J. Math. Kyoto Univ. (JMKYAZ)
33-3 (1993) 709-722

Kostant's formula and homology vanishing theorems for generalized Kac-Moody algebras

By

Satoshi NAITO

Introduction

A real $n \times n$ matrix $A=(a_{ij})_{i,j\in I}$ indexed by a set $I=\{1, 2, \dots, n\}$ is called a GGCM if it satisfies

(C1) either $a_{ii}=2$ or $a_{ii}\leq 0$;
(C2) $a_{ij}\leq 0$ if $i\neq j$, and $a_{ij}\in \boldsymbol{Z}$ if $a_{ii}=2$;
(C3) $a_{ij}=0$ implies $a_{ji}=0$.

Let $\mathfrak{g}(A)$ be a *generalized Kac-Moody algebra* (GKM algebra), over the complex number field $\boldsymbol{C}$, associated to a symmetrizable GGCM $A=(a_{ij})_{i,j\in I}$, with Cartan subalgebra $\mathfrak{h}$, simple roots $\Pi=\{\alpha_i\}_{i\in I}$, and simple coroots $\Pi^\vee=\{\alpha_i^\vee\}_{i\in I}$. And let $\mathfrak{g}(A)=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$ be the triangular decomposition with $\mathfrak{n}^\pm=\sum^\oplus_{\alpha\in\Delta^\pm}\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space attached to a root $\alpha\in\Delta^\pm$. See [5] (and also [1]) for the definition of generalized Kac-Moody algebras.

In the previous paper [11], we studied the $\mathfrak{h}$-module structure of the homology $H_j(\mathfrak{n}^-, L(\lambda))$ $(j\geq 0)$ of $\mathfrak{n}^-$ or the cohomology $H^j_c(\mathfrak{n}^+, L(\lambda))$ $(j\geq 0)$ of $\mathfrak{n}^+$ with coefficients in the irreducible highest weight $\mathfrak{g}(A)$-module $L(\lambda)$ with highest weight $\lambda\in\mathfrak{h}^*:=\mathrm{Hom}_C(\mathfrak{h}, \boldsymbol{C})$. (Remark that the cohomology $H^j_c(\mathfrak{n}^+, L(\lambda))$ $(j\geq 0)$ used in [11] is slightly different from the usual Lie algebra cohomology.) Then, we proved "Kostant's formula" under the following condition ($\hat{\mathrm{C}}1$) on the GGCM $A=(a_{ij})_{i,j\in I}$:

($\hat{\mathrm{C}}1$) either $a_{ii}=2$ or $a_{ii}=0$ $(i\in I)$.

Namely, we proved

Theorem A ([11]). *Let $\Lambda\in P^+:=\{\lambda\in\mathfrak{h}^* \mid \langle\lambda, \alpha_i^\vee\rangle\geq 0$ $(i\in I)$, and $\langle\lambda, \alpha_i^\vee\rangle\in\boldsymbol{Z}_{\geq 0}$ if $a_{ii}=2\}$. Denote by $\mathfrak{S}$ the set of all sums of distinct pairwise perpendicular elements from $\Pi^{im}:=\{\alpha_i\in\Pi \mid a_{ii}\leq 0\}$. And we put $\mathfrak{S}(\Lambda):=\{\lambda\in\mathfrak{S} \mid (\lambda|\Lambda)=0\}$, where $(\cdot|\cdot)$ is a standard bilinear form on $\mathfrak{h}^*$. Then, as $\mathfrak{h}$-modules $(j\geq 0)$,*

Prc. Jan. 21, 1992
Recceived January 21, 1992

$$H_c^j(\mathfrak{n}^+, L(\Lambda)) \cong H_j(\mathfrak{n}^-, L(\Lambda)) \cong \sum\nolimits^{\oplus}_{\beta\in\mathfrak{S}(\Lambda)} \sum^{\oplus}_{\substack{w\in W \\ \ell(w)=j-\mathrm{ht}(\beta)}} \boldsymbol{C}(w(\Lambda+\rho-\beta)-\rho),$$

where $\boldsymbol{C}(\mu)$ $(\mu\in\mathfrak{h}^*)$ *is the irreducible (one-dimensional)* $\mathfrak{h}$*-module with weight* μ. *Here,* ρ *is a fixed element of* $\mathfrak{h}^*$ *such that* $\langle\rho, \alpha_i^\vee\rangle=(1/2)\cdot a_{ii}$ $(i\in I)$, $\ell(w)$ *is the length of an element* w *of the Weyl group* W, *and for* $\beta=\sum_{i\in I} k_i\alpha_i$ $(k_i \in \boldsymbol{Z}_{\geq 0})\in\mathfrak{S}$, *we put* $\mathrm{ht}(\beta):=\sum_{i\in I} k_i$.

In the present paper, using the idea of L. Liu [10] for Kac-Moody algebras, we extend the above result so that the *nilpotent part* $\mathfrak{n}^+$ of the *Borel subalgebra* $\mathfrak{b}:=\mathfrak{h}\oplus\mathfrak{n}^+$ is allowed to be the *nilpotent part* of a *parabolic subalgebra* containing $\mathfrak{b}$.

Let us explain in more detail. Let I^{re} (resp. I^{im}) be the subset $\{i\in I \mid a_{ii}=2$ (resp. $a_{ii}\leq 0)\}$ of the indexing set I. And let J be a subset of I^{re}. We define a submatrix A_J of A by $A_J:=(a_{ij})_{i,j\in J}$, which is a *generalized Cartan matrix* (GCM). Note that there exists a certain subspace $\mathfrak{h}_J$ of $\mathfrak{h}$ with $\alpha_i^\vee\in\mathfrak{h}_J$ $(i\in J)$, such that the triple $(\mathfrak{h}_J, \{\alpha_i|\mathfrak{h}_J\}_{i\in J}, \{\alpha_i^\vee\}_{i\in J})$ is a *minimal realization* of the GCM A_J. Then, we can identify the Kac-Moody algebra $\mathfrak{g}(A_J)$ with the subalgebra $\mathfrak{g}_J$ of $\mathfrak{g}(A)$ generated by e_i, f_i $(i\in J)$, and $\mathfrak{h}_J$. Furthermore, $\mathfrak{g}_J=\mathfrak{h}_J\oplus\sum^{\oplus}_{\alpha\in\Delta_J}\mathfrak{g}_\alpha$, where $\Delta_J:=\Delta\cap\sum_{i\in J}\boldsymbol{Z}\alpha_i$ (or its restriction to $\mathfrak{h}_J$) is the root system of $(\mathfrak{g}_J, \mathfrak{h}_J)$. Now, we define the following subalgebras of $\mathfrak{g}(A)$:

$$\mathfrak{n}_J^+:=\sum\nolimits^{\oplus}_{\alpha\in\Delta_J^+}\mathfrak{g}_\alpha, \quad \mathfrak{n}_J^-:=\sum\nolimits^{\oplus}_{\alpha\in\Delta_J^+}\mathfrak{g}_{-\alpha}, \quad \mathfrak{u}^+:=\sum\nolimits^{\oplus}_{\alpha\in\Delta^+(J)}\mathfrak{g}_\alpha,$$

$$\mathfrak{u}^-:=\sum\nolimits^{\oplus}_{\alpha\in\Delta^+(J)}\mathfrak{g}_{-\alpha}, \quad \mathfrak{m}:=\mathfrak{n}_J^-\oplus\mathfrak{h}\oplus\mathfrak{n}_J^+, \quad \mathfrak{p}:=\mathfrak{m}\oplus\mathfrak{u}^+,$$

where $\Delta(J):=\Delta\setminus\Delta_J$, $\Delta_J^+:=\Delta^+\cap\Delta_J$, $\Delta^+(J):=\Delta^+\cap\Delta(J)$. We call $\mathfrak{p}=\mathfrak{m}\oplus\mathfrak{u}^+$ the *parabolic subalgebra* of $\mathfrak{g}(A)$ defined by J. Note that since the triple $(\mathfrak{h}, \{\alpha_i\}_{i\in J}, \{\alpha_i^\vee\}_{i\in J})$ is a *realization* (but not necessarily a minimal realization) of the GCM A_J, $\mathfrak{m}=\mathfrak{g}_J+\mathfrak{h}$ can be regarded as a Kac-Moody algebra associated to A_J, whose Cartan subalgebra is $\mathfrak{h}$.

Recall that the *Weyl group* W of $\mathfrak{g}(A)$ is defined to be the subgroup of $GL(\mathfrak{h}^*)$ generated by *fundamental reflections* r_i $(i\in I^{re})$. Now, let W_J be the subgroup of W generated by r_i's $(i\in J)$, which is the Weyl group of $\mathfrak{m}$. And we put $W(J):=\{w\in W \mid w(\Delta^-)\cap\Delta^+\subset\Delta^+(J)\}$ $(=\{w\in W \mid w^{-1}(\Delta_J^+)\subset\Delta^+\})$. Then, we will obtain the following theorem. (Here, as in [11], the cohomology $H_c^j(\mathfrak{u}^+, L(\Lambda))$ $(j\geq 0)$ is slightly different from the usual one, whereas the homology $H_j(\mathfrak{u}^-, L(\Lambda))$ $(j\geq 0)$ is the usual Lie algebra homology. See § 3 for the definition.)

Theorem. *Let* $\Lambda\in P^+$. *Assume that the* GGCM $A=(a_{ij})_{i,j\in I}$ *is symmetrizable and satisfies the condition* $(\hat{\mathrm{C}}1)$. *Then,*

$$H_c^j(\mathfrak{u}^+, L(\Lambda)) \cong H_j(\mathfrak{u}^-, L(\Lambda)) \cong \sum_{\beta\in\mathfrak{S}(\Lambda)}^{\oplus} \sum_{\substack{w\in W(J)\\ \ell(w)=j-\mathrm{ht}(\beta)}}^{\oplus} L_{\mathfrak{m}}(w(\Lambda+\rho-\beta)-\rho)\,,$$

as $\mathfrak{m}$*-modules* ($j\geq 0$). *Here, for* $\mu\in P_J^+ := \{\lambda\in\mathfrak{h}^* \mid \langle\lambda, \alpha_i^\vee\rangle\in \boldsymbol{Z}_{\geq 0}\ (i\in J)\}$, $L_{\mathfrak{m}}(\mu)$ *is the irreducible highest weight* $\mathfrak{m}$*-module with highest weight* μ.

Note that when $J=\phi$, this theorem is nothing but Theorem A, since in this case, $\mathfrak{u}^+=\mathfrak{n}^+$, $\mathfrak{u}^-=\mathfrak{n}^-$, $\mathfrak{m}=\mathfrak{h}$, and $W(J)=W$.

And in the last part of this paper, we prove a homology vanishing theorem for GKM algebras with coefficients in a *generalized Verma module,* as a consequence of our "Kostant's formula". This theorem generalizes the result of C. Sen [13], which is only for the class of Kac-Moody algebras and under the condition that the subset J of I is *of finite type* (i.e., the submatrix $A_J=(a_{ij})_{i,j\in J}$ of A is a *classical Cartan matrix* of finite type).

This paper is organized as follows. In § 1, we review some basic results for GKM algebras, especially the Weyl-Kac-Borcherds character formula. In § 2, we will introduce the algebra $\mathcal{F}$ of *formal* $\mathfrak{m}$*-characters,* where we can carry out certain formal operations. In § 3, we rewrite some results of L. Liu [10] for Kac-Moody algebras, which can be proved also for GKM algebras in just the same way that they are proved for Kac-Moody algebras. In § 4, we prove our main theorem stated above, combining the results of [10] and [11]. In § 5, as consequences of our main theorem, we obtain some vanishing theorems for the homology of GKM algebras.

Acknowledgments. I would like to express my sincere thanks to Professor Akira Kono for answering my questions about the singular cohomology of Kac-Moody groups.

§ 1. The category $\mathcal{O}$ and the character formula

In this section, we prepare fundamental results about GKM algebras for later use. For detailed accounts of this section, see [1] and [5].

We put $I:=\{1, 2, \dots, n\}$. Let $\mathfrak{g}(A)$ be the GKM algebra associated to a GGCM $A=(a_{ij})_{i,j\in I}$ with the Cartan subalgebra $\mathfrak{h}$.

Definition 1.1 ([5]). $\mathcal{O}$ is the category of all $\mathfrak{h}$-modules V satisfying the following:

(1) V admits a weight space decomposition $V=\sum_{\lambda\in\mathcal{P}(V)}^{\oplus} V_\lambda$, where $\mathcal{P}(V)$ is the set of all weights of V. And each weight space V_λ is finite-dimensional ($\lambda\in\mathcal{P}(V)$);

(2) there exist a finite number of elements $\lambda_i\in\mathfrak{h}^*$ ($1\leq i\leq s$) such that $\mathcal{P}(V) \subset \bigcup_{i=1}^{s} D(\lambda_i)$, where $D(\lambda_i) := \{\lambda_i-\beta \mid \beta\in Q_+=\sum_{j\in I}\boldsymbol{Z}_{\geq 0}\alpha_j\}$ ($1\leq i\leq s$).

Note that the category $\mathcal{O}$ is closed under the operations of taking submodules, quotients, finite direct sums, and finite tensor products.

Now, let $\mathcal{E}$ be the algebra over $\boldsymbol{C}$ consisting of all series of the form $\sum_{\lambda\in\mathfrak{h}^*} c_\lambda e(\lambda)$, where $c_\lambda \in \boldsymbol{C}$ and $c_\lambda = 0$ for λ outside a finite union of sets of the form $D(\mu)$ $(\mu\in\mathfrak{h}^*)$. Here, the elements $e(\lambda)$ are called *formal exponentials.* They are linearly independent and are in one-to-one correspondence with the elements $\lambda\in\mathfrak{h}^*$. And the multiplication in $\mathcal{E}$ is defined by $e(\lambda)\cdot e(\mu) := e(\lambda+\mu)$ $(\lambda, \mu\in\mathfrak{h}^*)$. Then, for $V=\sum^{\oplus}_{\lambda\in\mathfrak{h}^*} V_\lambda$ in $\mathcal{O}$, we define the *formal character* of V by $\mathrm{ch}\, V := \sum_{\lambda\in\mathfrak{h}^*} (\dim_{\boldsymbol{C}} V_\lambda) e(\lambda) \in \mathcal{E}$. Then, we know the following character formula.

Theorem 1.1 ([1] and [5]). *Assume that A is a symmetrizable* GGCM. *Let $(\cdot|\cdot)$ be a fixed standard bilinear form on $\mathfrak{h}^*$. For $\Lambda\in P^+$, we put*

$$S_\Lambda := e(\Lambda+\rho)\cdot\sum_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{\mathrm{ht}(\beta)}e(-\beta)\,,\quad R := \prod_{\alpha\in\Delta^+}(1-e(-\alpha))^{\mathrm{mult}(\alpha)}\,,$$

where $\mathrm{mult}(\alpha) := \dim_{\boldsymbol{C}} \mathfrak{g}_\alpha$ $(\alpha\in\Delta^+)$. *Then,*

$$e(\rho)\cdot R\cdot \mathrm{ch}\, L(\Lambda) = \sum_{w\in W} (\det\, w) w(S_\Lambda)\,,$$

with $w(e(\mu)) := e(w(\mu))$ $(\mu\in\mathfrak{h}^*)$.

Remark 1.1. The set $\{0\}\cup\Pi^{im}$ is contained in $\mathfrak{S}$ by definition. And, especially when A is a GCM, $\mathfrak{S}$ consists of only one element $0\in\mathfrak{h}^*$.

§ 2. The category $\mathcal{O}_J$ and the algebra $\mathcal{F}$

In this section, we explain the notion of the category $\mathcal{O}_J$ of $\mathfrak{m}$-modules. And then, we introduce the algebra $\mathcal{F}$ of "formal $\mathfrak{m}$-characters" of $\mathfrak{m}$-modules from the category $\mathcal{O}_J$. Note that when $J=\phi$, these are nothing but the category $\mathcal{O}$ and the algebra $\mathcal{E}$.

From now on, we always assume that the GGCM A is symmetrizable, and that J is a subset of $I^{re}=\{i\in I \mid a_{ii}=2\}$. We use notations in the Introduction.

Definition 2.1 (cf. [10, § 1]). $\mathcal{O}_J$ is the category of all $\mathfrak{m}$-modules M satisfying the following:

(1) Viewed as an $\mathfrak{h}$-module, M is an object of the category $\mathcal{O}$;

(2) Viewed as an $\mathfrak{m}$-module, M is a direct sum of irreducible highest weight $\mathfrak{m}$-modules $L_{\mathfrak{m}}(\lambda)$ with highest weight $\lambda\in P_J^+ = \{\mu\in\mathfrak{h}^* \mid \langle\mu, \alpha_i^\vee\rangle \in \boldsymbol{Z}_{\geq 0}\ (i\in J)\}$.

Clearly, the category $\mathcal{O}_J$ is closed under the operations of taking submodules, quotients, and finite direct sums. Moreover, a tensor product of two modules from $\mathcal{O}_J$ is again in the category $\mathcal{O}_J$, because $L_{\mathfrak{m}}(\lambda)\otimes_{\boldsymbol{C}} L_{\mathfrak{m}}(\mu)\in\mathcal{O}_J$ $(\lambda, \mu\in P_J^+)$ by [5, Theorem 10.7.b)] (note that the modules $L_{\mathfrak{m}}(\tau)$ $(\tau\in P_J^+)$ remain irreducible as $\mathfrak{g}_J$-modules). The main reason for our requirement that J is a subset of I^{re} comes from the fact that this theorem holds only for Kac-Moody algebras.

The following proposition plays a fundamental role in this paper.

Proposition 2.1 (cf. [10, § 1]). *For* $\Lambda\in P^{+}$, $L(\Lambda)$ *and* $(\Lambda^{j}\mathfrak{u}^{-})\otimes_{C}L(\Lambda)$ $(j\geq 0)$ *are in the category* $\mathcal{O}_{J}$, *where* $\Lambda^{j}\mathfrak{u}^{-}$ *is the exterior algebra of degree j over* $\mathfrak{u}^{-}$, *and is an* $\mathfrak{m}$*-module under the adjoint action* $(j\geq 0)$, *since* $[\mathfrak{m}, \mathfrak{u}^{-}]\subset\mathfrak{u}^{-}$.

Now, we define a certain algebra $\mathcal{F}$ over $\boldsymbol{C}$. The elements of $\mathcal{F}$ are series of the form $\sum_{\lambda\in P_{J}^{+}}c_{\lambda}m(\lambda)$, where $c_{\lambda}\in\boldsymbol{C}$ and $c_{\lambda}=0$ for λ outside a finite union of sets of the form $D(\mu)$ $(\mu\in\mathfrak{h}^{*})$. Here, the elements $m(\lambda)$ are called *formal* $\mathfrak{m}$*-exponentials*. They are linearly independent and are in one-to-one correspondence with the elements $\lambda\in P_{J}^{+}$.

For a module M in the category $\mathcal{O}_{J}$, we define the *formal* $\mathfrak{m}$*-character* $\mathrm{ch}_{\mathfrak{m}}M$ of M by $\mathrm{ch}_{\mathfrak{m}}M:=\sum_{\lambda\in P_{J}^{+}}[M:L_{\mathfrak{m}}(\lambda)]m(\lambda)$, where $[M:L_{\mathfrak{m}}(\lambda)]$ is the "multiplicity" of $L_{\mathfrak{m}}(\lambda)$ in M (see [5, Ch. 9, Lemma 9.6]). Note that $[M:L_{\mathfrak{m}}(\lambda)]$ $(\lambda\in P_{J}^{+})$ is finite since M is in the category $\mathcal{O}$ as an $\mathfrak{h}$-module. Therefore, $\mathrm{ch}_{\mathfrak{m}}M$ is an element of the algebra $\mathcal{F}$ for $M\in\mathcal{O}_{J}$. Then, the multiplication in $\mathcal{F}$ is defined as follows: for λ, $\mu\in P_{J}^{+}$, $m(\lambda)\cdot m(\mu):=\mathrm{ch}_{\mathfrak{m}}(L_{\mathfrak{m}}(\lambda)\otimes_{C}L_{\mathfrak{m}}(\mu))$. Thus, $\mathcal{F}$ becomes a commutative associative algebra over $\boldsymbol{C}$.

Following [10], we now define an algebra homomorphism $\Psi(\mathfrak{m}, \mathfrak{h})$: $\mathcal{F}\rightarrow\mathcal{E}$, by $\Psi(\mathfrak{m}, \mathfrak{h})(m(\lambda)):=\mathrm{ch}\,L_{\mathfrak{m}}(\lambda)\in\mathcal{E}$ $(\lambda\in P_{J}^{+})$. Then, we have

Lemma 2.1. *The mapping* $\Psi(\mathfrak{m}, \mathfrak{h})$: $\mathcal{F}\rightarrow\mathcal{E}$ *is injective.*

Proof (cf. [10, § 1]). Let $\sum_{\lambda\in P_{J}^{+}}c_{\lambda}m(\lambda)$ be a non-zero element of $\mathcal{F}$. Then, there exist $\mu_{i}\in\mathfrak{h}^{*}$ $(1\leq i\leq s)$ such that $\{\lambda\in P_{J}^{+}|\,c_{\lambda}\neq 0\}\subset\bigcup_{i=1}^{s}D(\mu_{i})$. By replacing the set $\{\mu_{i}\}_{i=1}^{s}$ with a suitable finite subset $\{\mu_{i}'\}_{i=1}^{t}$ of $\mathfrak{h}^{*}$ if necessary, we can assume that $\mu_{k}'-\mu_{l}'\notin Q=\sum_{j\in I}\boldsymbol{Z}\alpha_{j}$ $(1\leq k\neq l\leq t)$. Consider the subset $\bigcup_{i=1}^{t}\{\mathrm{ht}(\mu_{i}'-\lambda)|\,\lambda\in P_{J}^{+}$ with $c_{\lambda}\neq 0$, and $\lambda\in D(\mu_{i}')\}$ of $\boldsymbol{Z}_{\geq 0}$, and take $\lambda_{0}\in P_{J}^{+}$ which attains the minimum of this subset. Then, clearly λ_{0} is not a weight of $L_{\mathfrak{m}}(\lambda)(\lambda\in P_{J}^{+}\backslash\{\lambda_{0}\})$. Hence, $\Psi(\mathfrak{m}, \mathfrak{h})(\sum_{\lambda\in P_{J}^{+}}c_{\lambda}m(\lambda))\neq 0\in\mathcal{E}$. Thus we have shown the injectivity of $\Psi(\mathfrak{m}, \mathfrak{h})$. Q.E.D.

§ 3. Some results of L. Liu

In this section, we rewrite, in the case of GKM algebras, some of Liu's results on $\mathfrak{m}$-modules $H_{j}(\mathfrak{u}^{-}, L(\lambda))$ and $H_{C}^{j}(\mathfrak{u}^{+}, L(\lambda))$ $(j\geq 0)$ for Kac-Moody algebras. His proofs for these results require no modifications. For details, see [10].

The j-th homology $H_{j}(\mathfrak{u}^{-}, L(\lambda))$ of $\mathfrak{u}^{-}$ with coefficients in $L(\lambda)$ $(\lambda\in\mathfrak{h}^{*})$ is defined as the j-th homology of the $\mathfrak{m}$-module complex $\{(\Lambda^{j}\mathfrak{u}^{-})\otimes_{C}L(\lambda), d_{j}\}_{j}$, where the action of $\mathfrak{m}$ and the boundary operators d_{j} are defined in a usual way (see [3] and [9]). The j-th cohomology $H_{C}^{j}(\mathfrak{u}^{+}, L(\lambda))$ of $\mathfrak{u}^{+}$ with coefficients in $L(\lambda)$ is defined as the j-th cohomology of the $\mathfrak{m}$-module complex $\{\mathrm{Hom}_{C}^{\mathfrak{h}}(\Lambda^{j}\mathfrak{u}^{+}, L(\lambda)), d^{j}\}_{j}$, where $\mathrm{Hom}_{C}^{\mathfrak{h}}(\Lambda^{j}\mathfrak{u}^{+}, L(\lambda))$ is the $\mathfrak{h}$-semisimple part of $\mathrm{Hom}_{C}(\Lambda^{j}\mathfrak{u}^{+}, L(\lambda))$ (see § 5.1 for the definition), with the action of $\mathfrak{m}$ and the coboundary operators d^j being the restrictions of the usual ones. Note that this cohomology $H_C^j(\mathfrak{u}^+, L(\lambda))$ $(j\geq 0)$ of $\mathfrak{u}^+$ is different from the usual Lie algebra cohomology, which we denote by $H^j(\mathfrak{u}^+, L(\lambda))$ $(j\geq 0)$, since we have employed $\mathrm{Hom}_C^{\mathfrak{h}}(\Lambda^j\mathfrak{u}^+, L(\lambda))$ instead of $\mathrm{Hom}_C(\Lambda^j\mathfrak{u}^+, L(\lambda))$ as the space of j-cochains $(j\geq 0)$ (see [3] and [10]).

Then, we have the following, due to L. Liu.

Proposition 3.1 (cf. [10, § 4]). *For any $\Lambda\in P^+$ and $j\in \boldsymbol{Z}_{\geq 0}$, $H_C^j(\mathfrak{u}^+, L(\Lambda))$ is isomorphic to $H_j(\mathfrak{u}^-, L(\Lambda))$ as $\mathfrak{m}$-modules.*

So, from now on, we concentrate on $\mathfrak{m}$-modules $H_j(\mathfrak{u}^-, L(\Lambda))$ $(j\geq 0)$. Since $L(\Lambda)$ and $(\Lambda^j\mathfrak{u}^-)\otimes_C L(\Lambda)$ are in the category $\mathcal{O}_J$ by Proposition 2.1, $H_j(\mathfrak{u}^-, L(\Lambda))$ is also in $\mathcal{O}_J$, and so, is a direct sum of modules $L_{\mathfrak{m}}(\mu)$ $(\mu\in P_J^+)$ as an $\mathfrak{m}$-module. Furthermore, we have

Proposition 3.2 (cf. [10, § 5]). *Let $(\cdot|\cdot)$ be a fixed standard bilinear form on $\mathfrak{h}^*$. Then, for any $\Lambda\in P^+$ and $j\in \boldsymbol{Z}_{\geq 0}$, every $\mathfrak{m}$-irreducible component of $H_j(\mathfrak{u}^-, L(\Lambda))$ is of the form $L_{\mathfrak{m}}(\mu)$ $(\mu\in P_J^+)$ with $(\mu+\rho|\mu+\rho)=(\Lambda+\rho|\Lambda+\rho)$.*

§ 4. Kostant's formula for GKM algebras

In this section, we prove "Kostant's formula" for GKM algebras, which is a generalization of that in my previous paper [11]. Here, we assume that the symmetrizable GGCM $A=(a_{ij})_{i,j\in I}$ satisfies the following condition $(\hat{\mathrm{C}}1)$:

$(\hat{\mathrm{C}}1)$ either $a_{ii}=2$ or $a_{ii}=0$ $(i\in I)$.

And recall that J is a subset of I^{re}.

4.1. Necessary condition. Now, we review some results given in [11, Lemma 4.2] and its proof. Let $(\cdot|\cdot)$ be a standard bilinear form on $\mathfrak{h}^*$. Then, we have

Lemma 4.1 ([11]). *Let $\Lambda\in P^+$. If, for some j $(j\geq 0)$, μ is a weight of $(\Lambda^j\mathfrak{n}^-)\otimes_C L(\Lambda)$ and satisfies $(\mu+\rho|\mu+\rho)=(\Lambda+\rho|\Lambda+\rho)$, then*

(1) *there exist a $\beta_0\in\mathfrak{S}(\Lambda)$ and a $w_0\in W$, such that $\ell(w_0)+\mathrm{ht}(\beta_0)=j$ and $\mu=w_0(\Lambda+\rho-\beta_0)-\rho$;*

(2) *the multiplicity of μ in $(\Lambda^*\mathfrak{n}^-)\otimes_C L(\Lambda)$ is equal to one, where $\Lambda^*\mathfrak{n}^-=\sum_{j\geq 0}^{\oplus}\Lambda^j\mathfrak{n}^-$.*

Let us fix $\Lambda\in P^+$. From the above, we can prove the following.

Lemma 4.2. *Assume that $\mu\in\mathfrak{h}^*$ is a weight of $(\Lambda^j\mathfrak{u}^-)\otimes_C L(\Lambda)$ for some $j\in \boldsymbol{Z}_{\geq 0}$, and satisfies $(\mu+\rho|\mu+\rho)=(\Lambda+\rho|\Lambda+\rho)$. Then,*

(a) *there exist a $\beta\in\mathfrak{S}(\Lambda)$ and a $w\in W(J)$, such that $\ell(w)+\mathrm{ht}(\beta)=j$ and $\mu=w(\Lambda+\rho-\beta)-\rho$;*

(b) *the multiplicity of μ in $(\Lambda^j\mathfrak{u}^-)\otimes_C L(\Lambda)$ is equal to one.*

Proof. If $\mu\in\mathfrak{h}^*$ is a weight of $(\Lambda^j\mathfrak{u}^-)\otimes_C L(\Lambda)$, then μ is a weight of $(\Lambda^j\mathfrak{n}^-)\otimes_C L(\Lambda)$, since $(\Lambda^j\mathfrak{u}^-)\otimes_C L(\Lambda)$ can be regarded as a submodule of $(\Lambda^j\mathfrak{n}^-)\otimes_C L(\Lambda)$. Then, by Lemma 4.1, it follows that there exist a $\beta_0\in\mathfrak{S}(\Lambda)$ and a $w_0\in W$, such that $\ell(w_0)+\mathrm{ht}(\beta_0)=j$ and $\mu=w_0(\Lambda+\rho-\beta_0)-\rho$, and that the multiplicity of μ in $(\Lambda^*\mathfrak{n}^-)\otimes_C L(\Lambda)$ is equal to one. So, we have only to show that $w_0\in W(J)=\{w\in W|\, w(\Delta^-)\cap\Delta^+\subset\Delta^+(J)\}$. Now, recall that $w_0(\rho)-\rho=-\sum_{\alpha\in\Phi_{w_0}}\alpha$, where $\Phi_{w_0}=w_0(\Delta^-)\cap\Delta^+$ (see [11, Proposition 1.2.b)]). Express $\beta_0=\sum_{k=1}^m \alpha_{i_k}$, where $m=\mathrm{ht}(\beta_0)$, $\alpha_{i_k}\in\Pi^{im}$ $(1\le k\le m)$, and $i_r\neq i_t$ $(1\le r\neq t\le m)$. And take non-zero root vectors $E_k\in\mathfrak{g}_{-w_0(\alpha_{i_k})}$ $(1\le k\le m)$, $E_\alpha\in\mathfrak{g}_{-\alpha}$ $(\alpha\in\Phi_{w_0})$, and a non-zero weight vector $v\in L(\Lambda)_{w(\Lambda)}$. Then, it is clear that $0\neq(E_1\wedge\cdots\wedge E_m)\wedge(\Lambda_{\alpha\in\Phi_{w_0}}E_\alpha)\otimes v\in(\Lambda^*\mathfrak{n}^-)\otimes_C L(\Lambda)$ is a weight vector of weight μ (cf. the proof of [11, Lemma 4.2]). Since the multiplicity of μ in $(\Lambda^*\mathfrak{n}^-)\otimes_C L(\Lambda)$ is equal to one, and μ is a weight of $(\Lambda^j\mathfrak{u}^-)\otimes_C L(\Lambda)$ by assumption, it follows that $(E_1\wedge\cdots\wedge E_m)\wedge(\Lambda_{\alpha\in\Phi_{w_0}}E_\alpha)\otimes v\in(\Lambda^j\mathfrak{u}^-)\otimes_C L(\Lambda)$. Therefore, $\alpha\in\Delta^+(J)$ (if $\alpha\in\Phi_{w_0}$). Hence, $w_0\in W(J)$ by the definition of $W(J)$. Thus we have proved Lemma 4.2. Q.E.D.

By Proposition 3.2 and Lemma 4.2, we have the following.

Proposition 4.1. *Let $j\in\mathbf{Z}_{\ge0}$. If $L_{\mathfrak{m}}(\mu)$ $(\mu\in P_J^+)$ is an $\mathfrak{m}$-irreducible component of $H_j(\mathfrak{u}^-, L(\Lambda))$, then*

(a) *$\mu=w(\Lambda+\rho-\beta)-\rho$, for some $\beta\in\mathfrak{S}(\Lambda)$ and some $w\in W(J)$ such that $\ell(w)+\mathrm{ht}(\beta)=j$;*

(b) *$L_{\mathfrak{m}}(\mu)$ occurs with multiplicity one as $\mathfrak{m}$-irreducible components of $H_j(\mathfrak{u}^-, L(\Lambda))$.*

4.2. Sufficient condition. Here, we use the setting in § 2. Let $\Lambda\in P^+$. Before carrying out formal operations on formal $\mathfrak{m}$-characters in the algebra $\mathcal{F}$, we note that $w(\Lambda+\rho-\beta)-\rho$ varies if $w\in W$ or $\beta\in\mathfrak{S}$ varies (see the proof of [11, Proposition 4.2]).

Lemma 4.3. *For $w\in W(J)$ and $\beta\in\mathfrak{S}$, we have $w(\Lambda+\rho-\beta)-\rho\in P_J^+$.*

Proof. We have to show that $\langle w(\Lambda+\rho-\beta)-\rho, \alpha_i^\vee\rangle\in\mathbf{Z}_{\ge0}$ for $i\in J$. Since $w\in W(J)=\{w\in W|\, w^{-1}(\Delta_J^+)\subset\Delta^+\}$ and $i\in J\subset I^{re}$, it follows that $w^{-1}(\alpha_i)\in\Delta^+$. So, we have $w^{-1}(\alpha_i^\vee)\in(\Delta^\vee)^+$, where $\Delta^\vee=\Delta({}^tA)\subset\mathfrak{h}$ is the *dual root system* of $\mathfrak{g}(A)$ (see [5]). Moreover, $w^{-1}(\alpha_i^\vee)\in\sum_{j\in I^{re}}\mathbf{Z}\alpha_j^\vee$ since $J\subset I^{re}$. On the other hand, we have

$$\begin{aligned}\langle w(\Lambda+\rho-\beta)-\rho, \alpha_i^\vee\rangle &= \langle\Lambda+\rho-\beta, w^{-1}(\alpha_i^\vee)\rangle-\langle\rho, \alpha_i^\vee\rangle\\ &=\langle\Lambda, w^{-1}(\alpha_i^\vee)\rangle-\langle\beta, w^{-1}(\alpha_i^\vee)\rangle+\langle\rho, w^{-1}(\alpha_i^\vee)\rangle-1\,.\end{aligned}$$

Since $\Lambda\in P^+$ and β is a sum of elements from Π^{im}, we deduce that $\langle w(\Lambda+\rho-\beta)-\rho, \alpha_i^\vee\rangle\in\mathbf{Z}_{\ge0}$ from the above equality. Thus the assertion has been proved. Q.E.D.

Proposition 4.2. *For $\Lambda \in P^+$, there holds in the algebra $\mathcal{F}$,*

$$\sum_{j\geq 0}(-1)^j \mathrm{ch}_{\mathfrak{m}}(H_j(\mathfrak{u}^-, L(\Lambda)))$$
$$=\sum_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{\mathrm{ht}(\beta)}\sum_{w\in W(J)}(\det w)m(w(\Lambda+\rho-\beta)-\rho)\,.$$

Proof. Both sides of the above equality are clearly in the algebra $\mathcal{F}$ by Lemma 4.3. So, because $\Psi(\mathfrak{m}, \mathfrak{h})$: $\mathcal{F}\to\mathcal{E}$ is injective, we have only to show the following in the algebra $\mathcal{E}$ (cf. also Proposition 4.1).

(#)
$$\sum_{j\geq 0}(-1)^j \mathrm{ch}(H_j(\mathfrak{u}^-, L(\Lambda)))$$
$$=\sum_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{\mathrm{ht}(\beta)}\sum_{w\in W(J)}(\det w)\,\mathrm{ch}\,L_{\mathfrak{m}}(w(\Lambda+\rho-\beta)-\rho)\,.$$

By the well-known Euler-Poincaré principle, the left hand side of (#) is equal to

$$\sum_{j\geq 0}(-1)^j \mathrm{ch}(H_j(\mathfrak{u}^-, L(\Lambda)))=\sum_{j\geq 0}(-1)^j \mathrm{ch}((\Lambda^j\mathfrak{u}^-)\otimes_C L(\Lambda))$$
$$=(\sum_{j\geq 0}(-1)^j \mathrm{ch}(\Lambda^j\mathfrak{u}^-))\cdot \mathrm{ch}\,L(\Lambda)=\prod_{\alpha\in\Delta^+(J)}(1-e(-\alpha))^{\mathrm{mult}(\alpha)}\cdot \mathrm{ch}\,L(\Lambda)$$
$$=\frac{e(\rho)\cdot\prod_{\alpha\in\Delta^+}(1-e(-\alpha))^{\mathrm{mult}(\alpha)}}{e(\rho)\cdot\prod_{\alpha\in\Delta_J^+}(1-e(-\alpha))^{\mathrm{mult}(\alpha)}}\cdot \mathrm{ch}\,L(\Lambda)\,.$$

By Theorem 1.1, this is equal to

$$e(-\rho)\cdot R_J^{-1}\cdot\sum_{w\in W}(\det w)\sum_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{\mathrm{ht}(\beta)}e(w(\Lambda+\rho-\beta))\,,$$

where $R_J:=\prod_{\alpha\in\Delta_J^+}(1-e(-\alpha))^{\mathrm{mult}(\alpha)}$.

On the other hand, by Theorem 1.1 applied to the $\mathfrak{m}(=\mathfrak{g}_J+\mathfrak{h})$-module $L_{\mathfrak{m}}(w(\Lambda+\rho-\beta)-\rho)$, the right hand side of (#) is equal to

$$e(-\rho)\cdot R_J^{-1}\cdot\sum_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{\mathrm{ht}(\beta)}\sum_{w\in W(J)}(\det w)\times$$
$$\times\sum_{u\in W_J}(\det u)e(u(w(\Lambda+\rho-\beta)))$$
$$=e(-\rho)\cdot R_J^{-1}\cdot\sum_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{\mathrm{ht}(\beta)}\times$$
$$\times\sum_{w\in W(J),u\in W_J}(\det uw)e(uw(\Lambda+\rho-\beta))\,.$$

Now, we quote the fact that every $w\in W$ can be uniquely expressed in the form $w_J\cdot w(J)$, where $w_J\in W_J$ and $w(J)\in W(J)$. Note that this fact requires J to be a subset of I^{re}. (See [10, § 2] for the proof.) Therefore, the above is equal to

$$e(-\rho)\cdot R_J^{-1}\cdot\sum_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{\mathrm{ht}(\beta)}\sum_{w\in W}(\det w)e(w(\Lambda+\rho-\beta))$$
$$=e(-\rho)\cdot R_J^{-1}\cdot\sum_{w\in W}(\det w)\sum_{\beta\in\mathfrak{S}(\Lambda)}(-1)^{\mathrm{ht}(\beta)}e(w(\Lambda+\rho-\beta))\,.$$

Thus, we have proved the equality (#). This completes the proof of Proposition 4.2. Q.E.D.

By Propositions 4.1 and 4.2, we have the following.

Proposition 4.3. *Fix $j\in \mathbf{Z}_{\geq 0}$. And put $\mu:=w(\Lambda+\rho-\beta)-\rho$, where $\beta\in \mathfrak{S}(\Lambda)$ and $w\in W(J)$ such that $\ell(w)+ht(\beta)=j$. Then, $L_{\mathfrak{m}}(\mu)$ occurs as $\mathfrak{m}$-irreducible components of $H_j(\mathfrak{u}^-, L(\Lambda))$.*

Summarizing Propositions 3.1, 4.1, and 4.3, we obtain the following theorem.

Theorem 4.1 (Kostant's formula). *Let $\Lambda\in P^+$. And let $\mathfrak{g}(A)$ be the* GKM *algebra associated to a symmetrizable* GGCM $A=(a_{ij})_{i,j\in I}$ *satisfying* ($\hat{\mathrm{C}}1$). *We assume that the subset J of I is contained in $I^{re}=\{i\in I\,|\,a_{ii}=2\}$. Then, as $\mathfrak{m}$-modules ($j\geq 0$),*

$$H_c^j(\mathfrak{u}^+, L(\Lambda))\cong H_j(\mathfrak{u}^-, L(\Lambda))\cong \sum\nolimits_{\beta\in\mathfrak{S}(\Lambda)}^{\oplus} \sum_{\substack{w\in W(J)\\ \ell(w)=j-\mathrm{ht}(\beta)}}^{\oplus} L_{\mathfrak{m}}(w(\Lambda+\rho-\beta)-\rho)\,.$$

Here, the above sum is a direct sum of inequivalent irreducible highest weight $\mathfrak{m}$-modules.

Remark 4.1. Theorem 4.1 is a generalization of "Kostant's formula" for symmetrizable Kac-Moody algebras, which was proved by L. Liu in [10] without assuming that the subset J of I is of finite type (cf. Remark 1.1).

Remark 4.2. In our arguments, the assumption that J is a subset of I^{re} plays an essential role. So, we cannot remove it.

§ 5. Applications—some vanishing theorems

In this section, as applications of Theorem 4.1, we extend some classical results about the homology of symmetrizable Kac-Moody algebras to GKM algebras associated to symmetrizable GGCMs satisfying the condition ($\hat{\mathrm{C}}1$). Their proofs are very similar to the ones for Kac-Moody algebras given in [13] or [7].

5.1. Linear homomorphisms with compact support. Here, we introduce the notion of *linear homomorphisms with compact support,* following [10]. Let $V=\sum_{\lambda\in\mathfrak{h}^*}^{\oplus} V_\lambda$ and $W=\sum_{\mu\in\mathfrak{h}^*}^{\oplus} W_\mu$ be $\mathfrak{h}$-diagonalizable modules with finite-dimensional weight spaces. Then, a linear homomorphism f from V to W is called *with compact support* if $f(V_\lambda)=0$ for all but finitely many weights $\lambda\in\mathfrak{h}^*$ of V. We denote by $\mathrm{Hom}_{\mathbf{C}}^c(V, W)$ the space of all linear homomorphisms with compact support from V to W. In particular, we write V_c^* for $\mathrm{Hom}_{\mathbf{C}}^c(V, \mathbf{C})$, where $\mathbf{C}$ is the trivial $\mathfrak{h}$-module, while for a (possibly infinite-dimensional) vector space X over $\mathbf{C}$, X^* denotes the full dual $\mathrm{Hom}_{\mathbf{C}}(X, \mathbf{C})$. Then, we can easily show the following.

Proposition 5.1. *For $\mathfrak{h}$-diagonalizable modules V and W with finite-dimensional weight spaces, we have*

(1) $(V_c^*)_c^* \cong V$,

(2) $V_c^* \otimes_C W \cong \mathrm{Hom}_C^c(V, W)$.

Corollary 5.1. *Let V and W be $\mathfrak{m}$-modules. If, as $\mathfrak{h}$-modules, they are $\mathfrak{h}$-diagonalizable with finite-dimensional weight spaces, then*

$$\{(V_c^*) \otimes_C W\}^{\mathfrak{m}} \cong \mathrm{Hom}_{\mathfrak{m}}^c(V, W) := \mathrm{Hom}_{U(\mathfrak{m})}(V, W) \cap \mathrm{Hom}_C^c(V, W).$$

Here, $U(\mathfrak{m})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{m}$, and for an $\mathfrak{m}$-module X, we put $X^{\mathfrak{m}} := \{x \in X \mid m(x) = 0$ for all $m \in \mathfrak{m}\}$ (the space of $\mathfrak{m}$-invariants).

Remark 5.1. For the irreducible highest weight $\mathfrak{m}$-module $L_{\mathfrak{m}}(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$, $\{L_{\mathfrak{m}}(\lambda)\}_c^*$ is isomorphic to the irreducible lowest weight $\mathfrak{m}$-module with lowest weight $-\lambda$ as $\mathfrak{m}$-modules (see [5]). We simply write $L_{\mathfrak{m}}^*(\lambda)$ for it.

5.2. Homology vanishing theorem for GKM algebras with coefficients in a generalized Verma module. From now on, we assume that $A = (a_{ij})_{i,j \in I}$ is an $n \times n$ symmetrizable GGCM satisfying the condition ($\hat{\mathrm{C}}1$), and that J is a (fixed) arbitrary subset of $I^{re} = \{i \in I \mid a_{ii} = 2\}$. Note that since J is not necessarily of finite type, $L_{\mathfrak{m}}(\lambda)$ may be infinite-dimensional even if $\lambda \in P_J^+ = \{\mu \in \mathfrak{h}^* \mid \langle \mu, \alpha_i^\vee \rangle \in \boldsymbol{Z}_{\geq 0}\ (i \in J)\}$.

For $\lambda \in P_J^+$, we define the *generalized Verma module* $V_{\mathfrak{m}}(\lambda)$ with highest weight λ as follows: $V_{\mathfrak{m}}(\lambda) := U(\mathfrak{g}(A)) \otimes_{U(\mathfrak{p})} L_{\mathfrak{m}}(\lambda)$, where $\mathfrak{u}^+$ ($\subset \mathfrak{p}$) acts on $L_{\mathfrak{m}}(\lambda)$ trivially. This becomes a $U(\mathfrak{g}(A))$-module by left multiplication. Note that when $J = \phi$, the module $V_{\mathfrak{m}}(\lambda)$ is just the *Verma module* $V(\lambda) := U(\mathfrak{g}(A)) \otimes_{U(\mathfrak{b})} \boldsymbol{C}(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$, where $\boldsymbol{C}(\lambda)$ is the one-dimensional $\mathfrak{b}$-module on which $\mathfrak{h}$ acts by the weight λ and $\mathfrak{n}^+$ acts trivially. Then, as an application of Theorem 4.1, we obtain the following generalization of [13, Theorem 4.17].

Theorem 5.1. *Let $\lambda \in P_J^+$. And let $V_{\mathfrak{m}}(\lambda)$ be the generalized Verma module with highest weight λ. Then, as $\boldsymbol{C}$-vector spaces:*

(a) *If $\lambda \neq w(\rho - \beta) - \rho$ for any $\beta \in \mathfrak{S}$ and $w \in W(J)$, we have*

$$H_i(\mathfrak{g}(A), V_{\mathfrak{m}}(\lambda)) = 0 \quad \text{for all } i \geq 0.$$

(b) *If $\lambda = w_0(\rho - \beta_0) - \rho$ for some (necessarily unique) $\beta_0 \in \mathfrak{S}$ and some (necessarily unique) $w_0 \in W(J)$, we have*

$$H_i(\mathfrak{g}(A), V_{\mathfrak{m}}(\lambda)) \cong$$

$$\cong H_{i-(\ell(w_0)+\mathrm{ht}(\beta_0))}(\mathfrak{m}, L_{\mathfrak{m}}^*(w_0(\rho-\beta_0)-\rho)\otimes_C L_{\mathfrak{m}}(w_0(\rho-\beta_0)-\rho))$$

for all $i\geq 0$.

In particular, $H_i(\mathfrak{g}(A), V_{\mathfrak{m}}(\lambda))=0$ *unless* $i\geq\ell(w_0)+\mathrm{ht}(\beta_0)$.

Proof. First, note that $H_i(\mathfrak{g}(A), V_{\mathfrak{m}}(\lambda))\cong H_i(\mathfrak{p}, L_{\mathfrak{m}}(\lambda))$ $(i\geq 0)$ as $\boldsymbol{C}$-vector spaces, as is well-known (see [2, Proposition 4.2, p. 275]). Now, for the pair $(\mathfrak{p}, \mathfrak{u}^+)$ and a $\mathfrak{p}$-module $L_{\mathfrak{m}}(\lambda)$, there exists the Hochschild-Serre spectral sequence for homology $\{E_{p,q}^r, d_{p,q}^r\colon E_{p,q}^r\to E_{p-r,q+r-1}^r\}$ such that $E_n\cong H_n(\mathfrak{p}, L_{\mathfrak{m}}(\lambda))$ and $E_{p,q}^2\cong H_p(\mathfrak{m}, H_q(\mathfrak{u}^+, L_{\mathfrak{m}}(\lambda)))$ (see [2, p. 351] for example).

Since $\mathfrak{u}^+$ acts trivially on $L_{\mathfrak{m}}(\lambda)$, we clearly have

$$H_q(\mathfrak{u}^+, L_{\mathfrak{m}}(\lambda))\cong H_q(\mathfrak{u}^+, L(0))\otimes_C L_{\mathfrak{m}}(\lambda) \quad \text{as } \mathfrak{m}\text{-modules } (q\geq 0).$$

And, we can show that $H_q(\mathfrak{u}^+, L(0))\cong\{H_q(\mathfrak{u}^-, L(0))\}_C^*$ as $\mathfrak{m}$-modules $(q\geq 0)$ (cf. [10, § 4]). Therefore, by Theorem 4.1, we get

$$H_q(\mathfrak{u}^+, L(0))\cong{\sum}^{\oplus}_{\beta\in\mathfrak{S}}\ \sum^{\oplus}_{\substack{w\in W(J)\\ \ell(w)=q-\mathrm{ht}(\beta)}} L_{\mathfrak{m}}^*(w(\rho-\beta)-\rho) \text{ as } \mathfrak{m}\text{-modules.}$$

So, as $\mathfrak{m}$-modules $(q\geq 0)$,

$$H_q(\mathfrak{u}^+, L_{\mathfrak{m}}(\lambda))\cong{\sum}^{\oplus}_{\beta\in\mathfrak{S}}\ \sum^{\oplus}_{\substack{w\in W(J)\\ \ell(w)=q-\mathrm{ht}(\beta)}} L_{\mathfrak{m}}^*(w(\rho-\beta)-\rho)\otimes_C L_{\mathfrak{m}}(\lambda).$$

Here, we have the following claim:

CLAIM. $H_p(\mathfrak{m}, L_{\mathfrak{m}}^*(\mu)\otimes_C L_{\mathfrak{m}}(\lambda))=0$ for any $\mu\in P_J^+$ such that $\mu\neq\lambda$ $(p\geq 0)$.

Proof of the claim. Recall that there exists a subspace $\mathfrak{h}(J)$ of $\mathfrak{h}$ such that $\mathfrak{h}=\mathfrak{h}_J\oplus\mathfrak{h}(J)$ and that the simple roots α_i $(i\in J)$ vanish on $\mathfrak{h}(J)$. Then, we have the direct sum decomposition of $\mathfrak{m}$ as ideals:

$$\mathfrak{m}=\mathfrak{g}_J\oplus\mathfrak{h}(J).$$

Further, we see that $L_{\mathfrak{m}}(\mu)$ $(\mu\in P_J^+)$ is naturally isomorphic to the outer tensor product $L_J(\mu|\mathfrak{h}_J)\hat{\otimes}_C\boldsymbol{C}(\mu|\mathfrak{h}(J))$, where $L_J(\mu|\mathfrak{h}_J)$ is the irreducible highest weight $\mathfrak{g}_J(\cong\mathfrak{g}(A_J))$-module with highest weight $\mu|\mathfrak{h}_J\in(\mathfrak{h}_J)^*$, and $\boldsymbol{C}(\mu|\mathfrak{h}(J))$ is the irreducible (one-dimensional) $\mathfrak{h}(J)$-module with weight $\mu|\mathfrak{h}(J)\in(\mathfrak{h}(J))^*$. So, $L_{\mathfrak{m}}^*(\mu)$ is isomorphic to the outer tensor product $L_J^*(\mu|\mathfrak{h}_J)\hat{\otimes}_C\boldsymbol{C}(-\mu|\mathfrak{h}(J))$, where $L_J^*(\mu|\mathfrak{h}_J):=\{L_J(\mu|\mathfrak{h}_J)\}_C^*$ is $\mathfrak{g}_J$-module isomorphic to the irreducible lowest weight $\mathfrak{g}_J$-module with lowest weight $-\mu|\mathfrak{h}_J\in(\mathfrak{h}_J)^*$. Hence, by [9, Proposition 4.12], we have the following vector space isomorphism:

$$\mathrm{Tor}_p^{\mathfrak{m}}((L_{\mathfrak{m}}(\lambda))^t, L_{\mathfrak{m}}^*(\mu))\cong$$

$$\cong \sum_{r+s=p}^{\oplus} \mathrm{Tor}_r^{\mathfrak{g}_J}((L_J(\lambda|\mathfrak{h}_J))^t,\ L_J^*(\mu|\mathfrak{h}_J)) \otimes_{\boldsymbol{C}} \mathrm{Tor}_s^{\mathfrak{h}(J)}((\boldsymbol{C}(\lambda|\mathfrak{h}(J)))^t,\ \boldsymbol{C}(-\mu|\mathfrak{h}(J)))\,.$$

Since $\mathfrak{g}_J$ is isomorphic to the symmetrizable Kac-Moody algebra $\mathfrak{g}(A_J)$, we can easily deduce that $\mathrm{Tor}_p^{\mathfrak{m}}((L_{\mathfrak{m}}(\lambda))^t,\ L_{\mathfrak{m}}^*(\mu))=0$ for $\mu \in P_J^+$ with $\mu \neq \lambda$ $(p \geq 0)$ from [8, Corollary 2.13. (b$_1$)] and its proof. The claim now follows from [9, Propositions 4.2 and 4.3].

By the above claim and Lemma 4.3, we have

in Case (a), $E_{p,q}^2=0$ $(p,\ q \geq 0)$,

$$\text{in Case (b),}\quad E_{p,q}^2 \cong \begin{cases} 0 & (p \geq 0,\ q \neq \ell(w_0)+\mathrm{ht}(\beta_0))\,, \\ H_p(\mathfrak{m},\ L_{\mathfrak{m}}^*(w_0(\rho-\beta_0)-\rho) \otimes_{\boldsymbol{C}} L_{\mathfrak{m}}(w_0(\rho-\beta_0)-\rho)) & \\ \qquad (p \geq 0,\ q=\ell(w_0)+\mathrm{ht}(\beta_0))\,. & \end{cases}$$

Therefore, in Case (a), $H_n(\mathfrak{p},\ L_{\mathfrak{m}}(\lambda)) \cong E_n \cong E_{n,0}^2=0$ $(n \geq 0)$, and in Case (b),

$$\begin{aligned} H_n(\mathfrak{p},\ L_{\mathfrak{m}}(\lambda)) &\cong E_n \cong E^2_{n-\ell(w_0)-\mathrm{ht}(\beta_0),\ell(w_0)+\mathrm{ht}(\beta_0)} \\ &\cong H_{n-(\ell(w_0)+\mathrm{ht}(\beta_0))}(\mathfrak{m},\ L_{\mathfrak{m}}^*(w_0(\rho-\beta_0)-\rho) \otimes_{\boldsymbol{C}} L_{\mathfrak{m}}(w_0(\rho-\beta_0)-\rho)) \quad (n \geq 0)\,. \end{aligned}$$

Thus, we have proved the theorem. Q.E.D.

Corollary 5.2. *Let $\lambda \in P_J^+$ be such that $L_{\mathfrak{m}}(\lambda)$ is finite-dimensional. Then, as $\boldsymbol{C}$-vector spaces:*

(a) *If $\lambda \neq w(\rho-\beta)-\rho$ for any $\beta \in \mathfrak{S}$ and $w \in W(J)$, we have*

$$H_i(\mathfrak{g}(A),\ V_{\mathfrak{m}}(\lambda))=0 \quad \textit{for all } i \geq 0\,.$$

(b) *If $\lambda=w_0(\rho-\beta_0)-\rho$ for some $\beta_0 \in \mathfrak{S}$ and $w_0 \in W(J)$, we have*

$$H_i(\mathfrak{g}(A),\ V_{\mathfrak{m}}(\lambda)) \cong H_{i-(\ell(w_0)+\mathrm{ht}(\beta_0))}(\mathfrak{m},\ L_{\mathfrak{m}}(0)) \quad \textit{for all } i \geq 0\,.$$

Proof. Since $L_{\mathfrak{m}}(\lambda)$ is finite-dimensional by assumption, $L_{\mathfrak{m}}^*(\lambda) \otimes_{\boldsymbol{C}} L_{\mathfrak{m}}(\lambda)$ is completely reducible as an $\mathfrak{m}$-module by [5, Theorem 10.7.b)]. So, it is a direct sum of modules $L_{\mathfrak{m}}(\mu)$ with $\mu \in P_J^+$. And we know that $H_j(\mathfrak{m},\ L_{\mathfrak{m}}(\mu))=0$ for $\mu \in P_J^+$ such that $\mu \neq 0 \in \mathfrak{h}^*$ $(j \geq 0)$ (see the claim in the proof of Theorem 5.1). Therefore, we see that

$$H_j(\mathfrak{m},\ L_{\mathfrak{m}}^*(\lambda) \otimes_{\boldsymbol{C}} L_{\mathfrak{m}}(\lambda)) \cong H_j(\mathfrak{m},\ \{L_{\mathfrak{m}}^*(\lambda) \otimes_{\boldsymbol{C}} L_{\mathfrak{m}}(\lambda)\}^{\mathfrak{m}}) \quad (j \geq 0)\,.$$

Now, by Corollary 5.1 and the finite-dimensionality of $L_{\mathfrak{m}}(\lambda)$, we have

$$\begin{aligned} \{L_{\mathfrak{m}}^*(\lambda) \otimes_{\boldsymbol{C}} L_{\mathfrak{m}}(\lambda)\}^{\mathfrak{m}} &\cong \mathrm{Hom}_{\mathfrak{m}}^{c}(L_{\mathfrak{m}}(\lambda),\ L_{\mathfrak{m}}(\lambda))=\mathrm{Hom}_{U(\mathfrak{m})}(L_{\mathfrak{m}}(\lambda),\ L_{\mathfrak{m}}(\lambda)) \\ &\cong L_{\mathfrak{m}}(0)\,. \end{aligned}$$

Hence, the corollary immediately follows from Theorem 5.1. Q.E.D.

Remark 5.2. By [7, Proposition 1.9], we have

$$H_*(\mathfrak{m}, L_{\mathfrak{m}}(0)) \cong \Lambda^*(\mathfrak{m}/[\mathfrak{m}, \mathfrak{m}]) \otimes_{\boldsymbol{C}} H_*([\mathfrak{m}, \mathfrak{m}], \boldsymbol{C})$$

as graded vector spaces, where $\boldsymbol{C}$ is the one-dimensional trivial module. Here, the derived subalgebra $[\mathfrak{m}, \mathfrak{m}]$ of $\mathfrak{m}$ is clearly equal to $[\mathfrak{g}_J, \mathfrak{g}_J]$, which is isomorphic to $[\mathfrak{g}(A_J), \mathfrak{g}(A_J)]$. On the other hand, it is well-known that

$$H^i([\mathfrak{g}(A_J), \mathfrak{g}(A_J)], \boldsymbol{C}) \cong \{H_i([\mathfrak{g}(A_J), \mathfrak{g}(A_J)], \boldsymbol{C})\}^*,$$

where $H^i([\mathfrak{g}(A_J), \mathfrak{g}(A_J)], \boldsymbol{C})$ is the usual i-th Lie algebra cohomology of $[\mathfrak{g}(A_J), \mathfrak{g}(A_J)]$ with coefficients in the trivial module $\boldsymbol{C}$ ($i \geq 0$). Furthermore, by [6, Theorem 1.6], $H^i([\mathfrak{g}(A_J), \mathfrak{g}(A_J)], \boldsymbol{C})$ is isomorphic to the i-th singular cohomology $\hat{H}^i(K(A_J), \boldsymbol{C})$ of $K(A_J)$ with coefficients in the complex number field $\boldsymbol{C}$ as vector spaces ($i \geq 0$), where $K(A_J)$ is the "standard compact real form" of the *Kac-Moody* (*algebraic*) *group* $G(A_J)$ associated to the Kac-Moody algebra $\mathfrak{g}(A_J)$. (See also [12] for the definitions of $G(A_J)$ and $K(A_J)$.)

When the GCM A_J is of finite type, the singular cohomology $\hat{H}^i(K(A_J), \boldsymbol{C})$ ($i \geq 0$) is well-known, and each cohomology space is of course finite-dimensional. And, when A_J is of non-twisted affine type, $\hat{H}^i(K(A_J), \boldsymbol{C})$ can be easily determined by a standard spectral sequence argument from the structure theory of $K(A_J)$ (cf. [4, § 2.8]), and proves to be finite-dimensional ($i \geq 0$). More generally, V. G. Kac (and D. H. Peterson) claimed to have determined $\hat{H}^i(K(A_J), \boldsymbol{C})$ ($i \geq 0$) for an arbitrary GCM A_J, though the proofs have not yet appeared. According to their results (see [4, § 2.6]), $\hat{H}^i(K(A_J), \boldsymbol{C})$ is still finite-dimensional ($i \geq 0$). Then, $H_i([\mathfrak{g}(A_J), \mathfrak{g}(A_J)], \boldsymbol{C})$ is finite-dimensional, and we have

$$\begin{aligned} H_i([\mathfrak{g}(A_J), \mathfrak{g}(A_J)], \boldsymbol{C}) &\cong \{H^i([\mathfrak{g}(A_J), \mathfrak{g}(A_J)], \boldsymbol{C})\}^* \\ &\cong \{\hat{H}^i(K(A_J), \boldsymbol{C})\}^* \qquad (i \geq 0). \end{aligned}$$

In particular, under the conditions of Corollary 5.1, we may conclude that $H_i(\mathfrak{g}(A), V_{\mathfrak{m}}(\lambda))$ is finite-dimensional for all $i \geq 0$.

Remark 5.3. When the subset J of I is of finite type, $L_{\mathfrak{m}}(\lambda)$ is automatically finite-dimensional for $\lambda \in P_J^+$. And in this case, we see from the above corollary that $H_i(\mathfrak{g}(A), V_{\mathfrak{m}}(\lambda)) = 0$ unless $i \leq \ell(w_0) + \mathrm{ht}(\beta_0) + \dim_{\boldsymbol{C}} \mathfrak{m}$.

So, by putting $J = \phi$, we get the following corollary.

Corollary 5.3. *Let $\lambda \in \mathfrak{h}^*$ and $V(\lambda)$ be the Verma module with highest weight λ. Then, as $\boldsymbol{C}$-vector spaces:*

(a) *If $\lambda \neq w(\rho - \beta) - \rho$ for any $\beta \in \mathfrak{S}$ and $w \in W$, we have*

$$H_i(\mathfrak{g}(A), V(\lambda)) = 0 \quad \textit{for all } i \geq 0.$$

(b) *If $\lambda = w_0(\rho - \beta_0) - \rho$ for some $\beta_0 \in \mathfrak{S}$ and $w_0 \in W$, we have*

$$H_i(\mathfrak{g}(A),\ V(\lambda))\cong\Lambda^{i-\ell(w_0)-\mathrm{ht}(\beta_0)}(\mathfrak{h})\quad \textit{for all } i\geq 0\ .$$

In particular, $H_i(\mathfrak{g}(A),\ V(\lambda))=0$ *unless* $\ell(w_0)+\mathrm{ht}(\beta_0)\leq i\leq \ell(w_0)+\mathrm{ht}(\beta_0)+\dim_{C}\mathfrak{h}$.

DEPARTMENT OF MATHEMATICS
SHIZUOKA UNIVERSITY

References

[1] R. Borcherds, Generalized Kac-Moody algebras, J. Algebra, **115** (1988), 501-512.
[2] H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, 1956.
[3] H. Garland and J. Lepowsky, Lie algebra homology and the Macdonald-Kac formulas, Invent. Math., **34** (1976), 37-76.
[4] V. G. Kac, Constructing groups associated to infinite dimensional Lie algebras, in Infinite Dimensional Groups with Applications, MSRI publications Vol. 4, Springer-Verlag, 1985, 167-216.
[5] V. G. Kac, Infinite Dimensional Lie Algebras (3rd edition), Cambridge University Press, Cambridge, 1990.
[6] S. Kumar, Rational homotopy theory of flag varieties associated to Kac-Moody groups, in Infinite Dimensional Groups with Applications, MSRI publications Vol. 4, Springer-Verlag, 1985, 233-273.
[7] S. Kumar, A homology vanishing theorem for Kac-Moody algebras with coefficients in the category $\mathcal{O}$, J. Algebra, **102** (1986), 444-462.
[8] S. Kumar, Extension of the category $\mathcal{O}^{g}$ and a vanishing theorem for the Ext functor for Kac-Moody algebras, J. Algebra, **108** (1987), 472-491.
[9] J. Lepowsky, Generalized Verma modules, loop space cohomology and Macdonald-type identities, Ann. Sci. Éc. Norm. Sup., **12** (1979), 169-234.
[10] L. Liu, Kostant's formula for Kac-Moody Lie algebras, J. Algebra, **149** (1992), 155-178.
[11] S. Naito, Kostant's formula for a certain class of generalized Kac-Moody algebras, Tôhoku Math. J, **44** (1992), 567-580.
[12] D. H. Peterson and V. G. Kac, Infinite flag varieties and conjugacy theorems, Proc. Natl. Acad. Sci. USA, **80** (1983), 1778-1782.
[13] C. Sen, The homology of Kac-Moody Lie algebras with coefficients in a generalized Verma module, J. Algebra, **90** (1984), 10-17.