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Multiple points of fractional stable processes

By

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§ 1. Introductions and main results

Let $Z_\alpha(u)$, $-\infty<u<\infty$, be a real-valued Lévy symmetric α-stable process, $0<\alpha\leq 2$. Let $H>0$, $H\neq 1/\alpha$, and define $f(t, u)$ by

$$f(t, u)=|t-u|^{H-1/\alpha}-|u|^{H-1/\alpha}, \quad t\geq 0 \text{ and } u\in \mathrm{R}.$$

The real-valued process $X(t)$ determined by the integral representation

$$\int_{-\infty}^{\infty} f(t, u) Z_\alpha(du)$$

is called a *linear fractional stable process with parameter* (α, H). Cambanis-Maejima[5] have had detailed discussions on the distributional properties and limiting theorems on this class of processes.

Now, we consider a d-dimensional process $\mathrm{X}(t)=(X_1(t), \cdots, X_d(t))$, where the components $X_j(t)$ are independent and each $X_j(t)$ is linear fractional with parameter (α_j, H_j). In case $\alpha_j=2$ for all j, $\mathrm{X}(t)$ is Gaussian and has been called a fractional Brownian motion, see Mandelbrot-Van Ness[19]. We assume that $\alpha_j<2$ for all j in this paper and call $\mathrm{X}(t)$ a *d-dimensional fractional stable process* (We have suppressed the adjective "linear" because it is somewhat misleading). $\mathrm{X}(t)$ is non-Gaussian and non-Markovian; yet it is self-similar and has stationary increments in the following sense. For each $c>0$ and $b>0$

$$\mathrm{X}(ct)\overset{d}{=}(c^{H_1}X_1(t), \cdots, c^{H_d}X_d(t)) \quad \text{and}$$

$$\mathrm{X}(t+b)-\mathrm{X}(b)\overset{d}{=}\mathrm{X}(t)-\mathrm{X}(0),$$

where $\overset{d}{=}$ denotes the finite-dimensional equivalence of two processes. Fact: when $1/\alpha_j<H_j$ for all j, $\mathrm{X}(t)$ has a version of which all the paths are continuous; while in case $H_j<1/\alpha_j$ for some j, any version of $\mathrm{X}(t)$ is of everywhere discontinuous paths (indeed, the paths are unbounded on each time interval, see Maejima[18]).

Let $k\geq 2$ be a given integer. A point $x\in \mathrm{R}^d$ is a *k-multiple point* of the

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path $X(\cdot, \omega)$ if there exist $0<t_1<\cdots<t_k$ (depending on ω) such that $x=X(t_1, \omega)=\cdots=X(t_k, \omega)$. We will prove in this paper that

Theorem 1.1. *Let $X(t)=(X_1(t), \cdots, X_d(t))$ be a d-dimensional fractional stable process as described above. Suppose that $1\leq\alpha_j<2$, $1/\alpha_j<H_j<1$, and that we have chosen and fixed a continuous version of $X(t)$. Suppose moreover that*

$$\theta \overset{def}{=} \frac{k-1}{k}\sum_{j=1}^{d} H_j<1 \tag{1.1}$$

Then almost surely the path $X(\cdot, \omega)$ has k-multiple points. Futhermore, almost surely

$$dim\{(t_1, \cdots, t_k): t_1<\cdots<t_k, X(t_1)=\cdots=X(t_k)\}\geq k(1-\theta).$$

In the above, $dimE$ denotes the Hausdorff dimension of a Borel E. Various authors have studied multiple points of fractional Brownian motions; we cite Kono[14], Goldman[10], Rosen[24], Cuzick[6]. We also cite Evans[7], Fitzsimmons-Salisbury[8], Legall-Rosen-Shieh[17], Rogers[23], and Shieh[26] for recent result concerning this aspect of Levy and Markov processes. Theorem 1.1 represents a generalization of the above known results to certain important non-Gaussian non-Markovian stable processes.

Remark. Since $1/2<1/\alpha_j<H_j$ for all j, (1.1) is non-void only if

$d=2$, $k\geq 2$: $1<H_1+H_2<k/(k-1)$ and

$d=3$, $k=2$: $3/2<H_1+H_2+H_3<2$.

Based on the known results for Brownian and fractional Brownian motions, we believe that, for the process in Theorem1.1, almost all paths are simple (no intersections) in case $d\geq 4$.

The basic idea of proving Theorem 1.1, akin to that found in the most of the above literature, is to consider the k-parameter $d(k-1)$-dimensional random field

$$\bar{Z}(t_1, \cdots, t_k) \overset{def}{=} (X(t_2)-X(t_1), \cdots, X(t_k)-X(t_{k-1})), \quad 0<t_1<\cdots<t_k.$$

The view is that the set of $t_1<\cdots<t_k$ such that $X(t_1)=\cdots=X(t_k)$ (the set of "k-multiple times") is simply the zero set of $\bar{Z}$, and the best way to study level sets of a random field is through its local times. Let $[a_j, b_j]$ be k disjoint intervals in R_+, with $0<a_j<b_j<a_{j+1}<b_{j+1}<\infty$, $j=1, \cdots, k-1$. Another contribution of this paper is to prove that

Theorem 1.2. *Let $X(t)=(X_1(t), \cdots, X_d(t))$ be a d-dimensional fractional stable process as described above. Suppose that $1\leq\alpha_j<2$, $0<H_j<1$ and that* *the condition* (1.1) *holds. Let* $I=[a_1, b_1]\times\cdots\times[a_k, b_k]$. *There exists* (*almost surely*) *a jointly continuous* $\phi(\bar{x}, \bar{t})$, $\bar{x}=(x_1, \cdots, x_{k-1})\in R^{d(k-1)}$ *and* $\bar{t}=(t_1, \cdots, t_k)\in I$ *such that for all bounded continuous* $f(\bar{x})$

$$\int_{\mathrm{R}^{d(k-1)}} f(\bar{x})\phi(\bar{x}, \bar{t})d\bar{x}=\int_{a_1}^{t_1}\cdots\int_{a_k}^{t_k} f(\bar{Z}(s_1, \cdots, s_k))ds_1\cdots ds_k\,. \tag{1.2}$$

That is to say, $\bar{Z}$ *has jointly continuous local time* $\phi(\bar{x}, \bar{t})$ *over I. Let* $\phi(\bar{x}, B)$, *B being a Borel subset of I, denote that local time of* $\bar{Z}$ *over B, i.e. we replace the rectangle* $[a_1, t_1]\times\cdots\times[a_k, t_k]$ *on the right hand side of* (1.2) *by B. Let* λ *be* $0<\lambda<1-\theta$, *then the following uniform Hölder condition "in the set variable" holds almost surely. Let* $e(B)$ *denote the edge length of a k-dimensional cube B. Let K be any compact subset of* $\mathrm{R}^{d(k-1)}$. *There exist finitely positive* $C=C(\omega)$, $\varepsilon=\varepsilon(\omega)$ *such that*

$$\sup_{\bar{x}\in K}\ \sup_{B\subset I, e(B)<\varepsilon} |\phi(\bar{x}, B)|\leq C\mathrm{Leb}(B)^{\lambda}\,. \tag{1.3}$$

Remark. In Theorem 1.2, we do not assume that $1/\alpha_j<H_j$ for all j(which asserts the path continuity); the local time is defined for every measurable path. While for the application of Theorem 1.2 to Theorem 1.1, we do need the path continuity.

As that already appeared in the literature $\phi(\bar{x}, \bar{t})$ is called the *k-hold* (*self*-) *intersection local time* of X. We refer to Geman-Horowitz[9] and Rosen[10], Adler[1, Chapter 8] as excellent overviews on local time theory. In case X is Brownian or fractional Brownian motion, Geman-Horowitz-Rosen[10], Rosen[24] have studied multiple points from the intersection local time of $\mathrm{X}(t)$. The contiributions of this paper can be regarded as to show that we can extend fruitfully the results in [10, 24] to certain important non-Gaussian non-Markovian stable processes. However, we should mention our novelty in such generalizations. In the Brownian case [10, 24], the independent increments property is indispensable for the proof. To compensate this property for fractional Brownian motions, Rosen[24] used the local nondeterminism of Gaussian processes, a concept formulated firstly by Berman[3] and later in a different context by Pitt[22]. Pitt's formulation seems stronger; yet Geman-Horowitz [9, Theorem 24.3] pointed out that the two formulations are in fact equilvalent. Nolan[20] extented Berman's formulation from the Gaussian case to the symmetric stable case. Nolan's result is sufficient for his application to the joint continuity of (non-intersection) local times. While for our purpose in proving Theorem 1.2, we need certain stronger (i.e. that similar to Pitt's) local nondeterminism which we prove in Proposition 2.1 of § 2. See the remark below Proposition 2.1 for the more explicit comparision between Berman's and Pitt's fomulations. We also remark that it is not clear at all that Theorem 24.3 in [9] holds for non-Gaussian stable processes. In this sense, our Theorem 1.2 is not merely a routine generalization of [10, 24]. The proof of Theorem 1.1 is based on Theorem 1.2, a real-analytic lemma concerning the connections between the level sets and local times, and the "ergodicity" of linear fractional stable processes.

We commence to prove Theorem 1.2 in § 2. Then in § 3 we prove Theorem 1.1. We prove in the final § 4 the existence of k-multiple points for $X(T)$, where T is a compact subset of R^+ with $\dim T > \theta$ and θ is given in Theorem 1.1. This is again a generalization of those known in Kahane[12], Testard [30] and Shieh[27] for Gaussian and Markov processes. We remark that the study of "fine sample path property" of self-similar processes has been an expanding topic since the pioneering work of Vervaat[32]. The real-valued case has been quite well studied, see Kono-Maejima[15]; however, results of the multidimensional case seem to be few. Nolan[21] obtained some dimension results for the image, the graph, and the level sets of certain "index-β stable fields". It is hoped that our results contribute some steps in this aspect. We should mention the recent detailed study of Sato[25] on the structure of multidimensional self-similar processes with independent increments, of which the self-intersections of the paths seem to occur too.

§ 2. The proof of Theorem 1.2

The following "strong" local nondeterminism property of linear fractional stable processes is needed for the proof of Theorem 1.2.

Proposition 2.1. *Let* $1 \leq \alpha < 2$, $0 < H < 1$, $0 < \epsilon < T < \infty$. *Let* $f(t, u)$ *denote the kernel function of a linear fractional stable process with parameter* (α, H), *as mentioned in § 1. Then, for each integer* $p \geq 2$, *there exist positive* δ_p *and* C_p *such that for all* $v_1, \cdots, v_{p-1} \in R$

$$\|[f(t_p, \cdot) - f(t_{p-1}, \cdot)] - \sum_{j=1}^{p-1} v_i [f(t_j, \cdot) - f(t_{j-1}, \cdot)]\|_\alpha^\alpha \tag{2.1}$$

$$\geq C_p \|f(t_p, \cdot) - f(t_{p-1}, \cdot)\|_\alpha^\alpha,$$

whenever $\varepsilon \leq t_1 < \cdots < t_p \leq T$ *and* $(t_p - t_{p-1}) < \delta_p$. *In the above,* $t_0 = 0$ *and* $\|f(t, \cdot)\|_\alpha$ *denotes the* $L^\alpha(R, Leb)$ *norm of* $f(t, u)$ *with respect to the variable* u.

Remark. The local nondeterminism of Gaussian processes was formulated by Berman[3], in which the range for t_j is restrictd to $t_p - t_1 < \delta_p$ (the "locally" local nondetermism). Pitt[22] later reformulated Berman's definition, in which $t_1 < \cdots < t_p$ are ranging freely all over $[\varepsilon, T]$ (the "globally" local nondeterminism). Nolan[20] formulated the local nondeterminism for symmetric stable processes by using linear predictors to replace conditional variances, and pointed out ([20, Theorem 3.2]) that his formulation expressed in terms of kernel functions is exactly the display (2.1). He also proved that certain stable processes are locally nondeterministic; the time range is in the same consideration as Berman. Our Proposition 2.1 requires that $t_p - t_{p-1}$ is small yet allows that $t_1 < \cdots < t_{p-1}$ are freely ranging; thus Proposition 2.1 is a "adequate" non-Gaussian version of Pitt's local nondeterminism. As we have mentioned in § 1, it is unknown that the two formulations of Berman and Pitt are equivalent in the non-Gaussian case.

Proof of Proposition 2.1. In [16, Proposition 4.1], Kono and Shieh proved that linear fractional stable processes is locally nondeterministic in Nolan's sense; we find that the arguments in [16] can be modified to obtain Proposition 2.1. It seems beter to include a complete proof here. Note that (2.1) is equivalent to

$$\liminf_{(t_p-t_{p-1})\downarrow 0}\ \inf_{\varepsilon\le t_1<\cdots<t_{p-1}<t_p\le T}\ \inf_{v_1,\cdots,v_{p-1}\in R}$$

$$\cdot\left\{\frac{\|[f(t_p,\cdot)-f(t_{p-1},\cdot)]-\sum_{j=1}^{p-1}v_j[f(t_j,\cdot)-f(t_{j-1},\cdot)]\|_\alpha^\alpha}{\|f(t_p,\cdot)-f(t_{p-1},\cdot)\|_\alpha^\alpha}\right\}$$

$$=C_p>0\,.$$

Suppose on the contrary that $C_p=0$ for some p. Then there exist sequences t_j^n and a_j^n, $j=1,\cdots,p$ and $n=1,2,\cdots$, $\varepsilon\le t_1^n<t_2^n<\cdots<t_{p-1}^n<t_p^n\le T$, $t_p^n-t_{p-1}^n\downarrow 0$ as $n\uparrow\infty$ and $a_j^n\in\mathrm{R}$ such that

$$\lim_{n\to\infty}\frac{\|[f(t_p^n,\cdot)-f(t_{p-1}^n,\cdot)]-\sum_{j=1}^{p-1}a_j^n f(t_j^n,\cdot)\|_\alpha^\alpha}{\|f(t_p^n,\cdot)-f(t_{p-1}^n,\cdot)\|_\alpha^\alpha}=0\,. \tag{2.2}$$

Define

$$f(u)=|1-u|^{H-1/\alpha}-|u|^{H-1/\alpha},\quad u\in\mathrm{R}\,.$$

It is sasy to check that

$$f(t_p^n,u)=f(t_{p-1}^n,u)=(t_p^n-t_{p-1}^n)^{H-1/\alpha}f\left(\frac{u-t_{p-1}^n}{t_p^n-t_{p-1}^n}\right)\quad\text{and}$$

$$f(t_j^n,u)=(t_j^n)^{H-1/\alpha}f\left(\frac{u}{t_j^n}\right)$$

By a linear change of variable:

$$u=t_{p-1}^n+(t_p^n-t_{p-1}^n)u_1\,,$$

the numerator of (2.2) becomes

$$(t_p^n-t_{p-1}^n)^{H\alpha}\left\|f(u_1)-\sum_{j=1}^{p-1}a_j^n\left(\frac{t_j^n}{t_p^n-t_{p-1}^n}\right)^{H-1/\alpha}f\left(\frac{t_{p-1}^n+(t_p^n-t_{p-1}^n)u_1}{t_j^n}\right)\right\|_{L^\alpha(du_1)}^\alpha\,, \tag{2.3}$$

and the denominator of (2.2) becomes

$$(t_p^n - t_{p-1}^n)^{H\alpha}\|f(u_1)\|^{\alpha}_{L^\alpha(du_1)}\,.$$

Therefore, the limit in (2.2) is 0 means that the $L^\alpha(du_1)$ norm in (2.3) tends to 0 as $n\uparrow\infty$. Since $\alpha\geq 1$, there must exist subsequences $t_j^{n'}$ and $a_j^{n'}$ such that

$$(2.4)\qquad f(u_1)=\lim_{n'\to\infty}$$

$$\left\{\sum_{j=1}^{p-1} a_j^{n'}\left(\frac{t_j^{n'}}{t_p^{n'}-t_{p-1}^{n'}}\right)^{H-1/\alpha} f\left(\frac{t_{p-1}^{n'}+(t_p^{n'}-t_{p-1}^{n'})u_1}{t_j^{n'}}\right)\right\}.$$

for a.e. u_1. We argue that (2·3) is impossible. Observe that, for each $u_1>0$, the $f(\cdot)$ in the above summund is

$$\left[\frac{(t_{p-1}^{n'}-t_j^{n'})+(t_p^{n'}-t_{p-1}^{n'})u_1}{t_j^{n'}}\right]^{H-1/\alpha}-\left[\frac{t_{p-1}^{n'}+(t_p^{n'}-t_{p-1}^{n'})u_1}{t_j^{n'}}\right]^{H-1/\alpha}$$

Since $\varepsilon\leq t_j^{n'}<t_{p-1}^{n'}<t_p^{n'}\leq T$ and $0<(t_p^n-t_{p-1}^{n'})\downarrow 0$, we can extract further subsequences $t_j^{n''}$ along which the above display tends to a limit *independent on* $u_1>0$ as $n\uparrow\infty$. Thus, $f(u_1)$ determined by (2.4) is constant on $u_1>0$, which is obviously a contradiction. q.e.d,

The proof of Theorem 1.2. We proceed by the arguments adapted from Rosen [24] and Berman[4]. In the following, $U^l=(u_1^l,\cdots,u_{k-1}^l)\in\mathrm{R}^{d(k-1)}$ while each component $u_r^l=(u_{r1}^l,\cdots,u_{rd}^l)\in\mathrm{R}^d$; moreover, $T^l=(t_1^l,\cdots,t_k^l)\in I$. Let γ be any fixed real number such that

$$0<\gamma<\frac{1-\theta}{2\theta},$$

and let $B\subset I$ be any fixed cube. We will estimate, for all even m, the followding multiple integral

$$(2.5)\qquad \int_{\mathrm{R}^{d(k-1)m}}\int_{B^m}\prod_{l=1}^m|U^l|^\gamma E e^{i\Sigma_{l=1}^m \bar{Z}(T^l)\cdot U^l}dT^1\cdots dT^m dU^1\cdots dU^m\,,$$

where, $\cdot$ and $d(\cdot)$ denote the inner product and the Lebesgue integral in the appropriate Euclidean space.

From the definition of $\bar{Z}$, the independence assumption on the components of X, and the characteristic functions of a symmetric stable process, we have

$$(2.6)\qquad Ee^{i\Sigma_{l=1}^m\bar{Z}(T^l)\cdot U^l}=Ee^{i\Sigma_{l=1}^m\Sigma_{r=1}^{k-1}(\mathrm{X}(t_{r+1}^l)-\mathrm{X}(t_r^l))\cdot u_r^l}$$

$$=\prod_{j=1}^d Ee^{i\Sigma_{l=1}^m\Sigma_{r=1}^{k-1}(\mathrm{X}_j(t_{r+1}^l)-\mathrm{X}_j(t_r^l))u_{rj}^l}$$

$$=\prod_{j=1}^d e^{-\|\Sigma_{l=1}^m\Sigma_{r=1}^{k-1}(f_j(t_{r+1}^l,\cdot)-f_j(t_r^l,\cdot))u_{rj}^l\|^{\alpha_j}_{L^{\alpha_j}(\mathrm{R})}}\,.$$

Since $t_r^l \in [a_r, b_r]$ and $b_r < a_{r+1}$ by our assumption, we always have $t_r^l < t_{r+1}^{l'}$, $\forall\, l, l'$. By the symmetry in integrating t_r^l, it moreover suffices to consider the case $t_r^{l+1} > t_r^l$, $\forall\, l{:}1 \le l \le m$, in evaluating (2.5). Define $u_r^l \in R^d$ $r=1, \cdots, k$, $l=1, \cdots, m$ recursively by

$$u_r^l - v_r^{l-1} = u_r^l - u_{r-1}^l \, ,$$

$$u_r^0 = v_{r-1}^m \, ,$$

$$u_0^l = u_k^l = 0 \, ,$$

$$v_0^l = v_k^m = 0 \, .$$

We see that (2.6) is transformed into

$$Ee^{i\Sigma_{l=1}^m \bar{Z}} = \prod_{j=1}^d e^{-\|\Sigma_{l=1}^k \Sigma_{r=1}^m (f_j(t_r^{l+1}, \cdot) - f_j(t_r{}^l, \cdot)) v_{rj}^l\|_{L^{\alpha_j}(\mathrm{R})}^{\alpha_j}} \, .$$

where $t_r^{m+1} \equiv t_{r+1}^1$. Now we apply our local nondeterminism Proposition 2.1, with $p=mk$ there, and Nolan [20, Theorem 3.2 (b)(c)] to conclude that there exists a positive C_m such that

$$Ee^{\Sigma_{l=1}^m \bar{Z}(T^l) \cdot U^l} \le \prod_{j=1}^d e^{-C_m \Sigma_{r=1}^k \Sigma_{l=1}^m \|f_j(t_r^{l+1}, \cdot) - f_j(t_r^l, \cdot)\|_{L^{\alpha_j}(\mathrm{R})}^{\alpha_j} |v_{jr}^l|^{\alpha_j}} \, . \tag{2.7}$$

To assure (2.7), It should be noted that we may always assume that $t_k^m - t_k^{m-1}$ is small enough so that it is smaller than the δ given in the condition in Proposition 2.1, since we shall eventually be considering the integration over T^l and t_k^m, t_m^{m-1} are adjacent each other (This is the spot whee Berman's formulation is not sufficinent for our purpose; we *cannot* assume that $t_k^m - t_1^1 < \delta_{mk}$, since the latter quantity could become arbitrarily small as m becomes bigger). Then using the same scaling method as that in the proof of Proposition 2.1 to calculate the L^α norm for f_j, we can deduce from (2.7) that

$$Ee^{\Sigma_{l=1}^m \bar{Z}(T^l) \cdot U^l} \le \prod_{j=1}^d e^{-C_m \Sigma_{r=1}^k \Sigma_{l=1}^m (t_r^{l+1} - t_r^l)^{H_j \alpha_j} |v_{jr}^l|^{\alpha_j}} \, , \tag{2.8}$$

where the two constants C_m in (2.7) and (2.8) are different, although we have used the same notation.

Now, we integrate firstly over U^l in (2.8). To transform u_r^l into v_r^l, the following algebraic inequality in Resen [24, (2.12)] is tactical. For each $p=1, \cdots, k$, let

$$C_p = \{(r, l) | 1 \le r \le k, 1 \le l \le m, r \ne p\} \cup \{(p, m)\} \, ,$$

then, there exists some constant c independent on m such that

$$\prod_{l=1}^m |U^l| \le c^m \prod_{p=1}^k \Big(\prod_{(r,l) \in C_p} (1 + |v_r^l|^2) \Big)^{1/k}$$

$$\leq c^m \prod_{p=1}^{k} \Big(\prod_{(r,l)\in C_p} \prod_{j=1}^{d} (1+|v_{jr}^l|^2)^{1/k} \Big)$$

From the definition of v_r^l, we see that $\{v_r^l\colon (r, l)\in C_p\backslash\{(k, m)\}\}$ is a set of coordinates for $\mathrm{R}^{d(k-1)m}$. Therefore, by (2.8), the inequlity above, and Hölder inequality for the product of k functions we have (*confer* Rosen [24, (2.13)] for such techniques in the Gaussian case)

$$\int_{\mathrm{R}^{d(k-1)m}} \prod_{l=1}^{m} |U^l|^\gamma E e^{i\Sigma_{l=1}^m \bar{Z}(T^l)\cdot U^l} dU^1 \cdots dU^m$$

$$\leq c^m \prod_{p=1}^{k} \Big[\int_{\mathrm{R}^{d(k-1)m}} \prod_{(r,l)\in C_p} \prod_{j=1}^{d} (1+|v_{jr}^l|^2)^\gamma e^{-kC_m |t_r^{l+1}-t_r^l|^{H^j\alpha_j} |v_{jr}^l|^{\alpha_j}} \prod d(v_{jr}^l) \Big]^{1/k}$$

$$\leq (Const.)_m \prod_{p=1}^{k} \prod_{(r,l)\in C_p} (t_r^{l+1}-t_r^l)^{-(\Sigma_{j=1}^d H_j)(1+2\gamma)1/k}$$

$$\leq (Const.)_m \prod_{r=1}^{k} \prod_{l=1}^{m-1} (t_r^{l+1}-t_r^l)^{-(\Sigma_{j=1}^d H_j)(1+2\gamma)\frac{k-1}{k}} .$$

Therefore,

The multiple integral in (2.5)

$$\leq (Const.)_m \int_{B^m} \prod_{r=1}^{k} \prod_{l=1}^{m-1} (t_r^{l+1}-t_r^l)^{-(\Sigma_{j=1}^d H_j)(1+2\gamma)\frac{k-1}{k}} \prod_{(r,l)} dt_r^{\,l} .$$

We have, by our choice of γ and our definition of θ,

The multiple integral in (2.5) $= O(e(B))^{k(m-1)(1-\theta(1+2\gamma))}$.

Now, by those indicated in Geman-Horowitz [9, § 26] and Resen[24], the assertions of Theorem 1.2 follow directly from the above display. q.e.d.

§ 3. The proof of Theorem 1.1

Firstly, we cite the following lemma from Adler [1, Theorem 8.7.4].

Lemma 3.1. *Let* F: $\bar{t}\in I=[a_1, b_1]\times\cdots\times[a_p, b_p]\subset R_+^p \mapsto \bar{x}=F(\bar{t})\in R^q$ *be a continuous function. Assume that F has a jointly continuous local time* $\phi(\bar{x}, \bar{t})$ *which also satisfies a uniform Hölder condition as that (1.3). Then* $\dim F^{-1}(\bar{x})\geq p\lambda$ *for every* $\bar{x}\in R^q$ *such that* $\phi(\bar{x}, I)>0$.

The proof of Theorem 1.1. Let $I=[1, 2]\times[3, 4]\times\cdots\times[2k-1, 2k]$. By Theorem 1.2 and Lemma 3.1, we have

$P\{$there exist $t_j\in[2j-1, 2j]$ such that $\mathrm{X}(t_1)=\cdots=\mathrm{X}(t_k)\}$

$\geq P\{\bar{Z}^{-1}(\bar{0})\neq 0\}$

$\geq P\{\phi(\bar{0}, I)>0\}$,

where ϕ is the local time of $\bar{Z}$ over I given in Theorem 1.2. The last event in the above display is of positive probability, since it can be seen that

$$E\phi(\bar{0}, I)=\int_I \int_{R^d} p(t_1, \cdots, t_k; x, \cdots, x)dxdt_1\cdots dt_k>0\ ,$$

where $p(t_1, \cdots, t_k; x_1, \cdots, x_k)$ denotes the joint density function of $X(t_1), \cdots, X(t_k)$ at $(x_1, \cdots, x_k)$. Now, we consider the event

$$\Omega'=\{\omega: \text{the path } X(\cdot, \omega) \text{ has } k\text{-multiple points}\}\ .$$

We have shown at above that $P(\Omega')>0$. Takashima [28] noted that for each $c>0$, $c\neq 1$ the transformation S_c on the underlying probability space Ω which is induced by $S_c(X)(t)=(c^{-H_1}X_1(ct), \cdots, c^{-H_d}X_d(ct))$ is P-measure preserving, and he proved that S_c is ergodic (in fact, mixing). It is easy to see that Ω' is invariant with respect to S_c, i.e. $S_c^{-1}\Omega'\subset\Omega'$. Therefore, we must hcave $P(\Omega')=1$, which proves the almost sure existence of k-multiple points. The lower bound estimate on the Hausdorff dimension of the set of k-multiple times follows directly from Lemma 3.1 and our exponent of Hölder continuity of $\phi(\bar{x}, \bar{t})$ in the set variable.

§ 4. A related result

Once we have proved the existence of multiple points of $X(t)$, it becomes interesting to ask (i) for which subsets E of R^d can we find multiple points $x\in E$ with positive probability (such an E is referred as "not k-multiple polar") and (ii) for which time sets $T\subset R^+$ are thick enough to make that $\{X(t), t\in T\}$ has multiple points (i.e. $x=X(t_1)=\cdots=X(t_k)$ for k different $t_j\in T$), see Taylor[29]. The two problems have been investigated for Brownian motions, Markov processes and certain Gaussian fields, see Fitzsimmon-Salisbery[8], Kahane[12], Shieh[27], Tastard[30] and Tongring[31]. The first problem seems closely related to the potential theory of the processes which we do not know any development in this aspect for non-Markovian stable processes. Here we show that the arguments for proving Theorem 1.1 can be modified to have a result for the second problem.

Theorem 4.1. *Under the conditions of Theorem 1.1. Let a compact* $T\subset R^+$ *be with* $\dim T>\theta$. *Then, with positive probability* $\{X(t), t\in T\}$ *has k-multiple points.*

To prove the above theorem, since $\dim T>\theta$, where must exist some θ_1: $\theta<\theta_1\leq\dim T$ such that the Hausdorff θ_1-measure of T is positive. By Frostman's Lemma as mentioned in Kahane [13, p 130], there exists a probability measure ν on R_+ which is supported on T and satisfies Hölder condition that $\nu[a, a+h]\leq(const.)h^{\theta_1}$ for all a, h. Let $T_1, \cdots, T_k$ be k disjoint compact subsets of T, each $T_j\subset[a_j, b_j]$, $0<a_j<b_j<a_{j+1}<b_{j+1}$, and $\nu(T_j)>0$. Let ν_j be the restriction of ν to $[a_j, b_j]$ and let μ be the k-product of ν_j. Then, μ is a Borel mersure on $I\equiv[a_1, b_1]\times\cdots\times[a_k, b_k]$ which is supported on $T_1\times\cdots\times T_k$ and satisfies the Hölder condition that $\mu(J)\leq const.\mathrm{Leb}(J)^{\theta_1}$ for all rectangles $J\subset I$. Then we can use the same techniques as in the proof of Theorem 1.2, with Lebesgue measure there replaced by the non-atomic μ, to have jointly continuous local time ϕ_μ of $\bar{Z}$ with respect to the measure μ. This ϕ_μ also satisfies the uniform Hölder condition in the set variable, with exponent $\lambda<\theta_1-\theta$. We remark that the local time theory is usually stated with respect to Lebesgue measure; yet some basic formulations and properties still hold with respect a non-atomic Borel measure and in fact they have certain nice applications, see Pitt[22] and Geman-Horowitz[9]. Lemma 3.1 holds for ϕ_μ; note that μ is absolutely continuous with respect to Lebesgue measure. By Lemma 3.1 and noting that our local time is relative to μ which is supported on $T_1\times\cdots\times T_k$, we have

$$dim\{(t_1, \cdots, t_k)\in T_1\times\cdots\times T_k\subset T^k:\bar{Z}(t_1, \cdots, t_k)=\bar{0}\}\geq k(\theta_1-\theta)>0$$

whenever $\phi_\mu(\bar{0}, I)>0$. The latter holds with positive probability, as we can see from the first moment of $\phi_\mu(\bar{0}, I)$ which is given by

$$E\phi_\mu(\bar{0}, I)=\int_{T_1}\cdots\int_{T_k}\int_{R^d}p(t_1, \cdots, t_k; x, \cdots, x)dx\mu(dt_1)\cdots\mu(dt_k)>0\,.$$

Theorem 4.1 follows again from the correspondence between the k-multiple points of X and the zeroes of $\bar{Z}$.

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