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On the support of solution of a stochastic differential equation without drift

By

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Introduction

Let M be a σ-compact C^∞-manifold and $V_0, V_1, \cdots, V_r$ be C^∞-vector fields on M. Throughout this paper, we shall fix any point $x \in M$. Let $(w^1, \cdots, w^r)$ be an r-dimensional Wiener process. We denote by x_t, $0 \leq t < \tau$, the solution of the stochastic differential equation

$$(1) \qquad \begin{cases} dx_t = \sum_{k=1}^r V_k(x_t) \circ dw^k(t) + V_0(x_t) dt \\ x_0 = x \end{cases}$$

of Stratonovich type, where τ is the explosion time.

Stroock and Varadhan obtained the topological support of the distribution of $\{x_t\}_{t \geq 0}$ on the path space. Therefore, that of x_1 on M is known definitely. But sometimes the notion of topological support seems too rough to characterize a measure. Indeed, the topological support of a measure on M which consists of countable point masses may be M itself.

Recently, Aida-Kusuoka-Stroock [1] obtained the "support" of the distribution of x_1 assuming the Hörmander's condition. Under their assumption, there exists a smooth density function of x_1. Then, they determined the points on M at which the density function of x_1 becomes positive.

In this paper, we will generalize a part of the results in [1]. Especially when $V_0 \equiv 0$, we will obtain a simple image of the "support" of x_1.

In section 1, we will give a sufficient condition for a point on M to be of "positive density". The results of section 1 were essentially obtained in [2] and [1].

In section 2, we will define the "orbit" E of the family $\{V_0, \cdots, V_r\}$ of vector fields, which was proved to be a C^∞-submanifold of M by Sussmann [7]. Then, we will review the result of [5] that x_t stays in E up to the explosion.

In section 3, we will study the case without drift and show that x_1 is distributed everywhere on E. In fact, we will prove that the Radon-Nikodym density of the absolutely continuous part of the distribution of x_1 with respect

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to the volume on E is uniformly positive on any compact subset of E. We shall note that the distribution of x_1 is not necessarily absolutely continuous with respect to the volume on E. We will offer a simple counterexample to this.

1. Points of positive density

Let H be the space of $\boldsymbol{R}^r$-valued absolutely continuous functions $h=(h^1, \cdots, h^r)$ on $[0, +\infty)$ such that $h(0)=0$ and $\int_0^{+\infty}|\dot{h}(t)|^2 dt<\infty$ where $\dot{h}$ is the Radon-Nikodym derivative of h. We consider H as a Hilbert space with the inner product $(h_1, h_2)=\sum_{k=1}^r \int_0^{+\infty} \dot{h}_1^k(t) \dot{h}_2^k(t) dt$. For $h \in H$, we denote by $\varphi_t(h)$, $0 \leq t<\tau_h$, the solution of the ordinary differential equation

$$
\begin{cases} \dot{\varphi}_t(h)=\sum_{k=1}^r V_k(\varphi_t(h)) \dot{h}^k(t)+V_0(\varphi_t(h)) \\ \varphi_0(h)=x \end{cases} \tag{2}
$$

where τ_h is the explosion time. If $h \in H$ and $\tau_h>1$, then $\varphi_1(\tilde{h})$ is defined for $\tilde{h}$ in some neighborhood $\tilde{H}$ of h, and the map φ_1: $\tilde{H} \rightarrow M$ is differentiable in the sense of Fréchet. We denote the Fréchet differential of φ_1 at h by $D\varphi_1(h)$: $H \rightarrow T_{\varphi_1(h)}(M)$.

Let us assign on M any Riemannian metric g. We denote the Riemannian volume and the Riemannian distance on (M, g) by $vol_M(\cdot)$ and $d(\cdot, \cdot)$ respectivelly.

The aim of this section is to prove the following fact, which was essentially obtained in [2] and [1].

Theorem 1.1. *Suppose that $h \in H$ exists such that $\tau_h>1$ and* Image $(D\varphi_1(h))=T_y(M)$ *where $y=\varphi_1(h)$. Put for $\varepsilon>0$ that*

$$
\sigma_\varepsilon=\inf(\{t<\tau \mid d(x_t, \varphi_t(h)) \geq \varepsilon\} \cup\{\tau\}) .
$$

Then, there exist a neighborhood N of y and $c>0$ such that

$$
P(\sigma_\varepsilon>1, x_1 \in B) \geq c \cdot vol_M(B)
$$

for any Borel subset B of N.

We shall carry out the proof of Theorem 1.1 in the remainder of this section. Let us fix $h \in H$ in the assumption of Theorem 1.1. Since $\{\sigma_\varepsilon>1\}$ increases as ε increases, we may assume that ε is small enough so that $K=\{z \in M \mid d(z, \varphi_t(h)) \leq \varepsilon$ for some $t \in[0,1]\}$ is compact. Noting that $\sigma_\varepsilon>1$ implies $x_t \in K$ for all $t \in[0,1]$, we can see that $P(\sigma_\varepsilon>1, x_1 \in B)$ is invariant when we replace $V_0, \cdots, V_r$ by another family of vector fields $V_0', \cdots, V_r'$ such that $V_j=V_j'$ on K. Therefore we may and will assume that the supports of $V_0, \cdots, V_r$ are compact

to the end of this section. Let us notice that, under this condition, the processes we consider may not explode.

Let m be the dimension of M and put

$$D_\rho=\{u\in \boldsymbol{R}^m \| u|<\rho\}$$

for $\rho>0$. Let $w_j=(w_j^1, \cdots, w_j^r)$, $j=1, \cdots, m$ be the independent copies of w. For $u=(u^1, \cdots, u^m)\in D_1$, put

$$w^k(t; u)=(1-|u|^2)^{1/2}w^k(t)+\textstyle\sum_{j=1}^m u^j w_j^k(t)\,. \tag{3}$$

Then, for each $u\in D_1$, $(w^1(t; u), \cdots, w^r(t; u))$ also becomes a Wiener process. Let $C^1(D_1; M)$ be the space of C^1-mappings of D_1 into M, and we will assign on $C^1(D_1; M)$ the compact-open C^1-topology (*cf*. [3]). Then, we can define a continuous $C^1(D_1; M)$-valued process $\{\theta_t\}_{t\geq 0}$ which satisfies for each $u\in D_1$ the stochastic differential equation

$$\begin{cases} d\theta_t(u)=\sum_{k=1}^r V_k(\theta_t(u))\circ dw^k(t; u)+V_0(\theta_t(u))dt \\ \theta_0(u)=x \end{cases} \tag{4}$$

(see [4] for example). We shall introduce some notations:

$$\sigma_\varepsilon(u)=\inf\{t|d(\theta_t(u), \varphi_t(h))\geqq\varepsilon\}\,,$$

$$\sigma_{\rho,\varepsilon}=\inf\{\sigma_\varepsilon(u)|u\in D_\rho\}\,.$$

$$|J(u)|=(\det(g_{\theta_1(u)}(\xi_i, \xi_j)))^{1/2}$$

where $\xi_i=(d\theta_1)_u\left(\frac{\partial}{\partial u^i}\right)=\frac{\partial}{\partial u^i}(\theta_1(u))\in T_{\theta_1(u)}(M)$.

Since the law of $\{\theta_t(u)\}_{t\geq 0}$ is the same as that of $\{x_t\}_{t\geq 0}$ for each $u\in D_1$, we have

$$P(\sigma_\varepsilon>1, x_1\in B)=P(\sigma_\varepsilon(u)>1\,, \theta_1(u)\in B)\,, \quad u\in D_1\,. \tag{5}$$

Suppose $0<\rho<1$. Integrating (5) on D_ρ and applying the Fubini's theorem and the change of variables formula, we have

$$\begin{aligned} |D_\rho|P(\sigma_\varepsilon>1, x_1\in B)&=\int_{D_\rho}P(\sigma_\varepsilon(u)>1, \theta_1(u)\in B)du \\ &\geq\int_{D_\rho}P(\sigma_{\rho,\varepsilon}>1, \theta_1(u)\in B)du \\ &=E\left[\chi_{\{\sigma_{\rho,\varepsilon}>1\}}\cdot\int_{D_\rho}\chi_B(\theta_1(u))du\right] \\ &\geq E\left[\chi_{\{\sigma_{\rho,\varepsilon}>1\}}\cdot(\sup_{D_\rho}|J|)^{-1}\cdot\int_{D_\rho}|J(u)|\chi_B(\theta_1(u))du\right] \end{aligned}$$

$$\geq E[\chi_{\{\sigma_{\rho,\varepsilon}>1\}}\cdot(\sup_{D_\rho}|J|)^{-1}\cdot vol_M(B\cap\theta_1(D_\rho))]$$

$$\geq E[\chi_{A(\rho,\varepsilon,N)}\cdot(\sup_{D_\rho}|J|)^{-1}\cdot]\cdot vol_M(B)$$

where $|D_\rho|$ is the volume of D_ρ and

$$A(\rho, \varepsilon, B)=\{\sigma_{\rho,\varepsilon}>1, \theta_1(D_\rho)\supset B\}.$$

Thus we have proved the following.

Lemma 1.2. *Let N be a neighborhood of $y=\varphi_1(h)$. Suppose $0<\rho<1$ and $\varepsilon>0$. Then, for any Borel subset B of N.*

$$P(\sigma_\varepsilon>1, x_1\in B)\geq c_{\rho,\varepsilon,N}\cdot vol_M(B)$$

where

$$c_{\rho,\varepsilon,N}=|D_\rho|^{-1}E[(\sup_{D_\rho}|J|)^{-1}\cdot\chi_{A(\rho,\varepsilon,N)}]. \tag{6}$$

We shall show in the remainder of this section that, for any $\varepsilon>0$, $c_{\rho,\varepsilon,N}$ becomes positive for suitable ρ and N. We shall begin with a refinement of the inverse mapping theorem. Let us denote by $\|\cdot\|$ the operator norm of $m\times m$-matrices.

Lemma 1.3. *Let $0<\rho<1$ and $\delta>0$. Let f: $D_\rho\to\boldsymbol{R}^m$ be a C^∞-mapping such that $\|I-\partial f(u)\|<\delta$ for all $u\in D_\rho$ where I and ∂f are the identity matrix and the Jacobian matrix of f respectively. Then, $f(D_\rho)\supset\{z\in\boldsymbol{R}^m||z-f(0)|<(1-\delta)\rho\}$.*

Proof. We may assume that $f(0)=0$. Let us denote by ι the identity mapping on D_ρ. Then, we see by the assumption that

$$|(\iota-f)(u)-(\iota-f)(v)|<\delta|u-v|, \; u, v\in D_\rho.$$

Suppose that $|z|<(1-\delta)\rho$. Noting that the equation $f(u)=z$ is equivalent to $u=z+(\iota-f)(u)$, we shall solve it as follows.

First, let us show that a sequence $\{u_n\}_{n\geq 0}$ can be defined inductively by

$$u_0=0, \quad u_{n+1}=z+(\iota-f)(u_n), \quad n=0, 1, 2, \cdots. \tag{7}$$

and then it holds that

$$|u_{n+1}-u_n|\leq\delta^n|z|, \quad n=0, 1, 2, \cdots. \tag{8}$$

Indeed, $u_1=z$ by (7) and so (8) holds when $n=0$. If $\{u_0, \cdots, u_k\}$ satisfying (7) and (8) is defined, then $|u_k|\leq|u_0|+\sum_{j=0}^{k-1}|u_{j+1}-u_j|\leq\frac{|z|}{1-\delta}<\rho$ so that u_{k+1} can be defined by (7). Moreover, by (7) and (8), $|u_{k+1}-u_k|=|(\iota-f)(u_k)-(\iota$

$-f)(u_{k-1})| \leq \delta |u_k - u_{k-1}| \leq \delta^k |z|$.

Then, since $|u_0| + \sum_{n=0}^{\infty} |u_{n+1} - u_n| \leq |z| \sum_{n=0}^{\infty} \delta^n = \frac{|z|}{1-\delta} < \rho$, u_n converges to some $u \in D_\rho$. Then, by (7), $u = z + (\iota - f)(u)$ so $f(u) = z$.

We put

$$\theta_t(u; \boldsymbol{h}) = \varphi_t((1-|u|^2)^{1/2} h_0 + \textstyle\sum_{i=1}^{m} u^i h_i) \tag{9}$$

for $\boldsymbol{h} = (h_0, h_1, \cdots, h_m) \in H^{m+1}$, $u = (u^1, \cdots, u^m) \in D_1$ and $t \geq 0$. If we fix $u \in D_1$ and put

$$\tilde{V}_{k,0} = (1-|u|^2)^{1/2} V_k , \qquad \tilde{V}_{k,j} = u^j V_k , \qquad j = 1, \cdots, m ,$$

then, by (2), (3), (4) and (9), we have the stochastic differential eqution

$$\begin{cases} d\theta_t(u) = \sum_{k=1}^{r} \sum_{j=0}^{m} \tilde{V}_{k,j}(\theta_t(u)) \circ dw_j^k(t) + V_0(\theta_t(u)) dt \\ \theta_0(u) = x \end{cases}$$

where $w_0(t) = w(t)$ and the ordinary differential equation

$$\begin{cases} \dot{\theta}_t(u; \boldsymbol{h}) = \sum_{k=1}^{r} \sum_{j=0}^{m} \tilde{V}_{k,j}(\theta_t(u; \boldsymbol{h})) \dot{h}_j^k(t) + V_0(\theta_t(u; \boldsymbol{h})) \\ \theta_0(u; \boldsymbol{h}) = x . \end{cases}$$

Let $\mathrm{dist}(\cdot, \cdot)$ be any metric on $C^1(D_1; M)$ compatible with the compact-open C^1-topology. For $\boldsymbol{h} \in H^{m+1}$ and $\delta > 0$, define the event $G_\delta(\boldsymbol{h})$ by

$$G_\delta(\boldsymbol{h}) = \{\sup_{0 \leq t \leq 1} \mathrm{dist}(\theta_t, \theta_t(\cdot; \boldsymbol{h})) < \delta\} .$$

Then we can obtain the following generalization of the Stroock-Varadhan's support theorem (see [4], the proof of Theorem 5.7.6).

Lemma 1.4. *$P(G_\delta(\boldsymbol{h})) > 0$ for any $\boldsymbol{h} \in H^{m+1}$ and $\delta > 0$.*

By means of Lemmas 1.3 and 1.4, we can now prove the following.

Lemma 1.5. *Under the condition of Theorem 1.1, for any $\varepsilon > 0$, there exist $\rho > 0$ and a neighborhood N of y such that $P(A(\rho, \varepsilon, N)) > 0$.*

Proof. Take any local chart $z = (z^1, \cdots, z^m)$ on a neighborhood of y. By the assumption, we can take $h_1, \cdots, h_m \in H$ such that

$$\frac{d}{du} z^i(\varphi_1(h + uh_j))|_{u=0} = \delta^i_j , \quad i, j = 1, \cdots, m$$

where δ^i_j is the Kronecker's delta. Put $\boldsymbol{h} = (h, h_1, \cdots, h_m) \in H^{m+1}$. Then the Jacobian matrix of the mapping $z \circ \theta_1(\cdot; \boldsymbol{h}): D_1 \to \boldsymbol{R}^m$ at 0 is the identity matrix. If ρ is sufficiently small, then

$$\sup_{0\leq t\leq 1, u\in D_\rho} |\theta_t(u; \boldsymbol{h})-\varphi_t(h)|<\frac{\varepsilon}{2}.$$

By Lemma 1.3, $G_\delta(\boldsymbol{h})\subset A(\rho, \varepsilon, N)$ holds for sufficiently small ρ and δ and a small neighborhood N of y. Then, by Lemma 1.4, $P(A(\rho, \varepsilon, N))\geq P(G_\delta(\boldsymbol{h}))>0$.

Noting (6), $c_{\rho,\varepsilon,N}>0$ if and only if $P(A(\rho, \varepsilon, N))>0$. Therefore Lemmas 1.2 and 1.5 imply the assertion of Theorem 1.1.

2. Submanifold supporting the distribution of solution

We will introduce in this section the "orbit" E of $\{V_0, \cdots, V_r\}$ following [7], and review the result of [5].

Definition. Let E be the subset of M consisting of all $y\in E$ such that there exist $n\geq 1$, $k_1, \cdots, k_n\in\{0, \cdots, r\}$ and $\boldsymbol{t}=(t_1, \cdots, t_n)\in \boldsymbol{R}^n$ for which

$$\psi_{k_1,\cdots,k_n}(\boldsymbol{t})=\theta_{t_n}^{k_n}(\cdots(\theta_{t_2}^{k_2}(\theta_{t_1}^{k_1}(x))\cdots) \tag{10}$$

is well-defined and equal to y. Here, $\theta_t^k(x)$ is the integral curve of V_k such that $\theta_0^k(x)=x$.

We assign on E the strongest topology such that (10) becomes a continuous map of $\boldsymbol{t}$ into E for any $k_1, \cdots, k_n$.

The following fact was obtained by Sussmann [7].

Proposition 2.1. *There is a differentiable structure on E for which*

(i) *E is a σ-compact C^∞-submanifold of M,*

(ii) *for each $y\in E$, there exist $k_1, \cdots, k_n\in\{0, \cdots, r\}$ and $\boldsymbol{t}\in\boldsymbol{R}^n$ such that*

$$y=\psi_{k_1,\cdots,k_n}(\boldsymbol{t}) \qquad \textit{and} \quad \mathrm{Image}(d\psi_{k_1,\cdots,k_n}(\boldsymbol{t}))=T_y(E). \tag{11}$$

Remark. We can easily show that the differentiable structure on E compatible with the topology defined above and with which (i) holds is unique.

Throughout this paper, we consider E as a C^∞-submanifold of M with the differentiable structure introduced in Proposition 2.1. Then, $V_0, \cdots, V_r$ can be considered as C^∞-vector fields on E. Therefore it is obvious that $x_t\in E$ for sufficiently small t almost surely. In fact, the following was obtained in [5].

Propoition 2.2. (i) *$\varphi_t(h)\in E$ for any $h\in H$ and $t\in[0, \tau_h)$,* (ii) *$x_t\in E$ for all $t\in[0, \tau)$ almost surely.*

Remark. Only (ii) was proved in [5] explicitly but (i) can be shown more easily in the same way.

3. The case without drift

In this section, we shall study the case where $V_0 \equiv 0$ by combining the results in sections 1 and 2. Let us assign on E the Riemannian structure induced from M and denote by $vol_E(\cdot)$ the Riemannian volume on E. The aim of this section is to show the following fact.

Theorem 3.1. *Suppose $V_0 \equiv 0$. Then, for any compact subset K of E, there exists a positive constant c_K such that*

$$P(\tau > 1, x_1 \in B) \geq c_K \cdot vol_E(B) \tag{12}$$

for any Borel subset B of K.

Remark. Let $p(y)$ be the Radon-Nikodym density of the absolutely continuous part of the distribution of x_1 with respect to the volume on E. Then the assertion of Theorem 3.1 means that

$$\operatorname*{ess.inf}_{y \in K} p(y) > 0$$

for any compact subset K of E.

We shall prove theorem 3.1. The following lemma tells that Theorem 1.1, replaced M by E, can be applied for any points on E.

Lemma 3.2. *Suppose $V_0 \equiv 0$. Then, for each $y \in E$, there exists an $h \in H$ such that $\varphi_1(h) = y$ and* Image$(D\varphi_1(h)) = T_y(E)$.

Proof. Let $k_1, \cdots, k_n \in \{1, \cdots, r\}$ and $\boldsymbol{t} = (t_1, \cdots, t_n) \in \boldsymbol{R}^n$ satisfy (11). Define $h_1, \cdots, h_n \in H$ by

$$h_j^k(t) = \begin{cases} n \text{ if } \dfrac{j-1}{n} \leq t < \dfrac{j}{n} \text{ and } k_j = k \\ 0 \text{ if otherwise.} \end{cases}$$

Then we can easily see that

$$\phi_{k_1 \cdots, k_n}(\boldsymbol{s}) = \varphi_1(\textstyle\sum_{j=1}^n s_j h_j)$$

for any $\boldsymbol{s} = (s_1, \cdots, s_n)$ in some neighborhood of $\boldsymbol{t}$. Therefore, putting $h = \sum_{j=1}^n t_j h_j$, we have

$$\text{Image}(D\varphi_1(h)) \supset \text{Image}(d\varphi_{k_1, \cdots, k_n}(\boldsymbol{t})) = T_y(E)\,.$$

Proof of Theorem 3.1. Suppose that K is a compact subset of E. By Theorem 1.1 and Lemma 3.2, there exist a finite number of open subsets $N_1, \cdots, N_l$ of E and positive constants $c_1, \cdots, c_l$ such that $K \subset N_1 \cup \cdots \cup N_l$ and

$$P(\tau > 1, x_t \in B) \geq c_j \cdot vol_E(B)$$

if B is a Borel subset of N_j. Then (12) holds by putting $c_K=\min\{c_1, \cdots, c_l\}$.

We shall note that the distribution of x_1 is not necessarily absolutely continuous with respect to the volume on E. Indeed, we have the following counterexample.

Example. Let us consider the case where $M=\boldsymbol{R}^2$, $x=(0,0)$ and $r=2$. We denote the coordinate on $\boldsymbol{R}^2$ by (u, v). Take any smooth function f on $\boldsymbol{R}^1$ such that $f(u)=0$ (*resp.* $f(u)>0$) if $u\leq 1$ (*resp.* $u>1$). Put $V_1=\frac{\partial}{\partial u}$, $V_2=f(u)\frac{\partial}{\partial v}$ and $V_0=0$. Then $E=\boldsymbol{R}^2$ with its proper differentiable structure. However, the distribution of x_1 is a sum of two measures which are equivalent to the uniform measures on $(-\infty, 1)$ and on $\boldsymbol{R}^2$ respectively.

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References

[1] S. Aida, S. Kusuoka and D. W. Stroock, On the support of Wiener functionals, Preprint.
[2] G. Ben Arous et R. Léandre, Décroissance exponentielle du noyau de la chaleur sur la diagonale (I), Probab. Th. Rel. Fields, **90** (1991), 175-202.
[3] M. Hirsch, Differential topology, Springer,-Verlag, Berlin-Heidelberg-New York, 1976.
[4] H. Kunita, Stochastic flows and stochastic differential equations, Cambridge Univ. Press, 1990.
[5] T. Masuda, Certain invariant sets of stochastic flows generated by stochastic differential equations, To appear in J. Math. Kyoto Univ. **32** (1992).
[6] D. W. Stroock and S. R. S. Varadhan, On the support of diffusion processes with applications to the strong maximum principle, Proc. Sixth Berkeley Symp. Math. Statist. Prob. III. 333-359, Univ. California Press, Berkeley, 1972.
[7] H. J. Sussmann, Orbits of families of vector fields and integrability of distributions, Trans. Amer. Math. Soc. **180** (1973), 171-188.