.1.4) (The injectivity of $\tilde{\lambda}_m^*$)** Let $L$ be a line bundle on $X$ such that $\tilde{\lambda}_m^*(L)\cong\mathcal{O}_{(X_m)_\eta}$. First, we assume $s=1$. Since $\tilde{p}_m^*(L)$ is trivial on the generic fiber of $g_m$, using the upper semi-continuity of cohomology, the set

$$\{x\in T_m \mid h^0(g_m^{-1}(x),\ \tilde{p}_m^*(L))>0, \qquad h^0(g_m^{-1}(x),\ \tilde{p}_m^*(L^{-1}))>0\}$$

is equal to $T_m$. Noting the fact (cf. [16, p 54, Proof of Corollary 6]) that for a proper integral algebraic scheme $V$ over a field and a line bundle $M$ on $V$, if $h^0(V, M)>0$ and $h^0(V, M^{-1})>0$, then $M\cong\mathcal{O}_V$, we get $\tilde{p}_m^*(L)|_{g_m^{-1}(x)}\cong\mathcal{O}_{g_m^{-1}(x)}$ for $x\in T_m\setminus A_m$. Thus, if we set $M=((g_m)_*(\tilde{p}_m^*(L)))^{\vee\vee}$, then $\tilde{p}_m^*(L)$ is isomorphic to $g_m^*(M)$ on $X_m\setminus g_m^{-1}(A_m)$ (see [16, p 54, Proof of Corollary 6]). Since $s=1$, $q_m$ is flat. Hence, $g_m$ is also flat. By Sublemma (4.1.1) and flatness of $g_m$, $\operatorname{codim}(g_m^{-1}(A_m))\geqq 2$, which implies $\tilde{p}_m^*(L)\cong g_m^*(M)$ because $M$ is a line bundle. Thus, we complete the proof of (4.1.4) in the case where $s=1$.

For $s>1$, we proceed by induction on $s$. For this purpose, note the following fact (cf. [12, Lemma 2.1.2]).

**(4.1.5)** Let $h: V\to Z$ be a proper flat morphism of irreducible varieties such that all fibers are integral. Let $M$ be a line bundle on $V$. Then, the following are equivalent:

(1) $M$ is trivial on the generic fiber of $h$.

(2) $M$ is trivial on all geometric fibers of $h$ over a non-empty Zariski open set of $Z$.

(3) $M$ is trivial on all the geometric fibers of $h$.

(4) There is a line bundle $R$ on $Z$ such that $M\cong h^*(R)$.

Let $\Gamma$ be a general member of $|m_1 H|$. Then, by the hypothesis of induction, $L|_{f^{-1}(\Gamma)}$ is trivial. Hence, $L$ must be trivial using injectivity of the case $s=1$. $\square$

## 5. Restriction of semi-stable fiber spaces to curves

This section is devoted to proving Mumford-Mehta-Ramanathan's type theorem. Throughout this section, we use the notation in §4 freely and assume that the characteristic of the ground field $K$ is zero.

**Definition (5.1)** Let $d$ be a positive integer with $d\geqq 2$. Let $X\to S$ be a flat and surjective morphism from a $(d+1)$-dimensional non-singular projective variety $X$ to a $d$-dimensional nonsingular variety $S$ with $f_*\mathcal{O}_X=\mathcal{O}_S$. Let $L$ be an $f$-ample divisor on $X$ and $H$ a very ample divisor on $S$. For $m\in(\boldsymbol{Z}_{>0})^{d-1}$, we set

$$\nu_m(X/S, L, H)$$
$$=\begin{cases} 0, & \text{if } L-\dfrac{f^*(c_1(X/S, L))}{2} \text{ is nef on } (X_m)_\eta, \\ \min\left\{\deg(\Delta\to(S_m)_\eta)\,\middle|\, \begin{array}{l} \Delta \text{ is an irreducible curve on } (X_m)_\eta \text{ with} \\ \left(L-\dfrac{f^*(c_1(X/S, L))}{2}\cdot\Delta\right)_{(X_m)_\eta}<0. \end{array}\right\}, & \text{otherwise.}\end{cases}$$

We say $(X/S, L)$ is *$H$-bounded* if $\{\nu_m(X/S, L, H)\}_{m\in(Z_{>0})^{d-1}}$ is bounded.

We divide polarized fiber spaces into two cases.

(1) $(X/S, L)$ is $H$-bounded.

(2) $(X/S, L)$ is not $H$-bounded.

First, we treat the case (2).

**Proposition (5.2)** *With notation as in Definition (5.1), if $(X/S, L)$ is not $H$-bounded, then $(X/S, L)$ is $f^*(H)$-strongly $H$-semi-stable.*

*Proof.* We need the following Lemma.

**Lemma (5.3)** *Let $f: V\to C$ be a surjective morphism of a non-singular projective surface $V$ to a non-singular curve $C$ with all fibers connected. Let $F$ be a general fiber of $f$ and $L$ a divisor on $V$ such that $(L\cdot F)>0$. Assume that $L-K_{V/C}$ is nef. Then, for any $f$-dominant reduced and irreducible curve $\Delta$,*

$$\left(L-\frac{(L^2)}{2(L\cdot F)}F\cdot\Delta\right)\geqq\frac{-(\Delta\cdot F)(L^2)}{2(2(\Delta\cdot F)+(L\cdot F))}.$$

*In particular, if* $(L^2)\geqq 0$,

$$\left(L-\frac{(L^2)}{2(L\cdot F)}F\cdot\varDelta\right)\geqq\frac{-(L^2)}{4}.$$

*Proof.* The essential idea to get our inequality is found in [18]. Let $\iota:\tilde{\varDelta}\to\varDelta$ be the normalization of $\varDelta$. Then by Hurwitz formula,

$$\begin{aligned}(L+\varDelta\cdot\varDelta)&=((L-K_{V/C})+(K_{V/C}+\varDelta)\cdot\varDelta)\\&\geqq(K_{V/C}+\varDelta\cdot\varDelta)\\&=(K_V-f^*(K_C)+\varDelta\cdot\varDelta)\\&=\deg(K_{\varDelta})-(f^*(K_C)\cdot\varDelta)\\&\geqq\deg(K_{\tilde{\varDelta}})-\deg(\iota^*f^*(K_C))\\&\geqq 0.\end{aligned}$$

We set $a=(L+\varDelta\cdot\varDelta)$. Thanks to Hodge index theorem (cf. Corollary (3.3)),

$$\begin{vmatrix}(L^2) & (L\cdot F) & (L\cdot\varDelta)\\(F\cdot L) & (F^2) & (F\cdot\varDelta)\\(\varDelta\cdot L) & (\varDelta\cdot F) & (\varDelta^2)\end{vmatrix}\geqq 0,$$

which implies

$$2(a-(\varDelta^2))(\varDelta\cdot F)(L\cdot F)-(L\cdot F)^2(\varDelta^2)-(\varDelta\cdot F)^2(L^2)\geqq 0.$$

Thus, we obtain

$$(\varDelta^2)\leqq\frac{2a(\varDelta\cdot F)(L\cdot F)-(\varDelta\cdot F)^2(L^2)}{(L\cdot F)(2(\varDelta\cdot F)+(L\cdot F))}.$$

Hence, we have

$$\begin{aligned}\left(L-\frac{(L^2)}{2(L\cdot F)}F\cdot\varDelta\right)&=a-(\varDelta^2)-\frac{(L^2)(F\cdot\varDelta)}{2(L\cdot F)}\\&\geqq a\left(1-\frac{2(\varDelta\cdot F)}{2(\varDelta\cdot F)+(L\cdot F)}\right)+\frac{-(\varDelta\cdot F)(L^2)}{2(2(\varDelta\cdot F)+(L\cdot F))}\\&\geqq\frac{-(\varDelta\cdot F)(L^2)}{2(2(\varDelta\cdot F)+(L\cdot F))}.\end{aligned}$$ □

Let us go back to the proof of Proposition (5.2). Adding some multiple of $f^*(H)$, multiplying $L$ and using Proposition (2.7), we may assume that $L$ and $L-K_{X/S}$ are ample. Let $\varepsilon$ be a positive rational number. Take an element $m=(m_1,\cdots,m_{d-1})$ of $(\boldsymbol{Z}_{>0})^{d-1}$ such that

$$-\frac{(L^2\cdot f^*(H)^{d-1})}{4}+\varepsilon\nu_m(X/S,L,H)(H^d)_S>0.$$

Note that the numerical criterion of ampleness on a projective surface works well even over a non-closed field (see the argument of the proof of Lemma (2.19)). Hence, to show that

$$L-\frac{f^*(c_1(X/S,\ L))}{2}+\varepsilon f^*(H)$$

is ample on $(X_m)_\eta$, it is sufficient to see that

$$\Big(L-\frac{f^*(c_1(X/S,\ L))}{2}+\varepsilon f^*(H)\cdot\Delta\Big)_{(X_m)_\eta}>0$$

for a $(f_m)_\eta$-dominant irreducible curve $\Delta$ on $(X_m)_\eta$ such that

$$\Big(L-\frac{f^*(c_1(X/S,\ L))}{2}\cdot\Delta\Big)_{(X_m)_\eta}<0.$$

By Lemma (5.3), we get

$$\begin{aligned}
&\Big(L-\frac{f^*(c_1(X/S,\ L))}{2}+\varepsilon f^*(H)\cdot\Delta\Big)_{(X_m)_\eta}\\
&\geqq-\frac{(L^2)_{(X_m)_\eta}}{4}+\varepsilon(f^*(H)\cdot\Delta)_{(X_m)_\eta}\\
&=m_1\cdots m_{d-1}\Big(-\frac{(L^2\cdot f^*(H)^{d-1})}{4}+\varepsilon\deg(\Delta\to(S_m)_\eta)(H^d)_S\Big)\\
&\geqq m_1\cdots m_{d-1}\Big(-\frac{(L^2\cdot f^*(H)^{d-1})}{4}+\varepsilon\nu_m(X/S,\ L,\ H)(H^d)_S\Big)\\
&>0.
\end{aligned}$$

□

Next, we treat the case (1), that is, $(X/S,\ L)$ is $H$-bounded, which is similar to classical vector bundle situation. We use Mumford-Mehta-Ramanathan's method [12].

**Theorem (5.4)** *With notation as in Definition* (5.1), *if* $(X/S,\ L)$ *is* $H$*-bounded and* $H$*-semi-stable and* $f$ *is generically smooth, then* $(X/S,\ L)$ *is strongly* $H$*-semi-stable.*

*Proof.* First of all, we prepare some notions. Let $M$ be a subset of $(\boldsymbol{Z}_{>0})^s$. An element $m=(m_1,\ \cdots,\ m_s)$ of $M$ is called a *minimal element* of $M$ if for all $x=(x_1,\ \cdots,\ x_s)$ of $M$, we have $x_i\geqq m_i(i=1,\ \cdots,\ s)$. For $x\in\boldsymbol{Z}_{>0}$, we set

$$M(x)=\{(x_2,\ \cdots,\ x_s)\in(\boldsymbol{Z}_{>0})^{s-1}\,|\,(x,\ x_2,\ \cdots,\ x_s)\in M\}\ .$$

Inductively, we define a notion of *recursive infiniteness* of $M$ as follows. In the case $s=1$, $M$ is said to be recursively infinite if $M$ has infinitely many elements. In the case $s>1$, $M$ is said to be recursively infinite if the set

$$\{x\in\boldsymbol{Z}_{>0}\,|\,M(x)\text{ is recursively infinite}\}$$

has infinitely many elements.

For a recursively infinite subset $M$ of $(\boldsymbol{Z}_{>0})^s$, we have

**(5.4.1)** (1) Let $M=M_1\cup\cdots\cup M_k$ be a partition of $M$. Then, one of $M_1,\ \cdots,\ M_k$ is also recursively infinite.

(2) Let $m=(m_1,\ \cdots,\ m_s)$ be an element of $M$. Then, the set

$$M'=\{(x_1, \cdots, x_s)\in M \mid x_i \geqq m_i \quad \text{for all } i\}$$

is also recursively infinite.

*Proof of* (5.4.1) If $s=1$, (1) and (2) are trivial.

For $s>1$, we proceed by induction on $s$. Set $T=\{x\in \boldsymbol{Z}_{>0} \mid M(x) \text{ is recursively infinite}\}$. For $t\in T$, one of $M_1(t), \cdots, M_k(t)$ must be recursively infinite by the hypothesis of induction. Hence, there is an infinite subset $T'$ of $T$ and $j$ such that $M_j(x)$ is recursively infinite for $x\in T'$, which completes the proof of (1).

For the proof of (2), we set $W=\{x\in \boldsymbol{Z}_{>0} \mid x\geqq m_1 \text{ and } M(x) \text{ is recursively infinite}\}$. For $x\in W$, $M'(x)$ is recursively infinite by the hypothesis of induction. Hence, we have (2).

Let us start the proof of Theorm (5.4). We fix a sequence $(\alpha_1, \cdots, \alpha_{d-1})$ of positive integers. For $m=(m_1, \cdots, m_{d-1})\in(\boldsymbol{Z}_{>0})^{d-1}$, we set $\alpha(m)=(\alpha_1^{m_1}, \cdots, \alpha_{d-1}^{m_{d-1}})$.

We will prove that $L-f^*(c_1(X/S, L))/2$ is nef on $(X_{\alpha(m)})_\eta$ for some $m\in(\boldsymbol{Z}_{>0})^{d-1}$. Here, we assume that $L-f^*(c_1(X/S, L))/2$ is not nef on $(X_{\alpha(m)})_\eta$ for all $m\in(\boldsymbol{Z}_{>0})^{d-1}$. Since $L$ is $f$-ample, there is an integer $t$ such that

$$L-\frac{f^*(c_1(X/S, L))}{2}+tf^*(H),$$

say $A$, is ample. For $m=(m_1, \cdots, m_{d-1})\in(\boldsymbol{Z}_{>0})^{d-1}$, we define $\sigma_m$ as follows.

$$\sigma_m=\min\left\{\frac{(A\cdot D)_{(X_{\alpha(m)})_\eta}}{\alpha_1^{m_1}\cdots\alpha_{d-1}^{m_{d-1}}} \;\middle|\; \begin{array}{l} D \text{ is an effective divisor on } (X_{\alpha(m)})_\eta \text{ such that} \\ \left(L-\dfrac{f^*(c_1(X/S, L))}{2}\cdot D\right)_{(X_{\alpha(m)})_\eta}<0. \end{array}\right\}.$$

Let $D_m$ be an effective divisor on $(X_{\alpha(m)})_\eta$ such that

$$\sigma_m=\frac{(A\cdot D_m)_{(X_{\alpha(m)})_\eta}}{\alpha_1^{m_1}\cdots\alpha_{d-1}^{m_{d-1}}} \quad \text{and} \quad \left(L-\frac{f^*(c_1(X/S, L))}{2}\cdot D_m\right)_{(X_{\alpha(m)})_\eta}<0.$$

**Claim** (5.4.2) *$D_m$ is irreducible and reduced.*

Assume that there are non-zero effective divisors $E_1$ and $E_2$ on $(X_{\alpha(m)})_\eta$ such that $D_m=E_1+E_2$. Then,

$$\left(L-\frac{f^*(c_1(X/S, L))}{2}\cdot E_i\right)_{(X_{\alpha(m)})_\eta}<0$$

for an $i$. Since $A$ is ample,

$$(A\cdot D_m)_{(X_{\alpha(m)})_\eta}>(A\cdot E_i)_{(X_{\alpha(m)})_\eta},$$

which contradicts to the minimality of $\sigma_m$.

By Lemma (4.1), we can find a divisor on $\tilde{D}_m$ on $X$ such that $\lambda_{\alpha(m)}^*(\tilde{D}_m)\sim D_m$. Then,

$$(A\cdot D_m)_{(X_{\alpha(m)})_\eta}=\alpha_1^{m_1}\cdots\alpha_{d-1}^{m_{d-1}}(A\cdot\tilde{D}_m\cdot f^*(H)^{d-1}),$$

which asserts $\sigma_m$ is an integer.

**Claim** (5.4.3) *$\sigma_m$ is bounded.*

Let $\Delta_m$ be an irreducible and reduced curve on $(X_{\alpha(m)})_\eta$ such that

$$\nu_{\alpha(m)}(X/S,\ L,\ H)=\deg(\Delta_m\to(S_{\alpha(m)})_\eta)\quad\text{and}\quad\left(L-\frac{f^*(c_1(X/S,\ L))}{2}\cdot\Delta_m\right)_{(X_{\alpha(m)})_\eta}<0.$$

Then, we get

$$\begin{aligned}\sigma_m&\leqq\frac{(A\cdot\Delta_m)_{(X_{\alpha(m)})_\eta}}{\alpha_1^{m_1}\cdots\alpha_{d-1}^{m_{d-1}}}\\&<\frac{(tf^*(H)\cdot\Delta_m)_{(X_{\alpha(m)})_\eta}}{\alpha_1^{m_1}\cdots\alpha_{d-1}^{m_{d-1}}}\\&=t\nu_{\alpha(m)}(X/S,\ L,\ H)(H^d)_S,\end{aligned}$$

which implies our Claim (5.4.3).

Hence, by (1) of Lemma (5.4.1), we have a recursively infinite subset $M$ of $(\boldsymbol{Z}_{>0})^{d-1}$ such that $\sigma_m$ is constant for every $m\in M$. Moreover, by (2) of Lemma (5.4.1), we may assume that $M$ has the minimal element $m_0=(m_{0,1},\ \cdots,\ m_{0,d-1})$.

**Claim** (5.4.4) *For all $m\in M$, $\tilde{\lambda}^*_{\alpha(m_0)}(\tilde{D}_m)$ is linearly equivalent to an effective divisor.*

Let $m=(m_1,\ \cdots,\ m_{d-1})$ be an element of $M$. By the upper semi-continuity of the dimensions of cohomology groups, we have

$$H^0(g^{-1}_{\alpha(m)}(x),\ \mathcal{O}_X(\tilde{D}_m))\neq0\qquad\text{for all}\quad x\in T_{\alpha(m)}$$

and there is a dense Zariski open set $U_m$ of $T_{\alpha(m_0)}$ such that

$$h^0(g^{-1}_{\alpha(m_0)}(t),\ \mathcal{O}_X(\tilde{D}_m))-1=\dim|\tilde{\lambda}^*_{\alpha(m_0)}(\tilde{D}_m)|\qquad\text{for all}\quad t\in U_m.$$

Setting $k_m=\alpha_1^{m_1-m_{0,1}}\cdots\alpha_{d-1}^{m_{d-1}-m_{0,d-1}}$, let $t_1,\ \cdots,\ t_{k_m}$ be $k_m$-distinct general members of $U_m(K)$. Set $V_j=g^{-1}_{\alpha(m_0)}(t_j)$. Since $\sum_{j=1}^{k_m}V_j$ is a fiber of $g_{\alpha(m)}$, we obtain

$$H^0\left(\sum_{j=1}^{k_m}V_j,\ \mathcal{O}_X(\tilde{D}_m)\right)\neq0.$$

Thus, considering the injective homomorphism:

$$\mathcal{O}_{\sum_{j=1}^{k_m}V_j}(\tilde{D}_m)\longrightarrow\bigoplus_{j=1}^{k_m}\mathcal{O}_{V_j}(\tilde{D}_m),$$

we see that for some $j$, $H^0(V_j,\ \mathcal{O}_X(\tilde{D}_m))\neq0$. Hence we get our Claim (5.4.4) because $V_j$ is a fiber of $g_{\alpha(m_0)}$ over a point of $U_m$.

Since $(A\cdot\tilde{D}_m\cdot f^*(H)^{d-1})=\sigma_{m_0}$ and $(L-(f^*(c_1(X/S,\ L))/2)\cdot\tilde{D}_m\cdot f^*(H)^{d-1})<0$, by the same argument as in Claim (5.4.2), $\tilde{\lambda}^*_{\alpha(m_0)}(\tilde{D}_m)$ is linearly equivalent to an irreducible and reduced curve $\Delta$ such that

$$\left(L-\frac{f^*(c_1(X/S,\ L))}{2}\cdot\Delta\right)_{(X_{\alpha(m_0)})_\eta}<0.\tag{5.4.5}$$

On the other hand, by Proposition (2.8), $(L-(f^*(c_1(X/S,\ L))/2))|_{(X_{\alpha(m_0)})_\eta}$ is pseudo-effective. (Note that in the proof of Proposition (2.8), it is not essential that the defining field is algebraically closed.) To see that there are finitely many irreducible curves with the property (5.4.5), we need the following lemma.

**Lemma (5.5)** *Let $S$ be a non-singular projective surface over a field $k$, which is not necessarily algebraically closed. Let $D$ be a pseudo-effective divisor on $S$. Then, the number of integral curves $C$ with $(D\cdot C)<0$ is finite.*

*Proof.* Let $\bar{k}$ be the algebraic closure of $k$ and $C$ a curve on $S$. Then, since $(D\cdot C)_X=(D\cdot C)_{X_{\bar{k}}}$, we may assume that $k$ is algebraically closed. In this case, this lemma is an immediate consequence of Zariski decomposition of $D$ or the first step to prove the existence of Zariski decomposition on a surface (cf. [14, Lemma 3.6], [15] and [17]). □

Thus, we have a recursively infinite subset $M'$ of $M$ such that $\tilde{\lambda}^*_{\alpha(m_0)}(\tilde{D}_m)$ are linearly equivalent to each other for all $m\in M'$. Therefore, by Lemma (4.1), $\tilde{D}_m$ are linearly equivalent to each other for all $m\in M'$. We take a divisor $D$ on $X$ such that for all $m\in M'$, $\tilde{D}_m\sim D$. Here we need the following lemma.

**Lemma (5.6) (Enriques-Severi-Zariski's lemma for flat fiber spaces)** *Let $f: X\to S$ be a surjective and flat morphism of non-singular projective varieties with* $\dim S\geqq 2$. *Let $F$ be a reflexive sheaf on $X$ and $H$ a very ample divisor on $S$. Then, there is a positive integer $n_0$ such that for all $n\geqq n_0$ and a general member $T$ of $|nH|$, the natural homomorphism*

$$H^0(X,\ F)\longrightarrow H^0(f^{-1}(T),\ F|_{f^{-1}(T)})$$

*is surjective.*

*Proof.* Set $G=f_*(F)^{\check{}\check{}}$ and $d=\dim S$. First, we prove the following claim.

**Claim (5.6.1)** *There is a positive integer $n_0$ such that $H^1(S,\ G\otimes\mathcal{O}_S(-nH))=0$ for all $n\geqq n_0$.*

By Grothendieck duality (cf. [8, §III, Theorem 7.7]), we have

$$(H^1(S,\ G\otimes\mathcal{O}_S(-nH)))^{\check{}}\cong \mathrm{Ext}^{d-1}_S(G\otimes\mathcal{O}_S(-nH),\ \omega_S)\cong \mathrm{Ext}^{d-1}_S(G,\ \omega_S(nH)).$$

Since $H$ is ample, there is a positive integer $n_0$ such that for all $n\geqq n_0$, $p\neq 0$ and $q$,

$$H^p(S,\ \mathcal{E}xt^q_{\mathcal{O}_S}(G,\ \mathcal{O}_S)\otimes\omega(nH))=0.$$

Hence, considering the spectral sequence

$$H^p(S,\ \mathcal{E}xt^q_{\mathcal{O}_S}(G,\ \mathcal{O}_S)\otimes\omega(nH))\Longrightarrow \mathrm{Ext}^{p+q}_S(G,\ \omega_S(nH)),$$

we obtain

$$\mathrm{Ext}^{d-1}_S(G,\ \omega_S(nH))\cong H^0(S,\ \mathcal{E}xt^{d-1}_{\mathcal{O}_S}(G,\ \mathcal{O}_S)\otimes\omega(nH)).$$

On the other hand, since $G$ is reflexive and $d\geqq 2$, we get

$$\operatorname{depth}_x G_x \geqq 2 \quad \text{for all} \quad x \in S,$$

which implies that $\mathcal{E}xt_{\mathcal{O}_S}^{d-1}(G, \mathcal{O}_S)=0$. This proves our claim.

Since $f_*(F)$ is torsion free, there is a Zariski open set $U$ of $S$ such that $\operatorname{codim}_S(S \backslash U) \geqq 2$ and $f_*(F)|_U$ is locally free. Note the following facts.

(**5.6.2**) Let $V$ be a non-singular variety and $Q$ a coherent sheaf on $V$.

( i ) $Q$ is reflexive if and only if $Q$ is locally a second syzygy sheaf (cf. [12, Lemma 3.1]).

(ii) Let $W$ be a Zariski open set of $V$ with $\operatorname{codim}_V(V \backslash W) \geqq 2$ and $i: W \to V$ the inclusion map. If $Q$ is reflexive, then $i_*(i^*(Q))=Q$ (cf. [19]).

For $n \geqq n_0$, take a general member $T$ of $|nH|$ such that

(1) $T$ and $f^{-1}(T)$ are non-singular,

(2) $\operatorname{codim}_T(T \backslash T \cap U) \geqq 2$,

(3) $R^1 f_*(F) \otimes \mathcal{O}_S(-T) \to R^1 f_*(F)$ is injective and

(4) $G|_T$ and $F|_{f^{-1}(T)}$ are reflexive.

Then, we have that $\operatorname{codim}_X(X \backslash f^{-1}(U)) \geqq 2$ by flatness of $f$ and the restriction homomorphism

$$H^0(f^{-1}(T), F|_{f^{-1}(T)}) \longrightarrow H^0(f^{-1}(T \cap U), F|_{f^{-1}(T)})$$

is injective by torsion freeness of $F|_{f^{-1}(T)}$. Hence, in order to show that

$$H^0(X, F) \longrightarrow H^0(f^{-1}(T), F|_{f^{-1}(T)})$$

is surjective, it is sufficient to see that

$$H^0(f^{-1}(U), F) \longrightarrow H^0(f^{-1}(T \cap U), F|_{f^{-1}(T)})$$

is surjtctive. By (3), taking the direct image of the exact sequence:

$$0 \longrightarrow F \otimes \mathcal{O}_X(-f^*(T)) \longrightarrow F \longrightarrow F|_{f^{-1}(T)} \longrightarrow 0,$$

we have

$$0 \longrightarrow f_*(F) \otimes \mathcal{O}_S(-T) \longrightarrow f_*(F) \longrightarrow f_*(F|_{f^{-1}(T)}) \longrightarrow 0.$$

This shows that $f_*(F)|_T \cong f_*(F|_{f^{-1}(T)})$. Therefore, the surjectivity of

$$H^0(f^{-1}(U), F) \longrightarrow H^0(f^{-1}(T \cap U), F|_{f^{-1}(T)})$$

is equivalent to the surjectivity of

$$H^0(U, f_*(F)) \longrightarrow H^0(T \cap U, f_*(F)|_T).$$

On the other hand, by Claim (5.6.1), $H^0(S, G) \to H^0(T, G|_T)$ is surjective, which asserts that

$$H^0(U, f_*(F)) \longrightarrow H^0(T \cap U, f_*(F)|_T)$$

is surjective because $G$ is reflexive and $f_*(F)|_U = G|_U$. Thus, we have our lemma. □

Let us continue the proof of Theorem (5.4). Using Lemma (5.6) inductively and the property of the recursively infinite subset $M'$, there is an element $m=(m_1, \cdots, m_{d-1})$

of $M'$ such that for general members $H_i$ of $|\alpha_i^{m_i}H|(i=1, \cdots, d-1)$, the natural homomorphism

$$H^0(\mathcal{O}_X(D)) \longrightarrow H^0(\mathcal{O}_{f^{-1}(H_1\cap\cdots\cap H_{d-1})}(D))$$

is surjective. On the other hand, $H^0(\mathcal{O}_{f^{-1}(H_1\cap\cdots\cap H_{d-1})}(D))\neq 0$. Hence, $D$ is linearly equivalent to an effective divisor. Therefore, $(L-(f^*(c_1(X/S, L))/2)\cdot D\cdot f^*(H))<0$ contradicts to Proposition (2.15). Thus, we complete the proof of Theorem (5.4). □

Here, we state the main theorem of this paper, which is an immediate corollary of Theorem (3.1), Proposition (5.2) and Theorem (5.4).

**Theorem** (**5.7**) *Assume that the characteristic of the ground field $K$ is zero. Let $d$ be a positive integer with $d\geqq 2$. Let $f: X\to S$ be a flat morphism of a $(d+1)$-dimensional non-singular projective variety $X$ to a $d$-dimensional non-singular projective variety $S$ with $f_*\mathcal{O}_X=\mathcal{O}_S$. Let $L$ be an $f$-ample divisor on $X$ and $H$ an ample divisor on $S$. Assume that the polarized fiber space $(X/S, L)$ is $H$-semi-stable. Then, we have the following inequality of Bogomolov-Gieseker's type.*

$$(c_1^2(X/S, L)\cdot H^{d-2})\leqq 4(c_2(X/S, L)\cdot H^{d-2}).$$

## 6. Semi-stably polarized fiber space over a non-closed field

Let $k$ be a field, which is not necessarily algebraically closed, and $\bar{k}$ the algebraic closure of $k$. Every definition of §2 is well-defined in the category of algebraic schemes over $k$.

Let $X$ be an equi-dimensional projective scheme over $k$, $S$ a non-singular projective variety with ample divisor $H$ and $f: X\to S$ a fiber space. Let $L$ be a Cartier divisor on $X$ and $A$ a nef Cartier divisor on $X$. Assume that $L$ gives a weak polarization on $f: X\to S$. Then, we have

**Proposition** (**6.1**) *The notation as above. If $S$ is geometrically irreducible, then*

(1) *$(X/S, L)$ is $H$-semi-stable if and only if $(X_{\bar{k}}/S_{\bar{k}}, L_{\bar{k}})$ is $H_{\bar{k}}$-semi-stable.*

(2) *$(X/S, L)$ is $A$-strongly $H$-semi-stable if and only if $(X_{\bar{k}}/S_{\bar{k}}, L_{\bar{k}})$ is $A_{\bar{k}}$-strongly $H_{\bar{k}}$-semistable.*

*Proof.* First we note $c_i(X_{\bar{k}}/S_{\bar{k}}, L_{\bar{k}})=c_i(X/S, L)\otimes_k\bar{k}$, which is easily checked by [4, Example 6.2.9]. Hence, (2) is obvious by Lemma (2.19). The proof of (1) is very easy if we refer to the proof of Lemma (2.19). □

Hence, we have the following generalization.

**Theorem** (**6.2**) *Theorem* (3.1), *Proposition* (5.2), *Theorem* (5.4) *and Theorem* (5.7) *hold in the category of algebraic schemes over a field $k$ if $S$ is geometrically irreducible.*

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