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Moduli of stable bundles on blown up surfaces

By

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1. Introduction

The aim of this paper is to study the behavior of stable bundles on algebraic surfaces under the blowing-up. Stable rank 2 bundles on blown up surfaces have been considered in [FM], [B] and the relations between the moduli spaces were analyzed. In this paper we shall treat the higher rank case.

Let X be a smooth projective surface defined over an algebraically closed field k and let H be an ample divisor on X. Let π: $\tilde{X} \to X$ be the blow up of X at l distinct points p_i $(1 \leq i \leq l)$ and let E_i be the exceptional divisors. We define a divisor H_n on $\tilde{X}$ by the following

$$H_n = n\pi^* H - \sum_{i=1}^{l} E_i .$$

Then for sufficiently large n, H_n is ample.

We denote by $M_H(r, c_1, c_2)$ the moduli space of H-stable vector bundles of rank r on X with Chern classes c_1, c_2. On $\tilde{X}$, we consider two types of moduli spaces. Put $\tilde{c}_1 = \pi^* c_1$ and $\hat{c}_1 = \pi^* c_1 + \sum_{i=1}^{l} a_i E_i$ where $1 \leq a_i \leq r-1$. For large n we have moduli spaces of H_n-stable bundles $M_{H_n}(r, \tilde{c}_1, c_2)$ and $M_{H_n}(r, \hat{c}_1, c_2)$. Then our main result is the following

Theorem 1. (1) *For sufficiently large n, there exists an open immersion*

$$\phi: M_H(r, c_1, c_2) \hookrightarrow M_{H_n}(r, \tilde{c}_1, c_2)$$

defined by the pullback.
(2) *Assume that $M_H(r, c_1, c_2)$ has a universal family $\boldsymbol{E}$. Then there is a scheme $\boldsymbol{GE}$ over $M_H(r, c_1, c_2)$ whose fibers are the products of Grassmann varieties and we have an open immersion*

$$\hat{\phi}: \boldsymbol{GE} \hookrightarrow M_{H_n}(r, \hat{c}_1, c_2) .$$

This theorem generalizes the results in [B], [FM] to higher rank bundles. Unfortunately unlike these works we don't have an explicit lower bound for n.

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In the rank two case, we also obtain the following result concerning the generic smoothness of the moduli spaces.

Theorem 2. *Assume that* char $k=0$. *For given* c_1, *there exist integers* n_0 *and* c_0 *such that*

(1) $M_{H_n}(2, \tilde{c}_1, c_2)$ *is good for all* $n \geq n_0$ *and* $c_2 \geq c_0$;

(2) *If* $c_1. H$ *is odd, then* $M_{H_n}(2, \hat{c}_1, c_2)$ *is good for all* $n \geq n_0$ *and* $c_2 \geq c_0$.

For the difinition of the goodness, see § 3.

2. Stable bundles on a blown up surface

Let X be a smooth projective surface defined over an algebraically closed field k. We shall consider the stable bundles on the blow up $\tilde{X} \to X$ at l distinct points $p_1, \cdots, p_l$. We fix an ample divisor H on X. Then for sufficiently large n, $H_n = n\pi^* H - \sum_{i=1}^{l} E_i$ is an ample divisor on $\tilde{X}$ where $E_i = \pi^{-1}(p_i)$ are the exceptional divisors. For given $c_1 \in \mathrm{Pic} X$ and $c_2 \in \boldsymbol{Z}$, we denote by $M_H(r, c_1, c_2)$ the moduli space of H-stable rank r vector bundles E on X with $c_1(E) = c_1$, $c_2(E) = c_2$. Similarly we define $M_{H_n}(r, c_1', c_2')$ for $c_1' \in \mathrm{Pic}\, \tilde{X}$, $c_2' \in \boldsymbol{Z}$.

Definition. For fixed r, $c_2' \in \boldsymbol{Z}$ and $c_1' \in \mathrm{Pic}\, \tilde{X}$, a polarization H on X is said to be (r, c_1', c_2')-*suitable* if there exists an integer n_0 such that for all $n \geq n_0$, every vector budle E with the invariant (r, c_1', c_2') is H_n-stable if and only if E is H_{n_0}-stable.

We note that if $r=2$, the suitablity of H is equivalent to the condition that for sufficiently large n_1, n_2, H_{n_1} and H_{n_2} are equivalent in the sense of Qin [Q].

Proposition 2.1. *Every polarization* H *is* $(2, c_1', c_2')$-*suitable for arbitrary* c_1', c_2'.

Proof. We shall exploit the theory of chamber structures of Qin. For details we refer to [Q]. Let r_0 be the smallest real number such that for all $r \geq r_0$, H_r is ample. Assume that for arbitrary integer $k>0$, there exist integers $n(k)_i > k$ $(i=1, 2)$ such that two polarizations $H_{n(k)_1}$ and $H_{n(k)_2}$ are not equivalent. Then we would obtain a strictly increasing sequence $\{r(k) | k=1, 2, \cdots\}$ of real numbers $r(k) > \min(n(k)_1, n(k)_2)$ such that each $H_{r(k)}$ lies on the wall $W^{\zeta(k)}$ defined by some $\zeta(k) \in \mathrm{Num}(\tilde{X}) \otimes \boldsymbol{R}$. However, this obviously contradicts the following

· Claim. If r is a real number such that H_r is ample and lies on some wall W^{ζ}, then either for all $r' \geq r_0$, $H_{r'}$ lies on W^{ζ} or

$$r \leq \sqrt{l(4c_2' - c_1'^2 + 1)}\ .$$

To prove the claim, we write ζ as $\zeta=a\pi^*M+\sum_{i=1}^{l}b_iE_{di}$. Then $0=H_r.\zeta=(r\pi^*H-\sum E_i).(a\pi^*M+\sum_{i=l}^{l}bE_i)=raM.H+\sum b_i$. Hence we have $\sum b_i=-raM.\ H$. If $aM.H=0$, then $\sum b_i=0$ and hence $H_{r'}.\zeta=0$ for arbitrary $r'\geq r_0aMzH=0$, then $\sum b_i=0$ and hence $H_{r'}.\zeta=0$ for arbitrary $r'\geq r_0$. This leads to the first possibility. Assume that $aM.C\neq 0$. By definition of W^ζ we have

$$-(4c_2'-c_1'^2)\leq\zeta^2=a^2M^2-\sum b_i^2<0\,.$$

If $M^2>0$, then we have

$$\begin{aligned}-\frac{(4c_2'-c_1'^2)}{a^2M^2}&\leq 1-\frac{\sum b_i^2}{a^2M^2}\\ &\leq 1-\frac{(\sum b_i)^2}{la^2M^2}\\ &=1-\frac{r^2(M.H)^2}{lM^2}\\ &\leq 1-\frac{r^2H^2}{l}\,.\end{aligned}$$

Here, the second inequality follows from the Schwarz inequality $(\sum b_i)^2\leq l(\sum b_i^2)$ and the third one from the Hodge index theorem. Therefore we obtain

$$\begin{aligned}r&\leq\sqrt{\frac{1}{H^2}+\frac{l(4c_2'-c_1'^2+1)}{a^2M^2H^2}}\\ &\leq\sqrt{l(4c_2'-c_1'^2+1)}\,.\end{aligned}$$

Thus we are done in this case. If $M^2\leq 0$, then

$$-(4c_2'-c_1'^2)\leq -r^2a^2(M.H)^2\,.$$

This yields $r\leq\sqrt{4c_2'-c_1'^2}$, hence the proof has been completed.

Proposition 2.2. *For given c_1, c_2, there exists an integer n_0 such that for all $n\geq n_0$, the following hold:*

(1) *For any $E\in M_H(r, c_1, c_2)$, π^*E belongs to $M_{H_n}(r, \tilde{c}_1, c_2)$ for all $n\geq n_0$;*

(2) *If $\tilde{E}\in M_{H_n}(r, \tilde{c}_1, c_2)$ satisfies $\tilde{E}=\pi^*E$ for some bundle E on X, then E is H-stable;*

(3) *If H is $(r, \tilde{c}_1, c_2)$-suitable, then for any $\tilde{E}\in M_{H_n}(r, \tilde{c}_1, c_2)$, $(\pi_*\tilde{E})^{\vee\vee}$ is H-semistable. If furthermore $c_1.H\not\equiv 0$ (mod r') for any r' with $r'|r$, $(\pi_*\tilde{E})^{\vee\vee}$ is H-stable.*

Proof. Let E be a rank r vector bundle on X with $c_1(E)=c_1$, $c_2(E)=c_2$.

Assume that E is H-stable and let $F \subset \pi^* E$ be a subsheaf of rk $F = r'$. We write

$$c_1(F) = \pi^* M + \sum_{i=1}^{l} b_i E_i .$$

We choose m_0 so that

$$H_0 := m_0 \pi^* H - \sum_{i=1}^{l} E_i$$

becomes ample and for $n \geq m_0$ we get an ample divisor $H_n = (n - m_0)\pi^* H + H_0$.

We have to show the existence of n_0 such that $(r'\pi^* c_1 - r c_1(F)).H_n > 0$ for all $n \geq n_0$. If we put $\delta = (r'c_1 - rM).H$ and $n' = n - m_0$, this condition can be rewritten as

$$(*) \qquad n'\delta > (r c_1(F) - r'\pi^* c_1).H_0$$

for sufficiently large n'. Pushing down the inclusion $F \hookrightarrow \pi^* E$, we have $\pi_* F \hookrightarrow E$. Since E is H-stable, we conclude $\delta > 0$.

On the other hand, for fixed ample divisors H, H_0 we have

$$\sup\{c_1(F).H_0 | F \hookrightarrow \pi^* E,\ E \in MH(r, c_1, c_2)\} < \infty$$

since $M_H(r, c_1, c_2)$ is of finite type ([M]).

Therefore we conclude that there exists an integer n_0 depending only on the given invariants and ample divisors, such that $(*)$ holds for all $n' \geq n_0$. Thus (1) has been proved.

Next assume that $\tilde{E} = \pi^* E$ for some bundle E on X and led $F \hookrightarrow E$ be a subsheaf with rk$F = r'$. Then we have an inclusion $\pi^* F \hookrightarrow \tilde{E}$ and hence by H_n-stability of $\tilde{E}$,

$$\frac{c_1(F).H}{r'} = \frac{c_1(\pi^* F).H_n}{nr'} < \frac{c_1(\tilde{E}).H_n}{nr} = \frac{c_1(E).H}{r} .$$

This proves (2).

Finally assume that H is $(r, \tilde{c}_1, c_2)$-suitable and let n_0 be an integer such that every $\tilde{E} \in M_{Hn_0}(r, \tilde{c}_1, c_2)$ is H_n-stable for all $n \geq n_0$. Let $F \hookrightarrow (\pi_* \tilde{E})^{\vee\vee}$ be a coherent subsheaf of rank r'. Letting $\delta := (r'c_1 - r c_1(F)).H$, we shall show $\delta \geq 0$. We have a homomorphism

$$\pi^* F_{|U} \hookrightarrow \pi^* (\pi_* \tilde{E})^{\vee\vee}_{|U} \cong \tilde{E}_{|U}$$

where $U = \tilde{X} \setminus \cup E_i$. For sufficiently large integers b_i, the above morphism extends to an inclusion

$$\pi^* F \hookrightarrow \tilde{E}(\sum_{i=1}^{l} b_i E_i) .$$

If we let $Z=\sum_i b_i E_i$, we get a subsheaf $\pi^*F(-Z)$ of $\tilde{E}$. Since $\tilde{E}$ is H_n-stable for all $n\geq n_0$, we have

$$n\delta>(-r'\pi^*c_1+r\pi^*c_1(F)-r'Z).H_0\,.$$

It folows that $\delta\geq 0$ and hence $\pi_*\tilde{E}$ is H-semistiabe. If $c_1.H\not\equiv 0 \pmod{r'}$ for any $r'|r$, semistability implies stability and hence (3) follows.

Theorem 2.3. *For sufficiently large n, the pull back map defines an open immersion*

$$M_H(r,\,c_1,\,c_2)\hookrightarrow M_{H_n}(r,\,\tilde{c}_1,\,c_2)\,.$$

Proof. The above proposition shows that for sufficiently large n, there exists a morphism ϕ: $M_H(r,\,c_1,\,c_2)\to M_{Hn}(r,\,\tilde{c}_1,\,c_2)$ which is defined on closed points by the correspondence $E\mapsto\pi^*E$. This morphism is clearly injective. Moreover, by Lemma 5.8 in [FM] ϕ is also an open immersion. Therefore the theorem is proved.

Next we consider stable bundles E on $\tilde{X}$ with $c_1(E)=\hat{c}_1=\pi^*c_1+\sum_{i=1}^{l}a_iE_i$ where $1\leq a_i\leq r-1$. Let S be a scheme and let $\boldsymbol{E}$ be an S-family of rank r vector bundles E on X with $c_1(E)=c_1$, $c_2(E)=c_2$. We define a scheme $\boldsymbol{GE}$ over S as the following fibered product:

$$\boldsymbol{GE}=Gr(a_1,\,\boldsymbol{E})_{|x_1\times S}\times_S\cdots\times_S\times Gr(a_l,\,\boldsymbol{E})_{|x_l\times S}\,.$$

Here $G_r(a_i,\,\boldsymbol{E})$ is the Grassmann variety of quotient bundles with rank a_i of $\boldsymbol{E}^\vee$.

Let ψ: $\boldsymbol{GE}\to S$ and w_i: $\boldsymbol{GE}\to Gr(a_i,\,\boldsymbol{E})_{|x_i\times S}$ be the natural projections. On $\tilde{X}\times\boldsymbol{GE}$ we have the restriction map

$$(\pi\times\psi)^*\boldsymbol{E}^\vee\to\bigoplus_{i=1}^{l}(\pi\times\psi)^*\boldsymbol{E}^\vee_{|x_i\times S}\otimes\mathcal{O}_{E_i\times GE}\,.$$

Also we have the natural surjection

$$\bigoplus_{i=1}^{l}(\pi\times\psi)^*\boldsymbol{E}^\vee_{|x_i\times S}\otimes\mathcal{O}_{E_i\times GE}\to\bigoplus_{i=1}^{l}(\pi\times\psi_i)^*Q_{i|x_i\times S}\otimes\mathcal{O}_{E_i\times GE}$$

where Q_i is the universal quotient bundle for $Gr(a_i,\,\boldsymbol{E})$. Let $\boldsymbol{K}$ be the kernel of the composition of these maps. Then we obtain the following exact sequence on $\tilde{X}\times\boldsymbol{GE}$:

$$0\to\boldsymbol{K}\to(\pi\times\psi)^*\boldsymbol{E}^\vee\to\bigoplus_{i=1}^{l}(\pi\times\psi_i)^*Q_{i|x_i\times S}\otimes\mathcal{O}_{E_i\times GE}\to 0\,.$$

It is easy to see that $\boldsymbol{K}$ is locally free. Taking the dual of this sequence and using the isomorphism $\mathcal{E}xt^1((\pi\times\psi_i)^*Q_{i|x_i\times S}\otimes\mathcal{O}_{E_i\times GE},\ \mathcal{O}_{\tilde{X}\times GE})\cong(\pi\times\psi_i)^*$

$Q^{\vee}_{i|x_i\times S}\otimes\mathcal{O}_{E_i\times GE}$, we have the following exact sequence

$$0\to(\pi\times\psi)^*\boldsymbol{E}\to\hat{\boldsymbol{E}}\to\bigoplus_{i=1}^{l}(\pi\times\psi_i)^*Q^{\vee}_{i|x_i\times S}\otimes\mathcal{O}_{E_i\times GE}\to 0\,.$$

$\hat{\boldsymbol{E}}$ is a family of rank r bundles $\hat{E}$ on $\tilde{X}$ with $c_1(\hat{E})=\pi^*c_1+\sum_{i=1}^{l}a_iE_i$, $c_2(\hat{E})=c_2$ which are obtained from extensions of the form

$$0\to\pi^*E\to\hat{E}\to\bigoplus_{i=1}^{l}\mathcal{O}_{E_i}(-1)^{\oplus a_i}\to 0$$

where E is a member of S. Following [B], we call $\hat{\boldsymbol{E}}$ the standard family associated with S. Any rank r vector bundle $\hat{E}$ on $\tilde{X}$ with $c_1(\hat{E}).E_i\not\equiv 0$ (mod r) can be normalized so that $c_1(E)=\pi^*c_1+\sum_{i=1}^{l}a_iE_i(1\le a_i\le r-1)$ after tensoring by appropriate line bundles. We have

Lemma 2.4. *Let $\hat{E}$ be a normalized bundle. Then the following conditions are equivalent*

(1) *$\hat{E}$belongs to a standard family;*

(2) *$\hat{E}_{|E_i}\cong\mathcal{O}_{E_i}(-1)^{\oplus a_i}\oplus\mathcal{O}_{E_i}^{\oplus r-a_i}$ for $i=1,\cdots,l$.*

Proof. (1)$\to$(2): If $\hat{E}$ is a member of a standard family, there is a vector bundle E on X with $c_1(E)=c_1$, $c_2(E)=c_2$ and an exact sequence on $\tilde{X}$

$$(**)\qquad 0\to\pi^*E\to\hat{E}\to\bigoplus_{i=1}^{l}\mathcal{O}_{E_i}(-1)^{\oplus a_i}\to 0\,.$$

Restricting this sequence to E_i, we obtain

$$0\to\mathcal{T}or^1(\mathcal{O}_{E_i}(-1)^{\oplus a_i},\mathcal{O}_{E_i})\to\bigoplus_{i=1}^{r}\mathcal{O}_{E_i}\to\hat{E}_{|E_i}\to\mathcal{O}_{E_i}(-1)^{\oplus a_i}\to 0\,.$$

Since $\mathcal{T}or^1(\mathcal{O}_{E_i}(-1),\mathcal{O}_{E_i})\cong\mathcal{O}_{E_i}$, we obtain the sequence

$$0\to\mathcal{O}_{E_i}^{\oplus r-a_i}\to\hat{E}_{|E_i}\to\mathcal{O}_{E_i}(-1)^{\oplus a_i}\to 0\,.$$

Since the above sequence splits, it follows that $\hat{E}_{|E_i}\cong\mathcal{O}_{E_i}(-1)^{\oplus a_i}\oplus\mathcal{O}_{E_i}^{\oplus r-a_i}$.

(2)$\to$(1): Let $\tilde{E}$ be the kernel of the projection $\hat{E}\to\mathcal{O}_{E_i}(-1)^{\oplus a_i}$. Then we have

$$0\to\tilde{E}\to\hat{E}\to\mathcal{O}_{E_i}(-1)^{\oplus a_i}\to 0\,.$$

As before we see $\hat{E}_{|E_i}\cong\bigoplus_{i=1}^{r}\mathcal{O}_{E_i}$ for each i, so there is a vector bundle E on X such that $\tilde{E}=\pi^*E$. Pushing down this sequence to X, we have $E\cong\pi_*\hat{E}$ and hence dualizing $(**)$, we obtain

$$0\to\hat{E}^{\vee}\to\pi^*E^{\vee}\to\bigoplus_{i=1}^{l}\mathcal{O}_{E_i}^{\oplus a_i}\to 0\,.$$

Thus $\hat{E}$ belongs to a standard family.

The following can be proved similarly as in Proposition 2.2.

Propostion 2.5. *There exists an integer n_0 such that for $n \geq n_0$ the followin hold:*

(1) *For every $E \in M_H(r, c_1, c_2)$, $\hat{E}$ belongs to $M_{H_n}(r, \hat{c}_1, c_2)$;*

(2) *If H is $(r, \hat{c}_1, c_2)$-suitable and $\hat{E}$ is a normalized H_n-stable bundle, then $(\pi_* \hat{E})^{\vee\vee}$ is H-semistable. If we assume furthermore $c_1. H \not\equiv 0 (mod\ r')$ for any $r'|r$, thcen $(\pi_* \hat{E})^{\vee\vee}$ is H-stable.*

If $M_H(r, c_1, c_2)$ is a fine moduli space, we can construct a standard family GE associated with a universal family $\boldsymbol{E}$. Then we have the following

Theorem 2.6. *Suppose $M_H(r, c_1, c_2)$ has a universal amily $\boldsymbol{E}$. Then for sufficiently large n, we have an open immersion*

$$\hat{\phi}: GE(r, c_1, c_2) \hookrightarrow M_{H_n}(r, \hat{c}_1, c_2)\,.$$

Proof. In view of the above proposition, $\hat{\phi}$ is well defined and injective. The argument in te rank 2 case in [B] shows that this is an open immersion.

3. Generic smoothness of moduli spaces

In what follows we shall always assume char $k=0$. We study the images of ϕ and $\hat{\phi}$ defined in the previous section. In particular, we shall give some sufficient conditions for these images to be dense. These yield results concerning the generic smoothmess of the moduli space.

Definition. Let D be a divisor on a polarized surface (X, H). For given $c_1 \in \mathrm{Pic}\, X$ and an integer c_2, the moduli space $M_H(r, c_1, c_2)$ is called *D-good* if generic $E \in M_H(r, c_1, c_2)$ satisfies $H^2(\mathrm{ad}E(-D))=0$ where $\mathrm{ad}E$ denotes the adjoint bundle of E. If $D=0$, we simply say *good*. This condition is equivalent to saying that every component of $M_H(r, c_1, c_2)$ is generically reduced and has the expected dimension.

Recall that a rank r vector bundle $E = \bigoplus_{i=1}^{r} \mathcal{O}(a_i)$ $(a_1 \leq \cdots \leq a_r)$ on $\boldsymbol{P}^1$ is called *rigid* if $a_r - a_1 \leq 1$. The following follows from Proposition 2.2, Lemma 2.4 and Proposition 2.5.

Lemma 3.1. *For sufficiently large n, $\tilde{E} \in M_{H_n}(r, \tilde{c}_1, c_2)$ belongs to $Im\phi$ if and only if $\tilde{E}_{|E_i}$ is rigid for all i. The same result holds for $\hat{\phi}$ if H is $(r, \hat{c}_1, c_2)$-suitable and $c_1.H \not\equiv 0$ $(mod\ r')$ for any $r'|r$.*

Lemma 3.2. *Let X be an algebraic surface and let C be a smooth rational curve on X. Assume that a vector bundle E satisfies the conditions $c_1(E).C \leq$*

0 and $H^2(adE(-C))=0$. Then E can be deformed to a bundle which is rigid on C.

Proof. We note that for a bundle F on $\boldsymbol{P}^1$ with $\deg F \leq 0$, F is rigid if only if $F \cong \mathcal{O}(-1)^{\oplus a} \oplus \mathcal{O}^{\oplus b}$. Let $\mathcal{M}$ be the local moduli space of E. For a fixed F, let S_F be the subset of $\mathcal{M}$ consisting of E' such that $E'_{|C} \simeq F$. Then by the deformation theory of Brieskorn, if $\mathcal{M}$ induces a versal deformation of F, we have codim $S_F = h^1(\mathrm{End}\ F)$. Since the condition $H^2(\mathrm{ad}E(-C))=0$ implies that the restriction map $H^1(\mathrm{ad}E) \to H^1(\mathrm{ad}E_{|C})$ is surjective, $\mathcal{M}$ induces a versal deformation of $E_{|C}$. If we denote by S the subset of $\mathcal{M}$ consisting of bundles which restrict to nonrigid bundles on C, it is easy to see that codim $S>0$ and hence the claim is proved.

Let K_X and $K_{\tilde{X}}$ denote canonical divisors of X and $\tilde{X}$, respectively.

Proposition 3.3. *Assume that $K_X.H<0$. Then for sufficiently large n, Im ϕ is dense in $M_{H_n}(r, \tilde{c}_1, c_2)$. The same result holds for $\hat{\phi}$ if H is $(r, \hat{c}_1, c_2)$-suitable and $c_1.H \not\equiv 0$ (mod r') for any $r'|r$.*

Proof. We note that $H^0(K_{\tilde{X}}+E_i)=0$ for all i. So we have Hom $(E, E(K_{\tilde{X}}+E_i)) = H^0(\mathrm{End}\ E(K_{\tilde{X}}+E_i)) = H^0(\mathrm{ad}E(K_{\tilde{X}}+E_i))$. If $n \gg 0$, the assumption $K_X.H<0$ implies that $c_1(E(K_{\tilde{X}}+E_i)).H_n = c_1(E).H_n + h(nK_X.H+l+1) < c_1(E).H_n$, hence we have $H^2(\mathrm{ad}E(-E_i)) \cong H^0(\mathrm{ad}E(K_{\tilde{X}}+E_i))^\vee = 0$ by stability. Thus from Lemma 3.2 and the openness of stability, it follows that E can be deformed to an H_n-stable bundle which is rigid on every E_i. Therefore the proposition is an immediate consequence of Lemma 3.1.

Corollary 3.4. *$M_{H_n}(r, \tilde{c}_1, c_2)$ (resp. $M_{H_n}(r, \hat{c}_1, c_2)$) is good if and only if so is $M_H(r, c_1, c_2)$ under the same assumptions as in the above proposition.*

Proof. The first case is obvious. In the second case, the claim follows from the fact $\boldsymbol{GE}$ is a fibration over $M_H(r, c_1, c_2)$ whose fibers are the products of Grassmannians $Gr(a_i, r)$.

In the rank 2 case, following result is known ([O, Corollary 2.2]).

Proposition 3.5. *For fixed c_1 and D on a polarized surface (X, H), $M_H(2, c_1, c_2)$ is D-good for sufficiently large c_2.*

As an application of this, we obtain

Theorem 3.6. *Assume that n is sufficiently large. Then*
(1) *$M_{H_n}(2, \tilde{c}_1, c_2)$ is good for sufficiently large c_2;*
(2) *If $c_1.H$ is odd, $M_{H_n}(2, \hat{c}_1, c_2)$ is good for sufficiently large c_2.*

Proof. Let M denote either $M_{H_n}(2, \tilde{c}_1, c_2)$ or $M_{H_n}(2, \hat{c}_1, c_2)$. Applying Proposition 3.5 to $\tilde{X}$ and the exceptional divisors E_i, we see that for every i

and for generic $E \in M$, $h^2(\mathrm{ad}E(-E_i))=0$ for $c_2 \gg 0$. On the other hand, the same proposition applied to X and $D=0$ shows that $M_H(2, c_1, c_2)$ is good for $c_2 \gg 0$. Therefore the claim follows as in Corollary 3.4.

Let $\Delta=2rc_2-(r-1)c_1^2$(resp$\hat{\Delta}=2rc_2-(r-1)(\pi^* c_1+\sum_{i=1}^{l} a_i E_i)^2$) be the discriminant of a bundle E with $c_1(E)=c_1$, $c_2(E)=c_2$ (resp. a bundle $\hat{E}$ with $c_1(\hat{E})=\pi^* c_1+\sum_{i=1}^{l} a_i E_i$, $c_2(\hat{E})=c_2$).

Lemma 3.7. *Assume that H is $(r, \hat{c}_1, c_2)$-suitable and $\hat{\Delta} \leq (r-1)\sum_{i=1}^{l} a_i^2 +2r-1$. Then for sufficiently large n and $\hat{E} \in M_{H_n}(r, \hat{c}_1, c_2)$, $\pi_* \hat{E}$ is locally free and $R^1\pi_*\hat{E}=0$.*

Proof. Both $(\pi_*\hat{E})^{\vee\vee}/\pi_*\hat{E}$ and $R^1\pi_*\hat{E}$ are torsion sheaves supported by finite sets of points. Let $l_1=$length $((\pi_*\hat{E})^{\vee\vee}/\pi_*\hat{E})$ and $l_2=$length $(R^1\pi_*\hat{E})$. It is easy to see that $c_1((\pi_*\hat{E})^{\vee\vee})=c_1$. By the Riemann-Roch theorem, we have

$$\begin{aligned}\mathrm{ch}((\pi_*\hat{E})^{\vee\vee}) &= r+c_1+\frac{1}{2}(c_1^2-2c_2((\pi_*\hat{E})^{\vee\vee})) \\ &= r+c_1+\frac{1}{2}(c_1^2-2(c_2-l_1-l_2)) .\end{aligned}$$

Therefore $c_2((\pi_*\hat{E})^{\vee\vee})=c_2-l_1-l_2$ and hence we obtain

$$\Delta((\pi_*\hat{E})^{\vee\vee})=\hat{\Delta}-(r-1)\sum_{i=1}^{l} a_i^2-2r(l_1+l_2) .$$

By Proposition 2.5, $(\pi_*\hat{E})^{\vee\vee}$ is H-semistable for sufficiently large n. Since Bogomolov's inequality implies $\Delta((\pi_*\hat{E})^{\vee\vee})\geq 0$, it follows that if $\hat{\Delta}\leq(r-1)\sum_{i=1}^{l} a_i^2+2r-1$, then we have $l_1=l_2=0$ and the proof is complete.

As a corollary to the proof of the above lemma, we obtain the following sharpened Bogomolov's inequality for normalized H_n-stable bundles on $\tilde{X}$ (cf. [B, Theorem 10]).

Proposition 3.8. *Let H and $\hat{E}$ be as above. Then we have*

$$\hat{\Delta}\geq(r-1)\sum_{i=1}^{l} a_i^2 .$$

Finally we obtain the following criterion for $\hat{\phi}$ to be an isomorphism.

Proposition 3.9. *Assume that H is $(r, \hat{c}_1, c_2)$-sutable for $\hat{c}_1=\pi^* c_1+\sum_{i=1}^{l}(r-1)E_i$. If $l(r-1)^3\leq\hat{\Delta}\leq l(r-1)^3+2r-1$ and $c_1.H\not\equiv 0$ (mod r') for any $r'|r$, then $\hat{\phi}$ is an isomorphism.*

Proof. Let $\hat{E}$ be a member of $M_{H_n}(r, \tilde{c}_1, c_2)$. For each $1 \leq i \leq l$, let $\hat{E}_{|E_i} \cong \bigoplus_{j=1}^{r} \mathcal{O}(a_{ij})$, $(a_{i1} \leq \cdots \leq a_{ir})$. Consider the following exact sequence

$$0 \to \tilde{E} \to \hat{E} \to \mathcal{O}_{E_i}(a_{i1}) \to 0\,.$$

By our assumption and Lemma 3.7,

$$R^1 \pi_* \mathcal{O}_{E_i}(a_{i1}) = H^1(\boldsymbol{P}^1, \mathcal{O}(a_{i1})) \otimes \mathcal{O}_{p_i} = 0\,.$$

Thus we obtain $a_{i1} \geq -1$. Then it can be easily seen that $\hat{E}_{|E_i}$ are rigid for all i. By assumption and Proposition 2.5, it follows that $\hat{E}$ belongs to Im $\hat{\phi}$. Thus we conclude that $\hat{\phi}$ is surjective, hence an isomorphism. This completes the proof.

As an example, we shall consider the moduli of bundles on the blown-up projective plane. We recall that a coherent sheaf E on a smooth projective surface X is called exceptional if $\mathrm{Ext}^1(E, E) = 0$. If X is $\boldsymbol{P}^2$ and H is $\mathcal{O}(1)$, then an H- stable torsion free sheaf E is exceptional if and only if $\Delta(E) = r^2 - 1$. Such sheaves have been studied extensively by Drezet and Le Potier (cf. [DL]). They proved that every stable exceptional sheaves are locally free and they are determined up to isomorphism by their slopes $\mu = c_1/r$. Let $\mathcal{S}$ be the set of rational numbers which are slopes of stable exceptional bundles. Theorem A in [DL] gives the complete description of $\mathcal{S}$.

By Theorem 2.3, Theorem 2.6 and Proposition 3.3, we obtain

Proposition 3.10. *Let $\tilde{X}$ be the blow up of $\boldsymbol{P}^2$ at l distinct points and let E_i be the exceptional divisors. If $c_1/r \in \mathcal{S}$ and $1 \leq a_i \leq r-1$, then for sufficiently large n we have*

(1) *$M_{H_n}(r, \tilde{c}_1, c_2)$ is a reduced one point;*

(2) *$M_{H_n}(r, \tilde{c}_1, c_2)$ has a component isomorphic to $\prod_{i=1}^{l} Gr(a_i, r)$.*

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