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Pseudoconvex domains of general order and q-convex domains in the complex projective space

By

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Introduction

Let D be an open set in the n-dimensional complex projective space $\boldsymbol{P}^n$ and let $d(p)$ be the distance from $p \in D$ to the boundary ∂D associated with the Fubini metric of $\boldsymbol{P}^n$. Takeuchi [9] showed that if the set D $(\subset \boldsymbol{P}^n)$ is pseudoconvex (in the original sense) then the (continuous) function $-\log d(p)$ is strongly plurisubharmonic in D. Therefore, the set D is strongly 1-complete, because every continuous strongly plurisubharmonic function can be approximated by such functions of class C^2 (i.e., strongly 1-convex functions). On the other hand, Barth [1] showed that if A is a non-singular algebraic set in $\boldsymbol{P}^n$ whose irreducible components are at least $(n-q)$-dimensional, $1 \leq q \leq n$, then the domain $\boldsymbol{P}^n \backslash A$ is strongly q-convex.

An open set D in an n-dimensional complex manifold X is said to be pseudoconvex of order $n-q$, $1 \leq q \leq n$, if its complement $X \backslash D$ has the same continuity as an analytic set of pure dimension $n-q$ (for the precise definition, see § 3). The pseudoconvexity of order $n-q$ is a local property. Pseudoconvex open sets in the original sense are pseudoconvex of order $n-1$. An open set in X is pseudoconvex of order $n-q$ if it is weakly q-convex, but the converse is not valid even if $X = \boldsymbol{C}^n$ (see Diederich-Fornaess [2] and Matsumoto [7]). By Fujita [3] and Slodkowski [8], an open set in $\boldsymbol{C}^n$ is pseudoconvex of order $n-q$ if and only if it is exhausted by an (upper semi-continuous) $(q-1)$-plurisubharmonic function, which was first introduced by Hunt-Murray [5] as a generalization of a plurisubharmonic function (for the definition, see § 2). 0-plurisubharmonic functions are plurisubharmonic functions in the original sense and $(q-1)$-plurisubharmonic functions of class C^2 are weakly q-convex functions. We note that, when $1 < q < n$, $(q-1)$-plurisubharmonic functions cannot be approximated by such functions of class C^2.

In this paper, we first show by the method of Takeuchi [9] that if an open

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set D in $\boldsymbol{P}^n$ is pseudoconvex of order $n-q$, $1\leq q\leq n$, then $-\log d(p)$ is $(q-1)$-plurisubharmonic in D (§ 4, Theorem 1). The function $-\log d(p)$ is, further, strongly q-convex in the subdomain of D where it is of class C^2 (§ 5, Theorem 2). Therefore, if D is a pseudoconvex open set of order $n-q$ in $\boldsymbol{P}^n$ and if its boundary ∂D is a real submanifold of class C^2 in $\boldsymbol{P}^n$ (whose irreducible components may have different dimensions from each other), then D is strongly q-convex (§ 5, Corollary).

For example, an open set D in $\boldsymbol{P}^n$ is pseudoconvex of order $n-q$ if its complement $E=\boldsymbol{P}^n\backslash D$ is an analytic set (algebraic set) whose irreducible components are at least $(n-q)$-dimensional. Moreover, D is also pseudoconvex of order $n-q$ if for each $p\in\partial D$ there exists an analytic set S_p of pure dimension $n-q$ defined near p such that $p\in S_p$ and $S_p\subset E$. Therefore, our result includes the main result of Barth [1] stated above.

The method used in our proof of Theorem 1 is based on that of Takeuchi (Théorème I in [9]). A mistake in his proof is corrected.

1. Fubini metric of $\boldsymbol{P}^n$

Let $(x_0: x_1: \cdots : x_n)$ be a (fixed) homogeneous coordinates of $\boldsymbol{P}^n$ and let ds^2 be the Fubini (or Fubini-Study) metric of $\boldsymbol{P}^n$ determined by $(x_0: x_1: \cdots : x_n)$. If, for example, U_0 is the open set in $\boldsymbol{P}^n$ defined by $x_0\neq 0$ and if $(z_1, , \cdots, z_n)$ where $z_i=x_i/x_0$ is the inhomogeneous coordinates of U_0, then the metric ds^2 is written in the form

$$ds^2=\frac{\sum_{i=1}^n|dz_i|^2}{1+\sum_{i=1}^n|z_i|^2}-\frac{|\sum_{i=1}^n z_i d\bar{z}_i|^2}{(1+\sum_{i=1}^n|z_i|^2)^2} \tag{1}$$

on U_0. This is the well-known standard Kähler metric of $\boldsymbol{P}^n$.

If $(y_0: y_1: \cdots : y_n)$ is a homogeneous coordinates of $\boldsymbol{P}^n$ obtained by a unitary transformation of $(x_0: x_1: \cdots : x_n)$ (as a transformation of $\boldsymbol{C}^{n+1}$), then the Fubini metric ds^2 above has the same expression as (1) replaced z_i with w_i, where $w_i=y_i/y_0$. We call such $(w_1, \cdots, w_n)$ an *admissible* inhomogeneous coordinates of $\boldsymbol{P}^n$.

Lemma A ([9], Lemme 1). *Let $L_{n-1}\supset L_{n-2}\supset\cdots\supset L_0$ be a decreasing sequence of linear subspaces L_i of $\boldsymbol{P}^n$ such that $\dim L_i=i$. Then there exists an admissible inhomogeneous coordinates $(z_1, \cdots, z_n)$ of $\boldsymbol{P}^n$ such that each L_i is written by $z_{i+1}=z_{i+2}=\cdots=z_n=0$.*

For any two points a, b of $\boldsymbol{P}^n$, we can choose an admissible inhomogeneous coordinates $(z_1, \cdots, z_n)$ of $\boldsymbol{P}^n$, so that the (complex) line L which passes a and b is written by $z_2=\cdots=z_n=0$. Then a geodesic joining a and b in $\boldsymbol{P}^n$ is included in L ($\cong\boldsymbol{P}^1$). If α and β are the z_1-coordinates of a and b respectively, then the Fubini distance between a and b is given by

$$d(a, b)=\sin^{-1}\frac{|\alpha-\beta|}{\sqrt{(1+|\alpha|^2)(1+|\beta|^2)}}. \tag{2}$$

We will later use the following lemmas to prove our theorems:

Lemma 1. *For $\alpha, \beta\in \boldsymbol{C}$ with $\alpha\neq\beta$, we have*

$$\sin^{-1}\frac{|\alpha-\beta|}{\sqrt{(1+|\alpha|^2)(1+|\beta|^2)}}\leq 2\tan^{-1}\frac{|\alpha-\beta|}{2}.$$

Here the equality holds if and only if $\alpha=-\beta$ and $|\alpha|\leq 1$.

Lemma 2. *Let $\alpha\in \boldsymbol{C}$ and $|\alpha|\leq 1$. For $z\in \boldsymbol{C}$, put*

$$\delta(z, \alpha)=|z-\alpha|, \quad d(z, \alpha)=\sin^{-1}\frac{|z-\alpha|}{\sqrt{(1+|z|^2)(1+|\alpha|^2)}},$$

and suppose that $\delta(z\ \alpha)\leq\delta(-\alpha, \alpha)$ $(=2|\alpha|)$. Then it holds that $d(z, \alpha)\leq d(-\alpha, \alpha)$.

The proofs of Lemma 1 and 2 are elementary.

2. q-plurisubharmonic functions

Let D be an open set in $\boldsymbol{C}^n$. A function φ: $D\to\boldsymbol{R}\cup\{-\infty\}$ is said to be *subpluriharmonic* in D if it satisfies the following conditions ([3]):

(i) φ is upper semi-continuous on D.

(ii) For every domain $\varDelta$ such that $\varDelta\Subset D$ and for every pluriharmonic function u defined in a neighborhood of $\bar{\varDelta}$ such that $\varphi\leq u$ on $\partial\varDelta$ $(=\bar{\varDelta}\backslash\varDelta)$, we have $\varphi\leq u$ in $\varDelta$.

When $n=1$, the notion of subpluriharmonic function coincides with that of subharmonic function.

A function φ: $D\to\boldsymbol{R}\cup\{-\infty\}$ where $D\subset\boldsymbol{C}^n$ is said to be $(n-1)$-*plurisubharmonic* in D (in the sense of Hunt-Murray [5]), if it is upper semi-continuous on D and if it satisfies the condition replaced 'pluriharmonic function' with 'plurisuperharmonic function' in the condition (ii) above. By Slodkowski ([8], Lemma 4.4), φ is subpluriharmonic in D $(\subset\boldsymbol{C}^n)$ if and only if φ is $(n-1)$-plurisubharmonic in D (for the direct proof, see [4], Proposition 2).

By Fujita ([3], Proposition 3), the subpluriharmonicity of a funtion is a local property. Moreover, the property is invariant under biholomorphic mappings. Therefore, we can naturally define subpluriharmonic functions on complex manifolds.

We have immediately from Proposition 3 in [3] the following:

Lemma 3. *Let D be an n-dimensional complex manifold. An upper semi-continuous function φ:$D\to\boldsymbol{R}\cup\{-\infty\}$ is subpluriharmonic in D if and* *only if for each* $a \in D$ *there exists a neighborhood* $U(\subset D)$ *of* a *such that* φ *satisfies the following condition* $(*)$ *with respect to* U:

$(*)$ *For every domain* Δ *such that* $a \in \Delta$ *and* $\Delta \Subset U$ *and for every pluriharmonic function* u *defined in a neighborhood of* $\bar{\Delta}$ *such that* $\varphi \leq u$ *on* $\partial \Delta$, *we have* $\varphi(a) \leq u(a)$.

Let a be a point of a complex manifold D and let φ be a function with values in $\boldsymbol{R} \cup \{-\infty\}$ defined in a neighborhood V $(\subset D)$ of a. In this paper, we shall say that φ is *subpluriharmonic at* a if there exists a neighborhood U $(\subset V)$ of a such that φ satisfies the above condition $(*)$ with respect to U. Then a function $\varphi: D \to \boldsymbol{R} \cup \{-\infty\}$ is subpluriharmonic in D, if and only if φ is upper semi-continuous on D and subpluriharmonic at each point of D.

By Fujita ([3], Proposition 5), a real valued function φ of class C^2 defined in a neighborhood V of a is subpluriharmonic in V if and only if its Levi form $\partial\bar{\partial}\varphi$ has at least one non-negative eigenvalue at each point of V. Therefore, φ is subpluriharmonic at a if $\partial\bar{\partial}\varphi$ has at least one positive eigenvalue at a. Moreover, we can see, in view of the proof of Proposition 5 in [3], that φ is not subpluriharmonic at a if all the eigenvalues of $\partial\bar{\partial}\varphi$ are negative at a.

The following lemmas are the criterions of subpluriharmonicity of upper semi-continuous functions:

Lemma 4. *A function* φ *with values in* $\boldsymbol{R} \cup \{-\infty\}$ *defined in a neighborhood* V *of* a *is subpluriharmonic at* a *if there exists a non-singular analytic curve* L $(\subset V)$ *such that* $a \in L$ *and the restriction* $\varphi|_L$ *is subharmonic at* a.

Proof. If $\varphi' = \varphi|_L$ is subharmonic at a, we can take a neighborhood U' $(\subset V \cap L)$ of a so that, for every domain Δ' such that $a \in \Delta'$ and $\Delta' \Subset U'$ and for every harmonic function u' defined in a neighborhood of $\bar{\Delta}'$ such that $\varphi' \leq u'$ on $\partial\Delta'$, it holds that $\varphi'(a) \leq u'(a)$. We now choose a neighborhood U $(\subset V)$ of a so that $U \cap L \subset U'$. Let Δ be a domain such that $a \in \Delta$ and $\Delta \Subset U$, let u be a pluriharmonic function defined in a neighborhood of $\bar{\Delta}$ such that $\varphi \leq u$ on $\partial\Delta$, and denote by Δ' the connected component of $\Delta \cap L$ which contains a. Then $\Delta' \Subset U'$. Moreover, $u' = u|_L$ is harmonic in a neighborhood of $\bar{\Delta}'$ and we have $\varphi' \leq u'$ on $\partial\Delta'$. Therefore, we obtain $\varphi(a) = \varphi'(a) \leq u'(a) = u(a)$, which proves the subpluriharmonicity of φ at a.

Lemma 5. *Let* φ *and* ψ *be two functions with values in* $\boldsymbol{R} \cup \{-\infty\}$ *defined in a neighborhood* V *of* a *and suppose that* $\varphi(a) = \psi(a)$ *and* $\varphi \geq \psi$ *on* V. *Then* φ *is subpluriharmonic at* a *if so is* ψ.

Proof. If ψ is subpluriharmonic at a, we can take a neighborhood U $(\subset V)$ of a so that ψ satisfies the condition $(*)$ in Lemma 3 with respect to U. Then it is easy to see that φ also satisfies the condition $(*)$ with respect to the same U.

Next let D be an open set in $\boldsymbol{C}^n$ and let q be an integer with $1 \leq q \leq n$. A function $\varphi: D \to \boldsymbol{R} \cup \{-\infty\}$ is said to be $(q-1)$-*plurisubharmnic* in D (in the sense of Hunt-Murray [5]) if φ is upper semi-continuous on D and if the restriction $\varphi|_L$ is subpluriharmonic in $D \cap L$ for every q-dimensional linear subspace L such that $D \cap L \neq \emptyset$.

0-plurisubharmonic functions are plurisubharmonic functions (in the original sense). (n−1)-plurisubharmonic functions are subpluriharmonic functions.

The $(q-1)$-plurisubharmonicity of a function is also a local property. Moreover, the property is invariant under biholomorphic mappings (see Hunt-Murray [5] and Slodkowski [8], 1.11).

Let D be an n-dimensional complex manifold. A function φ: $D \to \boldsymbol{R} \cup \{-\infty\}$ is said to be $(q-1)$-plurisubharmonic in D if for every coordinate neighborhood (U, Ψ), $\Psi = (z_1, \cdots, z_n)$, of D, the composite $\varphi \circ \Psi^{-1}$ is $(q-1)$-plurisubharmonic in $\Psi(U)$ $(\subset \boldsymbol{C}^n)$ as a function of $(z_1, \cdots, z_n)$.

When D is an open set in $\boldsymbol{P}^n$, we have the following:

Lemma 6. *Let D be an open set in $\boldsymbol{P}^n$ and let q be an integer with $1 \leq q \leq n$. An upper semi-continuous function φ: $D \to \boldsymbol{R} \cup \{-\infty\}$ is $(q-1)$-plurisubharmonic in D if and only if $\varphi|_L$ is subpluriharmonic at a for every $a \in D$ and for every q-dimensional linear subspace L containing a.*

A real valued function φ of class C^2 defined in an n-dimensional complex manifold D is said to be strongly q-convex (and weakly q-convex) in D if its Levi form $\partial\bar{\partial}\varphi$ has at least $n-q+1$ positive (and non-negative) eigenvalues at each point of D (respectively).

Lemma B ([5], Lemma 2.6). *Let φ be a real valued function of class C^2 defined in an n-dimensional complex manifold D. Then φ is $(q-1)$-plurisubharmonic, $1 \leq q \leq n$, in D if and only if φ is weakly q-convex in D.*

3. Pseudoconvex domains of general order

Let D be an open set in an n-dimensional complex manifold X, let k be an integer with $1 \leq k \leq n-1$ and put $E = X \backslash D$. The set D is said to be *pseudoconvex of order k* in X if the following condition (**) is satisfied for every $b \in E$ and for every coordinate neighborhood $(U, (z_1, \cdots, z_n))$ which contains b as the origin:

(**) If the set

$$\{(z_1, \cdots, z_n) \in U;\ z_i = 0\ (1 \leq i \leq k),\ 0 < \sum_{i=k+1}^{n} |z_i|^2 < r\}$$

contains no points of E for some $r > 0$, then there exists $s > 0$ such that for each $(z_1', \cdots, z_k')$ with $|z_i'| < s$, $1 \leq i \leq k$, the set

$$\{(z_1, \cdots, z_n)\in U;\, z_i=z_i'\ (1\leq i\leq k),\ \sum_{i=k+1}^{n}|z_i|^2<r\}$$

contains at least one point of E.

Moreover, we say that every open set in X is pseudoconvex of order 0.

A complex manifold D is said to be strongly q-convex (and weakly q-convex) if one can choose a compact subset K $(\subset D)$ and a continuous function φ on D whose restriction $\varphi|_{(D\setminus K)}$ is strongly q-convex (and weakly q-convex) in $D\setminus K$ so that $\{p\in D;\ \varphi(p)<M\}\Subset D$ for every $M\in \boldsymbol{R}$ (respectively).

If an open set D in an n-dimensional complex manifold X is weakly q-convex, $1\leq q\leq n$, then D is pseudoconvex of order $n-q$ in X. However, the converse is not valid even if $X=\boldsymbol{C}^n$.

For open sets in $\boldsymbol{C}^n$, Fujita proved the following:

Theorem C ([3], Théorème 1). *Let D be a pseudoconvex open set of order $n-q$, $1\leq q\leq n$, in $\boldsymbol{C}^n$ and denote by $R(z)$ the Hartogs radius of D with respect to z_n at $z=(z_1, \cdots, z_n)\in D$. Then the function $-\log R(z)$ is $(q-1)$-plurisubharmonic in D.*

Remark 1. In [3], Fujita introduced the concept of 'pseudoconvex functions of order $n-q$' and showed that $-\log R(z)$ is pseudoconvex of order $n-q$ in D under the same assumption as Theorem C. It follows by definition that a function is $(q-1)$-plurisubharmonic if it is pseudoconvex of ordr $n-q$. Recently, Fujita [4] has proved that the converse is also valid.

4. Fubini boundary distances of domains in $\boldsymbol{P}^n$

Let D be an open set in $\boldsymbol{P}^n$ and let $d(p)=\inf_{b\in\partial D}d(p, b)$ be the Fubini distance from $p\in D$ to the boundary ∂D. Takeuchi ([9], Théorème I) showed that if an open set D in $\boldsymbol{P}^n$ is pseudoconvex (in the original sense) then the function $-\log d(p)$ is plurisubharmonic in D. In this section, we shall prove that if D $(\subset\boldsymbol{P}^n)$ is pseudoconvex of order $n-q$, $1\leq q\leq n$, then $-\log d(p)$ is $(q-1)$-plurisubharmonic in D.

We first prove the following:

Lemma 7. *Let b be a fixed point of $\boldsymbol{P}^q$, $q\geq 1$, and let $d_0(p)=d(p, b)$ be the Fubini distance between p and b for $p\in\boldsymbol{P}^q$. Then the function $-\log d_0(p)$ is subpluriharmonic in $\boldsymbol{P}^q\setminus\{b\}$.*

Proof. It is obvious that $-\log d_0(p)$ is continuous on $\boldsymbol{P}^q\setminus\{b\}$.

Let $a(\neq b)$ be a point of $\boldsymbol{P}^q$ and let L be the (complex) line which passes a and b. Then we can choose an admissible inhomogeneous coordinates $(z_1, \cdots, z_q)$ of $\boldsymbol{P}^q$ so that its domain U_0 $(\cong\boldsymbol{C}^q)$ contains both a and b and L is written by $z_2=\cdots=z_q=0$.

For $p\in U_0\cap L$, we denote by $d_1(p)$ the distance between p and b in L

associated with the Fubini metric of L ($\cong \boldsymbol{P}^1$) which is the restriction to L of that of $\boldsymbol{P}^q$. Then $d_1(p)=d_0(p)$ for every $p\in U_0\cap L$. If z and β are the z_1-coordinates of p and b respectively, it follows that

$$d_1(p)=\sin^{-1}\frac{|z-\beta|}{\sqrt{(1+|z|^2)(1+|\beta|^2)}}\,.$$

Now, the function $-\log d_1$ is subharmonic at a ($\neq b$). Hence the restriction $(-\log d_0)|_L$ is subharmonic at a and it follows from Lemma 4 that $-\log d_0$ is subpluriharmonic at a ($\neq b$). Therefore, we can by Lemma 3 conclude that $-\log d_0$ is subpluriharmonic in $\boldsymbol{P}^q\backslash\{b\}$.

Now we shall prove the following:

Theorem 1. *Let D be a pseudoconvex open set of order $n-q$, $1\leq q\leq n$, in $\boldsymbol{P}^n$ and let $d(p)=\inf_{b\in\partial D}d(p, b)$ be the Fubini distance from $p\in D$ to the boundary ∂D. Then the function $-\log d(p)$ is $(q-1)$-plurisubharmonic in D.*

Proof. Since the function $-\log d$ is continuous on D, it is sufficient by Lemma 6 to show that the restriction $\varphi=(-\log d)|_L$ is subpluriharmonic at a for every $a\in D$ and for every q-dimensional linear subspace L containing a. We take a point $b\in\partial D$ such that $d(a)=d(a, b)$.

Case (α) *where* $b\in L$.

For $p\in L$, we denote by $d_0(p)$ the distance between p and b in L associated with the Fubini metric of L ($\cong \boldsymbol{P}^q$) which is the restriction to L of that of $\boldsymbol{P}^n$. Then we have $d_0(a)=d(a)$ and $d_0(p)\geq d(p)$ for $p\in D\cap L$, and hence $-\log d_0(a)=\varphi(a)$ and $-\log d_0(p)\leq\varphi(p)$ for $p\in D\cap L$. Since by Lemma 7 the function $-\log d_0$ is subpluriharmonic at a ($\neq b$), it follows from Lemma 5 that φ is also subpluriharmonic at a.

Case (β) *where* $b\notin L$.

We can take an admissible inhomogeneous coordinates $(z_1, \cdots, z_n)$ of $\boldsymbol{P}^n$ such that its domain U_0 ($\cong \boldsymbol{C}^n$) contains both a and b and the $(q+1)$-dimensional linear subspace which includes b and L is written by $z_{q+2}=z_{q+3}=\cdots=z_n=0$. Further, we can by Lemma A choose the coordinates $(z_1, \cdots, z_n)$ as follows:

(i) The line which passes a and b is written by $z_1=\cdots=z_q=z_{q+2}=\cdots=z_n=0$.

(ii) If α and β are the z_{q+1}-coordinates of a and b respectively, then $\alpha=-\beta$ and $|\alpha|\leq 1$.

Here the condition (ii) is satisfied if only we take $(z_1, \cdots, z_n)$ so that the origin is the middle point of the shortest geodesic which joins a and b. Then the q-dimensional linear subspace $L(\ni a)$ is written in the form

$$z_{q+1}=k_1z_1+\cdots+k_qz_q+\alpha \quad (k_i\in\boldsymbol{C})\,,\quad z_{q+2}=\cdots=z_n=0\,.$$

We put $D_0=D\cap U_0$ and denote by Ψ the homeomorphism from U_0 to $\boldsymbol{C}^n$

which defines the coordinates of U_0. Identifying $p \in U_0$ with $\Psi(p)=(z_1, \cdots, z_n) \in \boldsymbol{C}^n$, we can now regard the set D_0 as a pseudoconvex open set of order $n-q$ in $\boldsymbol{C}^n$. Let $R(p)$ be the Hartogs radius of D_0 with respect to z_{q+1} at $p \in D_0$. Then it follows from Theorem C of Fujita that $-\log R$ is $(q-1)$-plurisubharmonic in D_0 and hence $(-\log R)|_L$ is subpluriharmonic in $D_0 \cap L$. Here the value $R(p)$ may be $+\infty$ for some $p \in D_0 \cap L$.

Let p' be a point of $D_0 \cap L$ such that $R(p') < +\infty$. Put $\Psi(p')=(z_1', \cdots, z_n')$, where $z_{q+1}'=k_1 z_1' + \cdots + k_q z_q' + \alpha$ and $z_{q+2}' = \cdots = z_n' = 0$, and define the line $L(p')$ $(\subset \boldsymbol{P}^n)$ passing p' by

$$z_1 = z_1', \cdots, z_q = z_q', z_{q+2} = \cdots = z_n = 0 .$$

If c is a point of $\partial D_0 \cap L(p')$ $(\neq \emptyset)$ such that the value $|z_{q+1}' - \gamma|$, where γ is the z_{q+1}-coordinate of c, is minimum, we can write $R(p') = |z_{q+1}' - \gamma|$. When $p'=a$, we can by Lemma 2 take the point b as this c and we have $R(a)=2|\alpha|$.

Since the inhomogeneous coordinates $(z_1, \cdots, z_n)$ of $\boldsymbol{P}^n$ is now admissible, the Fubini metric ds^2 of $\boldsymbol{P}^n$ is written in the form

$$ds^2 = \frac{(1+\sum_{i=1}^{q}|z_i'|^2)|dz_{q+1}|^2}{(1+\sum_{i=1}^{q}|z_i'|^2 + |z_{q+1}|^2)^2}$$

on $U_0 \cap L(p')$. If we change the parameter z_{q+1} to

$$w = w(z_{q+1}) = \frac{z_{q+1}}{\sqrt{1+\sum_{i=1}^{q}|z_i'|^2}}$$

on $U_0 \cap L(p')$, we obtain

$$ds^2 = \frac{|dw|^2}{(1+|w|^2)^2} .$$

Therefore, it holds that

$$d(p', c) = \sin^{-1}\frac{|w(z_{q+1}') - w(\gamma)|}{\sqrt{(1+|w(z_{q+1}')|^2)(1+|w(\gamma)|^2)}} ,$$

and it follows from Lemma 1 that

$$d(p', c) \leq 2\tan^{-1}\frac{|w(z_{q+1}') - w(\gamma)|}{2} = 2\tan^{-1}\frac{R(p')}{2\sqrt{1+\sum_{i=1}^{q}|z_i'|^2}} .$$

Since $d(p') \leq d(p', c)$, we further obtain the inequality

$$d(p') \leq 2\tan^{-1}\frac{R(p')}{2\sqrt{1+\sum_{i=1}^{q}|z_i'|^2}} . \tag{3}$$

When $p'=a$, it follows that $d(a)=d(a, b)=2\tan^{-1}|\alpha|$ and the equality holds in (3). Thus we have

$$(4)\qquad -\log R(p)\leq -\log 2-\frac{1}{2}\log(1+\sum_{i=1}^{q}|z_i|^2)-\log\tan\frac{d(p)}{2}$$

for every $p\in D_0\cap L$ with $R(p)<+\infty$, where $\Psi(p)=(z_1, \cdots, z_n)$. It is clear that this inequality (4) is also valid for $p\in D_0\cap L$ with $R(p)=+\infty$. Moreover, the equality holds for $p=a$.

To prove the subpluriharmonicity at a of $\varphi=(-\log d)|_L$, let $\varDelta$ be a domain such that $a\in\varDelta$ and $\varDelta\Subset D_0\cap L$, let u be a pluriharmonic function defined in a neighborhood of $\bar{\varDelta}$ and suppose that $\varphi(p)\leq u(p)$ on $\partial\varDelta$. Then it follows by (4) that

$$(5)\qquad (-\log R)|_L(p)+\log 2+\frac{1}{2}\log(1+\sum_{i=1}^{q}|z_i|^2)+\log\tan\frac{e^{-u(p)}}{2}\leq 0$$

on $\partial\varDelta$. Now, the left side of (5) is subpluriharmonic in a neighborhood of $\bar{\varDelta}$ and hence the inequality (5) is also valid for $p=a$. If we note that the equality holds in (4) for $p=a$, we obtain $\varphi(a)\leq u(a)$, which implies the subpluriharmonicity of φ at a.

5. q-convexity of domains with C^2-boundaries in $\boldsymbol{P}^n$

Let D be a pseudoconvex open set of order $n-q$, $1\leq q\leq n$, in $\boldsymbol{P}^n$ and let $d(p)=\inf_{b\in\partial D}d(p, b)$ be the Fubini distance from $p\in D$ to the boundary ∂D. Then by Theorem 1, the function $-\log d$ is $(q-1)$-plurisubharmonic in D. Moreover, if we denote by D' the set of all points of D near which d is of class C^2, it follows from Lemma B at once that $-\log d$ is weakly q-convex in D'. (Note that we do not assert here the existence of the set D'.)

If an open set D in $\boldsymbol{P}^n$ is pseudoconvex (in the original sense) then $-\log d$ is strongly plurisubharmonic in D (in the sense of [9], Théorème II). In particular, it is, in this case, strongly 1-convex in D'.

For pseudoconvex open sets of general order, we obtain the following:

Theorem 2. *Let D be a pseudoconvex open set of order $n-q$, $1\leq q\leq n$, in $\boldsymbol{P}^n$ and let $d(p)$ be the Fubini distance from $p\in D$ to the boundary ∂D. Denote by D' (if it exists) the set of all points of D near which $d(p)$ is of class C^2. Then the function $-\log d(p)$ is strongly q-convex in D'.*

Proof. If we suppose that $-\log d$ is not strongly q-convex in D', there exists a point a of D' and a q-dimensional linear subspace L $(\ni a)$ such that the Levi form of the restriction $\varphi=(-\log d)|_L$ has no positive eigenvalues at a.

We take $b\in\partial D$ such that $d(a)=d(a, b)$. Note that $0<d(a)<\pi/2$ because $a\in D'$. We shall distinguish two cases $b\in L$ and $b\notin L$ and lead a contradiction in each case.

Case (a) *where* $b\in L$.

We choose an admissible inhomogeneous coordinates $(z_1, \cdots, z_n)$ of $\boldsymbol{P}^n$ so that its domain U_0 $(\cong \boldsymbol{C}^n)$ contains both a and b and the line M which passes a and b is written by $z_2 = \cdots = z_n = 0$. If p is any point of $U_0 \cap M$ and if z and β are the z_1-coordinates of p and b respectively, we obtain the inequality

$$\varphi|_M(p) \geq -\log d(p, b) = -\log \sin^{-1} \frac{|z-\beta|}{\sqrt{(1+|z|^2)(1+|\beta|^2)}}, \tag{6}$$

where the equality holds for $p=a$.

Since the right side of (6) is a strongly subharmonic function of class C^2 in the subdomain of M $(\cong \boldsymbol{P}^1)$ defined by $z \neq \beta$, $-1/\bar{\beta}$, its Levi form has a positive eigenvalue at a. On the other hand, the eigenvalue of the Levi form of $\varphi|_M$ is non-positive at a, and it is easy to lead the contradiction to Lemma 5.

Case (β) *where* $b \notin L$.

We use the same notations as those in Case (β) in the proof of Theorem 1. Then by (4), we have

$$(-\log R)|_L(p) \leq -\log 2 - \frac{1}{2} \log(1 + \sum_{i=1}^{q} |z_i|^2) - \log \tan \frac{e^{-\varphi(p)}}{2}, \tag{7}$$

for every $p \in D_0 \cap L$ where $\Psi(p) = (z_1, \cdots, z_n)$. Moreover, the equality holds in (7) for $p=a$.

Now, since the Levi form of $\varphi = (-\log d)|_L$ has, by assumption, no positive eigenvalues at a, so does the Levi form of the third term in the right side of (7). Moreover, all the eigenvalues at a of the Levi form of the second term in the right side of (7) are negative. Therefore, all the eigenvalues of the Levi form of the right side of (7) are negative at a and hence the right side of (7) is not subpluriharmonic at a. On the other hand, the left side $(-\log R)|_L$ of (7) is subpluriharmonic at a, which contradicts to Lemma 5.

Let D be an open set in $\boldsymbol{P}^n$ and suppose that the boundary ∂D is a real submanifold of class C^2 in $\boldsymbol{P}^n$, where ∂D need not be connected and their irreducible components may have different dimensions from each other. Denote by $d(p)$ the Fubini distance from $p \in D$ to ∂D. Then there exists an open set Δ $(\subset \boldsymbol{P}^n)$ such that $\partial D \subset \Delta$ and d is of class C^2 on $D \cap \Delta$.

Remark 2. It is well-known that the function d is of class C^2 near ∂D if ∂D is a real submanifold of class C^3 in $\boldsymbol{P}^n$. Further, we can prove the fact if only it is of class C^2. When ∂D is a hypersurface in $\boldsymbol{R}^n$, see Krantz [6], p. 136.

The following is the direct result of Theorem 2:

Corollary. *If D is a pseudoconvex open set of order $n-q$, $1 \leq q \leq n$, in $\boldsymbol{P}^n$ and if its boundary ∂D is a real submanifold of class C^2 in $\boldsymbol{P}^n$, then the set D is strongly q-convex.*

Remark 3. 1) Under the assumption of the corollary, the real dimension of each irreducible component of the boundary ∂D is at least $2(n-q)$ (see, [7], Proposition).

2) If A is a non-singular algebraic set in $\boldsymbol{P}^n$ whose irreducible components are at least $(n-q)$-dimensional and if $D=\boldsymbol{P}^n\backslash A$, then D satisfies the assumption of the corollary above. Therefore, the corollary includes the result of Barth [1] asserting that the set D $(\subset\boldsymbol{P}^n)$ is strongly q-convex.

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