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Some remarks on inertial manifolds

By

Hirokazu NINOMIYA

1. Introduction

A recently developed theory of inertial manifolds reveals that the asymptotic behavior of solutions to some classes of semilinear evolution equations is controlled by a finite-dimensional system. More precisely such equations admit a finite-dimensional invariant manifold which attracts every solution exponentially (we call it *inertial manifold*) and the ordinary differential equation reduced on the manifold determines the asymptotic behavior of the original equation. For example it is known that there exists an inertial manifold for a reaction-diffusion equation and Kuramoto-Sivashinsky equation under some condition (see C. Foias, B. Nicolaenko, G. R. Sell and R. Temam [7], C. Foias, G. R. Sell and R. Temam [8], J. Mallet-Paret and G. R. Sell [12] and R. Temam [15]).

Although there are many works for the existence of the inertial manifold, it seems that three basic techniques work in them: Hadamard method (or the graph transformation method, see [12] and P. Constantin, C. Foias, B. Nicolaenko and R. Temam [3]), Lyapunov-Perron method (based on the variation of constants formula, see S. N. Chow and K. Lu [1] and [7], [8], [15]) and the elliptic regularization method (see A. Debussche [4] and so on). Our aim of this paper is to charactrize Lyapunov-Perron method from the point of dynamics of the semiflow (generated by solutions). Usually the technique to find a fixed point of some operator (which comes from the variation of constants formula) is called Lyapunov-Perron method; this fixed point in a suitable function space whose graph gives the inertial manifold. However in the classical adaptation of the Lyapunov-Perron method, it seems to be difficult to relate it to some property of the semiflow explicitly. C. Foias, G. R. Sell and R. Temam used in [8] some property of the flow namely *squeezing property* (or *cone property*) for the proof of the existence of the manifold. Through their researches we will use the squeezing property more effectively for the above purpose than in the related works including [8]. Since the squeezing property is charactrized in terms of differential inequalities, we hardly use the variation of constants formula in the proof. Furthermore we can also make clear a geometrical meaning of the exponential tracking in the squeezing property and prove it shortly.

We will also prove the regularity of the inertial manifold (C^1-manifold). Here we will show that the inertial manifold is a limit of a family of approximate inertial manifolds of Galerkin approximate equations in C^1-topology. This will be discussed

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in §4. In §5 we investigate how the inertial manifold varies when the nonlinear term is perturbed. Roughly speaking the difference of inertial manifolds is estimated by size of the perturbed term.

2. Main Results

Let H be a separable Hilbert space and denote its inner product and its norm by $(\cdot, \cdot)$ and $|\cdot|$ respectively. We consider the equation in H

$$(2.1)\qquad \begin{cases} \dfrac{du}{dt}+Au+R(t, u)=0\,, \\ u(t_0)=u_0\,. \end{cases}$$

We assume the following:

A1: A is a positive selfajoint operator and A^{-1} is compact.

It has eigenvalues λ_j and eigenfunctions w_j satisfying

$$Aw_j=\lambda_j w_j\,,$$
$$0<\lambda_1\leqq\lambda_2\leqq\lambda_3\leqq\cdots\,.$$

A2: There is a γ $(0\leqq\gamma\leqq 1/2)$ such that there are constants K_0, K_1 and K_2 such that for every t, h in $\mathbf{R}$ and u, v in $D(A^\gamma)$,

$$(2.2)\qquad \begin{cases} |R(t, u)|\leqq K_0\,, \\ |R(t, u)-R(t, v)|\leqq K_1|A^\gamma(u-v)|\,, \\ |R(t+h, u)-R(t, u)|\leqq K_2|h|\,, \end{cases}$$

where A^γ and $D(A^\gamma)$ denote the fractional power of A and its domain respectively.

A3: There exists an integer $N>0$ satisfying

$$(2.3)\qquad \lambda_{N+1}-\lambda_N>K_1(\lambda_{N+1}^{\gamma/2}+\lambda_N^{\gamma/2})^2\,.$$

Remark 2.1. The equation (2.1) has a unique solution $u(t)$ in $C([t_0, \infty); D(A^\gamma))\cap L^2_{loc}([t_0, \infty); D(A))$ for every $t_0\in\mathbf{R}$ and $u_0\in D(A^\gamma)$ under A1 and A2. Then we can define a semiflow

$$S(t, t_0): u(t_0)\in D(A^\gamma)\longrightarrow u(t)\in D(A^\gamma)\qquad \text{for } t\geqq t_0\,.$$

Under the above assumptions, we can obtain three theorems below.

Theorem 2.2. *Under* A1, A2 *and* A3, *there exists an manifold* M_t *for each* $t\in\mathbf{R}$ *which satisfies the following:*

(1) *Lipschitz property:*

There exists a Lipschitz function $\Phi(\cdot, \cdot)$ *from* $\mathbf{R}\times P_N D(A^\gamma)$ *into* $(I-P_N)D(A^\gamma)$ *such that for each* t *in* $\mathbf{R}$,

$$M_t=\text{graph}\,\Phi(t, \cdot)\,,$$

where P_N *is a projection operator onto the space spanned by* $w_1, w_2, w_3, \cdots, w_N$.

(ii) *Invariant property:*

$$M_t = S(t, t_0) M_{t_0} \quad \textit{for } t, t_0 \textit{ in } \mathbf{R}.$$

(iii) *Exponential tracking:*
For any solution $u(t) = S(t, t_0)u_0$, *there exists a* $v_0 \in M_{t_0}$ *such that*

$$|A^{\gamma}(S(t+t_0, t_0)u_0 - S(t+t_0, t_0)v_0)| < c_1 e^{-\nu t} \qquad (t \geqq 0),$$

where c_1 *and* ν *are positive constants independent of* t *and* t_0.

Remark 2.3. We call the manifold $\mathcal{M} = \{M_t\}_{-\infty<t<\infty}$ *inertial manifold* for (2.1). If the nonlinear term R is periodic with respect to t, then Φ is also periodic. For an autonomous equation (2.1), Φ is independent of time t. Then $\mathcal{M} = M_t$ for any t.

The constant ν is any number between ν_1 and ν_2 which are defined in (3.2) and c_1 is given by (3.23) in § 3.

Theorem 2.4. *In addition to* A2 *and* A3, *assume that* $R(t, u)$ *has a Fréchet derivative* $D_u R$ *which belongs to* $C^0(\mathbf{R} \times D(A^{\gamma}); \mathcal{L}(D(A^{\gamma}); H))$. *Then* M_t *is a* C'*-manifold.*

Next we will consider the perturbed equation of (2.1)

$$\text{(2.6)} \qquad \begin{cases} \dfrac{dv}{dt} + Av + \tilde{R}(t, v) = 0, \\[2ex] v(t_0) = v_0, \end{cases}$$

where we assume that $\tilde{R}(t, u)$ satisfies A1, A2 and A3 as $R(t, u)$ does. Under A1, A2 and A3, we have Lipschitz functions Φ, $\tilde{\Phi}$ whose graphs represent inertial manifolds for (2.1) and (2.6) respectively. Let us consider the equation on the manifold,

$$\text{(2.7)} \qquad \begin{cases} \dfrac{dp}{dt} + Ap + PR(t, p + \Phi(t, p)) = 0, \\[2ex] p(t_0) = p_{01} \in P_N D(A^{\gamma}), \end{cases}$$

$$\text{(2.8)} \qquad \begin{cases} \dfrac{dp}{dt} + Ap + P\tilde{R}(t, p + \tilde{\Phi}(t, p)) = 0, \\[2ex] p(t_0) = p_{02} \in P_N D(A^{\gamma}). \end{cases}$$

We denote a solution of (2.7) (resp. (2.8)) by $p_1(t, t_0, p_{01})$ (resp. $p_2(t, t_0, p_{02})$). Let us introduce the next equation.

$$\text{(2.9)} \qquad \lambda_{N+1} - \lambda_N - K_1(1 + l^{-1})\lambda_{N+1}^{\gamma} - K_1(1 + l)\lambda_N^{\gamma} = 0.$$

It is easily checked that (2.9) has two solutions, say l_1, $l_2 (l_1 > l_2 > 0)$. We assume

A4: There exists a constant K_3 satisfying

$$\text{(2.10)} \qquad |R(t, u) - \tilde{R}(t, u)| \leqq K_3 e^{-\eta t} \qquad \text{for any } t \in \mathbf{R},\ u \in D(A^{\gamma}),$$

where η is some number satisfying

(2.11) $$0\leqq\eta<\lambda_{N+1}-K_1(1+l_1^{-1})\lambda_{N+1}^{\gamma}.$$

Theorem 2.5. *Let the equations* (2.1) *and* (2.6) *satisfy* A1-A4. *Let* Φ, $\tilde{\Phi}$ *be the mapping representing the inertial manifolds for* (2.1) *and* (2.6) *respectively. Then for any* $t\in\boldsymbol{R}$, $p\in PD(A^{\gamma})$,

$$|A^{\gamma}(\Phi(t, p)-\tilde{\Phi}(t, p))|\leqq c_2K_3e^{-\eta t},$$

where c_2 *is a positive constant independent of* t, p. *Moreover, if*

(2.12) $$\lambda_N+K_1(1+l_2)\lambda_N^{\gamma}<\eta<\lambda_{N+1}-K_1(1+l_1^{-1})\lambda_{N+1}^{\gamma},$$

then for every solution of (2.6) *there exists a solution of* (2.1) *which approaches it exponentially. That is the following sense: For* $t_0\in\mathbf{R}$, $p_{02}\in PD(A^{\gamma})$, *there exists* $p_{01}\in P_ND(A^{\gamma})$ *such that for all* t *in* $\mathbf{R}$,

(2.13) $$|A^{\gamma}(p_1(t, t_0, p_{01})-p_2(t, t_0, p_{02}))|\leqq c_3K_3e^{-\eta t},$$

where c_3 *is a positive constant independent of* t, t_0 *and* p_{02}.

Remark 2.6. In particular, when $\eta=0$, we have the estimate:

$$|A^{\gamma}(\Phi(t, p)-\tilde{\Phi}(t, p))|\leqq c_2\sup_{t\in\mathbf{R},\, u\in D(A^{\gamma})}|R(t, u)-\tilde{R}(t, u)|.$$

The above Theorem 2.5 will be proved in §5 where the constants c_2 and c_3 are given by (5.3) and (5.7).

Remark 2.7. The squeezing property is first introduced by C. Foias, G.R. Sell and R. Temam [8]. In this paper, this property is modified and the condition for the existence of the manifold is improved by its modification. In addition this modification makes us to treat the regularity and the perturbation of the manifold easily. S.N. Chow and K. Lu proved the existence of C^1-manifold and the exponential attractivity by estimating an integral equation. Their gap condition is rather restrictive. On the other hand, P. Constantin, C. Foias, B. Nicolaenko and R. Temam [3] and M. Miklavčič [13] proved the existence of the manifold with the better gap condition than that of this paper. But they did not make mention of the regularity and the perturbation.

3. Proof of the existence of an inertial manifold

We simply write $P=P_N$, $Q=I-P_N$, Preparatory to a proof of Theorem 2.2, we will prove an elementary lemma below.

Lemma 3.1. *If* A3 *in* §2 *is satisfied, then there exist positive constants* l *and* θ *such that*

(3.1) $$\begin{cases}\lambda_N+K_1(1+\theta^{-1}l)\lambda_N^{\gamma}<\lambda_{N+1}-K_1(\theta^{-1}+l^{-1})\lambda_{N+1}^{\gamma},\\ 0<\theta<1.\end{cases}$$

Proof. We can see from the inequality of A3 that there exists a positive constant l satisfying

$$\lambda_{N+1}-\lambda_N-K_1(1+l^{-1})\lambda_{N+1}^{\gamma}-K_1(1+l)\lambda_N^{\gamma}>0\,.$$

It is clear that we can take $0<\theta<1$ satisfying

$$\lambda_{N+1}-\lambda_N-K_1(\theta^{-1}+l^{-1})\lambda_{N+1}^{\gamma}-K_1(1+\theta^{-1}l)\lambda_N^{\gamma}>0\,. \qquad \blacksquare$$

Set

$$\left\{\begin{array}{l}\nu_1=\lambda_N+K_1(1+\theta^{-1}l)\lambda_N^{\gamma}\,,\\ \nu_2=\lambda_{N+1}-K_1(\theta^{-1}+l^{-1})\lambda_{N+1}^{\gamma}\,.\end{array}\right. \tag{3.2}$$

We introduce a complete metric space:

$$\begin{aligned}\mathcal{F}=\{q(t)\in C^0((-\infty,\ t_0]\,;\ QD(A^{\gamma}))\,;&\\ |A^{\gamma}q(t)|\leqq\frac{e^{-\gamma}K_0}{1-\gamma}\lambda_{N+1}^{\gamma-1}e^{\nu(t_0-t)}\,,&\\ |A^{\gamma}(q(t+h)-q(t))|\leqq B_1|h|e^{\nu(t_0-t)}\qquad \text{for all } t\leqq t+h\leqq t_0\}&\end{aligned} \tag{3.3}$$

with the metric:

$$d_{\nu}(q_1,\ q_2)=\sup_{t\leq t_0}|A^{\gamma}(q_1(t)-q_2(t))|e^{\nu(t-t_0)}\,,$$

where ν is any number such that $\nu_1<\nu<\nu_2$. The positive constant B_1 will be determined later. Fix any t_0 in $\mathbf{R}$, $p_0\in PD(A^{\gamma})$. We define a map T_{t_0,p_0} on $\mathcal{F}$ in the following way. For $q(t)\in\mathcal{F}$, consider the equation

$$\left\{\begin{array}{l}\dfrac{d}{dt}p+Ap+PR(t,\ p+q(t))=0\,,\\ p(t_0)=p_0\,.\end{array}\right. \tag{3.4}$$

This finite-dimensional equation has a unique solution $p(t)=p(t,\ t_0,\ p_0,\ q)$ for all $t\leqq t_0$. Using the solution $p(t)$, we define the following mapping:

$$T_{t_0,p_0}(q)(t)=-\int_{-\infty}^{t}e^{-A(t-s)}QR(s,\ p(s)+q(s))ds\,. \tag{3.5}$$

Hereafter we often write $T(q)(t)$ without its subscripts t_0 and p_0. Since the function $R(t,\ p(t)+q(t))$ is Lipschitz, (3.5) is equivalent to

$$\frac{d}{dt}T(q)+AT(q)+QR(t,\ p(t)+q(t))=0 \tag{3.6}$$

(see D. Henry [10]).

Proposition 3.2. *For all* $t_0\in\mathbf{R}$ *and* $p_0\in PD(A^{\gamma})$, T_{t_0,p_0} *is a contraction mapping from* $\mathcal{F}$ *into* $\mathcal{F}$.

Before the proof of Proposition 3.2, we give definitions of several sets and three key lemmas. Define Π, C_l, $\Omega(B)$ and $\Omega'(B)$ as follows:

$$(3.7)\quad \begin{cases} \Pi=\{(w,z)\in\mathbf{R}^2;\ w\geqq 0,\ z\geqq 0\}, \\ C_l=\{(w,z)\in\mathbf{R}^2;\ z\geqq lw\geqq 0\}(\subset\Pi), \\ \Omega(B)=\left\{(w,z)\in\mathbf{R}^2;\ z\geqq lw\geqq 0,\ z\geqq\dfrac{\theta}{1-\theta}B\right\}(\subset\Pi), \\ \Omega'(B)=\{(w,z)\in\mathbf{R}^2;\ z\geqq lw\geqq 0,\ z\geqq\theta B\}(\subset\Pi). \end{cases}$$

In the next three lemmas, it is assumed that the functions $w(t)$, $z(t)$ are non negative and belong to $C^0((-\infty, t_0], \mathbf{R})\cap C^1((-\infty, t_0), \mathbf{R})$.

Lemma 3.3. *Assume that two functions $w(t)$, $z(t)$ satisfy the following differential inequalities:*

$$(3.8)\quad \begin{cases} \dfrac{1}{2}\dfrac{d}{dt}w^2\geqq-(\lambda_N+K_1\lambda_N^\gamma)w^2-K_1\lambda_N^\gamma wz & \text{in } \Pi, \\ \dfrac{1}{2}\dfrac{d}{dt}z^2\leqq-(\lambda_{N+1}-K_1\lambda_{N+1}^\gamma)z^2+K_1\lambda_{N+1}^\gamma wz & \text{in } C_l. \end{cases}$$

If $(w(t_1), z(t_1))\in C_l$ for some $t_1(\leqq t_0)$, then $(w(t), z(t))\in C_l$ for any $t\leqq t_1$ and the following estimate holds:

$$(3.9)\quad 0\leqq z(t)\leqq z(t_2)e^{-\nu(t-t_2)} \qquad \text{for } t_2\leqq t\leqq t_1.$$

Moreover if $z(t)$ is bounded in $(-\infty, t_0]$, then

$$(3.10)\quad 0\leqq z(t)\leqq lw(t)\leqq le^{\nu(t_0-t)}w(t_0) \qquad \text{for all } t\leqq t_0.$$

Lemma 3.4. *Assume that two functions $w(t)$, $z(t)$ satisfy the following differential inequalities:*

$$(3.11)\quad \begin{cases} \dfrac{1}{2}\dfrac{d}{dt}w^2\geqq-(\lambda_N+K_1\lambda_N^\gamma-\nu)w^2-K_1\lambda_N^\gamma wz-K_1\lambda_N^\gamma Bw & \text{in } \Pi, \\ \dfrac{1}{2}\dfrac{d}{dt}z^2\leqq-(\lambda_{N+1}-K_1\lambda_{N+1}^\gamma-\nu)z^2+K_1\lambda_{N+1}^\gamma wz+K_1\lambda_{N+1}^\gamma Bz & \text{in } \Omega(B), \end{cases}$$

where B is a positive constant. If there exists $t_1(\leqq t_0)$ such that

$$(w(t_1), z(t_1))\in\Omega(B),$$

then $(w(t), z(t))\in\Omega(B)$ for any $t\leqq t_1$ and

$$(3.12)\quad 0\leqq z(t)\leqq z(t_2)e^{-(\nu_2-\nu)(t-t_2)} \qquad \text{for } t_2\leqq t\leqq t_1$$

where ν_2 is as in (3.2). Furthermore if $z(t)$ is bounded in $(-\infty, t_0]$ and $w(t_0)\leqq \theta B/(1-\theta)l$, then

$$(3.13)\quad 0\leqq w(t)\leqq\frac{\theta}{(1-\theta)l}B,\quad 0\leqq z(t)\leqq\frac{\theta}{(1-\theta)}B \qquad \text{for all } t\leqq t_0.$$

Lemma 3.5. *Assume that two functions $w(t)$, $z(t)$ satisfy*

$$(3.14)\quad \begin{cases} \frac{1}{2}\frac{d}{dt}w^2 \geqq -(\lambda_N+K_1\lambda_N^{\gamma}-\nu)w^2-K_1\lambda_N^{\gamma}Bw & \text{in } \Pi, \\ \frac{1}{2}\frac{d}{dt}z^2 \leqq -(\lambda_{N+1}-\nu)z^2+K_1\lambda_{N+1}^{\gamma}wz+K_1\lambda_{N+1}^{\gamma}Bz & \text{in } \Omega'(B), \end{cases}$$

and that $z(t)$ is bounded in $(-\infty, t_0]$ and $w(t_0)\leqq\theta l^{-1}B$. Then $z(t)\leqq\theta B$ for all $t\leqq t_0$.

Remark 3.6. These lemmas will be applied to the difference of solutions $u(t)$, $v(t)$ to the equation (2.1) or equations specified later. For example,

$$w(t)=|A^{\gamma}P(u(t)-v(t))|, \qquad z(t)=|A^{\gamma}Q(u(t)-v(t))|,$$

or

$$w(t)=e^{\nu(t-t_0)}|A^{\gamma}P(u(t)-v(t))|, \qquad z(t)=e^{\nu(t-t_0)}|A^{\gamma}Q(u(t)-v(t))|.$$

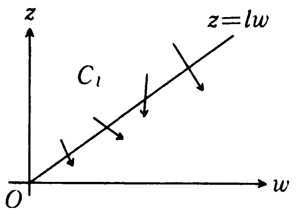

Fig. 1. Squeezing Property.

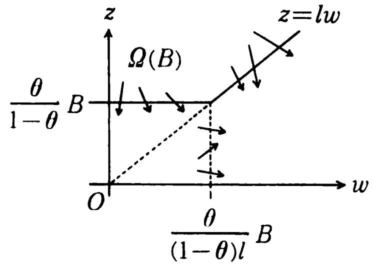

Fig. 2. Modified Squeezing Property.

If the flow for those equations satisfies the condition of Lemmas 3.3, we call the flow has *squeezing property* (or *cone property*). We also call the flow has *modified squeezing property* if the flow satisfies that of Lemma 3.4 or Lemma 3.5.

Proof of Lemma 3.3. If $(w, z)\in C_l$, then (3.8) yields

$$\frac{1}{2}\frac{d}{dt}(z^2-l^2w^2)\leqq-(\lambda_{N+1}-\lambda_N-K_1(1+l^{-1})\lambda_{N+1}^{\gamma}-K_1(1+l)\lambda_N^{\gamma})z^2$$

This inequality implies the invariance of the cone $\{(w, z)\in\mathbf{R}^2;\ 0\leqq z\leqq lw\}$ (see Fig. 1). Therefore it is seen as above that if $(w(t_1), z(t_1))\in C_l$, then $(w(t), z(t))\in C_l$ for any $t\leqq t_1$. In this region, the following inequality holds:

$$\frac{1}{2}\frac{d}{dt}z^2\leqq-(\lambda_{N+1}-K_1(1+l^{-1})\lambda_{N+1}^{\gamma})z^2 .$$

Thus we get a desired inequality (3.9).

We will next prove the latter part of this lemma. If there exists $t_1(\leqq t_0)$ such that $(w(t_1), z(t_1))$ in C_l, then by letting $t_2\to-\infty$ in (3.9) we get $z(t)=0$ and $w(t)=0$ for any $t\leqq t_1$. In such a case, the inequality $z(t)\leqq lw(t)$ also holds for all $t\leqq t_1$. Consequently $z(t)\leqq lw(t)$ for any $t\leqq t_0$. Then the first inequality of (3.8) yields

$$\frac{1}{2}\frac{d}{dt}w^2\geqq-(\lambda_N+K_1\lambda_N^{\gamma}(1+l))w^2\geqq-\nu w^2.$$

Integrating above inequality, we have that $(w(t), z(t))$ satisfies inequality (3.10) for any $t\leqq t_0$. ■

Proof of Lemma 3.4. Let (w, z) be in $\Omega(B)$. Since $z\geqq\theta B/(1-\theta)$ holds, we may replace B by $(\theta^{-1}-1)z$ in (3.11). Then we have

$$(3.15)\qquad \begin{cases} \dfrac{1}{2}\dfrac{d}{dt}(z^2-l^2w^2)\leqq-(\nu_2-\nu_1)z^2+\nu(z^2-l^2w^2), \\ \dfrac{1}{2}\dfrac{d}{dt}z^2\leqq-(\nu_2-\nu)z^2. \end{cases}$$

In view of the flow on the boundary $\partial\Omega(B)$, these inequalities imply the invariance of

$$\Pi\setminus\Omega(B)=\left\{(w, z)\in\mathbf{R}^2;\ 0\leqq z\leqq lw \text{ or } 0\leqq z\leqq\frac{\theta}{1-\theta}B\right\} \qquad \text{(see Fig. 2)}.$$

If $(w(t_1), z(t_1))\in\Omega(B)$, then its invariance says that $(w(t), z(t))$ belongs to $\Omega(B)$ for $t\leqq t_1$. It follows from the second inequality of (3.15) that

$$z(t)\leqq z(t_2)e^{-(\nu_2-\nu)(t-t_2)} \qquad \text{for any } t_2\leqq t\leqq t_1.$$

Next, we prove the latter part of this lemma. If there exists $t_1(\leqq t_0)$ such that $(w(t_1), z(t_1))\in\Omega(B)$, we obtain

$$z(t_1)\leqq z(t_2)e^{-(\nu_2-\nu)(t_1-t_2)} \qquad \text{for any } t_2(\leqq t_1).$$

Since $z(t)$ is bounded, we have $z(t_1)=0$ by letting $t_2\to-\infty$. This contradicts the hypothesis $(w(t_1), z(t_1))\in\Omega(B)$. Thus we can say that

$$\begin{cases} \text{either} \quad 0\leqq z(t)\leqq lw(t), \\ \text{or} \qquad\qquad z(t)\leqq\dfrac{\theta}{1-\theta}B. \end{cases}$$

In the set

$$\left\{(w, z)\in\mathbf{R}^2;\ 0\leqq z\leqq lw \text{ and } w\geqq\frac{\theta}{(1-\theta)l}B\right\},$$

we see from (3.11) that

$$\frac{1}{2}\frac{d}{dt}w^2\geqq(\nu-\lambda_N-K_1(1+\theta^{-1}l)\lambda_N^{\gamma})w^2>0.$$

This inequality means if $w(t_0)\leqq\theta B/(1-\theta)l$, then $w(t)\leqq\theta B/(1-\theta)l$ for all $t\leqq t_0$. Consequently, we obtain

$$0\leqq z(t)\leqq\frac{\theta}{1-\theta}B. \qquad ■$$

Proof of Lemma 3.5. We can prove this lemma in the same manner as in Lemma 3.4, so we omit it.

Proof of Proposition 3.2. First we will prove that T maps $\mathcal{F}$ into $\mathcal{F}$. We will use the following lemma in order to estimate the bounds of $T(q)$.

Lemma 3.7. *For any* $\tau<0$,

$$|(AQ)^{\gamma}e^{\tau AQ}|_{\mathcal{L}(QH)}\leqq\begin{cases}\gamma^{\gamma}e^{-\gamma}|\tau|^{-\gamma} & \text{for } -\gamma\lambda_{N+1}^{-1}\leqq\tau<0\,,\\ \lambda_{N+1}^{\gamma}e^{\tau\lambda_{N+1}} & \text{for } \tau<-\gamma\lambda_{N+1}^{-1}\,.\end{cases}$$

If $0\leqq\gamma<1$,

$$\int_{-\infty}^{0}|(AQ)^{\gamma}e^{\tau AQ}|_{\mathcal{L}(QH)}d\tau\leqq\frac{e^{-\gamma}}{1-\gamma}\lambda_{N+1}^{\gamma-1}\,.$$

Proof. See R. Temam [15]. ■

The next immediately follows from Lemma 3.7:

$$|A^{\gamma}T(q)(t)|\leqq\int_{-\infty}^{t}|A^{\gamma}e^{-A(t-s)}Q|_{\mathcal{L}(QH)}K_0ds$$
$$\leqq\frac{e^{-\gamma}K_0}{1-\gamma}\lambda_{N+1}^{\gamma-1}\,.$$

Set $\Delta_t p=p(t+h)-p(t)$, $\Delta_t q=T(q)(t+h)-T(q)(t)$. Those satisfy the next equations;

$$\frac{d}{dt}\Delta_t p+A\Delta_t p+PR(t+h,\ p(t+h)+q(t+h))-PR(t,\ p(t)+q(t))=0\,,$$

$$\frac{d}{dt}\Delta_t q+A\Delta_t q+QR(t+h,\ p(t+h)+q(t+h))-QR(t,\ p(t)+q(t))=0\,.$$

Taking the scalar product between above equations and $A^{2\gamma}\Delta_t p$, $A^{2\gamma}\Delta_t q$ respectively and using Poincaré inequality yield

$$\text{(3.16)}\quad\begin{cases}\frac{1}{2}\frac{d}{dt}|A^{\gamma}\Delta_t p|^2\geqq-\lambda_N|A^{\gamma}\Delta_t p|^2\\ \qquad -K_1\lambda_N^{\gamma}\Big(|A^{\gamma}\Delta_t p|+\Big(B_1e^{\nu(t_0-t)}+\frac{K_2}{K_1}\Big)|h|\Big)|A^{\gamma}\Delta_t p|\,,\\ \frac{1}{2}\frac{d}{dt}|A^{\gamma}\Delta_t q|^2\leqq-|A^{\gamma+1/2}\Delta_t q|^2\\ \qquad +K_1\lambda_{N+1}^{\gamma-1/2}\Big(|A^{\gamma}\Delta_t p|+\Big(B_1e^{\nu(t_0-t)}+\frac{K_2}{K_1}\Big)|h|\Big)|A^{\gamma+1/2}\Delta_t q|\,.\end{cases}$$

Put $w(t)=e^{\nu(t-t_0)}|A^{\gamma}\Delta_t p|$, $z(t)=e^{\nu(t-t_0)}|A^{\gamma}\Delta_t q|$. Then the first inequality of (3.16) is

$$\text{(3.17)}\qquad \frac{1}{2}\frac{d}{dt}w^2\geqq(\nu-\lambda_N-K_1\lambda_N^{\gamma})w^2-K_1\Big(B_1+\frac{K_2}{K_1}\Big)|h|w\,.$$

On the other hand from the second of (3.16) we have

$$\text{(3.18)}\qquad\begin{cases}\frac{1}{2}\frac{d}{dt}z^2\leqq-X^2+K_1\lambda_{N+1}^{\gamma-1/2}\Big(w+\Big(B_1+\frac{K_2}{K_1}\Big)|h|\Big)X\,,\\ X=e^{\nu(t-t_0)}|A^{\gamma+1/2}\Delta_t q|\,.\end{cases}$$

Let Ω_1' be $\Omega'(B_1)$ with $B_1\geqq\theta(B_1+K_2/K_1)|h|$ i.e.,

$$\Omega_1'=\Big\{(w,\ z)\in\mathbf{R}^2\,;\ z\geqq lw\geqq0,\ z\geqq\theta\Big(B_1+\frac{K_2}{K_1}\Big)|h|\Big\}\,.$$

Recall (3.1). In Ω_1', we have

$$\begin{aligned} e^{\nu(t-t_0)}|A^{r+1/2}\Delta_t q| &\geqq \lambda_{N+1}^{1/2} e^{\nu(t-t_0)}|A^r \Delta_t q| \\ &\geqq K_1 \lambda_{N+1}^{r-1/2}(\theta^{-1}+l^{-1})z(t) \\ &\geqq K_1 \lambda_{N+1}^{r-1/2}\Big(w(t)+\Big(B_1+\frac{K_2}{K_1}\Big)|h|\Big). \end{aligned}$$

The right hand side of (3.18) decreases as X increases if

$$X \geqq \frac{1}{2} K_1 \lambda_{N+1}^{r-1/2}\Big(w+\Big(B_1+\frac{K_2}{K_1}\Big)|h|\Big).$$

This fact and the inequality $|A^{r+1/2}\Delta_t q| \geqq \lambda_{N+1}^{1/2}|A^r \Delta_t q|$ yield

$$\frac{1}{2}\frac{d}{dt}z^2 \leqq -(\lambda_{N+1}-\nu)z^2+K_1\lambda_{N+1}^r wz+K_1\lambda_{N+1}^r\Big(B_1+\frac{K_2}{K_1}\Big)|h|z \qquad \text{in } \Omega_1'. \tag{3.19}$$

We must show

$$w(t_0-h) \leqq \theta l^{-1}\Big(B_1+\frac{K_2}{K_1}\Big)|h|.$$

If it is valid, then we get

$$z(t) \leqq \theta\Big(B_1+\frac{K_2}{K_1}\Big)|h| \qquad \text{for } t \leqq t_0-h$$

by applying Lemma 3.5 to (3.17) and (3.19). We can estimate as follows:

$$\begin{aligned} w(t_0-h) &= e^{-\nu h}|A^r(p_0-p(t_0-h))| \\ &\leqq e^{-\nu h}\Big(|A^r(p_0-e^{Ah}p_0)|+\int_{t_0-h}^{t_0}|A^r e^{-A(t_0-h-s)}PR(s,\ p+q)|ds)\Big) \\ &\leqq (\lambda_N|A^r p_0|+K_0\lambda_N^r)|h|. \end{aligned}$$

Thus take B_1 satisfying

$$B_1 \geqq \max\Big\{\frac{\theta K_2}{(1-\theta)K_1},\ (\lambda_N|A^r p_0|+K_0\lambda_N^r)\theta^{-1}l\Big\} \tag{3.20}$$

and we obtain that T_{t_0,p_0} maps from $\mathcal{F}$ into itself.

Finally we will prove that T is a contraction map. Let q_1 and q_2 be in $\mathcal{F}$. Put $\Delta_q p=p(t,\ t_0,\ p_0,\ q_1)-p(t,\ t_0,\ p_0,\ q_2)$, $\Delta_q q=T_{t_0,p_0}(q_1)(t)-T_{t_0,p_0}(q_2)(t)$ which satisfy the following inequalities:

$$\frac{1}{2}\frac{d}{dt}|A^r\Delta_q p|^2 \geqq -\lambda_N|A^r\Delta_q p|^2-K_1\lambda_N^r(|A^r\Delta_q p|+|A^r(q_1-q_2)|)|A^r\Delta_q p|,$$

$$\frac{1}{2}\frac{d}{dt}|A^r\Delta_q q|^2 \leqq -|A^{r+1/2}\Delta_q q|^2+K_1\lambda_{N+1}^{r-1/2}(|A^r\Delta_q p|+|A^r(q_1-q_2)|)|A^{r+1/2}\Delta_q q|.$$

Set $w_q(t)=e^{\nu(t-t_0)}|A^r\Delta_q p|$, $z_q(t)=e^{\nu(t-t_0)}|A^r\Delta_q q|$ and

$$\Omega_2=\{(w,\ z)\in\mathbf{R}^2;\ z \geqq lw \geqq 0,\ z \geqq \theta d_\nu(q_1,\ q_2)\}.$$

By using Lemma 3.5 and $w(t_0)=0$, we can prove similarly as in the above that $z_q(t) \leqq$

$\theta d_\nu(q_1, q_2)$. ■

Applying the contraction mapping theorem, we can find a fixed point in $\mathcal{F}$ which we denote by $q(t, t_0, p_0)$. We also denote a unique solution of the next equation by $p(t, t_0, p_0)$,

$$\left\{\begin{array}{l} \frac{d}{dt}p+Ap+PR(t, p+q(t, t_0, p_0))=0\,, \\ \\ p(t_0)=p_0\,. \end{array}\right. \tag{3.21}$$

Proposition 3.8. *$p(t, t_0, p_0)$ and $q(t, t_0, p_0)$ are locally Lipschitz with respect to t_0 and p_0.*

Proof. Set

$$w_\xi(t)=e^{\nu(t-t_0)}|A^\gamma(p(t, t_0, p_0+\xi)-p(t, t_0, p_0))|,$$

$$z_\xi(t)=e^{\nu(t-t_0)}|A^\gamma(q(t, t_0, p_0+\xi)-q(t, t_0, p_0))|.$$

Two functions w_ξ, z_ξ satisfy

$$\frac{1}{2}\frac{d}{dt}w_\xi^2\geqq(\nu-\lambda_N-K_1\lambda_N^\gamma)w_\xi^2-K_1\lambda_N^\gamma w_\xi z_\xi \qquad \text{in } II\,,$$

$$\frac{1}{2}\frac{d}{dt}z_\xi^2\leqq-(\lambda_{N+1}-K_1\lambda_{N+1}^\gamma-\nu)z_\xi^2+K_1\lambda_{N+1}^\gamma w_\xi z_\xi \qquad \text{in } C_l\,.$$

It follows from the boundedness of $z_\xi(t)$ and the squeezing property that $z_\xi(t)\leqq lw_\xi(t)\leqq l|A^\gamma\xi|$ for all $t\leqq t_0$, which proves Lipschitz continuity with respect to p_0. As for Lipschitz continuity with respect to t_0, we can prove the following similarly:

$$\begin{aligned} &e^{\nu(t-t_0)}|A^\gamma(q(t, t_0, p_0)-q(t, t_0-h, p_0))| \\ &\quad\leqq le^{\nu(t-t_0)}|A^\gamma(p(t, t_0, p_0)-p(t, t_0-h, p_0))| \\ &\quad\leqq le^{-\nu h}|A^\gamma(p(t_0-h, t_0, p_0)-p_0)| \qquad \text{(by using Lemma 3.3)} \\ &\quad\leqq l(\lambda_N|A^\gamma p_0|+K_0\lambda_N^\gamma)|h| \qquad \text{for } t\leqq t+h\leqq t_0\,. \end{aligned}$$ ■

We will show that this fixed point $q(t, t_0, p_0)$ is represented by a graph from $PD(A^\gamma)$ into $QD(A^\gamma)$.

Lemma 3.9. *$q(t_1, t_0, p_0)=q(t_1, t_0, p_1)$ for $t_1\leqq t_0$ where $p_1=p(t_1, t_0, p_0)$.*

Proof. Put $\Delta p=p(t, t_0, p_0)-p(t, t_1, p_1)$, $\Delta q=q(t, t_0, p_0)-q(t, t_1, p_1)$. The squeezing property shows that $|A^\gamma\Delta q|\leqq l|A^\gamma\Delta p|$. This lemma follows from substituting $t=t_1$ into Δp, Δq i.e.,

$$|A^\gamma(q(t_1, t_0, p_0)-q(t_1, t_1, p_1))|\leqq l|A^\gamma(p(t_1, t_0, p_0)-p(t_1, t_1, p_1))|=0\,.$$ ■

Set

$$\Phi(t, p)=q(t, t, p)\,.$$

Lemma 3.9 implies that

$$\Phi(t, p(t, t_0, p_0))=q(t, t_0, p_0).$$

Therefore it turns out that $p(t, t_0, p_0)$, $q(t, t_0, p_0)$ are solutions of

$$\left\{\begin{array}{l}\frac{d}{dt}p+Ap+PR(t, p+\Phi(t, p))=0\,, \\ p(t_0)=p_0\,, \\ \frac{d}{dt}q+Aq+QR(t, p+\Phi(t, p))=0\,.\end{array}\right. \tag{3.22}$$

We can easily check that

$$|A^{\gamma}(\Phi(t+h, p)-\Phi(t, p))|\leqq\frac{\theta K_2}{(1-\theta)K_1}|h|,$$

$$|A^{\gamma}(\Phi(t, p_1)-\Phi(t, p_2))|\leqq l|A^{\gamma}(p_1-p_2)|.$$

Here we used Lemma 3.3, 3.4 and (3.21).

Now consider the exponential attractivity. Let $u(t)$ be any solution of (1.1) with the initial value u_0. Define

$$D_{t,u_0}=\{p+\Phi(t, p)\in D(A^{\gamma})\,;\ |A^{\gamma}(Qu(t)-\Phi(t, p))|\geqq l|A^{\gamma}(Pu(t)-p)|\}.$$

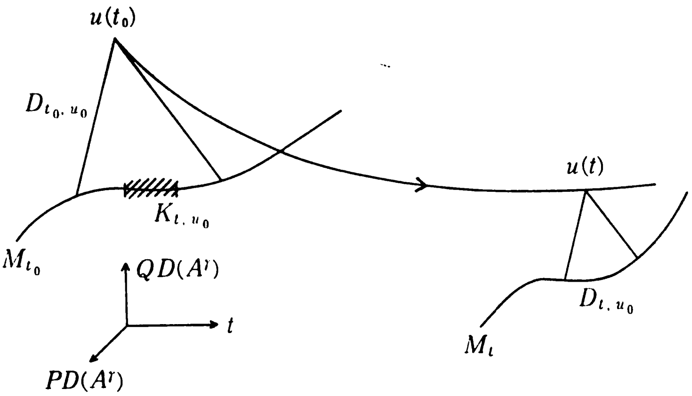

Fig. 3. Exponential Tracking.

This set is not empty because $Pu(t)+\Phi(t, Pu(t))\in D_{t,u_0}$. On the other hand, the semiflow is invertible on the inertial manifold. Setting $\tilde{S}(t_1, t_2)=S(t_1, t_2)|_{M_{t_2}}$, we have

$$\tilde{S}(t_1, t_2)\tilde{S}(t_2, t_3)=\tilde{S}(t_1, t_3) \qquad \text{for any } t_1, t_2, t_3\in\mathbf{R}\,.$$

We set $K_{t,u_0}=P\tilde{S}(t_0, t)D_{t,u_0}$ for $t\geqq t_0$. By applying the squeezing property in Lemma 3.3 to the difference of two solutions, we get

$$\tilde{S}(t_1, t_2)D_{t_2,u_0}\subset D_{t_1,u_0} \qquad \text{for } t_0\leqq t_1\leqq t_2\,.$$

Thus $K_{t_2,u_0}\subset K_{t_1,u_0}\subset K_{t_0,u_0}$ for $t_0\leqq t_1\leqq t_2$ (see Fig. 3). It is obvious that K_{t,u_0} is a closed bounded set with finite dimension. Set $K_{u_0}=\cap_{t\geqq 0}K_{t,u_0}$ which is a nonempty compact set. We can choose an element p_0 in K_0. Take the solution $v(t)$ on the manifold as

$$v(t)=\tilde{S}(t, t_0)(p_0+\Phi(t_0, p_0))\in K_{t, u_0}$$
$$=p(t, t_0, p_0)+\Phi(t, p(t, t_0, p_0)).$$

Because of the definition of K_{t, u_0},

$$|A^{r}Q(u(t)-v(t))|\geqq l|A^{r}P(u(t)-v(t))| \qquad \text{for any } t\geqq t_0,$$
$$|A^{r}Q(u(t)-v(t))|\leqq |A^{r}Q(u(t_0)-v(t_0))|e^{-\nu(t-t_0)} \quad \text{(by the squeezing property)}$$
$$\leqq\Big(|A^{r}Qu_0|+K_0\frac{e^{-\gamma}}{1-\gamma}\lambda_{N+1}^{\gamma-1}\Big)e^{-\nu(t-t_0)}.$$

Thus we can conclude

$$|A^{r}(u(t)-v(t))|\leqq c_1 e^{-\nu(t-t_0)},$$

where

$$c_1=(1+l^{-2})^{1/2}\Big(|A^{r}(I-P_N)u_0|+K_0\frac{e^{-\gamma}}{1-\gamma}\lambda_{N+1}^{\gamma-1}\Big). \tag{3.23}$$ ■

4. Regularity of an inertial manifold

Consider the Galerkin approximate equation

$$\frac{d}{dt}u_M+Au_M+P_MR(t, u_M)=0, \tag{4.1}$$

where $M\geqq N+1$. We can construct an inertial manifold for this approximate equation (4.1) with the same dimension N. Define the subspace of $\mathcal{F}$,

$$\mathcal{F}_M=\Big\{q_M(t)\in C^0((-\infty, t_0]\,;\, P_MQD(A^{r}))\,;$$
$$|A^{r}q_M(t)|\leqq\frac{e^{-\gamma}K_0}{1-\gamma}\lambda_{N+1}^{\gamma-1}e^{\nu(t_0-t)},$$
$$|A^{r}(q_M(t+h)-q_M(t))|\leqq B_1|h|e^{\nu(t_0-t)} \quad \text{for all } t\leqq t+h\leqq t_0\Big\}$$

where B_1 satisfies (3.20). Let $q_M(t)$ be in $\mathcal{F}_M$. Consider the next equation

$$\begin{cases} \dfrac{d}{dt}p_M+Ap_M+PR(t, p_M+q_M(t))=0, \\ p_M(t_0)=p_0. \end{cases}$$

We define a mapping on $\mathcal{F}_M$ using a solution $p_M(t)=p_M(t, t_0, p_0, q_M)$ of above equation as follows:

$$T_M(q_M)(t)=-\int_{-\infty}^{t}e^{-A(t-s)}P_MQR(s, p_M(s)+q_M(s))ds. \tag{4.2}$$

We can check similarly as in §3 that T_M is a contraction mapping. We denote the fixed point of T_M by $q_M(t, t_0, p_0)$. Let $p_M(t, t_0, p_0)$ be a unique solution of the following equation:

$$\begin{cases} \dfrac{d}{dt}p+Ap+PQ(t,\ p+q_M(t,\ t_0,\ p_0))=0, \\ p(t_0)=p_0. \end{cases} \tag{4.3}$$

We claim that the sequences $\{p_M\}_{M\geq N+1}$, $\{q_M\}_{M\geq N+1}$ are Cauchy sequences in $\mathfrak{F}$ (uniformly in p_0). Indeed since $P_M T|_{\mathfrak{F}_M}=T_M$,

$$\begin{aligned} d_\nu(q,\ q_M)&=\sup_{t\leq t_0} e^{\nu(t-t_0)}|A^\gamma(T(q)(t)-T_M(q_M)(t))| \\ &\leqq d_\nu(T(q),\ T(q_M))+\sup_{t\leq t_0} e^{\nu(t-t_0)}|A^\gamma(T(q_M)(t)-T_M(q_M)(t))| \\ &\leqq \theta d_\nu(q,\ q_M)+\sup_{t\leq t_0} e^{\nu(t-t_0)}|A^\gamma Q_M T(q_M)(t)|. \end{aligned}$$

Thus we get

$$\begin{aligned} d_\nu(q,\ q_M)&\leqq \frac{1}{1-\theta}\sup_{t\leq t_0} e^{\nu(t-t_0)}|A^\gamma Q_M T(q_M)(t)| \\ &\leqq \frac{1}{1-\theta}\sup_{t\leq t_0} e^{\nu(t-t_0)}\left|A^\gamma\int_{-\infty}^{t} e^{-A(t-s)}Q_M R(s,\ p_M(s)+q_M(s))ds\right| \\ &\leqq \frac{2K_0}{1-\theta}(\lambda_{M+1}-\nu)^{\gamma-1} \qquad \text{(by Lemma 3.7).} \end{aligned} \tag{4.4}$$

We will estimate $|A^\gamma(p_M-p)|$. From (3.21) and (4.3), we can see that

$$\frac{1}{2}\frac{d}{dt}|A^\gamma(p_M-p)|^2\geqq-(\lambda_N+K_1\lambda_N^\gamma)|A^\gamma(p_M-p)|^2-K_1\lambda_N^\gamma|A^\gamma(q_M-q)||A^\gamma(p_M-p)|$$

holds. Multiplying the above inequality by $e^{\nu(t-t_0)}$ and using (4.4), we deduce

$$e^{\nu(t-t_0)}|A^\gamma(p_M-p)|\leqq l^{-1}\frac{2K_0}{1-\theta}(\lambda_{M+1}-\nu)^{\gamma-1}. \tag{4.5}$$

This and (4.4) implies that our claim is valid.

Next we will show that the existence of Fréchet derivatives $D_{p_0}p_M(t,\ t_0,\ p_0)$ and $D_{p_0}q_M(t,\ t_0,\ p_0)$. Differentiate (4.1) with respect to the initial deta p_0 and decompose it as follows:

$$\begin{cases} \dfrac{d}{dt}\rho_M\xi+A\rho_M\xi+PD_uR(t,\ p_M+q_M)(\rho_M\xi+\sigma_M\xi)=0, \\ \rho_M(t_0,\ t_0,\ p_0)\xi=\xi, \\ \dfrac{d}{dt}\sigma_M\xi+A\sigma_M\xi+P_M D_uR(t,\ p_M+q_M)(\rho_M\xi+\sigma_M\xi)=0, \end{cases} \tag{4.6}$$

where $\rho_M(t)=D_{p_0}p_M$, $\sigma_M(t)=D_{p_0}q_M$. We observe that this formal proceeding is justified below. Let $\tilde{\mathfrak{F}}_M$ be

$$\{\sigma_M\in C^0((-\infty,\ t_0];\ \mathcal{L}(P_M D(A^\gamma);\ (I-P_M)QD(A^\gamma)));\ |A^\gamma\sigma_M|_{op}\leqq 2le^{\nu(t_0-t)}\},$$

where $|\cdot|_{op}$ denotes the operator norm, that is, for any $L\in\mathcal{L}(D(A^\gamma);\ H)$,

$$|L|_{op}=\sup_{u\in D(A^\gamma),\ |A^\gamma u|\leq 1}|Lu|.$$

For any $\sigma_M \in \tilde{\mathfrak{F}}_M$, let $\rho_M\xi = \rho_M(t, t_0, p_0)\xi$ be a solution of

$$\begin{cases} \dfrac{d}{dt}\rho_M\xi + A\rho_M\xi + PD_uR(t, p_M+q_M)(\rho_M\xi+\sigma_M\xi)=0\,, \\ \rho_M(t_0, t_0, p_0)\xi=\xi\,. \end{cases}$$

This solution $\rho_M\xi$ is linear in ξ. We define the operator

$$\tilde{T}_M(\sigma_M)\xi = -\int_{-\infty}^{t} e^{-A(t-s)}P_M QD_uR(s, p_M(s)+q_M(s))(\rho_M\xi+\sigma_M\xi)ds\,,$$

Using Lemma 3.5, we have

$$e^{\nu(t-t_0)}|A^r\tilde{T}_M(\sigma_M)\xi| \leqq l|A^r\xi| \qquad \text{for any } \sigma_M \in \tilde{\mathfrak{F}}_M\,,$$

$$e^{\nu(t-t_0)}|A^r(\tilde{T}_M(\sigma_{M,1})\xi - \tilde{T}_M(\sigma_{M,2})\xi)| \leqq \theta \sup_{t\leq t_0} e^{\nu(t-t_0)}|A^r(\sigma_{M,1}\xi-\sigma_{M,2}\xi)|\,,$$

for any ν in the interval (ν_1, ν_2) and $\sigma_{M,1}\sigma_{M,2} \in \tilde{\mathfrak{F}}_M$.

Therefore there exists a fixed point $\sigma_M\xi = \tilde{T}_M(\sigma_M)\xi$ satisfying

$$\frac{d}{dt}\sigma_M\xi + A\sigma_M\xi + P_M QD_uR(t, p_M+q_M)(\rho_M\xi+\sigma_M\xi)=0\,.$$

Hence the equation (4.6) is justified.

We will show continuity of ρ_M, σ_M with respect to p_0.

Lemma 4.1. *Assume that $f(t, p)$ belongs to $C^0(\mathbf{R}\times PD(A^r); H)$. Fix any p_0 in $PD(A^r)$. For any $\varepsilon>0$, there exists a positive δ such that*

$$\sup_{t\leq t_0} e^{(\nu-\nu_1)(t-t_0)/2}|D_uR(t, f(t, p_0+p)) - D_uR(t, f(t, p_0))|_{op} \leqq \varepsilon$$

for all $p \in PD(A^r)$ satisfying $|A^r p| \leqq \delta$.

Proof. Set $t_\varepsilon = t_0 - (2/(\nu-\nu_1))\log 2K_1\varepsilon$, which is less than t_0 if ε is sufficiently small. We obtain

$$\sup_{t\leq t_\varepsilon} e^{(\nu-\nu_1)(t-t_0)/2}|D_uR(t, f(t, p_0+p)) - D_uR(t, f(t, p_0))|_{op}$$
$$\leqq 2K_1 e^{(\nu-\nu_1)(t_\varepsilon-t_0)/2} \leqq \varepsilon\,.$$

On the other hand, the mapping

$$(t, p) \longrightarrow e^{(\nu-\nu_1)(t-t_0)/2}|D_uR(t, f(t, p_0+p)) - D_uR(t, f(t, p_0))|_{op}$$

is uniformly continuous in the interval $[t_\varepsilon, t_0]\times\{p\in PD(A^r);\ |A^r p| \leqq 1\}$. Thus we can find a positive number δ such that for any p satisfying $|A^r p| \leqq \delta$,

$$\sup_{t_\varepsilon \leq t \leq t_0} e^{(\nu-\nu_1)(t-t_0)/2}|D_uR(t, f(t, p_0+p)) - D_uR(t, f(t, p_0))|_{op} \leqq \varepsilon\,. \qquad \blacksquare$$

Lemma 4.2. *The solutions $\rho_M(t, t_0, p_0)$, $\sigma_M(t, t_0, p_0)$ of (4.6) are continuous with respect to $p_0 \in PD(A^r)$.*

Proof. Set

$$w(t)=e^{\nu(t-t_0)}|A^{\gamma}(\rho_M(t, t_0, p_0+p_1)\xi-\rho_M(t, t_0, p_0)\xi)|,$$

$$z(t)=e^{\nu(t-t_0)}|A^{\gamma}(\sigma_M(t, t_0, p_0+p_1)\xi-\sigma_M(t, t_0, p_0)\xi)|.$$

Those functions $w(t)$, $z(t)$ satisfy the following inequalities:

$$\frac{1}{2}\frac{d}{dt}w^2\geqq-\lambda_N w^2-K_1\lambda_N^{\gamma}(w+z)w-K_1\lambda_N^{\gamma}B_2 w \qquad \text{in } \Pi,$$

$$\frac{1}{2}\frac{d}{dt}z^2\leqq-\lambda_{N+1}z^2+K_1\lambda_{N+1}^{\gamma}(w+z)z+K_1\lambda_{N+1}^{\gamma}B_2 z \qquad \text{in } \Omega(B_2).$$

where

$$B_2=\frac{(1+l)|A^{\gamma}\xi|}{K_1}\sup_{t\leq t_0} e^{(\nu-\nu_1)(t-t_0)/2}|D_uR(t, p(t, t_0, p_0+p_1)+q(t, t_0, p_0+p_1)) -D_uR(t, p(t, t_0, p_0)+q(t, t_0, p_0))|_{op}.$$

Applying Lemma 3.4 to the above inequalities and using Lemma 4.1, we conclude the proof. ■

Next we will prove that ρ_M, σ_M converge. Put $M\geqq M'\geqq N+1$. We set

$$w(t)=e^{\nu(t-t_0)}|A^{\gamma}(\rho_M(t, t_0, p_0)\xi-\rho_{M'}(t, t_0, p_0)\xi)|,$$

$$z(t)=e^{\nu(t-t_0)}|A^{\gamma}(\sigma_M(t, t_0, p_0)\xi-\sigma_{M'}(t, t_0, p_0)\xi|.$$

We can easily check that these satisfy

$$\frac{d}{dt}w^2\geqq-\lambda_N w^2-K_1\lambda_N^{\gamma}(w+z)w-K_1\lambda_N^{\gamma}B_3 w \qquad \text{in } \Pi,$$

$$\frac{d}{dt}z^2\leqq-\lambda_{N+1}z^2+K_1\lambda_{N+1}^{\gamma}(w+z)z+K_1\lambda_{N+1}^{\gamma}B_3 z \qquad \text{in } \Omega(B_3),$$

where

$$B_3=\frac{(1+l)|A^{\gamma}\xi|}{K_1}\sup_{t\leq t_0} e^{(\nu-\nu_1)(t-t_0)/2}|D_uR(t, p_M(t, t_0, p_0)+q_M(t, t_0, p_0)) -D_uR(t, p_{M'}(t, t_0, p_0)+q_{M'}(t, t_0, p_0))|_{op}.$$

Lemma 3.4 implies that

$$w(t)\leqq\frac{\theta}{(1-\theta)l}B_3, \qquad z(t)\leqq\frac{\theta}{1-\theta}B_3.$$

If B_3 tends to 0 as $M, M'\to\infty$, we can get that ρ_M, σ_M converge compact uniformly in p_0. Indeed, if not, there exist ε, $t_j(\leqq t_0)$, M_j and p_{0_j} such that

$$e^{(\nu_1-\nu)(t_0-t_j)/2}|DR(t_j, p_{M_j}(t_j, t_0, p_{0j})+q_{M_j}(t_j, t_0, p_{0_j})) -DR(t_j, p(t_j, t_0, p_{0_j})+q(t_j, t_0, p_{0_j}))|_{op}\geqq\varepsilon.$$

This inequality shows that t_j $(j=1, 2, \cdots)$ is bounded. We may assume that t_j converges to t^* and p_{0_j} converges to p_0^* by taking a subsequence of $\{t_j\}$, if necessary. There exists ε' such that

(4.7) $$|D_uR(t_j,\ p_{M_j}(t_j,\ t_0,\ p_{0j})+q_{M_j}(t_j,\ t_0,\ p_{0j}))$$
$$-D_uR(t_j,\ p(t_j,\ t_0,\ p_{0j})+q(t_j,\ t_0,\ p_{0j}))|_{op}\geqq\varepsilon'\,.$$

Let $M_j\to\infty$ in (4.4). We can get by using (4.4) and (4.5)

$$p_{M_j}(t_j,\ t_0,\ p_{0j})+q_{M_j}(t_j,\ t_0,\ p_{0j})\longrightarrow p(t^*,\ t_0,\ p_0^*)+q(t^*,\ t_0,\ p_0^*)\,,$$
$$p(t_j,\ t_0,\ p_{0j})+q(t_j,\ t_0,\ p_{0j})\longrightarrow p(t^*,\ t_0,\ p_0^*)+q(t^*,\ t_0,\ p_0^*)\,.$$

Considering the above fact and the continuity of D_uR implies that the left side of (4.7) must tend to zero, which is a contradiction. Thus $\rho_M(t,\ t_0,\ p_0)$ and $\sigma_M(t,\ t_0,\ p_0)$ have their limits $\rho(t,\ t_0,\ p_0)$ and $\sigma(t,\ t_0,\ p_0)$ respectively.

Next we will show that ρ_M and σ_M are derivative of $p_M,\ q_M$ respectively. Set

$$w(t)=e^{\nu(t-t_0)}|A^\gamma p_M(t,\ t_0,\ p_0+s\xi)-p_M(t,\ t_0,\ p_0)-s\rho_M(t,\ t_0,\ p_0)\xi|\,,$$
$$z(t)=e^{\nu(t-t_0)}|A^\gamma q_M(t,\ t_0,\ p_0+s\xi)-q_M(t,\ t_0,\ p_0)-s\sigma_M(t,\ t_0,\ p_0)\xi|\,.$$

These satisfy

$$\frac{d}{dt}w^2\geqq-\lambda_N w^2-K_1\lambda_N^\gamma(w+z)w-K_1\lambda_N^\gamma B_4 w \qquad \text{in } \Pi\,,$$

$$\frac{d}{dt}z^2\leqq-\lambda_{N+1}z^2+K_1\lambda_{N+1}^\gamma(w+z)z+K_1\lambda_N^\gamma B_4 z \qquad \text{in } \Omega(B_4)\,,$$

where we let B_4 be

$$\frac{(1+l)|A^\gamma s\xi|}{K_1}\sup_{t\leq t_0}e^{(\nu-\nu_1)(t-t_0)/2}\int_0^1|D_uR(t,\ \zeta(p_M(t,\ t_0,\ p_0+s\xi)+q_M(t,\ t_0,\ p_0+s\xi))$$
$$+(1-\zeta)(p_M(t,\ t_0,\ p_0)+q_M(t,\ t_0,\ p_0)))-D_uR(t,\ p_M(t,\ t_0,\ p_0)+q_M(t,\ t_0,\ p_0))|_{op}d\zeta.$$

Here we used a mean value formula for any continuously differentiable function f,

$$f(u)-f(v)=\int_0^1 D_uf(\zeta u+(1+\zeta)v)d\zeta(u-v)\,.$$

Lemma 3.5 implies that $w(t)\leqq\theta B_4/(1-\theta)l,\ z(t)\leqq\theta B_4/(1-\theta)$. We can also get $B_4=o(|s|)$ compact uniformly in p_0 as $s\to0$. We conclude that $p_M,\ q_M$ have Fréchlet derivatives by using the fact that $\rho_M,\ \sigma_M$ is continuous with respect to p_0. Hence we get the following lemma:

Lemma 4.3. *$p_M(t,\ t_0,\ p_0),\ q_M(t,\ t_0,\ p_0)$ have Fréchlet derivatives in p_0.*

By the mean value formula, we have

$$p_M(t,\ t_0,\ p_0+s\xi)-p_M(t,\ t_0,\ p_0)=\int_0^1\rho_M(t,\ t_0,\ p_0+s\zeta\xi)s\xi d\zeta\,,$$

$$q_M(t,\ t_0,\ p_0+s\xi)-q_M(t,\ t_0,\ p_0)=\int_0^1\sigma_M(t,\ t_0,\ p_0+s\zeta\xi)s\xi d\zeta\,.$$

Letting $M\to\infty$, we have the following equalities:

$$(4.8)\qquad \begin{cases} p(t,\ t_0,\ p_0+s\xi)-p(t,\ t_0,\ p_0)=\int_0^1 \rho(t,\ t_0,\ p_0+s\zeta\xi)s\xi d\zeta\,. \\ q(t,\ t_0,\ p_0+s\xi)-q(t,\ t_0,\ p_0)=\int_0^1 \sigma(t,\ t_0,\ p_0+s\zeta\xi)s\xi d\zeta\,. \end{cases}$$

Hence we deduce from the contiuity of ρ, σ in p_0 and (4.8)

$$\begin{cases} |A^{\gamma}(p(t,\ t_0,\ p_0+s\xi)-p(t,\ t_0,\ p_0)-\rho(t,\ t_0,\ p_0)s\xi)| \\ \qquad \leqq \int_0^1 |A^{\gamma}(\rho(t,\ t_0,\ p_0+s\zeta\xi)-\rho(t,\ t_0,\ p_0))s\xi|\,d\zeta \\ \qquad \leqq o(|s|)\,, \\ |A^{\gamma}(q(t,\ t_0,\ p_0+s\xi)-q(t,\ t_0,\ p_0)-\sigma(t,\ t_0,\ p_0)s\xi)| \\ \qquad \leqq o(|s|)\,. \end{cases}$$

Thus we have proved that $p(t,\ t_0,\ p_0)$ and $q(t,\ t_0,\ p_0)$ have Fréchlet derivatives.

5. Proof of Theorem 2.5

First, we note that we can choose l, θ and ν satisfying

$$\begin{cases} \lambda_N+K_1(1+\theta^{-1}l)\lambda_N^{\gamma}\leqq\nu\leqq\lambda_{N+1}-K_1(\theta^{-1}+l^{-1})\,, \\ 0<\theta<1\,, \\ \eta\leqq\nu\,. \end{cases}$$

Recall that $p_1(t,\ t_0,\ p_0)$, $p_2(t,\ t_0,\ p_0)$ are the solutions to (2.7) and (2.8) respectively. For simplicity of our notations, we set

$$(5.1)\qquad \begin{cases} p_1(t)=p_1(t,\ t_0,\ p_0), & p_2(t)=p_2(t,\ t_0,\ p_0)\,, \\ q_1(t)=\Phi(t,\ p_1(t)), & q_2(t)=\tilde{\Phi}(t,\ p_2(t))\,, \\ p(t)=p_1(t)-p_2(t), & q(t)=q_1(t)-q_2(t)\,. \end{cases}$$

Then we get the following equations by (2.7) and (2.8):

$$\frac{dp}{dt}+Ap+PR(t,\ p_1+\Phi(t,\ p_1))-P\tilde{R}(t,\ p_2+\tilde{\Phi}(t,\ p_2))=0\,,$$

$$\frac{dq}{dt}+Aq+QR(t,\ p_1+\Phi(t,\ p_1))-Q\tilde{R}(t,\ p_2+\tilde{\Phi}(t,\ p_2))=0\,.$$

Therefore putting

$$(5.2)\qquad w(t)=e^{\nu(t-t_0)}|A^{\gamma}p(t)|,\quad z(t)=e^{\nu(t-t_0)}|A^{\gamma}q(t)| \qquad \text{for any } t\leqq t_0$$

and using the same arguement in §3, we have

$$\frac{d}{dt}w^2\geqq-\lambda_N w^2-K_1\lambda_N^{\gamma}(w+z)w-K_1\lambda_N^{\gamma}B_6 w \qquad \text{in } \Pi,$$

$$\frac{d}{dt}z^2 \leqq -\lambda_{N+1}z^2 + K_1\lambda_{N+1}^{\gamma}(w+z)z + K_1\lambda_{N+1}^{\gamma}B_5 z \qquad \text{in } \Omega(B_5),$$

where $B_5=(K_3/K_1)e^{-\eta t_0}$. Using Lemma 3.5 yields

$$z(t) \leqq \frac{\theta K_3}{(1-\theta)K_1}e^{-\eta t_0} \qquad \text{for any } t \leqq t_0 .$$

Particularly when we put $t=t_0$, we get

$$|A^{\gamma}(\Phi(t_0,\ p_0)-\tilde{\Phi}(t_0,\ p_0))| \leqq c_2 K_3 e^{-\eta t_0}$$

where

$$c_2 = \frac{\theta}{(1-\theta)K_1} \tag{5.3}$$

(see (5.1) and (5.2)). Let us prove the last part of Theorem 2.5. The condition (2.12) assure that we can find l, θ satisfying $\nu_1<\eta<\nu_2$ where ν_1, ν_2 are as in (3.2). We put

$$p_1(t\,;\,t_i)=p_1(t,\ t_i,\ p_2(t_i)) \qquad (i=1,\ 2).$$

Then we can easily check that

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}|A^{\gamma}\bar{p}|^2 \geqq -(\lambda_N+K_1(1+l)\lambda_N^{\gamma})|A^{\gamma}\bar{p}|^2 \geqq -\nu_1|A^{\gamma}\bar{p}|^2 , \\ \bar{p}(t)=p_1(t\,;\,t_1)-p_1(t\,;\,t_2). \end{cases}$$

Thus for $t \leqq t_1 \leqq t_2$,

$$|A^{\gamma}(p_1(t\,;\,t_1)-p_1(t\,;\,t_2))| \leqq |A^{\gamma}(p_2(t_1)-p_1(t_1\,;\,t_2))|e^{\nu_1(t_1-t)} . \tag{5.4}$$

On the other hand we estimate the difference of the solutions of (2.7) and (2.8). Considering A4, we see that $\tilde{p}(t)=p_2(t)-p_1(t\,;\,t_2)$ satisfies

$$\frac{1}{2}\frac{d}{dt}|A^{\gamma}\tilde{p}|^2+(\lambda_N+K_1(1+l)\lambda_N^{\gamma})|A^{\gamma}\tilde{p}|^2 \geqq -K_3 e^{-\eta t}\lambda_N^{\gamma}|A^{\gamma}\tilde{p}| .$$

Through the observation of the flow of

$$\tilde{w}(t)=e^{\eta t}|A^{\gamma}\tilde{p}(t)|$$

and the fact $\tilde{w}(t_2)=0$, we have

$$\tilde{w}(t) \leqq \frac{\theta}{(1-\theta)l}\frac{K_3}{K_1} \qquad \text{for any } t \leqq t_2 .$$

Hence

$$|A^{\gamma}\tilde{p}(t_1)| \leqq \frac{\theta K_3}{(1-\theta)lK_1}e^{-\eta t_1} \qquad \text{for any } t_1 \leqq t_2 . \tag{5.5}$$

Substituting (5.5) to (5.4) yields

$$|A^{\gamma}(p_1(t\,;\,t_1)-p_1(t\,;\,t_2))| \leqq \frac{\theta K_3}{(1-\theta)lK_1}e^{-(\eta-\nu_1)t_1-\nu_1 t} \qquad \text{for } t \leqq t_1 \leqq t_2 . \tag{5.6}$$

It is shown that $p_1(t\,;\,t_2)$ converges compact uniformly when t_2 tends to infinity. The limit function $\hat{p}_1(t)=\lim_{t_2\to\infty}p_1(t\,;\,t_2)$ is also the solution of (2.7). By letting $t_1=t$ and

$t_2\to\infty$ in (5.6), we obtain

$$|A^{\gamma}(p_2(t)-\hat{p}_1(t))|\leqq c_3K_3e^{-\eta t},$$

where

(5.7) $$c_3=\frac{\theta}{(1-\theta)lK_1}\left(=\frac{c_2}{l}\right). \qquad \blacksquare$$

6. Example

Consider the next example, Kuramoto-Sivashinsky equation,

(6.1) $$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial^4 u}{\partial x^4}+\dfrac{\partial^2 u}{\partial x^2}+u\dfrac{\partial u}{\partial x}=0 & \text{in } -\infty<x<\infty,\ t>0, \\ u(0,x)=u_0(x) & \text{in } -\infty<x<\infty, \\ u(t,x+L)=u(t,x) & \text{in } -\infty<x<\infty,\ t>0, \\ u(t,-x)=u(t,x) & \text{in } -\infty<x<\infty,\ t>0. \end{cases}$$

Let H be

$$\left\{u\in L^2\left(-\frac{L}{2},\frac{L}{2}\right);\ u(x)=-u(-x)\right\}.$$

This equation has an absorbing set $\mathcal{B}$, that is, a compabt set which attracts solutions in a finite time. Especially,

$$\mathcal{B}\subset\left\{u\in H;\ u\in H^2\left(-\frac{L}{2},\frac{L}{2}\right),\ |u|\leqq\rho_0,\ |u_x|\leqq\rho_1\right\},$$

where constants ρ_0, ρ_1 depend only on L (see B. Nicolaenko, B. Scheuer and R. Temam [14] and its refeɹences)

$$\rho_0=O(L^{5/2}),$$
$$\rho_1=O(L^{7/2}).$$

We define

$$A_0u=(P_{N_0}u)_{xx},$$
$$Au=u_{xxxx}+u_{xx}-A_0u,$$
$$R(u)=\varphi\left(\frac{|u|}{2\rho_0}\right)\varphi\left(\frac{|u_x|}{2\rho_1}\right)(uu_x-A_0u),$$

where $N_0=[L/2\pi]$ ($[\cdot]$ is denoted by Gauss' symbol) and φ is a smooth function satisfying $|\varphi'|\leqq 3$ and

$$\varphi(x)=\begin{cases} 1 & \text{if } x\leqq 1, \\ 0 & \text{if } x\geqq\dfrac{3}{2}. \end{cases}$$

Then we can easily check that

$$|R(u)-R(v)|\leqq c_4L^4|u-v|+c_5L^3|A^{1/4}(u-v)|$$

(see P. Constantin, C. Foias, B. Nicolaenko and R. Temam [3] and [7]). It is easily shown that the condition A1 and A2 are satisfied. We can find some number N satisfying

$$\lambda_{N+1}-\lambda_N-4c_4L^4-2c_5L^3(\lambda_{N+1}^{1/4}+\lambda_N^{1/4})>0\,.$$

because $\lambda_N=O(N^4)$, $\lambda_{N+1}-\lambda_N=O(N^3)$. Since $R(\cdot)$ is C^1, we can construct the C^1-inertial manifold.

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DEPARTMENT OF MATHFMATICS
KYOTO UNIVERSITY

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Added in proofs: After submission, the author knew the following article:

F. Demengel and J. M. Ghidaglia, Some remarks on the smoothness of inertial manifolds, Nonlinear Analysis T. M. A. **16** (1991), 79-87.

They prove the existence of the inertial manifold and its regularity in the case where the linear term is a sum of a selfadjoint operator and a skew-symmetric operator. The sufficient condition for the existence and the regularity of the manifold is more restrictive than the corresponding condition A3 in § 2.