J. Math. Kyoto Univ. (JMKYAZ)
33-3 (1993) 697-707

On the Dirichlet problem for the nonlinear equation of the vibrating string II

By

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0. Introduction

In the present paper we continue the investigations we have begun in [1]. We discussed the solvability in $L_2(\Omega)$ of the Dirichlet problem for the nonlinear equation of the vibrating string

$$(1)\qquad \begin{aligned} &u_{xy}+f(x, y, u)=0\,, \qquad (x, y)\in\Omega \\ &u|_{\partial\Omega}=0 \end{aligned}$$

where Ω was a bounded domain strictly convex relative to the lines $x=const$, $y=const$. Using results obtained in [2] we have proved existence and uniqueness of weak solutions of (1) in the case $\Gamma=\partial\Omega\in C^\infty$, Ω satisfies some symmetry conditions ($\Omega\in E(m, n)$ for some $m, n\in N$, $n>m$), $f(x, y, u)$ is continuous in (x, y, u), monotone in u and satisfies some estimates (see [1]).

In the present paper using some topological methods we discuss the existence and uniqueness of weak solutions of (1) without assumption of f to be monotone in u. For simplicity we consider here only domains with analytic boundary, although most results are valid also in the case when $\Gamma\in C^\infty$ or Γ is piecewise smooth. In the paper we essentially use some notations introduced in [1] and results of work [3].

1. Main notations

Let Ω be a bounded domain convex relative to the lines $x=const$, $y=const$. We shall assume that $\Gamma=\partial\Omega$ is an analytic curve and the curvature of Γ at those points where the tangent is parallel to one of the coordinate axes is positive.

We shal consider here the following problem

$$(2)\qquad Au+f(x, y, u)=0\,, \qquad (x, y)\in\Omega$$

where $A=\overline{A_0}$ in $L_2(\Omega)$, $A_0u=u_{xy}$, $u\in D(A_0)=C^\infty(\Omega\cup\Gamma)\cap \overset{o}{W}{}_2^1(\Omega)$. As in [1],

Received January 16, 1992

[3] we consider some diffeomorphism F of the boundary: $F=T^{-}\circ T^{+}$ where the diffeomorphism T^{+} assigns to a point of the boundary another boundary point with the same coordinate y, while the diffeomorphism T^{-} assigns to a point of the boundary another boundary point with the same coordnate x. The diffeomorphism F is analytic and preserves the orientation of the boundary.

Let $\Gamma=\{(x(s), y(s))|0\leq s<l\}$ be a natural parametrization of Γ, s be parameter of arc's length, l be total length of Γ. For each point $P\in T$ we assign its coordinate $S(P)\in[0, l)$. Then the diffeomorphism F can be lifted [4] to a map f: $\boldsymbol{R}\rightarrow\boldsymbol{R}$, i.e. there exists increasing function $f:\boldsymbol{R}\overset{on}{\rightarrow}\boldsymbol{R}$ such that $0\leq f(0)<l$ and

$$f(s+l)=f(s)+l\,,\quad s\in\boldsymbol{R}\,;\quad S(FP)=f(S(P))\ (\mathrm{mod}\ l)\,,\quad P\in\Gamma$$

As far as F is analytic then the lift f is an analytic function. It is known [4] that if we denote $f_1(s)=f(s)$, $f_{k+1}=f(f_k(s))$, $k\in\boldsymbol{N}$, then independently of the choice of $s\in\boldsymbol{R}$ there exists a limit

$$\lim_{n\rightarrow\infty}\frac{f_n(s)}{nl}\overset{\mathrm{def}}{=}\alpha(f)\in[0, 1]$$

which is called the rotation number of F [4]. Because of the analyticity of F the following cases are possible [3]:

(A) $\alpha(F)=\frac{m}{n}$ is a rational number, and $F^n=I$, where I is the identity mapping of Γ onto itself.

(B) $\alpha(F)=\frac{m}{n}$ is a rational number, $(m, n)=1$, and the set of fixed points of F^n on Γ is finite.

(C) $\alpha(F)$ is an irrational number, and F^n has no fixed points on Γ for any $n\in\boldsymbol{N}$.

2. Case (A)

This case is considered in [1]. The solvability of (2) has been proved if $f(x, y, u)$ is continuous in (x, y, u), monotone in u, and satisfies some estimates. Assuming $f(x, y, u)=g(u)-f(x, y)$ and using general topological methods we shall derive an abstract solvability condition if $g(u)$ is continuos and satisfies some estimates.

Let the conditions (A) hold. Then (see [3]) $\boldsymbol{A}=\boldsymbol{A}^*$, $\dim N(\boldsymbol{A})=\infty$, the range $R(\boldsymbol{A})$ is closed in $\boldsymbol{L}_2(\Omega)$, $\boldsymbol{L}_2(\Omega)=N(\boldsymbol{A})\oplus R(\boldsymbol{A})$. Besides, as it has been pointed out in [1], $\boldsymbol{A}_R{}^{-1}=(\boldsymbol{A}|_{R(\boldsymbol{A})})^{-1}$: $R(\boldsymbol{A})\rightarrow R(\boldsymbol{A})$ is a compact operator.

We denote by P_N, P_R orthogonal projections from $\boldsymbol{L}_2(\Omega)$ onto $N(\boldsymbol{A})$, $\boldsymbol{R}(\boldsymbol{A})$ respectively. Consider the following equation

(3) $\quad \boldsymbol{A}u+g(u)=f$

where $f\in \boldsymbol{L}_2(\Omega)$, $g(u)\in \boldsymbol{L}_2(\Omega)$ for any $u\in \boldsymbol{L}_2(\Omega)$. This equation can be re-written in the form

(4) $\quad \boldsymbol{A}u_1+P_R g(u)=f_1$

(5) $\quad P_N g(u)=f_2$

where $f_1=P_R f$, $f_2=P_N f$, $u_1=P_R u$, $u_2=P_N u$, or in the other form

(6) $\quad u_1=\boldsymbol{A}_R^{-1}(f_1-P_R g(u_1+u_2))$

(7) $\quad P_N g(u_1+u_2)=f_2$

Let $g(u)$ be a continuous function form $\mathbf{C}$ into itself and for some constant $C>0$ the following inequality hold:

(8) $\quad |g(u)|\leq M|u|+C\,, \qquad u\in \mathbf{C}$

Here $M>0$ is some positive constant satisfying

(9) $\quad M<\frac{1}{|\lambda_1|}$

where λ_1 is the eigenvalue of $\boldsymbol{A}_R^{-1}$ with the largest absolute value.

Let u_2 be an arbitrary function from $N(\boldsymbol{A})$. Then nonlinear operator

(10) $\quad \boldsymbol{B}(v)=\boldsymbol{A}_R^{-1}(f_1-P_R g(v+v_2))$

is compact operator from $R(\boldsymbol{A})$ into $R(\boldsymbol{A})$. Besides from (8), (9) it follows that $\boldsymbol{B}$ maps any closed ball $B_r\cap R(\boldsymbol{A})=\{u\in R(\boldsymbol{A})| \|u\|_{L_2(\Omega)}\leq r\}$ into itself if

(11) $\quad r>|\lambda_1|\cdot\frac{\|f_1\|_{L_2}+M\|u_2\|_{L_2}+C}{1-|\lambda_1|M}$

So by the Schauder fixed point theorem we obtain that for any $u_2\in N(\boldsymbol{A})$ there exists $u_1\in R(\boldsymbol{A})$ such that (6) holds. We denote this function $u_1=\boldsymbol{H}(u_2)$. Then

$$\boldsymbol{H}(u_2)=\boldsymbol{A}_R^{-1}(f_1-P_R g(\boldsymbol{H}(u_2)+u_2))$$

$$\|\boldsymbol{H}(u_2)\|_{L_2(\Omega)}\leq|\lambda_1|\cdot M\cdot\|u_2\|_{L_2(\Omega)}+|\lambda_1|\frac{\|f_1\|_{L_2(\Omega)}+C}{1-|\lambda_1|M}$$

So the operator $\boldsymbol{H}$: $N(\boldsymbol{A})\rightarrow R(\boldsymbol{A})\cap \boldsymbol{D}(\boldsymbol{A})$ is bounded. Thus the sufficient condition for the solvability of the problem (3) is the olvability in $N(\boldsymbol{A})$ of the following equation

(12) $\quad P_N g(\boldsymbol{H}(u_2)+u_2)=f_2$

Assume g satisfies

(13) $$|g(u)-g(v)|\leq M\cdot|u-v|, \qquad u, v\in\mathbf{C}$$

where $M<\frac{1}{|\lambda_1|}$. Then $\boldsymbol{B}$ is a contractive operator in $R(\boldsymbol{A})$. So for any $u_2\in N(\boldsymbol{A})$ there exists unique solution $u_1=\boldsymbol{H}(u_2)$ of the equation (6). Therefore solvability of (12) in $N(\boldsymbol{A})$ is a necessary and sufficient condition for the solvability of (3).

Thus we have obtained

Theorem 1. *If g is a continuous function and* (8) *holds then the solvability of* (12) *in $N(\boldsymbol{A})$ is a sufficient condition for the solvability of* (3) *in $\boldsymbol{L}_2(\Omega)$.*

If g satisfies (13) *then problem* (3) *is solvable in $\boldsymbol{L}_2(\Omega)$ if and only if equation* (12) *is solvable in $N(\boldsymbol{A})$.*

Remark 1. In fact, (8), (13) coincide with the inequalities under the fulfillment of which the existence and uniqueness of the 2π-periodic solution of the following problem

$$\begin{cases} u_{tt}-u_{xx}+g(u)=f(x, t) \\ u(0, t)=u(\pi, t)=0 \end{cases}$$

are usually proved (see, for example, [5]).

Remark 2. All the arguments remain valid if we consider the domain with non-analytic boundary, for example, $\Gamma=\bigcup_{i=1}^{n}\Gamma_i$, $\Gamma_i\in \boldsymbol{C}^2$.

Remark 3. All the arguments remain valid if instead of (3) we consider the folowing problem

$$\boldsymbol{A}u+\boldsymbol{K}(u)=f$$

where $\boldsymbol{K}(u)$ is some nonlinear opêrator in $\boldsymbol{L}_2(\Omega)$.

3. Case (B)

Let the conditions (B) hold. Then $\alpha(F)=\frac{m}{n}$, set Φ of all fixed points of F^n on Γ is finite. A fixed point $P\in\Phi$ is called "simple" [3] if $f_n(S(P))\neq 1$.

Let all fixed poioss $P\in\Phi$ of F^n be simple. Then from [3] it follows that $\boldsymbol{A}$ is a symmetric operator, $N(\boldsymbol{A})=\{0\}$, $\dim N(\boldsymbol{A}^*)=\infty$, the ranges $R(\boldsymbol{A})$, $R(\boldsymbol{A}^*)$ are closed in $\boldsymbol{L}_2(\Omega)$ and

$$\boldsymbol{L}_2(\Omega)=N(\boldsymbol{A}^*)\oplus R(\boldsymbol{A})=R(\boldsymbol{A}^*)$$

Besides, there exists a constant $C>0$ such that

(14) $$\|u\|_{\overset{o}{W}_2^1(\Omega)} \leq C\cdot\|\boldsymbol{A}u\|_{L_2(\Omega)}, \qquad u\in D(\boldsymbol{A})$$

So $\boldsymbol{D}(\boldsymbol{A})\subset \overset{o}{\boldsymbol{W}}_2^1(\Omega)$ and $\boldsymbol{K}=\boldsymbol{A}_R^{-1}\circ P_R$ is a compact operator in $\boldsymbol{L}_2(\Omega)$. The character of the problems (2), (3) in the case (B) is quite different than in the case (A). To show this we consider the simplest example.

Let $g(u)=-\lambda\cdot u$ where $\lambda\neq 0$ is not eigenvalue of $\boldsymbol{A}$. Then in the case (A) problem (3) can be written in the form

$$\begin{cases}\boldsymbol{A}u_1-\lambda u_1=f_1\\ -\lambda u_2=f_2\end{cases}$$

So $u_2=\frac{-f_2}{\lambda}$; $u_1=(\boldsymbol{A}-\lambda I)^{-1}f_1$; $u=u_1+u_2=(\boldsymbol{A}-\lambda I)^{-1}f_1-\frac{f_2}{\lambda}$ is unique solution of (3). Thus in the case (A) solution of (3) exists for any $f\in \boldsymbol{L}_2(\Omega)$, this solution is unique and depends continuously of f.

Let the case (B) hold. Consider $g(u)=-\lambda\cdot u$ where $1/\lambda$ is not an eigenvalue of the operator $\boldsymbol{K}=\boldsymbol{A}_R^{-1}\circ P_R$. Then the equation (3) can be written in the form

(15) $$\begin{cases}\boldsymbol{A}(u_1+u_2)-\lambda u_1=f_1\\ -\lambda u_2=f_2\end{cases}$$

where $u_1=P_Ru=P_{R(A)}u$, $u_2=P_{N^*}u=P_{N(A^*)}u$, $f_1=P_Rf$, $f_2=P_{N^*}f$. So $u_2=-\frac{f_2}{\lambda}$ and we obtain

(16) $$\boldsymbol{A}u_1-\lambda u_1=f_1+\frac{\boldsymbol{A}f_2}{\lambda}$$

We denote $v=\boldsymbol{A}u_1\in R(\boldsymbol{A})$. Then $u_1=\boldsymbol{K}v$ and equation (16) can be rewritten in the form

$$v-\lambda\boldsymbol{K}v=f_1+\frac{\boldsymbol{A}f_2}{\lambda}$$

Since $1/\lambda$ is not eigenvalue of $\boldsymbol{K}$ then

(18) $$v=(I-\lambda\boldsymbol{K})^{-1}\left(f_1+\frac{\boldsymbol{A}f_2}{\lambda}\right)$$

Assume

(19) $$(I-\lambda\boldsymbol{K})^{-1}\left(f_1+\frac{\boldsymbol{A}f_2}{\lambda}\right)\in R(\boldsymbol{A})$$

Then using (18), (19) we obtain

$$u_1 = \boldsymbol{A}_R^{-1} v = \boldsymbol{K} v = \frac{1}{\lambda}(v - (I - \lambda \boldsymbol{K}) v) = \left(\frac{1}{\lambda}\left(v - \left(f_1 + \frac{\boldsymbol{A} f_2}{\lambda}\right)\right)\right) \in R(\boldsymbol{A})$$

Thus the system (15) is solvable if and only if (19) holds.

Let $f(x, y) \in \boldsymbol{L}_2(\Omega)$ and $g(u) \in \boldsymbol{L}_2(\Omega)$ for any $u \in \boldsymbol{L}_2(\Omega)$. Consider again

(3) $\quad \boldsymbol{A} u + g(u) = f$

This equation can be rewritten in the form

(20) $\quad u = \boldsymbol{A}_R^{-1} \circ P_R(f - g(u)) = \boldsymbol{K}(f_1 - g(u))$

(21) $\quad P_{N^*} g(u) = f_2$

If g satisfies (8), (9) where $\lambda_1 = \lambda_1(\boldsymbol{K})$ is the eigenvalue of $\boldsymbol{K}$ with the largest absolute value, then for any $f_1 \in R(\boldsymbol{A})$ there exists a solution $u = \boldsymbol{H}(f_1) \in D(\boldsymbol{A})$ of the equation (20). So the following condition

(22) $\quad P_{N^*} g(\boldsymbol{H}(f_1)) = f_2$

is a sufficient condition for the solvability of (3).

If g satisfies (13) where $M < \frac{1}{|\lambda_1(\boldsymbol{K})|}$ then the solution $u = \boldsymbol{H}(f_1)$ of (20) is unique and the equation (3) is solvable if and only if (22) holds.

Consider the equation

(2) $\quad \boldsymbol{A} u + f(x, y, u) = 0$

where $f(x, y, u)$ satisfies

(23) $\quad \|f(x, y, u)\|_{L_2(\Omega)} \leq M \cdot \|u\|_{L_2(\Omega)} + C, \qquad u \in \boldsymbol{L}_2(\Omega)$

or

(24) $\quad \|f(x, y, u) - f(x, y, v)\|_{L_2(\Omega)} \leq M \cdot \|u - v\|_{L_2(\Omega)}, \qquad u, v \in \boldsymbol{L}_2(\Omega)$

where $M < \frac{1}{|\lambda_1(\boldsymbol{K})|}$. Then the equation

(25) $\quad u = -\boldsymbol{A}_R^{-1} \circ P_R f(x, y, u) = -\boldsymbol{K} f(x, y, u)$

posseses a solution $u_f \in D(\boldsymbol{A})$. This solution is unique if (24) holds. So the following condition

(26) $\quad P_{N^*} f(x, y, u_f) = 0$

is sufficient for the solvability of (2). If (24) holds then (26) is a necessary and sufficient condition for the solvability of (2).

If we draw analogy to he theory of linear equations in the spaces of finite dimensions then equation (2) in the case (B) corresponds to the linear equation

$$Ax=f$$

where $x\in \boldsymbol{R}^n$, $f\in R^m$, $m>n$, $\mathrm{rank}(A)=n$.

So it seems to us that it is impossible to find "regular" conditions for the functions $f(x, y, u)$, $g(u)$, $f(x, y)$ to resolve the equations (2), (3) in the case (B).

Remark 1. The analyticity of the boundary is not essential in the case (B). All the arguments remain valid if we consider domain with a boundary $\Gamma\in \boldsymbol{C}^2$ such that $\alpha(F_\Gamma)=\frac{m}{n}$ is a rational number, the set of fixed points of F^n on Γ is finite, all fixed points of F^n are simple.

Remark 2. Let the boundary Γ of the domain Ω satisfies (B). Then any mapping of the form

$$(27) \qquad x_1=f(x), \qquad y_1=g(y), \qquad (x, y)\in\Omega$$

transforms domain Ω onto some domain Ω_1 with boundary Γ_1 which satisfies the conditions (B) and $\alpha(F_\Gamma)=\alpha(F_{\Gamma_1})=\frac{m}{n}$. Besides the set of fixed points of F^n is transformed by (27) onto the set of fixed points of $F_1{}^n$.

So for the domains which satisfy (B) it is possible to prove a theorem analogous to Theorem 6 in [1]. However to transform Ω onto Ω_1 by mapping of the form (27) it is not sufficient that $\alpha(F_\Gamma)=\alpha(F_{\Gamma_1})=\frac{m}{n}$ and the sets of fixd points of $F_\Gamma{}^n$, $F_{\Gamma_1}^n$ are finite. It is necessary also that the sets of fixed points of $F_\Gamma{}^n$ and $F_{\Gamma_1}^n$ have the same number of elements.

4. Case (C)

Let the case (C) hold. We shall consider the solvability of the problem (2). This case is the most complicated and the properties of the operator $\boldsymbol{A}$ are investigated less than in the cases (A), (B).

We shall assume here that the following condition holds:

(I) The diffeomorphism F is analytically conjgate to the shift

$$R_{\alpha(F)}: s\mapsto(s+\alpha(F)) \pmod{l}$$

It means that there exists an increasing analytic function $g(s)$: $\boldsymbol{R}\xrightarrow{on}\boldsymbol{R}$ such that

$$(28) \qquad g(f(s))=g(s)+\alpha(F), \qquad s\in\boldsymbol{R}$$

where $f(s)$ is lift of diffeomorphism F [4].
According to the Denjoy's Theorem [4], if $\alpha(F)$ is irrational then there exits $g(s)$ satisfying (28). However it can be constructed such diffeomorphism F with an analytic lift $f(t)$ that $\alpha(F)$ is irrational and the conjugating function $g(s)$ is not absolutely continuous function [3], [7].

Let $[a, b]$, $[c, d]$ be the projections of $\Omega\cup\Gamma$ onto the x and y axes respectively. Following [3] we set

$$\sigma_1(x)=(x-a)\cdot(b-x)\,, \qquad \sigma_2(y)=(y-c)\cdot(d-y)$$

$$\beta(k, j)=\frac{1}{2}\max(0, 2j-k-1)$$

M.V. Fokin [3] introduced the spaces $\boldsymbol{H}_k(\Omega)$, $k=0, 1, \cdots$ obtained by closing $\boldsymbol{C}^\infty(\Omega\cup\Gamma)\cap \overset{o}{\boldsymbol{W}}{}_2^1(\Omega)$ in the norm

$$|u|_k^2=\|u\|^2_{L_2(\Omega)}+\sum_{j=1}^{k}(\|\sigma_1^{\beta(k,j)}D_x^j u\|^2_{L_2(\Omega)}+\|\sigma_2^{\beta(k,j)}D_y^j u\|^2_{L_2(\Omega)})$$

It is easy to see that $\boldsymbol{H}_0(\Omega)=\boldsymbol{L}_2(\Omega)$, $\boldsymbol{H}_1(\Omega)=\overset{o}{\boldsymbol{W}}{}_2^1(\Omega)$, $\boldsymbol{W}_2^k(\Omega)\cap \overset{o}{\boldsymbol{W}}{}_2^1(\Omega)\subset\boldsymbol{H}_k(\Omega)$ $\subset \overset{o}{\boldsymbol{W}}{}_2^1(\Omega)$, $k=2, 3, \cdots$.

From the results of paper [3] it follows

Theorem 2. *Let the condtions* (*C*), (*I*) *hold. Then*

1) $\boldsymbol{A}=\boldsymbol{A}^*$, $N(\boldsymbol{A})=\{0\}$.

2) *The range* $R(\boldsymbol{A})$ *of the operator* $\boldsymbol{A}$ *is closed in* $\boldsymbol{L}_2(\Omega)$ *if and only if there exists a constant* $C>0$ *such that for any* n, m

$$(29) \qquad \left|\alpha(F)-\frac{m}{n}\right|\geq\frac{C}{n^2}\,, \qquad m, n\in\boldsymbol{N}$$

3) *If there exists such* $p\in\boldsymbol{N}$, $C>0$ *that for any* m, n

$$(30) \qquad \left|\alpha(F)-\frac{m}{n}\right|\geq\frac{C}{n^p}\,, \quad m, n\in\boldsymbol{N}$$

then for any $k\geq p-2$ *it follows that* $\boldsymbol{W}_2^k(\Omega)\subset R(\boldsymbol{A})$ *and for some* $C(k)>0$

$$(31) \qquad |\boldsymbol{A}^{-1}u|_{k-(p-2)}\leq C(k)\cdot\|u\|_{W_2^k(\Omega)}\,, \quad u\in \boldsymbol{W}_2^k(\Omega)$$

It is known [7], [8] that estimate (30) for irrational $\alpha(F)$ can hold for all m, n only if $p\geq 2$. The set of numbers α for which (30) hols for $p=2+\epsilon$ has full measure for any $\epsilon>0$. There exist the transcendental numbers of Liouville [6] α that for any p, $C>0$ there exist $m, n\in\boldsymbol{N}$ such that (30) does not hold.

Let for any $m, n\in\boldsymbol{N}$ inequality (29) hold. Then from Theorem 2 it follows that $\boldsymbol{A}=\boldsymbol{A}^*$, $N(\boldsymbol{A})=\{0\}$, $R(\boldsymbol{A}\}=\boldsymbol{L}_2(\Omega)$ and

$$(32) \qquad |\boldsymbol{A}^{-1}u|_k \leq C(k)\cdot\|u\|_{W_2^k(\Omega)}, \qquad u\in \boldsymbol{W}_2^k(\Omega)$$

Assume $f(x, y, u)$ satisfies (24) for some constant $M>0$ such that

$$(33) \qquad M<\frac{1}{C(0)}$$

Besides, we assume that $f(x, y, u)$ is defined for all $(x, y)\in\Omega$, $u\in\mathbf{C}$ and

$$(34) \qquad f(x, y, 0)\in \boldsymbol{L}_2(\Omega)$$

Then from (24), (34) it follows that $f(x, y, u(x, y))\in \boldsymbol{L}_2(\Omega)$ for any $u\in \boldsymbol{L}_2(\Omega)$ and equation (2) can be rewritten in the form

$$(35) \qquad u=-\boldsymbol{A}^{-1}f(x, y, u)\overset{\text{def}}{=}\boldsymbol{B}u$$

Using (24), (32), (33) we obtain that $\boldsymbol{B}$ is a contractive operator in $\boldsymbol{L}_2(\Omega)$. So there exists unique solution $u\in \boldsymbol{D}(\boldsymbol{A})$ of equation (2).

Assume $f(x, y, u)$ satisfies the following inequality

$$(36) \qquad \|f(x, y, u(x, y))-f(x, y, u(x, y))\|_{W_2^1(\Omega)}\leq M_1\cdot\|u-v\|_{\overset{o}{W}_2^1(\Omega)}$$

for any $u, v\in \overset{o}{\boldsymbol{W}}_2^1(\Omega)$ and

$$(37) \qquad M_1<\frac{1}{C(1)}$$

$$(38) \qquad f(x, y, 0)\in \boldsymbol{W}_2^1(\Omega)$$

Then $\boldsymbol{B}$ is a contractive operator in $\overset{o}{\boldsymbol{W}}_2^1(\Omega)$. Hence there exists unique solution $u\in \overset{o}{\boldsymbol{W}}_2^1(\Omega)$ of equation (2).

So we have proved

Theorem 3. *Let the conditions* (C), (I) *hold. Assume* $\alpha(F)$ *satisfies* (29) *for any* $m, n\in \boldsymbol{N}$. *Then*

(1) *If* $f(x, y, u)$ *satisfies* (24), (33), (34) *then equation* (2) *possesses unique solution in* $\boldsymbol{L}_2(\Omega)$.

(2) *If* $f(x, y, u)$ *satisfies* (36), (37), (38) *then equation* (2) *possesses unique solution in* $\overset{o}{\boldsymbol{W}}_2^1(\Omega)$.

Unfortunately, estimate (32) does not allow us to prove the regularity of the solution if $f(x, y, u)$ is regular. For example, if $f(x, y, u)=-\epsilon u-g(x, y)$, $g\in \boldsymbol{C}^\infty(\Omega\cup\Gamma)$, $\epsilon>0$ is small, then

(39) $$u = \boldsymbol{A}^{-1}g + \epsilon \boldsymbol{A}^{-1}u = \boldsymbol{B}u$$

From (32) it follows that $\boldsymbol{A}^{-1}g \in \boldsymbol{C}^{\infty}(\Omega \cup \Gamma)$. However as far as $\boldsymbol{H}_k \supset \boldsymbol{W}_2^k(\Omega) \cap \overset{o}{\boldsymbol{W}}_2^1(\Omega)$, $\boldsymbol{H}_k \neq \boldsymbol{W}_2^k(\Omega) \cap \overset{o}{\boldsymbol{W}}_2^1(\Omega)$ for any $k > 1$, then from (32), (38) and the general topological methods it does not follow that the operator $\boldsymbol{B}$ has a fixed point in $\boldsymbol{H}_k(\Omega)$ (or $\boldsymbol{W}_2^k(\Omega)$, or $\boldsymbol{C}^k(\Omega \cup \Gamma)$) if $k > 1$.

If $\alpha(F)$ satisfies (30) for some natural number $p > 2$ then $R(\boldsymbol{A})$ is not closed in $\boldsymbol{L}_2(\Omega)$ and estimate (31) does not allow us to prove the existence of a fixed point in $\boldsymbol{L}_2(\Omega)$ of the operator $\boldsymbol{B}u = -\boldsymbol{A}^{-1}f(x, y, u)$ using only the general topological methods.

Remark 1. All the arguments remain valid if instead of (2) we consider the equation

$$\boldsymbol{A}u + \boldsymbol{K}(u) = 0$$

where $\boldsymbol{K}(u)$ is some (nonlinear) operator in $\boldsymbol{L}_2(\Omega)$.

Acknowledgements. I would like to express my thanks to Prof. T. Nishida for the useful discussions and to the Japan Society for the Promoton of Science for the opportunity to visit Japan and to conduct my research at Kyoto University.

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