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On the coordinate-free description of the conformal blocks

By

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Introduction

Recent study of conformal field theory has provided new insights and techniques concerning the moduli space of stable curves. Among them is an extensive study on an interplay between the representation theory of infinite-dimensional Lie algebras and the theory of vector bundles on the moduli space of stable curves. The basic idea is easy. Consider the moduli space of framed stable curves (stable curves with coordinates). While it is clear that the Lie algebra of infinitesimal change of coordinates acts on this moduli space, interesting point is that even *singular* change of coordinates acts on this moduli space. The effect of singular coordinate change is to deform the shape of curves. Thus the Lie algebra of infinitesimal changes of coordinates, possibly with singularities, serves as a Lie algebra of infinitesimal symmetry of the moduli space. The algebra plays an important role in the conformal field theory. In certain circumstances, however, the universal central extension of the algebra, called the Virasoro Lie algebra, is more convenient to use. It gives a symmetry of the determinant line bundle on the moduli space. It is also possible to consider the algebra of infinitesimal change of trivialization, possibly with singularities, as an algebra of infinitesimal symmetry of the moduli space of framed vector bundles. The central extension of this algebra is called an affine Lie algebra. Similar to the theory of vector bundles on a symmetric space, an interplay of Lie algebra representation and vector bundles on the moduli space thus arises.

Following this idea, Tsuchiya, Ueno, and Yamada [TUY] constructed vector bundles, called sheaves of vacua, over base schemes of local universal families of framed stable curves. They further showed that there is a projective connection on each such vector bundle. But we may ask what kind of a role the use of coordinates plays. We want a coordinate-free description of the theory to study the mechanism of this procedure deeper. In fact, it is proved in [TUY] that the choice of coordinate is unessential, but it is not at all clear a priori. In this paper we give a simple and coordinate-free description of the construction given there, and clarify the nature of the theory. In the way we deal with the notion of normal ordering, which is used in the construction of the projective

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connection, by the effective use of explicit representation of the Atiyah algebra of a determinant bundle, developed by Beilinson and Schechtman [BS].

As a result, we construct the sheaf of vacua on a base space of family of pointed stable curves, without the presence of coordinates, and we describe the infinitesimal symmetry of this sheaf by the infinitesimal symmetry of determinant line and the cotangent space of marked points.

Let us explain the content of this paper more precisely.

We first discuss the linear algebra of the ring $\mathbf{C}((t))$ of formal Laurent power series. In differential geometric approach, a loop algebra is an algebra of functions on a circle with values in a Lie algebra. In stead of using a circle, we use here the "infinitesimal circle", or the formal scheme $\operatorname{Spf}\mathbf{C}((t))$, an object whose ring of functions consists of formal Laurent power series. The difference between the usual circle and the infinitesimal one comes from the difference of the topology used to complete the ring of regular functions $\mathbf{C}(t)$ on $\mathbf{C}\backslash\{0\}$, namely, the C^{∞}-topology on $S^1 = \{z \in \mathbf{C}; |z| = 1\}$ and the t-adic filtration topology. But the difference is not so serious. All we do in this paper is to take a topological base and express the traces of operators by using them, just as in the differential-geometric case [SW]. Thus the result of this paper may well hold in the differential-geometric case also, with a slight modifications concerning the topology. The advantage of our approach is that we can use a strong tool of the formal Čech cohomology [BS], and that it gives an algebraic construction of the theory. The latter suggests that we may remove the restriction of the ground field being $\mathbf{C}$. Technical tools concerning the special kind of topological bases of the topological vector space $\mathbf{C}((t))$ are summed up in section 0. They are defined so that matrices we have to take determinants are essentially upper triangular ones (cf. [KNTY], [SW].)

In section 1 we define, for each family $X \to S$ of proper Gorenstein curves with marked points $Q_1, \ldots, Q_n$, a sheaf of affine Lie algebras and develop an "infinitesimal" analogue of representation theory of affine Lie algebras. We also introduce in this section the sheaf of covacua $\mathscr{V}_\lambda$ on S attached to the family, introduced in [TUY]. Not so much is known about this sheaf, but it has a few nice properties, such as the behavior under the normalization of the curves (factorization property), and the existence of a natural projective connection. (There is a deep relation between these two properties. See [Segal].)

We next mention the definition and the symmetry of the determinant line bundle. It is surprising that the notion of the determinant line bundle of a complex of sheaves defined in algebraic geometry is well suited to the description of what is called an "infinite determinant".

The infinitesimal symmetry, or the Atiyah algebra, of the determinant line bundle $\det \mathbf{R}\pi_* \Theta_X$, is studied by Beilingson and Shechtmann ([BS]). In section 3, we extend their result to make it applicable for a family of curves with singularities. Preliminary remarks about the Atiyah algebras are summarized in section 2. There we also describe the weight algebra which describes the symmetry of cotangent bundles along the marked points Q_i.

Finally, the projective connection on the sheaf of covacua is introduced in section 4. The essential part of this construction is to give the Sugawara form in a coordinate-free manner using the Virasoro algebra defined in section 3.

The goal of this paper is to prove the following.

Theorem 4.2. *A central extension of the tangent bundle obtained by composing the algebra of infinitesimal symmetries of the determinant line bundle and that of the relative cotangent bundles of the curve along the divisors Q_i acts on the sheaf of covacua.*

Thus we find an explicit relation of the projective connection on $\mathscr{V}_\lambda$ and the symmetry of the determinant line bundle. (Here $\mathscr{T}'_S$ is the sheaf of tangent vectors on S whose corresponding deformations of fibers preserves singularity. See (0.1).)

The reader may see that our definition of the projective connection is much more simpler than that of the original definition in [TUY], done after a lengthy induction argument and complicated coordinate-dependent formulae. Our definition is suitable for calculation and may be useful for the further study of the sheaf of covacua. We hope our result will help understanding the relation between this theory and the theory of geometric quantization of moduli space of stable bundles, developed, for example in [Hitchin].

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§ 0. Preliminaries

0.0. Basic settings. Throughout this paper, $\pi: X \to S$ denotes a proper Gorenstein morphism of schemes over $\mathbf{C}$ with integral fibers of dimension one (that is, a family of proper algebraic curves, possibly with singularities.) The assumption of π to be Gorenstein implies that it is flat and that the relative dualizing sheaf $\omega_{X/S}$ for π on X exists and is invertible. (See [Ha].) We assume that S is smooth over $\mathbf{C}$. In particular, the tangent sheaf $\mathscr{T}_S$ of S is $\mathscr{O}_S$-flat. Let $q_i: S \to X (i \in [1, n])$ be n mutually disjoint sections of π, and $Q_i = \operatorname{image} q_i$. We assume that π is smooth on some neighbourhood of each Q_i. Denote the formal neighbourhood of Q_i by U_i, and that of the divisor $D = \bigcup_{i=1}^n Q_i$ by $U: U = \bigcup_{i=1}^n U_i$. Denote the open set $X \backslash \operatorname{supp} D$ by $\dot{X}$, $\dot{U} = U \cap \dot{X}$. Note that $\dot{X}$ is affine over S.

0.1. Subsheaves of tangent sheaves. We begin with explaining some notations. Consider an exact sequence

$$0 \to \mathscr{T}_{X/S} \to \mathscr{T}_X \xrightarrow{d\pi} \pi^* \mathscr{T}_S.$$

The map $d\pi$ is surjective on U because π is smooth around U. We pull the above sequence back by the inclusion $\pi^{-1}\mathscr{T}_S \subset \pi^*\mathscr{T}_S$ (recall that $\mathscr{T}_S$ is $\mathscr{O}_S$-flat.), and we obtain an exact sequence

$$0 \to \mathscr{T}_{X/S} \to \mathscr{T}_{X,\pi} \xrightarrow{d\pi} \pi^{-1}\mathscr{T}_S\ .$$

The sheaf $\mathscr{T}_{X,\pi}$ is the sheaf of germs of tangent vector fields whose horizontal components are constant along the fibers of π [BS]. We define furthermore the subsheaf $\mathscr{T}_{X,\pi,D}$ of $\mathscr{T}_{X,\pi}$ as

$$\mathscr{T}_{X,\pi,D} = \{\tau \in \mathscr{T}_{X,\pi} | \tau \text{ preserves } D\} .$$

The sections of $\mathscr{T}_{X,\pi,D}$ are those of $\mathscr{T}_{X,\pi}$ which are tangential to D.

For later use, we define subsheaves $\mathscr{T}'_S$, $\mathscr{T}'_{X,\pi}$, and $\mathscr{T}'_{X,\pi,D}$, of $\mathscr{T}_S$, $\mathscr{T}_{X,\pi}$, and $\mathscr{T}_{X,\pi,D}$ respectively as follows.

$$\begin{aligned} \mathscr{T}'_S &= \pi_*(\mathrm{Image}\,(\mathscr{T}_X \to \pi^*\mathscr{T}_S)) \subset \pi_*\pi^*\mathscr{T}_S \cong \mathscr{T}_S \\ \mathscr{T}'_{X,\pi} &= d\pi^{-1}(\pi^{-1}\mathscr{T}'_S) \\ \mathscr{T}'_{X,\pi,D} &= \mathscr{T}'_{X,\pi} \cap \mathscr{T}_{X,\pi,D} \end{aligned}$$

Roughly speaking, $\mathscr{T}'_S$ is the sheaf of tangent vectors on S whose corresponding deformation of the fiber of π (given by the Kodaira-Spencer theory) preserves the singularity. As an example, consider a local universal desingularization $X \to S$ of a curve X_0 at an ordinary double point P. Namely, $\dim S = 1$, and there exists a divisor F on X, a neighbourhood V of F, a coordinate s on S, x, $y \in \Gamma(V, \mathcal{O}_V)$ with $xy = s$ satisfying,

$$X \backslash F \approx (X_0 \backslash P) \times S \qquad \text{(diffeomorphic)}$$

$$\begin{tikzcd} V \arrow[r, "\approx"] \arrow[d, "\pi"'] & \{(X, Y, Z) | XY = Z\} \subset \mathbf{A}^3 \arrow[d, "s"] \\ S \arrow[r, "s"] & \mathbf{A}^1 \end{tikzcd}$$

Then $\mathscr{T}'_{X/S}$ is generated by $s(\partial/\partial s)$ over $\mathcal{O}_S$.

Note that we have exact sequences

$$0 \to \mathscr{T}_{X/S}(-D) \to \mathscr{T}'_{X,\pi,D} \to \pi^{-1}\mathscr{T}'_S \to 0 ,$$

$$0 \to \mathscr{T}_{X/S} \to \mathscr{T}'_{X,\pi} \to \pi^{-1}\mathscr{T}'_S \to 0 .$$

0.2. Linear algebra of formal power series and residues. For any closed point $s_0 \in S$, we can choose a neighbourhood S_0 of s_0 such that there exists a formal coordinate $t_i \in \Gamma(S_0, \pi_*\mathcal{O}_{U_i})$ around each Q_i. (We always assume that $t_i(Q_i) = 0$.) The choice of these coordinates enables us to trivialize sheaves such as

$$\pi_*\mathcal{O}_{\dot{U}} \cong \bigoplus_{i=1}^{n} \mathcal{O}_S((t_i)) .$$

$$\pi_*\omega_{\dot{U}/S} \cong \bigoplus_{i=1}^{n} \mathcal{O}_S((t_i))dt_i .$$

We equip these sheaves with the $\mathscr{I}_D$-adic topology.

We have a canonical residue pairing between the above two sheaves, which makes them topological duals.

$$(\xi|\eta) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \operatorname{Res}_{t_i=0} \xi_i(t_i)\eta_i(t_i)$$

$$\left[\begin{aligned} &\xi = (\xi_i(t_i)) \in \pi_*\omega_{\dot{U}/S} \cong \bigoplus_{i=1}^{n} \mathcal{O}_S((t_i))dt_i \\ &\eta = (\eta_i(t_i)) \in \pi_*\mathcal{O}_{\dot{U}} \cong \bigoplus_{i=1}^{n} \mathcal{O}_S((t_i)) \end{aligned}\right]$$

Using residues, we may define various operators on $\pi_*\mathcal{O}_{\dot{U}}$ and $\pi_*\omega_{\dot{U}/S}$ as follows. Consider the product $\dot{U} \times_S \dot{U}$ over S and denote by Δ the diagonal. Each section $\check{r}(t_1, t_2)$ of $(\pi \times \pi)_*(p_2^*\omega(*\Delta)|_{\dot{U}\times_S\dot{U}}) \stackrel{\text{def}}{=} \varinjlim_k (\pi \times \pi)_*(p_2^*\omega(+k\Delta)|_{\dot{U}\times_S\dot{U}})$ defines the following operators on $\pi_*\mathcal{O}_{\dot{U}}$ and $\pi_*\omega_{\dot{U}/S}$.

$$((\operatorname{Res}_0 \check{r})f)_i(t_i) \stackrel{\text{def}}{=} \sum_j \operatorname{Res}_{u_j=0} \check{r}_{ij}(t_i, u_j)f(u_j)\,,$$

$$((\operatorname{Res}^0 \check{r})\omega)_i(t_i) \stackrel{\text{def}}{=} \sum_j \operatorname{Res}_{u_j=0} \check{r}_{ji}(u_j, t_i)\omega(u_j)\,,$$

$$((\operatorname{Res}_\Delta \check{r})f)_i(t_i) \stackrel{\text{def}}{=} \operatorname{Res}_{u_i=t_i} \check{r}_{ii}(t_i, u_i)f(u_i)\,,$$

$$((\operatorname{Res}^\Delta \check{r})\omega)_i(t_i) \stackrel{\text{def}}{=} \operatorname{Res}_{u_i=t_i} \check{r}_{ii}(u_i, t_i)\omega(u_i)\,,$$

$$(\operatorname{Res}_1 \check{r}) \stackrel{\text{def}}{=} \operatorname{Res}_0 \check{r} + \operatorname{Res}_\Delta \check{r}\,,$$

$$(\operatorname{Res}^1 \check{r}) \stackrel{\text{def}}{=} \operatorname{Res}^0 \check{r} + \operatorname{Res}^\Delta \check{r}$$

Where $f \in \pi_*\mathcal{O}_{\dot{U}}$, $\omega \in \pi_*\omega_{\dot{U}/S}$, $\check{r}_{ij} = \check{r}|_{U_i \times U_j}$, $\ldots$, e.t.c.

Note that $\operatorname{Res}_\Delta(\check{r})$, $\operatorname{Res}^\Delta(\check{r})$ are differential operators.

These operators satisfy the following adjoint relations.

$$(\omega|\operatorname{Res}_0(\check{r})f) = (\operatorname{Res}^1(\check{r})\omega|f)$$

$$(\omega|\operatorname{Res}_1(\check{r})f) = (\operatorname{Res}^0(\check{r})\omega|f)$$

$$(\omega|\operatorname{Res}_\Delta(\check{r})f) = (-\operatorname{Res}^\Delta(\check{r})\omega|f)$$

The following lemma is important for our later arguments.

Lemma 0.2.1. *For each $\check{r}$, there exists an integer p such that*

$$\operatorname{Ker}(\operatorname{Res}_0(\check{r})) \supset \pi_*(\mathcal{O}_U(-pD))\,,$$

$$\operatorname{Image}(\operatorname{Res}_1(\check{r})) \subset \pi_*(\mathcal{O}_U(+pD))\,.$$

A similar property holds for Res^0 *and* Res^1.

0.3. Two ways of expansion. It is worthwhile to give an explanation for these operators in the following way. For the sake of simplicity, in this subsection we consider the case $S = \operatorname{Spec}\mathbf{C}$, and $n = 1$ and fix some formal coordinate t of U.

Put

$$\mathscr{E} = \left\{ \sum_{k,l \in \mathbf{Z}} a_{k,l} z^k w^l dw \,\middle|\, \begin{array}{l} a_{k,l} \in \mathbf{C}, \text{ there exists an integer } N \text{ such that} \\ a_{k,l} = 0 \text{ for all } k, l \text{ satisfying } k + l < N \end{array} \right\}$$

It is regarded as an algebra of operators on the vector space $\mathbf{C}((t))$ as follows.

$$(\operatorname{Res} e.f)(t) = \operatorname{Res}_{u=0} e(t, u) f(u) \qquad (e \in \mathscr{E}, f \in \mathbf{C}((t))) .$$

Each element $\check{r}(z, w)$ of $(\pi \times \pi)_*(p_2^* \omega_{X/S}(*\Delta)|_{\dot{U} \times \dot{U}})$ admits two ways of expansion, corresponding to the cases $|z| > |w|$ and $|z| < |w|$. We thus have two ways of regarding $\check{r}$ as an element of $\mathscr{E}$, which we denote by $I_{|z|>|w|}\check{r}$ and $I_{|z|<|w|}\check{r}$. For example,

$$I_{|z|>|w|} \frac{dw}{z - w} = \sum_{k=0}^{\infty} z^{-k-1} w^k dw ,$$

$$I_{|z|<|w|} \frac{dw}{z - w} = \sum_{k=0}^{\infty} z^{k-1} w^{-k} dw .$$

In these terms, Res_0 and Res_1 is expressed as

$$\operatorname{Res}_0 \check{r} = \operatorname{Res} I_{|z|>|w|} \check{r}(z, w)$$

$$\operatorname{Res}_1 \check{r} = \operatorname{Res} I_{|z|<|w|} \check{r}(z, w)$$

We may regard $(\pi \times \pi)_*(p_2^* \omega_{X/S}(*\Delta))|_{\dot{U} \times \dot{U}})$ as a subalgebra of $\mathscr{E}$ by means of $I_{|z|>|w|}$ or $I_{|z|<|w|}$.

$$I_{|z|>|w|},\ I_{|z|<|w|} \colon (\pi \times \pi)_*(p_2^* \omega_{X/S}(*\Delta))|_{\dot{U} \times \dot{U}}) \hookrightarrow \mathscr{E}$$

This may be useful for construction of element of $(\pi \times \pi)_*(p_2^* \omega_{X/S}(*\Delta))|_{\dot{U} \times \dot{U}})$ by giving some element of $\mathscr{E}$. (This is not misleading, because if $I_{|z|>|w|}\check{r}_1 = I_{|z|<|w|}\check{r}_2$, then we see $\check{r}_1$ and $\check{r}_2$ are regular on Δ, and then we have actually $\check{r}_1 = \check{r}_2$.)

0.4. The notion of frames. Taking a sufficiently small affine open subscheme $S_1 = \operatorname{Spec} B_1$ of S such that we have a formal coordinate t_j along each Q_j on $\pi^{-1}(S_1)$. Note that

$$\Gamma(S_1; \pi_* \mathcal{O}_{\dot{U}}) \cong \bigoplus_{j=1}^{n} B_1((t_j)) .$$

We equip this vector space with a $\bigoplus t_j B_1((t_j))$-adic topology.

We introduce the following notions concerning special bases of $\pi_* \mathcal{O}_{\dot{U}}$ and its subsheaves.

I. A *local frame* of $\pi_* \mathcal{O}_{\dot{U}}$ (resp. $\pi_* \mathcal{O}_{\dot{X}}$, resp. $\pi_* \mathcal{O}_U$) defined over S_1 is a family $\{f^\mu\}_{\mu \in M}$ of elements of $\Gamma(S_1; \pi_* \mathcal{O}_{\dot{U}})$ (resp. $\Gamma(S_1; \pi_* \mathcal{O}_{\dot{X}})$, resp. $\Gamma(S_1; \pi_* \mathcal{O}_U)$) satisfying the following conditions.

(1) The index set M is an ordered set isomorphic to $(\mathbf{Z}, \geq)$ (resp. $(-\mathbf{N}, \geq)$, resp. $(\mathbf{N}, \geq)$).

(2) $\{f^\mu\}_{\mu \in M}$ is a topological base. Namely, every element f of $\Gamma(S_1; \pi_* \mathcal{O}_{\dot{U}})$ (resp. $\Gamma(S_1; \pi_* \mathcal{O}_{\dot{X}})$, resp. $\Gamma(S_1; \pi_* \mathcal{O}_U)$) is expressed uniquely as a convergent linear combination of $\{f^\mu\}_{\mu \in M}$ with coefficients in $\Gamma(S_1; \mathcal{O}_S)$:

$$f = \sum_{\mu \in M} a_\mu f^\mu .$$

(3) $f^\mu \to 0$ as $\mu \to +\infty$.

Note that the topology of $\Gamma(S_1; \pi_* \mathcal{O}_{\dot{X}})$ is discrete. A frame of $\Gamma(S_1; \pi_* \mathcal{O}_{\dot{X}})$ is thus the same as a basis of this vector space.

II. A triple $(\{e^\mu\}_{\mu \in M}, \{\xi^\nu\}_{\nu \in N}, \{\eta^\kappa\}_{\kappa \in K})$ of local frames of the three sheaves $\pi_* \mathcal{O}_{\dot{U}}$, $\pi_* \mathcal{O}_{\dot{X}}$, $\pi_* \mathcal{O}_U$ (respectively) is said to be *consistent* if the matrix expressing $\{\xi^\nu\}$, $\{\eta^\kappa\}$ by a linear combination of $\{e^\mu\}$ is "essentially upper triangular" one. More precisely, it is consistent if there exists $\mu_0 \in M$, $\nu_0 \in N$, $\mu_1 \in M$, $\kappa_1 \in K$ such that the following conditions are satisfied.

$\mu_0 < \mu_1$,

$\# N_{\geq \nu_0} < \infty$, $\quad \# K_{\geq \kappa_1} < \infty$, $\quad \#(M_{\geq \mu_0} \cap M_{\leq \mu_1}) < +\infty$,

$M_{\leq \mu_0} \cong N_{\leq \nu_0}$, $\quad M_{\geq \mu_1} \cong K_{\geq \kappa_0}$, $\quad$ and using this identification, we have

$\xi^\nu = e^\nu +$ finite linear combination of $\{e^{\mu'}\}_{\mu' > \nu}$ $\quad$ (if $\nu \geq \nu_0$)

$\eta^\kappa = e^\kappa +$ convergent linear combination of $\{e^{\kappa'}\}_{\kappa' < \mu}$ $\quad$ (if $\mu \leq \mu_0$)

Where we denote

$$N_{\geq \nu_0} = \{\nu \in N; \nu \geq \nu_0\} , \qquad K_{\leq \kappa_1} = \{\kappa \in K; \kappa \leq \kappa_1\}, \ \ldots, \text{ etc} ,$$

and $\#$ denotes the number of elements of a set.

III. A local frame $\{e^\mu\}_{\mu \in M}$ of $\pi_* \mathcal{O}_{\dot{U}}$ is said to be *good* if it consists of a disjoint union of a local frame of $\pi_* \mathcal{O}_{\dot{X}}$ and a local frame of $\pi_* \mathcal{O}_{\dot{U}}$ and a finite number of functions. In precise, it is good if there exists a partition of M

$$M \supset N, \ K , \qquad N \cap K = \{0\} ,$$

$$M = N' \sqcup K' \sqcup M' \sqcup \{0\} ,$$

$$M' < N' < 0 < K' , \qquad \# M' < +\infty ,$$

$$\text{where we denote } N' = N \backslash \{0\} , \quad K' = K \backslash \{0\} ,$$

such that $\{e^\nu\}_{\nu \in N}$, $\{e^\kappa\}_{\kappa \in K}$ is a subset (necessarily a frame) of $\pi_* \mathcal{O}_{\dot{X}}$, $\pi_* \mathcal{O}_U$, respectively.

Note that e^0 is necessarily an invertible element of B_1, which we assme to be 1 in the following. The triple of frames $(\{e^\mu\}_{\mu \in M}, \{e^\nu\}_{\nu \in N}, \{e^\kappa\}_{\kappa \in K})$ defined by a good local frame is said to be *very consistent*.

Now we have the following fundamental

Claim 0.4.1. *If the base scheme S is Noetherian, then for any closed point $x_0 \in S_1$, there always exists a consistent triple of local frames $(\{e^\mu\}_{\mu \in M}, \{\xi^\nu\}_{\nu \in N}, \{\eta^\kappa\}_{\kappa \in K})$ defined on some affine open neighbourhood of x_0.*

Indeed, define $\{e^{\mu}\}_{\mu \in M}$, $\{\eta^{\kappa}\}_{\kappa \in K}$ in the following way.

$$M = \mathbf{Z} \times [1, n] \qquad \text{with the lexicographic order},$$

$$K = \{(k, i) \in M | k \geq 0\},$$

$$e^{(k,i)}|_{U_j} = t_j^k \cdot \delta_{ij} \qquad \text{for } (k, i) \in M,$$

$$\eta^{(k,i)} = e^{(k,i)} \qquad \text{for } (k, i) \in K.$$

$\{\xi^{\nu}\}_{\nu \in N}$ may be chosen in the following way. The base change theorem implies that there is an integer k_0 such that for all pairs (k, i) with $k < k_0$, $i \in [1, n]$, there is a function $f^{(k,i)}$, regular over $\dot{X}$, with a form

$$(*) \qquad f = ae^{(k,i)} + \text{higher order terms}, \qquad a \in \Gamma(S_1; \mathcal{O}_S), \qquad a(x_0) \neq 0.$$

We choose such functions for finite such pairs: $(k, i) \in \{k_0 - 1, k_0\} \times [1, n]$. Shrinking S_1 if necessary, we may assume that all the leading coefficients of $f^{(k,i)}$ are 1. Taking appropriate product of them, we have a family $\{\xi^{(k,i)}\}_{k<k_0}$ of functions of the form $(*)$ defined on a common open neighbourhood of x_0. These functions, together with finite number of some extra functions (added to span the lower order functions), gives a required frame.

Similarly, we can show the following

Claim 0.4.2. *If the base scheme S is Noetherian, then for any closed point $x_0 \in S_1$, there always exists a good frame $\{e^{\mu}\}$ of $\pi_* \mathcal{O}_{\dot{U}}$ defined on some affine open neighbourhood of x_0.*

0.5. The notion of dual frames. We define the notion of a local frame of $\pi_* \omega_{\dot{U}/S}$ in the same way as above. Note that $\pi_* \mathcal{O}_{\dot{U}}$ and $\pi_* \omega_{\dot{U}/S}$ are the dual topological vector space of each other by virtue of the residue pairing.

Definition 0.5.1. *The frames $\{f^{\mu}\}_{\mu \in M}$ of $\pi_* \mathcal{O}_{\dot{U}}$ (resp. $\pi_* \mathcal{O}_{\dot{X}}$, resp. $\pi_* \mathcal{O}_U$) and $\{f_{\mu}\}_{\mu \in M}$ of $\pi_* \omega_{\dot{U}/S}$ (resp. $\pi_* \omega_{\dot{U}/S}/\pi_* \omega_{\dot{X}/S}$, resp. $\pi_* \omega_{\dot{U}/S}/\pi_* \omega_{U/S}$) are said to be dual to each other if relations*

$$(f_{\nu} | f^{\nu}) = \delta_{\nu}^{\mu} \qquad (\text{for all } \mu, \nu \in M)$$

hold.

It is easy to prove the following:

Lemma 0.5.2. *For any local frame of $\pi_* \mathcal{O}_{\dot{U}}$ (resp. $\pi_* \mathcal{O}_{\dot{X}}$, resp. $\pi_* \mathcal{O}_U$), there exists a unique local frame of $\pi_* \omega_{\dot{U}/S}$ (resp. $\pi_* \omega_{\dot{U}/S}/\pi_* \omega_{\dot{X}/S}$, resp. $\pi_* \omega_{\dot{U}/S}/\pi_* \omega_{U/S}$) dual to it.*

Furthermore, we see from the residue theorem, that a dual of a very consistent triple of local frames is also very consistent, and the dual of a good frame of $\pi_* \mathcal{O}_{\dot{U}}$ is also good, if we define these notions for $\omega_{X/S}$ in a similar manner as in the preceding subsection.

§ 1. Non-twisted affine Lie algebras and their representations

In this section, we define a sheaf of non-twisted affine Lie algebras $\mathfrak{g}'$ on S associated to a finite-dimensional simple Lie algebra $\mathring{\mathfrak{g}}$, and construct sheaves of its highest weight representations $\mathscr{M}_\lambda$, $\mathscr{L}_\lambda$. This can be done by rewriting the constructions in [Kac], Chapter 7, in the language of sheaves, with paying attention to coordinate-freeness. The 'sheaf of covacua' is then defined by posing a 'gauge' condition' on $\mathscr{L}_\lambda$.

1.1. Non-twisted affine Lie algebra $\mathfrak{g}'$. The realization of non-twisted affine Lie algebras using explicit formula for the cocycle, is easily extended to the sheaf version as follows. We fix a finite dimensional simple Lie algebra $\mathring{\mathfrak{g}}$ and a non-degenerate symmetric invariant bilinear form $(\ ,\ )$ on it.

Definition 1.1.1. *We define a sheaf* $\mathfrak{g}'$ *of Lie algebras over* $\mathcal{O}_S$ *by introducing a* ($\mathcal{O}_S$-*linear*) *bracket on the* $\mathcal{O}_S$-*module*

$$\mathfrak{g}' = \mathring{\mathfrak{g}} \otimes_{\mathbf{C}} \pi_* \mathcal{O}_{\dot{U}} \oplus \mathcal{O}_S \mathfrak{c}$$

as

(1) $\mathfrak{c}$ *is a central element of* $\mathfrak{g}'$.
(2) $[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + (X, Y)(df|g)\mathfrak{c}$ $\quad (X, Y \in \mathring{\mathfrak{g}}, f, g \in \pi_* \mathcal{O}_{\dot{U}})$.
Where $(|)$ *is the canonical pairing introduced in* 0.2.

We sometimes denote the element $X \otimes f$ of $\mathfrak{g}'$ as $X[f]$. In this notation, the commutation relation (2) above is rewritten as

$$(2') \qquad [X[f], Y[g]] = [X, Y][fg] + (X, Y)(df|g)\mathfrak{c}$$

The sheaf $\mathring{\mathfrak{g}} \otimes_{\mathbf{C}} \pi_* \mathcal{O}_{\dot{U}}$ of $\mathring{\mathfrak{g}}$-valued functions defined on $\dot{U}$ will be denoted as $\mathring{\mathfrak{g}} \otimes \pi_* \mathcal{O}_{\dot{U}}$. It has a canonical structure of Lie algebra obtained by the pointwise commutator. Our algebra $\mathfrak{g}'$ is a central extension of $\mathring{\mathfrak{g}} \otimes \pi_* \mathcal{O}_{\dot{U}}$.

Remark 1.1.2 $\mathring{\mathfrak{g}} \otimes \pi_* \mathcal{O}_U$, $\mathring{\mathfrak{g}} \otimes \pi_* \mathcal{O}_{\dot{X}}$ are Lie subalgebras of $\mathfrak{g}'$. Note that for any element ξ of $\mathring{\mathfrak{g}} \otimes \pi_* \mathcal{O}_U$, its value $\xi(Q_i)$ along Q_i is well-defined $\mathring{\mathfrak{g}}$-valued function on S. Namely, it is the pull back of the function ξ by the section Q_i. Similary, the value of the section of $\mathring{\mathfrak{g}} \otimes \pi_* \mathcal{O}_{\dot{X}}$ along each sections of π has also meaning and is also a $\mathring{\mathfrak{g}}$-valued function on S.

Next let us consider a triangular decomposition of $\mathfrak{g}'$. We fix a Cartan subalgebra $\mathring{\mathfrak{h}}$ of $\mathring{\mathfrak{g}}$ and a triangular decomposition $\mathring{\mathfrak{g}} = \mathring{\mathfrak{n}}_+ \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}_-$ of $\mathring{\mathfrak{g}}$.

We define the subalgebras $\mathfrak{n}_+$, $\mathfrak{p}_+$, $\widehat{\mathfrak{p}_+}$, $\mathfrak{h}$ of $\mathfrak{g}'$ as follows

$$\mathfrak{n}_+ = \{\xi \in \mathring{\mathfrak{g}} \otimes \pi_* \mathcal{O}_U | \xi(Q_i) \in \mathring{\mathfrak{n}}_+ \otimes_{\mathbf{C}} \mathcal{O}_S \quad \text{for all } i\}$$

$$\mathfrak{p}_+ = \{\xi \in \mathring{\mathfrak{g}} \otimes \pi_* \mathcal{O}_U | \xi(Q_i) \in (\mathring{\mathfrak{n}}_+ + \mathring{\mathfrak{h}}) \otimes_{\mathbf{C}} \mathcal{O}_S \quad \text{for all } i\}$$

$$\widehat{\mathfrak{p}_+} = \mathfrak{p}_+ + \mathcal{O}_S \mathfrak{c}$$

$$\mathfrak{h} = \mathring{\mathfrak{h}} \otimes_{\mathbf{C}} \pi_*((\pi^{-1}\mathcal{O}_S)|_{\dot{U}}) + \mathcal{O}_S \mathfrak{c} \cong \bigoplus_{i=1}^{n} (\mathring{\mathfrak{h}} \otimes_{\mathbf{C}} \mathcal{O}_S) \oplus \mathcal{O}_S \mathfrak{c}$$

Note that $\widehat{\mathfrak{p}_+} = \mathfrak{n}_+ \oplus \mathfrak{h}$. The notion corresponding to $\mathfrak{n}_-$ can not be defined in a coordinate free manner. But the existence of these subalgebras in the above definition is enough for construction of highest weight representations.

1.2. The representations $\mathscr{M}_\lambda$, $\mathscr{L}_\lambda$, and the sheaf of covacua $\mathscr{V}_\lambda^\dagger$. Put

$$P_{\mathbf{C}} = \{\lambda | \lambda = (\vec{\lambda}, c), \vec{\lambda} = (\lambda_1, \dots, \lambda_n) \in (\mathring{\mathfrak{h}}^\vee)^n,\ c \in \mathbf{C}\}.$$

Each $\lambda \in P_{\mathbf{C}}$ is regarded as an element of $(\widehat{\mathfrak{p}_+})^* = \mathscr{H}om(\widehat{\mathfrak{p}_+}, \mathcal{O}_S)$ as follows.

$$\lambda(\xi) = \sum_{i=1}^{n} \lambda_i(\xi(Q_i)) \qquad (\xi \in \mathfrak{p}_+); \qquad \lambda(\mathfrak{c}) = c$$

where we regard each λ_i as an element of $(\mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}_+)^\vee$ by extending it trivially on $\mathring{\mathfrak{n}}_+$.

We introduce the Verma module.

Construction-Definition 1.2.1. *For $\lambda \in P_{\mathbf{C}}$, we introduce the following $\widehat{\mathfrak{p}_+}$-module structure on $\mathcal{O}_S$ and denote it as $(\mathcal{O})_\lambda$.*

$$\xi . f = \lambda(\xi) \cdot f \qquad (\xi \in \widehat{\mathfrak{p}_+}, f \in \mathcal{O}_S).$$

Then, the sheaf $\mathscr{M}_\lambda$ of Verma modules with the highest weight λ is defined by an induced module

$$\mathscr{M}_\lambda = \mathfrak{U}(\mathfrak{g}') \otimes_{\mathfrak{U}(\widehat{\mathfrak{p}_+})} (\mathcal{O}_S)_\lambda,$$

where $\mathfrak{U}(\cdot)$ denotes the universal enveloping algebra.

We denote the section of $\mathscr{M}_\lambda$ corresponding to the section $1 \in (\mathcal{O}_S)_\lambda$ (the highest weight vector) by $|\lambda\rangle$.

By the use of formal coordinates $\{t_i\}_{i=1}^n$ on U, $\mathfrak{g}'$ may by trivialized as

$$\mathfrak{g}' \cong \bigoplus_{i=1}^{n} \mathring{\mathfrak{g}} \otimes_{\mathbf{C}} \mathcal{O}_S((t_i)) \oplus \mathcal{O}_S \mathfrak{c}.$$

With the help of a triangular decomposition of $\mathfrak{g}'$ derived by using these trivializations, we obtain a trivialization of $\mathscr{M}_\lambda$:

$$\mathscr{M}_\lambda \cong M_\lambda \otimes_{\mathbf{C}} \mathcal{O}_S$$

where M_λ in the formula above is the usual Verma module, that is, the Verma module in the case $S = \operatorname{Spec} \mathbf{C}$. In particular, $\mathscr{M}_\lambda$ is a quasi-coherent $\mathcal{O}_S$-module.

Next we define the 'smallest' highest weight module $\mathscr{L}_\lambda$. To do that, we first deal with the way how sheaves of tangent vectors act on $\mathfrak{g}'$. Recall the definition of $\mathscr{T}_{X,\pi}$ (section 0.1). We have the following

Lemma 1.2.2. *$\pi_* \mathscr{T}_{\dot{U},\pi}$ acts on $\mathfrak{g}'$ by the action determined as follows.*

(1) $\tau.(X \otimes f) = X \otimes (\tau.f)$ $(\tau \in \pi_* \mathscr{T}_{\dot{U},\pi},\ X \in \mathring{\mathfrak{g}},\ f \in \pi_* \mathcal{O}_{\dot{U}})$

(2) $\tau.(g\mathfrak{c}) = (d\pi(\tau).g) \cdot \mathfrak{c}$ $(g \in \mathcal{O}_S)$

Corollary 1.2.3. *$\pi_* \mathscr{T}_{\dot{U},\pi}$ acts on $\mathfrak{U}(\mathfrak{g}')$ as a derivation.*

Lemma 1.2.4. *The subalgebra $\pi_* \mathscr{T}_{U,\pi,D}$ of $\pi_* \mathscr{T}_{\dot{U},\pi}$ preserves $\widehat{\mathfrak{p}_+}$. Furthermore, for any element ξ of $\widehat{\mathfrak{p}_+}$ and an element τ of $\pi_* \mathscr{T}_{U,\pi,D}$, the following formula holds.*

$$\tau.(\xi - \lambda(\xi)) = \tau.\xi - \lambda(\tau.\xi)\,.$$

Therefore, we can define the action of $\pi_* \mathscr{T}_{U,\pi,D}$ on $\mathscr{M}_\lambda$ as follows

$$\tau.(\Psi|\lambda\rangle) = (\tau.\Psi)|\lambda\rangle$$

Using the trivialization of $\mathscr{M}_\lambda$, the same argument as in [Kac], proposition 9.3, gives the following:

Proposition 1.2.5. *$\mathscr{M}_\lambda$ has a unique, maximal $\pi_*(\mathscr{T}_{U/S}(-D))$-invariant $\mathfrak{g}'$-submodule $\mathscr{M}_\lambda^1$.*

Definition 1.2.6. $\mathscr{L}_\lambda = \mathscr{M}_\lambda / \mathscr{M}_\lambda^1$

Definition 1.2.7. *We define the sheaf of covacua associated to a weight λ to be the sheaf*

$$\mathscr{V}_\lambda = \mathscr{L}_\lambda / (\mathring{\mathfrak{g}} \otimes \pi_* \mathscr{O}_{\dot{X}}) \mathscr{L}_\lambda\,.$$

We refer to $(\mathring{\mathfrak{g}} \otimes \pi_* \mathscr{O}_{\dot{X}}) \mathscr{L}_\lambda$ as the gauge condition.

Note that our $\mathscr{V}_\lambda$ has no apparent symmetries. Namely, no Lie algebra (such as $\mathring{\mathfrak{g}}$, $\mathfrak{g}'$, ...) acts on this sheaf $\mathscr{O}_S$-linearly in a canonical way. But it carries geometric information about X, encoded by the gauge condition.

§2. A quick review on an Atiyah algebra

2.1. The Atiyah algebra of a line bundle. In this and the next subsections, we forget about our general assumption (section 0.0) and review the general theory of Atiyah algebras. See [BS] for more details.

Let $\mathscr{F}$ be a locally free sheaf of rank one on a scheme S over $\mathbf{C}$. The *Atiyah algebra* of $\mathscr{F}$ is, by definition, the sheaf of first order differential operators on $\mathscr{F}$.

$$\mathscr{A}_{\mathscr{F}} = \mathrm{Diff}_1\, \mathscr{F}\,.$$

It has a natural structure of Lie algebra, and is an extension of $\mathscr{T}_S$ by $\mathscr{O}_S$.

$$0 \to \mathscr{O}_S \to \mathscr{A}_{\mathscr{F}} \xrightarrow{\mathrm{Symb}_1} \mathscr{T}_S \to 0$$

We refer the above sequence as the *fundamental sequence* of $\mathscr{A}_{\mathscr{F}}$.

2.2. The definition of Atiyah algebra. Keeping the preceding subsection in mind, we define the general notion of Atiyah algebra (more precisely, $\mathscr{O}_S$-Atiyah algebra in the sense of [BS]) as a Lie algebra $\mathscr{A}$ with a fixed extension sequence (which we call fundamental),

$$0 \to \mathscr{O}_S \xrightarrow{i_{\mathscr{A}}} \mathscr{A} \xrightarrow{p_{\mathscr{A}}} \mathscr{T}_S \to 0 \tag{1}$$

We have two operations on Atiyah algebras, namely, summation and scalar multiplication.

The sum of two Atiyah algebras $\mathscr{A}$, $\mathscr{B}$ is defined by

$$\mathscr{A} + \mathscr{B} = (\mathscr{A} \times_{\mathscr{T}_S} \mathscr{B})/\{(i_{\mathscr{A}}(f), -i_{\mathscr{B}}(f)) | f \in \mathcal{O}_S\} .$$

Its fundamental sequence is given by $i_{\mathscr{A}+\mathscr{B}} = (i_{\mathscr{A}}, 0) = (0, i_{\mathscr{B}})$ and $p_{\mathscr{A}+\mathscr{B}} = p_{\mathscr{A}} = p_{\mathscr{B}}$.

It is clear that the sum of Atiyah algebras of line bundles $\mathscr{F}_1$, $\mathscr{F}_2$ is isomorphic to (hence we may identify with) the Atiyah algebra of the tensor product $\mathscr{F}_1 \otimes \mathscr{F}_2$.

Next we define the scalar multiple of an Atiyah algebra.

Let $\lambda \in \mathbf{C}$ and $\mathscr{A}$ be an Atiyah algebra. The λ-multiple of $\mathscr{A}$ is defined by

$$\lambda\mathscr{A} = (\mathcal{O}_S \oplus \mathscr{A})/(\lambda, 1)\mathcal{O}_S , \qquad \text{with } i_{\lambda\mathscr{A}} = (id, 0) , \quad p_{\lambda\mathscr{A}} = (0, p_{\mathscr{A}}) .$$

If $\lambda \neq 0$, it coincides with the Lie algebra $\mathscr{A}$, equipped with another fundamental sequence

$$0 \to \mathcal{O}_S \xrightarrow{\lambda^{-1} i_{\mathscr{A}}} \mathscr{A} \xrightarrow{p_{\mathscr{A}}} \mathscr{T}_S \to 0 .$$

For $\lambda = 0$, the 0-multiplie of $\mathscr{A}$ is just a direct sum $\mathcal{O}_S \oplus \mathscr{T}_S$ with the standard fundamental sequence.

It is easy to see that with these summation and scalar multiplication, the category of Atiyah algebras forms a "$\mathbf{C}$-vector space in category" [BS].

2.3. Action of an Atiyah algebra on a vector bundle. Let $\mathscr{V}$ be a locally free sheaf of finite rank (a vector bundle) over S, and $\mathscr{A}$ an Atiyah algebra with the fundamental sequence given by (1).

We say we have an action of $\mathscr{A}$ on $\mathscr{V}$ if the following conditions are satisfied.

(1) each section α of $\mathscr{A}$ acts on $\mathscr{V}$ as a first order differential operator $\Phi(\alpha)$

(2) the principal symbol of $\Phi(\alpha)$ coincides with $p_{\mathscr{A}}(\alpha) \otimes \mathrm{id}_{\mathscr{V}}$.

(3) $\Phi(i_{\mathscr{A}}(1)) = 1$.

It is easy to prove the following

Claim 2.3.1. *Given a vector bundle $\mathscr{V}$, the following data are equivalent.*

(1) *An action of $\mathscr{A}$ on $\mathscr{V}$ for some Atiyah algebra $\mathscr{A}$.*

(2) *An action of the tangent bundle $\mathscr{T}_S$ (flat connection) on projective bundle* $\mathbf{P}(\mathscr{V})$.

We call these data a projective connection on $\mathscr{V}$.

It is worth noting the following fact ([BS]).

Claim 2.3.2. *Given an action of an Atiyah algebra $\mathscr{A}$ on a $\mathscr{V}$, we can determine the first Chern class of $\mathscr{V}$ solely in terms of $\mathscr{A}$.*

This holds because we can consider a connection on $\mathscr{V}$ and its curvature purely in terms of $\mathscr{A}$.

2.4. The scalar multiple of an Atiyah algebra of a line bundle. We explain the notion of scalar multiple in the case when $\mathscr{A}$ is the Atiyah algebra of a line bundle $\mathscr{F}$.

If λ is a positive integer, $\lambda\mathscr{A}_{\mathscr{F}}$ is regarded as the Atiyah algebra of the λ-th tensor power $\mathscr{F}^{\otimes\lambda}$. In fact, a local section τ of $\mathscr{A}_{\mathscr{F}}$ acts on $\mathscr{F}^{\otimes\lambda}$ by

$$\tau.(s_1 \otimes \cdots \otimes s_n) = \tau.s_1 \otimes s_2 \otimes \cdots \otimes s_n + s_1 \otimes \tau.s_2 \otimes \cdots \otimes s_n + \cdots + s_1 \otimes s_2 \otimes \cdots \otimes \tau.s_n\,.$$

And under this action, $i_{\mathscr{A}_{\mathscr{F}}}(1)$ acts on $\mathscr{F}^{\otimes\lambda}$ by λ-multiple.

The same argument shows that if λ is a negative integer, $\lambda\mathscr{A}_{\mathscr{F}}$ is identified with the Atiyah algebra of $(\mathscr{F}^{\otimes(-\lambda)})^*$, and $0\mathscr{A}_{\mathscr{F}}$ is clearly the Atiyah algebra of $\mathcal{O}_S = \mathscr{F}^{\otimes 0}$.

For general λ, take a local generating section s of $\mathscr{F}$ and consider the formal λ-th power s^λ of s. A section τ of $\mathscr{A}_{\mathscr{F}}$ acts on fs^λ ($f \in \mathcal{O}_S$) by

$$\tau.(fs^\lambda) = (\mathrm{Symb}_1(\tau).f + \lambda((\tau.s)/s)f)s^\lambda\,.$$

The above formula suggests that the algebra of infinitesimal symmetry of the formal λ-th power $\mathscr{F}^{\otimes\lambda}$ is well-defined and identified with $\lambda\mathscr{A}_{\mathscr{F}}$, although $\mathscr{F}^{\otimes\lambda}$ itself is not a well-defined object. We may thus deal with $\lambda\mathscr{A}_{\mathscr{F}}$ as if $\mathscr{F}^{\otimes\lambda}$ exists and it is the Atiyah algebra of this "line bundle".

2.5. The weight algebras $\mathscr{A}_{q_i^*\omega_{X/S}}$, $\mathscr{A}_{\mathscr{F}}$. Return to our original settings (section 0.0). We have sections q_i of π. Since π is assumed to be smooth on some neighbourhood of $Q_i = \mathrm{Image}\, q_i$, the dualizing sheaf $\omega_{X/S}$ is identified with the sheaf of relative Kähler differentials there. For later use we introduce here the weight algebra, a linear combination of the Atiyah algebra of line bundles $q_i^*\omega_{X/S}$.

There is a surjection (in the usual sheaf theoretic sense) of $\pi_*\mathscr{T}_{U_i,\pi,D}$ onto $\mathscr{A}_{q_i^*\omega_{X/S}}$. Indeed, there is an action of $\pi_*\mathscr{T}_{U_i,\pi,D}$ on $q_i^*(\omega_{X/S})$ determined by the action (Lie derivative) of $\pi_*\mathscr{T}_{U_i,\pi,D}$ on $\pi_*\omega_{U_i/S}$. Namely, the action of a section τ_{U_i} of $\pi_*\mathscr{T}_{U_i,\pi,D}$ on a section $q_i^*\omega$ of $q_i^*\omega_{X/S}$ is given by the following.

$$\tau_{U_i}.(q_i^*\omega) = q_i^*(\mathrm{Lie}(\tau_{U_i}).\omega)\,.$$

In terms of local coordinate t_i of U_i, it is expressed as a first order differential operator.

$$\left(\sum_{k=0}^{\infty} a_k(s)t_i^{k+1}\frac{\partial}{\partial t_i} + b(s)\frac{\partial}{\partial s}\right).(c(s)q_i^*dt_i) = b(s)\frac{\partial c(s)}{\partial s}q_i^*dt_i + a_0(s)c(s)q_i^*dt_i\,.$$

Put "symbolically"

$$\mathscr{F} = \bigotimes_{i=1}^n q_i^*(\omega_{X/S})^{-\Delta_{\lambda_i}}\,.$$

Here Δ_{λ_i} is arbitrary complex number, but later in section 4.2, we will set it to be the value of the Casimir operator of $\mathring{\mathfrak{g}}$ on the irreducible highest weight module V_{λ_i} of $\mathring{\mathfrak{g}}$ with highest weight λ_i.

We thus have a surjection $\pi_*\mathscr{T}_{U,\pi,D} \to \mathscr{A}_{\mathscr{F}}$, in other words, $\pi_*\mathscr{T}_{U,\pi,D}$ acts on $\mathscr{F}$. We refer the Atiyah algebra $\mathscr{A}_{\mathscr{F}} = \sum(-\Delta_{\lambda_i})\mathscr{A}_{q_i^*\omega_{X/S}}$ as the weight algebra.

§ 3. The Atiyah algebra of the determinant line bundle

The determinant line bundle, defined by Grothendieck [SGA] (see also [KM]), may easily be described by the use of the notion of semi-infinite-forms ([KNTY], [SW]). We review it and describe in these terms the structure of the Atiyah algebra of the determinant line bundle. The construction of the explicit representation of the Atiyah algebra is essentially due to Beilinson and Schechtman [BS]. We modify their approach to make it applicable to the case in which π is not necessarily smooth. Actually, our construction gives a slightly restricted Atiyah algebra, written $\mathscr{A}'_{\det \mathbf{R}\pi_* \mathscr{O}_X}$, defined by $\mathrm{Symb}_1^{-1}(\mathscr{T}'_S)$. (See section 0.1 for the definition of $\mathscr{T}'_S$.) For the sake of brevity, we introduce the elements of the algebras in a rather ad-hoc manner. See [BS], especially section 2.8, for more sophisticated and natural approach using the language of derived categories.

3.1. The determinant line bundles $\det \mathbf{R}\pi_* \mathscr{O}_X$ **and** $\det \mathbf{R}\pi_* \omega_{X/S}$. Using a consistent triple of local frames (see section 0.2), we may present a locally generating section of the determinant line bundle $\det \mathbf{R}\pi_* \mathscr{O}_X \cong \det [\pi_* \mathscr{O}_{\dot{X}} \oplus \pi_* \mathscr{O}_U \to \pi_* \mathscr{O}_{\dot{U}}]$ defined on S_1, written symbolically as

$$(**)\qquad \left(\bigwedge_{\nu\in N} \xi^\nu\right) \otimes \left(\bigwedge_{\kappa\in K} \eta^\kappa\right) \otimes \left\{\bigwedge_{\mu\in M} e^\mu\right\}^{-1}.$$

The dual of the $\det \mathbf{R}\pi_* \mathscr{O}_X$ is expressed by another determinant line bundle, $\det \mathbf{R}\pi_* \omega_{X/S}$, by the Serre duality. It is isomorphic to

$$\det\left[\pi_* \omega_{\dot{U}/S} \to \frac{\pi_* \omega_{\dot{U}/S}}{\pi_* \omega_{\dot{X}/S}} \oplus \frac{\pi_* \omega_{\dot{U}/S}}{\pi_* \omega_{U/S}}\right].$$

Using the dual frames, the section of $\det \mathbf{R}\pi_* \omega_{X/S}$ corresponding to the trivialization of $\det \mathbf{R}\pi_* \mathscr{O}_X$ defined by the local generating section (**) is also expressed by semi-infinite forms as

$$\left(\bigwedge_{\nu\in N} \xi_\nu\right) \otimes \left(\bigwedge_{\kappa\in K} \eta_\kappa\right) \otimes \left\{\bigwedge_{\mu\in M} e_\mu\right\}^{-1}.$$

3.2. ω**-extension of** $\mathscr{T}_{X/S}(-D)$. Consider the product $X_{\mathrm{reg}} \times_S X_{\mathrm{reg}}$ over S, where we denote by X_{reg} the open set of X where π is smooth. We identify sheaves on X_{reg} with that on $X_{\mathrm{reg}} \times_S X_{\mathrm{reg}}$ with support on the diagonal Δ_{reg}. We have the following exact sequence.

$$0 \to p_2^* \omega \to p_2^* \omega(+2\Delta_{\mathrm{reg}}) \xrightarrow{\delta} \mathit{Diff}^1_{X/S} \to 0 .$$

Where $\mathit{Diff}^1_{X/S}$ denotes the sheaf of first order differential operators along the fiber of π. Recall that ω coincides with $\Omega^1_{X/S}$, the sheaf of relative 1-forms, on X_{reg}. The map δ is given by the following formula.

$$(\delta(r(z, w)dw).f)(t) = \mathrm{Res}_{t_2=t}\, r(t, t_2) f(t_2) .$$

We pull this sequence back by the inclusion $\mathscr{T}_{X/S}(-D) \subset \mathit{Diff}^1_{X/S}$ to obtain

$$0 \to p_2^* \omega \to B_{0,D} \xrightarrow{\delta} \mathscr{T}_{X/S}(-D) \to 0 ,$$

where we denote the sheaf $\delta^{-1}(\mathcal{T}_{X/S}(-D))$ by $B_{0,D}$. We push this sequence out by the projection $p_2^*\omega \to p_2^*\omega/p_2^*\omega(-\Delta_{\mathrm{reg}}) \cong \omega$ and obtain

$$0 \to \omega \to A_{0,D}^{-1} \xrightarrow{\delta} \mathcal{T}_{X/S}(-D) \to 0\,.$$

Here $A_{0,D}^{-1} \overset{\mathrm{def}}{=} B_{0,D}/p_2^*\omega(-\Delta_{\mathrm{reg}})$ is called the ω-extension of $\mathcal{T}_{X/S}(-D)$. Its support is contained in Δ_{reg} and we regard the above sequence as an exact sequence of sheaves on X_{reg}.

Using the above exact sequence we define the complex $A_{0,D}$ of sheaves on X_{reg} by the middle column in following diagram.

$$\begin{tikzcd}
 & & A_{0,D}^{0} \arrow[r, equal] & \mathcal{T}_{X,\pi,D} & \\
0 \arrow[r] & \omega \arrow[r] & A_{0,D}^{-1} \arrow[r] \arrow[u] & \mathcal{T}_{X/S,D} \arrow[r] \arrow[u] & 0 \\
 & \mathcal{O} \arrow[r, equal] \arrow[u] & A_{0,D}^{-2} \arrow[u] & &
\end{tikzcd}$$

Note that $\mathcal{T}_{X,\pi,D}$ acts on $A_{0,D}^{-1}$ as an infinitesimal symmetry of $(\pi\colon X \to S, \{q_i\})$.

3.3. The Virasoro algebra. We define the sheaf of Virasoro algebras as

$$\mathcal{VIR} = \pi_*(A_{0,D}^{-1}|_{\dot{U}})/d(\pi_*\mathcal{O}_{\dot{U}})\,.$$

It has a canonical structure of Lie algebra with the following bracket.

$$[r_1, r_2]_{\mathcal{VIR}} = \delta(r_1).r_2\,.$$

Note that there is an exact sequence

$$0 \to d(\pi_*\mathcal{O}_{\dot{U}}) \to \pi_*\omega_{\dot{U}/S} \xrightarrow{j} \mathcal{VIR} \xrightarrow{\delta} \pi_*\mathcal{T}_{\dot{U}/S} \to 0\,.$$

$\mathcal{VIR}$ has the following canonical subsheaves of Lie algebras

$$\mathcal{VIR}_0 = \pi_*(A_{0,D}^{-1}|_U)/d(\pi_*\mathcal{O}_U)$$

$$\mathcal{VIR}_{\dot{X}} = \left\{ \begin{array}{l|l} \text{the class of} & \delta(\check{r}) \in \pi_*\mathcal{T}_{\dot{X}/S} \\ \check{r} \in (\pi \times \pi)_*(B_0|_{\dot{U}\times\dot{U}}) & (\mathrm{Res}_1\, \check{r}).\pi_*\mathcal{O}_{\dot{X}} = 0 \\ \text{in } \mathcal{VIR} & (\mathrm{Res}^1\, \check{r}).\pi_*\omega_{\dot{X}/S} = 0 \end{array} \right\}$$

The map $\delta\colon \mathcal{VIR}_{\dot{X}} \to \pi_*\mathcal{T}_{\dot{X}/S}$ is surjective. In fact, take a good local frame $\{e^\mu\}_{\mu\in M}$ of $\pi_*\mathcal{O}_{\dot{U}}$ and its dual frame $\{e_\mu\}_{\mu\in M}$ of $\pi_*\omega_{\dot{U}/S}$. Then for any section $\tau_{\dot{X}}$ of $\pi_*\mathcal{T}_{\dot{X}/S}$, the element $\check{r}$ of $(\pi\times\pi)_*(B_0|_{\dot{U}\times\dot{U}})$ defined by

$$\check{r}(t,u) = \sum_{\nu\in N} (\tau_{\dot{X}}.\xi^\nu)(t)\xi_\nu(u)$$

gives the element of $\mathcal{VIR}_{\dot{X}}$ with $\delta(r) = \tau_{\dot{X}}$. (See the last part of section 0.3.)

3.4. Atiyah algebra $\mathscr{A}'_{\det \mathbf{R}\pi_*\mathcal{O}_X}$. In this subsection, we show that the presentation of $\mathscr{A}'_{\det \mathbf{R}\pi_*\mathcal{O}_X}$ admits a Čech-cohomological expression. If π is smooth, (so that X coincides with X_{reg},) it is the 0-th direct image $R^0\pi_*(A^{\bullet}_{0,D})$ of the complex $A^{\bullet}_{0,D}$, as you may find the proof in [BS]. Although the sheaf corresponding to $B_{0,D}$ is not defined all over X in the case when π is singular, the "formal Čech" presentation of the cohomology group, which is effectively used in [BS], extends to this case.

The set of cocycles is given by the following.

$$\mathscr{Z} = \{(\tau_{\dot{X}}, \tau_U; r_{\dot{U}}) | \tau_{\dot{X}} \in \pi_*\mathscr{T}'_{\dot{X},\pi}, \tau_U \in \pi_*\mathscr{T}'_{U,\pi,D}, r_{\dot{U}} \in \mathscr{V}\mathscr{I}\mathscr{R}; \tau_{\dot{X}} - \tau_U = \delta(r_{\dot{U}})\} .$$

It has a structure of Lie algebra as follows

$$[(\tau_{\dot{X}}, \tau_U; r_{\dot{U}}), (\sigma_{\dot{X}}, \sigma_U; s_{\dot{U}})] = ([\tau_{\dot{X}}, \sigma_{\dot{X}}], [\tau_U, \sigma_U]; \tau_{\dot{X}}.s_{\dot{U}} - \sigma_U.r_{\dot{U}})$$

The set of coboundaries is

$$\mathscr{B} = \{(\delta(r_{\dot{X}}), \delta(r_U); r_{\dot{X}} - r_U) | r_{\dot{X}} \in \mathscr{V}\mathscr{I}\mathscr{R}_{\dot{X}}, r_U \in \mathscr{V}\mathscr{I}\mathscr{R}_0\}$$

It is an ideal of $\mathscr{Z}$.

The cohomology group $\mathscr{Z}/\mathscr{B}$ gives a presentation of $\mathscr{A}'_{\det \mathbf{R}\pi_*\mathcal{O}_X}$. To see this, first remark that there is an exact sequence

$$(***) \qquad 0 \to \mathcal{O}_S \xrightarrow{i} \mathscr{Z}/\mathscr{B} \xrightarrow{\pi_*} \mathscr{T}'_S \to 0 ,$$

where the map π_* is defined by

$$\pi_*([\tau_{\dot{X}}, \tau_U; r_{\dot{U}}]) = \pi_*(\tau_{\dot{X}}) = \pi_*(\tau_U) ,$$

whereas the map i is determined by the following diagram.

$$\begin{array}{ccc} \pi_*\omega_{\dot{U}/S} & \xrightarrow{j} & \mathscr{Z} \\ {\scriptstyle \sum \mathrm{Res} = (\cdot|1)} \downarrow & & \downarrow {\scriptstyle \text{projection}} \\ \mathcal{O}_S & \xrightarrow{i} & \mathscr{Z}/\mathscr{B} \end{array}$$

Next we give the action of an element $(\tau_{\dot{X}}, \tau_U; r_{\dot{U}})$ of $\mathscr{Z}$ on $\det \mathbf{R}\pi_*\mathcal{O}_X$ by the following equation

$$(1 + \varepsilon(\tau_{\dot{X}}, \tau_U; r_{\dot{U}})).\left(\left(\bigwedge_{\nu \in N} \xi^\nu\right) \otimes \left(\bigwedge_{\kappa \in K} \eta^\kappa\right) \otimes \left\{\bigwedge_{\mu \in M} e^\mu\right\}^{-1}\right)$$

$$= \left(\bigwedge_{\nu \in N} (1 + \varepsilon\tau_{\dot{X}})\xi^\nu\right) \otimes \left(\bigwedge_{\kappa \in K} (1 + \varepsilon\tau_U \eta^\kappa\right) \otimes \left\{\bigwedge_{\mu \in M} (1 + \varepsilon(\tau_U - \mathrm{Res}_0\, \check{r}_{\dot{U}}))e^\mu\right\}^{-1} ,$$

where ε is the dual number, that is, $\varepsilon^2 = 0$.

Using lemma 0.2.1 we see that $\{(1 + \varepsilon\tau_{\dot{X}})\xi^\nu\}$, $\{(1 + \varepsilon\tau_U)\eta^\kappa\}$, $\{1 + \varepsilon(\tau_U - \mathrm{Res}_0\, \check{r}_{\dot{U}}))e^\mu\}$ is a consistent triple of frames defined on $\mathbf{C}[\varepsilon] \otimes S$, and hence the right hand side of the above formula is well-defined. If we take a very consistent triple of frames and choose a local trivialization defined by it, the above action is expressed as a first-order differential operator

$$\pi_*(\tau_U) + \sum_{\nu \in N'} (e_\nu | \mathrm{Res}_1 (\check{r}_{\dot{U}}) e^\nu) + \sum_{\kappa \in K'} (e_\kappa | \mathrm{Res}_0 (\check{r}_{\dot{U}}) e^\kappa) + (e_0 | \mathrm{Res}_0 (\check{r}_{\dot{U}}) e^0)$$
$$+ \sum_{\mu \in M'} (e_\mu | (\mathrm{Res}_0 (\check{r}_{\dot{U}}) - \tau_U) e^\mu) .$$

By this formula, it is easy to see that the action of $\mathscr{L}$ on $\det \mathbf{R}\pi_* \mathcal{O}_X$ factors through an action of $\mathscr{L}/\mathscr{B}$, and that the sequence (∗∗∗) can be identified with the fundamental sequence

$$0 \to \mathcal{O}_S \to \mathscr{A}'_{\det \mathbf{R}\pi_* \mathcal{O}_X} \xrightarrow{\mathrm{Symb}_1} \mathscr{T}'_S \to 0 .$$

§ 4. The projective connection on the sheaf of covacua

4.1. The Sugawara form. Here in this section, we formulate the notion of the Sugawara form (see for example [Kac], Chap. 12) in a coordinate-free manner using the result of the last section.

We fix a basis $\{J^a\}_{a=1}^{\dim \mathring{\mathfrak{g}}}$ and its dual basis $\{J_a\}_{a=1}^{\dim \mathring{\mathfrak{g}}}$ with respect to the invariant bilinear form $(\ , \ )$. We denote the value of an operator $(1/2)\sum \mathrm{ad}\, J^a \,\mathrm{ad}\, J_a$ by g^*. In what follows, we fix a weight $\lambda = (\vec{\lambda}, c) \in P_{\mathbf{C}}$ and assume that $c + g^* \neq 0$. Put $c_v = c(\dim \mathring{\mathfrak{g}})/(c + g^*)$.

We define the Sugawara form $T[r]$ associated to an element r of $\mathscr{V}\mathscr{I}\mathscr{R}$. Recall that the sheaf $\mathscr{V}\mathscr{I}\mathscr{R}$ is a quotient of $(\pi \times \pi)_*(B_{0,D}|\dot{U} \times \dot{U})$.

Definition 4.1.1. *Let $\{e^\mu\}_{\mu \in M}$ be a local frame of $\pi_* \mathcal{O}_{\dot{U}}$, $\{e_\mu\}_{\mu \in M}$ its dual frame. Let $\check{r}$ be a section of $(\pi \times \pi)_*(B_{0,D}|_{\dot{U} \times \dot{U}})$. We define the energy-momentum tensor $T[\check{r}]$ as*

$$T[\check{r}] = \frac{-1}{2(c + g^*)} \sum_{\mu \in M} \left(\sum_{a=1}^{\dim \mathring{\mathfrak{g}}} J^a[e_\mu \circ \delta(\check{r})] J_a[e^\mu] - c(\dim \mathring{\mathfrak{g}})(e_\mu | \mathrm{Res}_0 (\check{r}) e^\mu) \right) .$$

The sum converges strongly on the representation space $\mathscr{M}_\lambda$ in the sense that the sum is always finite when it acts on an element of $\mathscr{M}_\lambda$.

To show the last claim of the above definition, we first note that the commutation relation of $\mathfrak{g}'$ (definition 1.1.1), and the following equality

$$(d(\delta(\check{r}) \circ \omega) | f) = -(\omega | \delta(\check{r}).f) \qquad \text{for } f \in \pi_* \mathcal{O}_{\dot{U}} \,, \quad \omega \in \pi_* \omega_{\dot{U}/S} \,.$$

implies the following identity which gives another expression for the summand.

$$\sum_{a=1}^{\dim \mathring{\mathfrak{g}}} J^a[e_\mu \circ \delta(\check{r})] J_a[e^\mu] - c(\dim \mathring{\mathfrak{g}})(e_\mu | \mathrm{Res}_0 (\check{r}) e^\mu)$$
$$= \sum_{a=1}^{\dim \mathring{\mathfrak{g}}} J_a[e^\mu] J^a[e_\mu \circ \delta(\check{r})] - c(\dim \mathring{\mathfrak{g}})(e_\mu | \mathrm{Res}_1 (\check{r}) e^\mu) .$$

Then using lemma 0.2.1, we find that the summand is equal to

$$\sum_{a=1}^{\dim \mathring{\mathfrak{g}}} J^a[e_\mu \circ \delta(\check{r})] J_a[e^\mu]$$

when μ is large enough, and to

$$\sum_{a=1}^{\dim \mathring{\mathfrak{g}}} J_a[e^\mu] J^a[e_\mu \circ \delta(\check{r})]$$

when μ is small enough.

We refer the above situation simply as $T[\check{r}]$ is a strong-operator limit of elements of $\mathfrak{U}(\mathfrak{g}')$. It is clear from the definition that $T[\check{r}]$ acts also on $\mathscr{M}_\lambda^1$, $\mathscr{L}_\lambda$. It is easy to show the following lemmas.

Lemma 4.1.2. *$T[\check{r}]$ does not depend on the choice of a local frame $\{e^\mu\}$.*

Let $\{f^{\mu'}\}$ be another local frame. Then $\{e^\mu\}$ and $\{f^{\mu'}\}$ are related to each other by an essentially upper triangular matrix $\{a_\mu^{\mu'}\}$.

$$f^{\mu'} = \sum_\mu a_\mu^{\mu'} e^\mu .$$

The dual frames are related to each other by the following relation.

$$e_\mu = \sum_{\mu'} a_\mu^{\mu'} f_{\mu'} .$$

Using these relations, the lemma easily follows, similarly to the proof of the base-independence of trace of an linear operator on a finite dimensional vector space.

Lemma 4.1.3. *If $\check{r} \in p_2^* \omega_{X/S} (= \operatorname{Ker} \delta)$, then*

$$T[\check{r}] = \frac{c_v}{2} \mathrm{Trace}_{\pi_* \mathcal{O}_{\dot{U}}} (\mathrm{Res}_0 \, \check{r}) = \frac{c_v}{2} \sum_{j=1}^N \mathrm{Res}_{t_j=0} \, \check{r}_j(t_j, t_j) = \frac{c_v}{2} (\check{r}_{|\Delta} | 1) .$$

The above lemma suggests in particular that for any $\check{r}$ of $(\pi \times \pi)_* (B_{0,D}|_{\dot{U} \times \dot{U}})$, $T[\check{r}]$ depends only on the class r of $\check{r}$ in $\mathscr{V}\mathscr{I}\mathscr{R}$. We may thus write $T[r]$ instead of $T[\check{r}]$.

Lemma 4.1.4. *The action of $\tau \in \pi_* \mathscr{T}_{\dot{U}/S}$ on $T[r]$ is given by the following formula.*

$$\tau.(T[r]) = T[\tau.r] .$$

Where we consider $T[r]$ as a strong-operator limit of elements of $\mathfrak{U}(\mathfrak{g}')$. (Recall we have an action of $\pi_* \mathscr{T}_{\dot{U}/S}$ on $\mathfrak{U}(\mathfrak{g}')$. (Lemma 1.2.4).)

Let us denote by $T[\check{r}; \{e^\mu\}, \{e_\mu\}]$ the Sugawara form associated to $\check{r}$, calculated by means of a frame $\{e^\mu\}$ and its dual frame $\{e_\mu\}$. For each τ in $\pi_* \mathscr{T}_{\dot{U}/S}$, $1 + \varepsilon\tau$ defines a fiber preserving infinitesimal automorphism of $\dot{U}$. Applying this automorphism to $T[\check{r}; \{e^\mu\}, \{e_\mu\}]$, and computing, we find

$$(1 + \varepsilon\tau).T[\check{r}; \{e^\mu\}, \{e_\mu\}] = T[(1 + \varepsilon\tau).\check{r}; \{(1 + \varepsilon\tau).e^\mu\}, \{(1 + \varepsilon\tau).e_\mu\}] .$$

But since $\{(1 + \varepsilon\tau).e^\mu\}$ is a local frame and $\{(1 + \varepsilon\tau).e_\mu\}$ is its dual, we deduce the lemma from lemma 4.1.2.

The derivation of the following commutation relation between the Sugawara form and an element of the affine Lie algebra is essentially well known.

Lemma 4.1.5. $[T[r], X[f]] = X[\delta(r).f] \qquad (X \in \mathring{\mathfrak{g}}, f \in \pi_* \mathcal{O}_{\dot{U}})$.

The proof is obtained by rewriting the [Kac, lemma 12, 3] using local coordinates along Q_i.

Thus, for each element ξ of $\mathfrak{U}(\mathfrak{g}')$, we obtain,

$$[T[r], \xi] = \delta(r).\xi\,.$$

This commutation relation, together with lemma 4.1.4, implies

Lemma 4.1.6. $[T[r_1], T[r_2]] = T[\delta(r_1).r_2] = T[[r_1, r_2]_{\mathcal{VJR}}]$.

So the Sugawara form gives a representation of the Virasoro algebra on highest weight modules of affine Lie algebras.

4.2. The existence of a projective connection: main theorem. We are now in a position to give a projective connection on $\mathcal{V}_\lambda$.

Theorem.

$$\frac{c_v}{2}\mathscr{A}'_{\det \mathbf{R}\pi_*\mathcal{O}_X} + \sum_{i=1}^{n} \Delta_{\lambda_i}\mathscr{A}'_{q_i^*\omega_{X/S}}$$

acts on $\mathcal{V}_\lambda$. *(Here* $\{\Delta_{\lambda_i}\}_{i=1}^n$ *are the values given in section* 2.5, *that is, the value of the Casimir operator of* $\mathring{\mathfrak{g}}$ *on the irreducible highest weight module* V_{λ_i} *of* $\mathring{\mathfrak{g}}$ *with highest weight* λ_i.)

It follows that there is an action of $\mathcal{T}'_S$ *on* $\mathbf{P}(\mathcal{V}_\lambda)$. *In other words, there is a projective connection on* $\mathcal{V}_\lambda$.

We may put the statement in another way. Let $\mathcal{F}$ be a line bundle defined in section 2.5. Then the above theorem says that the Lie algebra $\mathscr{A}'_{\det \mathbf{R}\pi_*\mathcal{O}_X}$ acts on $\mathcal{V}_\lambda \otimes \mathcal{F}$, and that the element $I = i_{\mathscr{A}'_{\det \mathbf{R}\pi_*\mathcal{O}_X}}(1)$ of $\mathscr{A}'_{\det \mathbf{R}\pi_*\mathcal{O}_X}$ acts on $\mathcal{V}_\lambda \otimes \mathcal{F}$ as a multiplication by $c_v/2$. (We refer the latter fact as "the central charge of the action is $c_v/2$".)

Proof of the theorem. We first define an action of $\mathcal{L}$ on $\mathcal{M}_\lambda \otimes \mathcal{F}$ and $\mathcal{L}_\lambda \otimes \mathcal{F}$.

(1) $(\tau_{\dot{X}}, \tau_U; r_{\dot{U}}).(\Psi|\lambda\rangle \otimes s) = (\tau_U.\Psi)|\lambda\rangle \otimes s + T[r_{\dot{U}}]\Psi|\lambda\rangle \otimes s + \Psi|\lambda\rangle \otimes \tau_U.s\,,$

where $\Psi \in \mathfrak{U}(\mathfrak{g}')$ and $s \in \mathcal{F}$.

Using lemmas 4.1.5 and 4.1.6, we can easily check that this is indeed an action of a Lie algebra.

The commutation relation of a Sugawara form with elements of $\mathfrak{U}(\mathfrak{g}')$ gives another formula representing the action.

(2) $(\tau_{\dot{X}}, \tau_U; r_{\dot{U}}).\Psi|\lambda\rangle \otimes s = (\tau_{\dot{X}}.\Psi)|\lambda\rangle \otimes s + \Psi T[r_{\dot{U}}]|\lambda\rangle \otimes s + \Psi|\lambda\rangle \otimes \tau_U.s$

This formula implies that the action preserves the gauge condition and hence induces the action of $\mathscr{L}$ on $\mathscr{V}_\lambda$. By calculation we conclude that $\mathscr{B}$ acts trivially on $\mathscr{V}_\lambda \otimes \mathscr{F}$ and so $\mathscr{A}'_{\det \mathbf{R}\pi_*\mathcal{O}_X} = \mathscr{L}/\mathscr{B}$ acts on $\mathscr{V}_\lambda \otimes \mathscr{F}$. In fact, $(\delta(r_{\dot{X}}), 0, r_{\dot{X}})$ $(r_{\dot{X}} \in \mathscr{V}\mathscr{I}\mathscr{R}_{\dot{X}})$ acts on $\mathscr{V}_\lambda \otimes \mathscr{F}$ trivially, in view of (1) and the following formula for a Sugawara form using good frame of $\pi_* \mathcal{O}_{\dot{U}}$.

$$T[r_{\dot{X}}] = \sum_{\mu \in M \backslash N} J^a[\delta(r_{\dot{X}}) \circ e_\mu] J_a[e^\mu] + \sum_{\mu \in N} J_a[e^\mu] J^a[\delta(r_{\dot{X}}) \circ e_\mu] .$$

(We omit the summation symbol for a for simplicity.) Note that $\delta(r_{\dot{X}}) \circ e_\mu$ is regular on $\dot{X}$ if $\mu \in M \backslash N$ and that e^μ is regular on $\dot{X}$ if $\mu \in N$.

Similarly, $(0, \delta(r_U), -r_U)$ $(r_U \in \mathscr{V}\mathscr{I}\mathscr{R}_0)$ acts on $\mathscr{V}_\lambda \otimes \mathscr{F}$ trivially, in view of (2) and the following identity.

$$T[r_U] = \sum_{(m,i), m \geq 1} J^a[\delta(r_U) \circ h_{(m,i)}] J_a[h^{(m,i)}] + \sum_{(m,i), m \leq -1} J_a[h^{(m,i)}] J^a[\delta(r_U) \circ h_{(m,i)}]$$

$$+ \sum_i J^a[\delta(r_U) \circ h_{(0,i)}] J_a[h^{(0,i)}] ,$$

where we fix local coordinates t_i of U_i and set

$$h^{(m,i)}|_{U_j} = t_i^m \delta_{ij} , \qquad h_{(m,i)}|_{U_j} = t_i^{-m-1} dt_i \delta_{ij} .$$

Indeed,

$$\begin{aligned}
&(0, \delta(r_U), -r_U).\Psi|\lambda\rangle \\
&= -\Psi T[r_U]|\lambda\rangle \otimes s + \Psi|\lambda\rangle \otimes \tau_U.s \\
&= \Psi\left(-\sum_i J^a[\delta(r_U) \circ h_{(0,i)}] J_a[h^{(0,i)}]|\lambda\rangle + |\lambda\rangle \otimes \delta(r_U).s \right) \\
&= \Psi\left(\sum_i \lambda_i(\delta(r_U)_i \circ \mathrm{d}t_i)(0) \otimes s + |\lambda\rangle \otimes \delta(r_U).s \right) \\
&= 0
\end{aligned}$$

Lemma 4.1.3 shows that the central charge of the action is $c_v/2$.

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