J. Math. Kyoto Univ. (JMKYAZ)
34-2 (1994) 475-493

The absence of absolutely continuous spectra of 1-dimensional Schrödinger operators

By

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1. Introduction

In [8] B. Simon and T. Spencer have studied Jacobi matrices and Schrödinger operators. They have shown that under some conditions these operators have no absolutely continuous spectrum. Also they have shown that under other conditions there is no negative absolutely continuous spectrum of Jacobi matrices. In this paper we apply these ideas to show that under some conditions there is no negative absolutely continuous spectrum of Schrödinger operators (only in case of one dimension).

In Section 2, we give the main result of this paper. The potential of Schrödinger operator is characterized by having an inferior function which has enumerably infinitely many wells. Roughly speaking, we prove that Schrödinger operator has no negative absolutely continuous spectrum, under the conditions that the distances between wells diverge to infinity faster than logarithmic order.

In Section 3, we show the exponential estimate of spectral projection $P_{(a,b)}(H)$ of Schrödinger operator H for any $a<b<0$, $a, b \notin \sigma_{\mathrm{PP}}(H)$. This fact is used to show the trace class perturbation in Section 4.

2. Notation and Main Theorem

Definition 2.1. Let T be a compact operator from a Hilbert space H to a Hilbert space H', and let $\alpha_n(T)$ be eigen values of operator $|T|=\sqrt{T^*T}$. Then

$$\mathbf{B}_1(\mathsf{H}, \mathsf{H}')=\{T : \sum_n \alpha_n(T)<\infty\}$$

is called trace class and its norm is

$$\|T\|_{\mathbf{B}_1(\mathsf{H}, \mathsf{H}')}=\sum_n \alpha_n(T).$$

Notation. In this paper we use

Received July 6, 1993

1. $L^2=L^2(\mathbf{R})$, $H^2=H^2(\mathbf{R})$,
2. $\|\cdot\|=\|\cdot\|_{L^2}$,
3. $\|\cdot\|_0=\|\cdot\|_{\mathbf{B}(L^2,L^2)}$,
4. $\|\cdot\|_1=\|\cdot\|_{\mathbf{B}_1(L^2,L^2)}$,
5. $\|\cdot\|_\infty=\|\cdot\|_{L^\infty}$,
6. for $A, B\subset\mathbf{R}$, $\mathrm{d}(A,B)=\inf\{|x-y|;\, x\in A,\, y\in B\}$,

for convenience' sake.

For the subset S of $\mathbf{R}$, $\chi_S(x)$ represents the characteristic function, that is

$$\chi_S(x)=\begin{cases} 1 & x\in S \\ 0 & \textit{otherwise.}\end{cases}$$

In this paper we deal with 1-dimensional Schrödinger operaters which have bounded potentials and are essential self-adjoint in L^2. We use the same symbol for both a differential operator and its self-adjoint extension.

Here we intoduce the special potential V_0. For the sequence $\{x_i, y_i\in\mathbf{R};\, i\in\mathbf{Z}\}$ such that

$$-\infty\leftarrow\cdots<x_{i-1}<y_{i-1}<x_i<y_i<\cdots\rightarrow\infty,$$

we use such notations as

$$z_i=(y_i+x_{i+1})/2,\qquad d_i=x_{i+1}-y_i,\qquad w_i=y_i-x_i,$$
$$d_{i*}=\min\{d_{i-1}, d_i\},\qquad d_i^*=\max\{d_{i-1}, d_i\},$$
$$I_i=[z_{i-1}, z_i),\qquad \chi_i(x)=\chi_{I_i}(x),$$

and for $v>0$,

$$V_{oi}(x)=-v\chi_{[x_i,y_i]}(x)$$
$$V_o(x)=\sum_{i\in Z}V_{oi}(x)$$

Theorem 2.2. *Let* $H=-\dfrac{d^2}{dx^2}+V(x)$ *be a self-adjoint operator on* L^2. *Suppose* $V(x)$ *satisfies following conditions.*

(a) *V is a real L^∞ function and $-v\leq V$ a. e. on R.*

(b) *There exist bounded intervals $J_i=(y_i, x_{i+1})$ such that for all $i\in\mathbf{Z}$. $x_i<y_i$ and $V(x)\geq 0$ a. e. on J_i*

(c) *d_i and $w_i\,(i\in\mathbf{Z})$ satisfy next two assumptions.*

(A.1) $\inf_{i\in Z} d_i\equiv d>0$ *exists.*

(A.2) *For any $\varepsilon>0$*

$$\sum_{i\in Z}(w_i+d_i^*)\exp(-\varepsilon d_{i*})<\infty.$$

Then

$$\sigma_{ac}(H)\cap(-\infty,0)=\emptyset$$

where $\sigma_{ac}(H)$ *is the absolutely continuous spectrum of* H.

Remark. Condition (b) and condition (c) are equivalent to the next condition.
(d) There exists the special potential V_o that $V\geq V_o$ *a. e.* and that V_o satisfies the assumptions (A.1) and (A.2).

Example.

$$V(x)=\sin\sqrt{|x|}$$

$$V_o(x)=\begin{cases}-1 & (2n+1)^2\pi^2\leq|x|\leq(2n+2)^2\pi^2\\ 0 & (2n)^2\pi^2\leq|x|\leq(2n+1)^2\pi^2\end{cases}\quad n=0,1,2,\cdots$$

$V_o(x)$ satisfies the assumptions (A1) and (A.2), then the operator $-\dfrac{d^2}{dx^2}+V(x)$ has no negative absolutely continuous spectrum.

Notation. Set $\lambda=\sqrt{v+1}$. For any operator A we denote the operator $A+\lambda^2$ by A_λ.

3. Exponential estimate of spectral projections

In this section we will prove the property of the spectral projection which we need to prove Lemma 4.1. To this end we use some lemmas shown next.

Lemma 3.1. ([4] p.290 (b)). *Let* $\{A_n\}_{n=1}^{\infty}$ *and* A *be self-adjoint operators on* H *and suppose that* $A_n\to A$ *in the strong resolvent sense. Then*
If $a, b\in\mathbf{R}$. $a<b$ *and* $a, b\notin\sigma_{PP}(A)$, *then* $P_{(a,b)}(A_n)\varphi\to P_{(a,b)}(A)\varphi$ *for all* $\varphi\in\mathsf{H}$.

Lemma 3.2. *For any bounded interval* $I=(x_0, y_0)$, *let* $w_i(x)$ $(i=1,2)$ *be continuous functions on* I *with* $0<w_1\leq w_2$. *For real constants* a, b_1, b_2 *with* $a\neq 0$ *and* $a(b_1-b_2)\leq 0$, *let* $u_i(i=1,2)$ *be the non-trivial real solutions of equations*

$$\begin{cases}\dfrac{d^2}{dx^2}u_i(x)=w_i^2(x)u_i(x) \text{ on } I\\ u_i(x_0)=a\\ u_i'(x_0)=b_i.\end{cases}$$

Then

$$(1) \qquad \begin{cases} 0 \leq u_1^{(n)} \leq u_2^{(n)} \\ \text{or } 0 \geq u_1^{(n)} \geq u_2^{(n)} \end{cases} \qquad (n=0,\ 1,\ 2)$$

to the point where $u_1=0$ or $u_2=0$, *where the suffices* n *represent* n *times derivatives.*

Remark. Let u be a non trivial-solution of equation

$$\frac{d^2}{dx^2}u(x)=q(x)u(x) \text{ on an interval } I$$

where q is a positive real continuous function.

Then number of elements of $(q(0)^{-1} \cup q'(0)^{-1}) \cap I$ is at most one.

Proof. If $u_1u_2 \geq 0$, then $(u_1'u_2-u_1u_2')'=u_1''u_2-u_1u_2''=(w_1^2-w_2^2)u_1u_2 \leq 0$, and the Wronskian $u_1'u_2-u_1u_2'$ is non-increasing. Its value at $x=x_0$ is non-positive. Hence

$$(u_1/u_2)'=(u_1'u_2-u_1u_2')/u_2^2 \leq 0$$

so that

$$u_1(x)/u_2(x) \leq u_1(x_0)/u_2(x_0)=1.$$

Then part of inequalities (1) at the case $n=0$ hold. The other cases follow from the equations.

Lemma 3.3. *Let* $C(x)$ *be a bounded continuous function with* $0<c_1 \leq C(x)$. *And let* $A_C=-\frac{d^2}{dx^2}+C(x)$ *be a self-adjoint operator on* L_2. *For any* a_1, b_1, a_2, $b_2 \in \mathbf{R}$ *with* $a_1<b_1<a_2<b_2$, *set*

$$\chi_1(x)=\chi_{[a_1,b_1]}(x), \quad \chi_2(x)=\chi_{[a_2,b_2]}(x).$$

Then

$$(2) \qquad |(\chi_1 g_1, A_C^{-1}\chi_2 g_2)| \leq C_1 \exp\{-\sqrt{c_1}(a_2-b_1)\}\|\chi_1 g_1\|\,\|\chi_2 g_2\|$$

for all g_1, $g_2 \in L^2$ *where* C_1 *is some positive constant depending only on* c_1 *and* $C_1 \nearrow \infty$ *as* $c_1 \searrow 0$.

Proof. Let $u_\pm$ be non-trivial real independent solutions of the equation

$$\left(-\frac{d^2}{dx^2}+C(x)\right)u=0$$

with the conditions

$$u_+(a_2)=\exp(-\sqrt{c_1}a_2),\ u'_+(a_2)\leq-\sqrt{c_1}\exp(-\sqrt{c_1}a_2)$$
$$u_-(b_1)=\exp(-\sqrt{c_1}b_1),\ u'_-(b_1)\geq-\sqrt{c_1}\exp(-\sqrt{c_1}b_1)$$

Then by Lemma 3.2 and its remark,

$$\begin{cases} u_+(x)\geq e^{-\sqrt{c_1}x} & (x\leq a_2) \quad u_+(x)\leq e^{-\sqrt{c_1}x} \quad (x>a_2) \\ u_-(x)\geq e^{\sqrt{c_1}x} & (x\geq b_1) \quad u_-(x)\leq e^{\sqrt{c_1}x} \quad (x<b_1) \\ u'_+(x)\geq-\sqrt{c_1}e^{-\sqrt{c_1}x} & (x\leq a_2) \quad u'_-(x)\geq\sqrt{c_1}e^{\sqrt{c_1}x} \quad (x\geq b_1) \end{cases}$$

Then integral kernel of operator A_C is

$$E(x,\xi)=\frac{1}{\Delta}\begin{cases} u_+(x)u_-(\xi) & (\xi\leq x) \\ u_-(x)u_+(\xi) & (\xi>x) \end{cases}$$

where $\Delta=u'_-(x)u_+(x)-u'_+(x)u_-(x)\geq 2\sqrt{c_1}$

Then

$$\begin{aligned}(A_C^{-1}\chi_2 g_2)(x)&=\int_{-\infty}^{\infty}E(x,\xi)\chi_2(\xi)g_2(\xi)d\xi \\ &=\frac{u_+(x)}{\Delta}\int_{-\infty}^{x}u_-(\xi)\chi_2(\xi)g_2(\xi)d\xi \\ &+\frac{u_-(x)}{\Delta}\int_{x}^{\infty}u_+(\xi)\chi_2(\xi)g_2(\xi)d\xi.\end{aligned}$$

Substituting this in the left hand side of (2), we get

$$\begin{aligned}|(\chi_1 g_1, A_C^{-1}\chi_2 g_2)&=\frac{1}{\Delta}\left|\int_{a_1}^{b_1}u_-(x)\chi_1(x)g_1(x)dx\cdot\int_{a_2}^{b_2}u_+(\xi)\chi_2(\xi)g_2(\xi)d\xi\right| \\ &\leq\frac{1}{2\sqrt{c_1}}\left\{\int_{a_1}^{b_1}u_-(x)dx\right\}^{\frac{1}{2}}\|\chi_1 g_1\|\left\{\int_{a_2}^{b_2}u_+(\xi)d\xi\right\}^{\frac{1}{2}}\|\chi_2 g_2\| \\ &\leq\frac{1}{2\sqrt{c_1}}\left\{\int_{a_1}^{b_1}\exp(2\sqrt{c_1}x)dx\right\}^{\frac{1}{2}}\|\chi_1 g_1\|\left\{\int_{a_2}^{b_2}\exp(-2\sqrt{c_1}\xi)\right. \\ &\qquad\left.d\xi\right\}^{\frac{1}{2}}\|\chi_2 g_2\| \\ &=\frac{\|\chi_1 g_1\|\,\|\chi_2 g_2\|}{4c_1}\{\exp(2\sqrt{c_1}b_1)-\exp(2\sqrt{c_1}a_1)\}^{\frac{1}{2}} \\ &\quad\times\{\exp(-2\sqrt{c_1}a_2)-\exp(-2\sqrt{c_1}b_2)\}^{\frac{1}{2}} \\ &=\frac{\|\chi_1 g_1\|\,\|\chi_2 g_2\|}{4c_1}\exp\{-\sqrt{c_1}(a_2-b_1)\} \\ &\quad\times[1-\exp\{-2\sqrt{c_1}(b_1-a_1)\}]^{\frac{1}{2}}\ [1-\exp\{-2\sqrt{c_1}\end{aligned}$$

$$(b_2-a_2)\}]^{\frac{1}{2}}$$
$$\leq C_1\|\chi_1 g_1\|\ \|\chi_2 g_2\|\exp\{-\sqrt{c_1}(a_2-b_1)\}$$

where C_1 is more than or equal to $1/4c_1$. So the lemma is proved.

Lemma 3.4. *let we (x) be a continuous function on an interval* $I=[-x_0, x_0]$ $(x_0>0)$. *Set* $w_1=\inf_{x\in I} w(x)$, $w_2=\sup_{x\in I} w(x)$. *For real constants a,b with* $ab>0$, *let u be a real solution of the equation*

$$\begin{cases} \dfrac{d^2}{dx^2}u(x)=w(x)u(x) & \text{on } I \\ u(-x_0)=a \\ u'(-x_0)=b. \end{cases}$$

Then

$$|u'(x)|\leq C_2|u(x)| \quad \text{on} \quad [0, x_0] \tag{3}$$

$$\int_0^{x_0} u^2(x)dx\leq C_3^2\left(\int_0^{x_0} u(x)dx\right)^2 \tag{4}$$

where C_2, C_3 *are positive numbers depending only on* w_1, w_2 *and* x_0.

Remark. When $ab<0$ and boundary conditions are given at the point $x=x_0$, we can show the same estimates in the interval $[-x_0, 0]$.

Proof. Set u_i $(i=1, 2)$ be real solutions of the equations

$$\begin{cases} \dfrac{d^2}{dx^2}u_i(x)=w_i u_i(x) & \text{on } I \\ u_i(-x_0)=a \\ u_i'(-x_0)=b. \end{cases}$$

Then

$$u_i(x)=a\cosh w_i(x+x_0)+b\sinh w_i(x+x_0)\ (i=1, 2)$$

By Lemma 3.2, for any $x\in I$

$$\begin{aligned} \frac{u'(x)}{u(x)} &\leq \frac{u_2'(x)}{u_1(x)} \\ &\leq \frac{w_2(a\sinh 2w_2x_0+b\cosh 2w_2x_0)}{a\cosh w_1x_0+b\sinh w_1x_0} \\ &= \frac{w_2(a'\sinh 2w_2x_0+b'\cosh 2w_2x_0)}{a'\cosh w_1x_0+b'\sinh w_1x_0} \end{aligned} \tag{5}$$

where $a'=a/(a^2+b^2)$ and $b'=b/(a^2+b^2)$. Since the denominator of (5) is not zero when $(a', b')\in S^1$, (5) has supremum which is independent of a and b. Then (3) is proved.

Also

$$\int_0^{x_0} u^2(x)\,dx \leq \int_0^{x_0} u_2^2(x)\,dx \tag{6}$$

$$\left(\int_0^{x_0} u_1(x)\,dx\right)^2 \leq \left(\int_0^{x_0} u(x)\,dx\right)^2. \tag{7}$$

The right hand side of (6) and left hand side of (7) are both homogeneous polynomials of order 2 w. r. t. a and b, and the latter is not zero when $ab>0$. So the ratio of them has supremum which is independent of a and b. Then we can complete the proof.

Lemma 3.5. *Let $\rho_\varepsilon *$ be the mollifier. Then for any $u\in L^\infty$ and for any $f\in L^2$,*

$$\|(\rho_\varepsilon * u-u)f\|\to 0 \quad (as\ \varepsilon\to 0)$$

Lemma 3.6. *Let $H=-\dfrac{d^2}{dx^2}+V(x)$ be a self-adjoint operator on L^2 and let $V(x)$ satisfy the next conditions.*

(a) *$V(x)$ is a real L^∞ function and $u_{\min}\leq V(x)\leq u_{\max}$ a. e. on $\mathbf{R}$.*

(b) *There exists a bounded interval $I=(x', y')$ such that $V(x)\geq 0$ a.e. on I.*

Then for all a, b with $a<b<0$ and a, $b\notin\sigma_{\mathrm{PP}}(H)$, for any positive constant δ_0 with $16\delta_0<|I|$ and for a subinterval $J=[x'', y'']$ of I with $\delta=d(J, I^c)>8\delta_0$, there exists $C>0$ such that

$$\|\chi_J P_{(a,b)}(H)\|_0 \leq C\exp\left(-\frac{\sqrt{-b}\,\delta}{2}\right)$$

where C depends only on b, $v_{\min}$, $v_{\max}$, δ_0.

Remark. When I is unbounded, we can show the same estimate of any bounded interval I' in I. Then $I'\to I$. We can show the unbounded case. For example if $I=\mathbf{R}$, the estimate shows that $P_{(a,b)}(H)=0$. It means that there is no negative spectrum.

Proof. Set $V_L(x)=\chi_{(-L,L)}(x)\,V(x)$ for $L>0$ and set $V_{L\varepsilon}(x)=\rho_\varepsilon * V_L(x)$ where $\rho_\varepsilon *$ is the mollifier. Then $V_{L\varepsilon}$ is non-negative and continuous on the interval $(x'+\varepsilon, y'-\varepsilon)$. And set $H_L=-\dfrac{d^2}{dx^2}+V_L$ and $H_{L\varepsilon}-\dfrac{d^2}{dx^2}+V_{L\varepsilon}$. By the resolvent equation, for all $f\in L^2$

$$(H_L-\sqrt{-1})^{-1}f-(H-\sqrt{-1})^{-1}f$$
$$=(H_L-\sqrt{-1})^{-1}\{\chi_{(-\infty,-L]}+\chi_{[L,\infty)}\}V(H-\sqrt{-1})^{-1}f,$$
$$(H_{L\varepsilon}-\sqrt{-1})^{-1}f-(H_L-\sqrt{-1})^{-1}f$$
$$=(H_{L\varepsilon}-\sqrt{-1})^{-1}(V_L-V_{L\varepsilon})(H_L-\sqrt{-1})^{-1}f.$$

Since $(H-\sqrt{-1})^{-1}f\in H^2$ and $\|(H_L-\sqrt{-1})^{-1}\|_0\leq 1$, then

$$\|(H_L-\sqrt{-1})^{-1}\{\chi_{(-\infty,-L]}+\chi_{[L,\infty)}\}V(H-\sqrt{-1})^{-1}f\|$$
$$\leq\|(H_L-\sqrt{-1}^{-1})\|_0\,\|\{\chi_{(-\infty,-L]}+\chi_{[L,\infty)}\}V(H-\sqrt{-1})^{-1}f\|\to 0\ (\text{as } L\to\infty).$$

$(H_L-\sqrt{-1})^{-1}f\in H^2$ and $\|(H_{L\varepsilon}-\sqrt{-1})^{-1}\|_0\leq 1$, by Lemma 3.5,

$$\|(H_{L\varepsilon}-\sqrt{-1})^{-1}(V_L-V_{L\varepsilon})(H_L-\sqrt{-1})^{-1}f\|$$
$$\leq\|(H_{L\varepsilon}-\sqrt{-1})^{-1}\|_0\,\|(V_L-V_{L\varepsilon})(H-\sqrt{-1})^{-1}f\|\to 0\ (\text{as } \varepsilon\to 0).$$

So $H_L\to H$ in the strong resolvent sense and $H_{L\varepsilon}\to H_L$ in the strong resolvent sense for all $L>0$. By Lemma 3.1, for all $f\in L_2$,

$$\|P_{(a,b)}(H_L)f-P_{(a,b)}(H)f\|\to 0\ (\text{as } L\to\infty)$$
$$\|P_{(a,b)}(H_{L\varepsilon})f-P_{(a,b)}(H_L)f\|\to 0\ (\text{as } \varepsilon\to\infty)$$

then

$$\|\chi_J P_{(a,b)}(H_L)f-\chi_J P_{(a,b)}(H)f\|\to 0\ (\text{as } L\to\infty)$$
$$\|\chi_J P_{(a,b)}(H_{L\varepsilon})f-\chi_J P_{(a,b)}(H_L)f\|\to 0\ (\text{as } \varepsilon\to\infty)$$

Therefore it is enough to show that for all $f\in L^2$ for ε sufficiently small,

$$\|\chi_J P_{(a,b)}(H_{L\varepsilon})f\|\leq C\exp\left(-\frac{\sqrt{-b}\,\delta}{2}\right)\|f\|. \tag{8}$$

Here ε is fixed to satisfy $8(\delta_0+\varepsilon)\leq\delta$. Since the potential function of operator $H_{L\varepsilon}$ has compact support, the negative spectrum of $H_{L\varepsilon}$ is discrete, and the range of $P_{(a,b)}(H_{L\varepsilon})$ has finite dimension. So let $\{\varphi_n\}_{n=1}^{\infty}$ be a complete orthonomal system in L^2 such that $H_{L\varepsilon}\varphi_n=\lambda_n\varphi_n$ for $1\leq n\leq N_L$ and $\max(a, -u_{\min})\leq\lambda_n<b$, where N_L is the dimension of Range$\{P_{(a,b)}(H_{L\varepsilon})\}$.

For any $f\in L^2$

$$P_{(a,b)}(H_{L\varepsilon})f=\sum_{\lambda n\in(a,b)}(f,\varphi_n)\varphi_n$$
$$=\sum_{n=1}^{N_L}(f,\varphi_n)\varphi_n$$

Then

$$\begin{aligned}\|\chi_J P_{(a,b)}(H_{L\varepsilon})f\| &= \|\sum_{n=1}^{N_L}(f,\ \varphi_n)\chi_J\varphi_n\| \\ &\leq \sum_{n=1}^{N_L}|(f,\ \varphi_n)|\ \|\chi_J\varphi_n\| \\ &\leq \Big(\sum_{n=1}^{N_L}|(f,\ \varphi_n)^2|\Big)^{\frac{1}{2}}\Big(\sum_{n=1}^{N_L}\|\chi_J\varphi_n\|^2\Big)^{\frac{1}{2}} \\ &\leq \|f\|\Big(\sum_{n=1}^{N_L}\|\chi_J\varphi_n\|^2\Big)^{\frac{1}{2}}. \qquad (9)\end{aligned}$$

Here we define cut-off function $\widetilde{\varphi}_n$ of φ_n to apply Lemma 3.3 to (9). Considering the remark of Lemma 3.2, there is at most one point x_n in the set $(\varphi_n^{-1}(0)\cup\varphi_n'(0))\cap I$.

Set

$$(a_n^-,\ a_n^+)=\begin{cases}\left(x'+\frac{\delta}{2},\ y'-\frac{\delta}{2}\right) & \text{if } x_n\in\left(x',\ x'+\frac{\delta}{4}\right)\cup\left((y'-\frac{\delta}{4},\ y'\right) \quad \text{(case 1)}\\ \left(x'+\frac{\delta}{8},\ y'-\frac{\delta}{8}\right) & \text{if } x_n\in\left[x'+\frac{\delta}{4},\ y'-\frac{\delta}{4}\right] \text{ or } x_n \text{ does not exist.} \quad \text{(case 2)}\end{cases}$$

Let $\psi_n(x)\in C_0^2(\mathbf{R})$ have the properties that

(i) $0\leq\psi_n(x)\leq 1$

(ii) $\mathrm{supp}\psi_n(x)=[a_n^--\delta_0,\ a_n^++\delta_0]$

(iii) $\psi_n(x)=1$ on $[\alpha_n^-,\ a_n^+]$

(iv) $|\psi_n'(x)|\leq\frac{c}{\delta_0}$ $\quad |\psi_n'(x)|\leq\frac{c}{\delta_0^2}$ for some $c>0$.

Set $\widetilde{\varphi}_n(x)=\psi_n(x)\varphi_n(x)$. Set $\widetilde{A}_n=A_{V_{L\varepsilon}-\lambda_n}$ in Lemma 3.3,

$$\begin{aligned}\widetilde{A}_n\widetilde{\varphi}_n &= -\psi_n''\varphi_n-2\psi_n'\varphi_n'-\psi_n\varphi_n''+(V_{L\varepsilon}-\lambda_n)\psi_n\varphi_n \\ &= -\psi_n''\varphi_n-2\psi_n'\varphi_n'+\psi_n(H_{L\varepsilon}-\lambda_n)\varphi_n \\ &= -\psi_n''\varphi_n-2\psi_n'\varphi_n'\end{aligned}$$

where the last equality follows from the fact that $(H_{L\varepsilon}-\lambda_n)\varphi_n=0$.

Since $\mathrm{supp}\,(\widetilde{A}_n\widetilde{\varphi}_n)\subset[a_n^--\delta_0,\ a_n^-]\cup[a_n^+,\ a_n^++\delta_0]$ and $d\,(\mathrm{supp}\,(\widetilde{A}_n\widetilde{\varphi}_n),\ \chi_J)\geq\delta/2$, setting $C_1=-1/4b\,(>-1/4\lambda_n)$ and applying Lemma 3.3., we get

$$\begin{aligned}\|\chi_J\varphi_n\|^2 &= (\chi_J\varphi_n,\ \chi_J\varphi_n)=(\varphi_n,\ \chi_J\varphi_n) \\ &= (\widetilde{\varphi}_n,\ \chi_J\varphi_n)=(\widetilde{A}_n^{-1}\widetilde{A}_n\widetilde{\varphi}_n,\ \chi_J\varphi_n)\end{aligned}$$

$$\leq C_1 \exp\left(-\frac{\sqrt{-\lambda_n}\,\delta}{2}\right)\|\widetilde{A}_n \widetilde{\varphi}_n\| \, \|\chi_J \varphi_n\| . \tag{10}$$

Set

$$\chi_1^- (x) = \chi_{(x'+\frac{\delta}{2}-\delta_0, x'+\frac{\delta}{2})} (x)$$

$$\chi_1^+ (x) = \chi_{(y'-\frac{\delta}{2}, y'-\frac{\delta}{2}+\delta_0)} (x)$$

$$\chi_2^- (x) = \chi_{(x'+\frac{\delta}{8}-\delta_0, x'+\frac{\delta}{8})} (x)$$

$$\chi_2^+ (x) = \chi_{(y'-\frac{\delta}{8}, y'-\frac{\delta}{8}+\delta_0)} (x)$$

Then

$$\begin{aligned}
\|\widetilde{A}_n \widetilde{\varphi}_n\| &= \|\psi_n'' + 2\psi_n' \varphi_n''\| \\
&\leq \|\psi_n''\|_\infty \{\|\chi_i^- \varphi_n\| + \|\chi_i^+ \varphi_n\|\} + 2\|\psi_n'\|_\infty \{\|\chi_i^- \varphi_n'\| + \|\chi_i^+ \varphi_n'\|\} \\
&\leq C_4 \{\|\chi_i^- \varphi_n\| + \|\chi_i^+ \varphi_n\|\} \qquad (11)
\end{aligned}$$

where i is decided as the case i. The last inequality follows from Lemma 3.4. (3) and $C_4 = c(1+2\delta_0 C_2)/\delta_0^2$.

The potential of $H_{L\varepsilon} - \lambda_n$ in positive continous on the interval $[a_n^- - 2\delta_0, a_n^- + \delta_0]$ (or $[a_n^+ - \delta_0, a_n^+ + 2\delta_0]$) and x_n is not in there for all n. So we can apply Lemma 3.4 and get

$$\|\chi_i^{\pm} \varphi_n\| \leq |(\chi_i^{\pm}, \varphi_n)| \quad (i=1, 2) \tag{12}$$

By combining (10), (11) and (12), we get

$$\begin{aligned}
\sum_{n=1}^{N_L} \|\chi_J \varphi_n\|^2 &\leq \sum_{n=1}^{N_L} C_1^2 C_4^2 \exp(-\sqrt{-b}\,\delta) \{\|\chi_i^- \varphi_n\| + \|\chi_i^+ \varphi_n\|\}^2 \qquad (i=1,2) \\
&\leq 2C_1^2 C_4^2 \exp(-\sqrt{-b}\,\delta) \sum_{n=1}^{N_L} \{\|\chi_i^- \varphi_n\|^2 + \|\chi_i^+ \varphi_n\|^2\} \qquad (i=1,2) \\
&\leq 2C_1^2 C_3^2 C_4^2 \exp(-\sqrt{-b}\,\delta) \sum_{n=1}^{N_L} \{|(\chi_i^-, \varphi_n)|^2 + |(\chi_i^+, \varphi_n)|^2\} \qquad (i=1,2) \\
&\leq 2C_1^2 C_3^2 C_4^2 \exp(-\sqrt{-b}\,\delta) \sum_{n=1}^{N_L} \{\|\chi_i^-\|^2 + \|\chi_i^+\|^2\} \\
&= C^2 \exp(-\sqrt{-b}\,\delta)
\end{aligned}$$

where $C = 2\sqrt{2}\,\delta_0 C_1 C_3 C_4$. Combining this inequality and (9) we get (8), and we can complete the proof.

4. Trace class perturbation

Lemma 4.1. *Let H be the same operator as in Theorem 2.2 and $H_i = -\frac{d^2}{dx^2} + V_i$ be self-*adjoint operators in L^2, *where $V_i = \chi_i V$. Set*

$$K = \sum_{i \in Z} \chi_i H_{i\lambda}^{-1} \chi_i.$$

Then for all a,b with $a < b < 0$ and $a, b \notin \sigma_{PP}(H)$

$$(H_\lambda^{-1} - K) P_{(a,b)}(H) \in \mathbf{B}_1.$$

To prove this lemma we show a few lemmas first.

Lemma 4.2. *Let $T = -\frac{d^2}{dx^2} + V$ be a self-adjoint operator where $V = -v\chi_{[x',y']}$ for $v > 0$. Suppose $f \in L^2$ and $g \in L^\infty$, set*

$$x'' = \min\{\inf\{\operatorname{supp} f\}, x'\} \qquad y'' = \max\{\sup\{\operatorname{supp} f\}, y'\}$$

where x'' is possibly $-\infty$ and y'' is possibly ∞.

And suppose $\operatorname{supp} g \cap (x'', y'') = \emptyset$, *set* $d(\operatorname{supp} g, [x'', y'']) = \delta$. *Then*

$$\| g T_\lambda^{-1} f \| \leq C \| g \|_\infty \| f \| e^{-\lambda\delta}$$

where C depends only on v.

Proof. T_λ has constant potential in $(-\infty, x')$, (x', y') and (y', ∞). By considering the solutions of each interval and connecting them C^1 -smoothly, we get

$$u_+(x) = \begin{cases} A_1 e^{\lambda x} + A_2 e^{-\lambda x} & (x \leq x') \\ B_1 e^x + B_2 e^{-x} & (x' < x < y') \\ e^{-\lambda x} & (y'' \leq x) \end{cases}$$

$$u_-(x) = \begin{cases} e^{\lambda x} & (y \leq x') \\ C_1 e^x + C_2 e^{-x} & (x' < x < y') \\ D_1 e^{\lambda x} + D_2 e^{-\lambda x} & (y' \leq x) \end{cases}$$

where

$$(13) \quad \begin{cases} A_1 = (\lambda^2 - 1)\exp\{-\lambda(x'+y')\}\{e^l - e^{-l}\}/4\lambda & B_1 = -(\lambda-1)\exp(-\lambda y' - y')/2 \\ A_2 = \exp(-\lambda l)\{(\lambda+1)^2 e^l - (\lambda-1)^2 e^{-l}\}/4\lambda & B_2 = (\lambda+1)\exp(-\lambda y' + y')/2 \\ D_1 = A_2 & C_1 = (\lambda+1)\exp(\lambda x' - x')/2 \\ D_2 = (\lambda^2-1)\exp\{\lambda(x'+y')\}\{e^l - e^{-l}\}/4\lambda & C_2 = -(\lambda-1)\exp(\lambda x' + x')/2 \end{cases}$$

with $l = y' - x'$.

Remark. $A_1, A_2, B_2, C_1, D_1, D_2>0,\ B_1, C_2<0.$

Then the integral kernel of operator T_λ is

$$E(x,\xi)=\frac{1}{2\lambda A_2}\begin{cases} u_+(x)u_-(\xi) & (\xi\leq x)\\ u_-(x)u_+(\xi) & (\xi>x)\end{cases}$$

Therefore for any $f\in L^2$, the solution of equation $T_\lambda u=f$ is

$$\begin{aligned} u(x)&=\int_{-\infty}^{\infty}E(x,\xi)f(\xi)d\xi\\ &=\int_{-\infty}^{x}u_+(x)u_-(\xi)f(\xi)d\xi+\int_{x}^{\infty}u_-(x)u_+(\xi)f(\xi)d\xi\end{aligned}$$

At first, assume that $\mathrm{supp}(g)\subset(-\infty, x'']$ and set $g_0=\sup\{\mathrm{supp}\ g\}$ or $-\infty$ if $g\equiv 0$.

then

$$\begin{aligned}\|g(x)u(x)\|&=\frac{1}{2\lambda A_2}\left(\int_{-\infty}^{g_0}(ge^{\lambda x})^2dx\right)^{\frac{1}{2}}\cdot\left|\int_{x''}^{x'}(A_1e^{\lambda\varepsilon}+A_2e^{-\lambda\varepsilon})\ f(\xi)d\xi\right.\\ &\qquad\left.+\int_{x'}^{y'}(B_1e^{\xi}+B_2e^{-\xi})f(\xi)d\xi+\int_{y'}^{y''}e^{-\lambda\xi}f(\xi)d\xi\right|\\ &\leq\frac{\|g\|_\infty}{2\lambda A_2}\left(\int_{-\infty}^{g_0}e^{2\lambda x}dx\right)^{\frac{1}{2}}\cdot\left\{\int_{x''}^{x'}|(A_1e^{\lambda\xi}+A_2e^{-\lambda\xi})f(\xi)|d\xi\right.\\ &\qquad\left.+\int_{x'}^{y'}|(B_1e^{\xi}+B_2e^{-\xi})f(\xi)|d\xi+\int_{y'}^{y''}|e^{-\lambda\xi}f(\xi)|d\xi\right\}\\ &\leq\frac{\|g\|_\infty\|f\|}{2\lambda\sqrt{2\lambda}A_2}\exp(\lambda g_0)\left\{\left(\int_{x''}^{x'}(A_1e^{\lambda\xi}+A_2e^{-\lambda\xi})^2d\xi\right)^{\frac{1}{2}}\right.\\ &\qquad\left.+\left(\int_{x'}^{y'}(-B_1e^{\xi}+B_2e^{-\xi})^2d\xi\right)^{\frac{1}{2}}+\left(\int_{y'}^{y''}e^{-2\lambda\xi}d\xi\right)^{\frac{1}{2}}\right\}\end{aligned}$$

$$\begin{aligned}&\leq\frac{\|g\|_\infty\|f\|}{2\lambda\sqrt{2\lambda}A_2}\exp(\lambda g_0)\left\{\left(2\int_{x''}^{x'}(A_1^2e^{2\lambda\xi}+A^2e^{-\lambda\xi})d\xi\right)^{\frac{1}{2}}\right.\\ &\qquad\left.+\left(2\int_{x'}^{y'}(B_1^2e^{2\xi}+B^2e^{-2\xi})d\xi\right)^{\frac{1}{2}}+\frac{1}{\sqrt{2\lambda}}e^{-\lambda y'}\right\}\end{aligned}\tag{14}$$

where the third inequality follows from Schwarz inequality and fourth from the fact that for $a, b\in\mathbf{R}$, $(a+b)^2\leq 2(a^2+b^2)$.

By (13)

$$
(15)\qquad \begin{cases}
\dfrac{A_1}{A_2}=(\lambda^2-1)\exp(-2\lambda x')\,(e^l-e^{-l})/\{(\lambda+1)^2e^l-(\lambda-1)^2e^{-l}\} \\
\qquad \leq \dfrac{\lambda-1}{2}\exp(-2\lambda x') \\
-\dfrac{B_1}{A_2}=2\lambda(\lambda-1)\exp(-\lambda x'-y')/(\lambda+1)^2e^l-(\lambda-1)^2e^{-l}\} \\
\qquad \leq \dfrac{\lambda-1}{2}\exp(-\lambda x'-y') \\
\dfrac{B_2}{A_2}=2\lambda(\lambda+1)\exp(-\lambda x'+y')/\{(\lambda+1)^2e^l-(\lambda-1)^2e^{-l}\} \\
\qquad \dfrac{\lambda+1}{2}\exp(-\lambda x'+y'-l)
\end{cases}
$$

By combining (14) and (15), we get

$$
\begin{aligned}
\|g(x)u(x)\| \leq \frac{\|g\|_\infty\|f\|}{2\lambda\sqrt{2\lambda}} &\Big[\Big\{\frac{\lambda^2+1}{2}(1-\exp\{-2(y'-x')\})\Big\}^{\frac{1}{2}}\exp\{-\lambda(x'-g_0)\} \\
&+\frac{1}{\sqrt{2\lambda}}\exp\{-\lambda(y'-g_0)\}+\Big\{1-\exp\{-2\lambda(x'-x'')\}) \\
&\times\Big(\frac{(\lambda-1)^2}{4}\exp\{-2\lambda(x'-g_0)\}+\exp\{-2\lambda(x''-g_0)\}\Big)\Big\}^{\frac{1}{2}}/\sqrt{\lambda}\Big] \\
&\leq C\|g\|_\infty\|f\|\exp\{-\lambda(x''-g_0)\}
\end{aligned}
$$

where $C=\dfrac{1}{2\lambda\sqrt{\lambda}}\Big[\Big\{\dfrac{(\lambda-1)^2}{4}+1\Big\}^{\frac{1}{2}}\dfrac{1}{\sqrt{\lambda}}+\Big\{\dfrac{\lambda^2+1}{2}\Big\}^{\frac{1}{2}}+\dfrac{1}{\sqrt{2\lambda}}\Big]$.

When supp $g\in[y'', \infty)$, the same is true by symmetricity. In general case supp $g\in(-\infty, x'']\cup[y'', \infty)$ by hypothesis. Decomposing g into g_1+g_2 with supp $g_1\in(-\infty, x']$ and supp $g_2\in[y'', \infty)$ and applying the above argument, we can complete the proof.

Lemma 4.3. *Let T be the same operator as before and $\widetilde{T}=-\dfrac{d^2}{dx^2}+\widetilde{V}$ be a self-adjoint operator in L^2, where $\widetilde{V}$ is real-valued L^∞ function and $V(x)\leq\widetilde{V}(x)$ a. e. $x\in\mathbf{R}$. Then $\widetilde{T}$ satisfies the same estimate that T satisfies in Lemma 4.2.*

Proof. By Kato's inequality, for any $u\in L^1_{loc}$ with $\dfrac{d^2}{dx^2}u\in L^1_{loc}$,

$$
\frac{d^2}{dx^2}|u|\geq \mathrm{Re}\Big\{\frac{\overline{u}}{|u|}\Big(\frac{d^2}{dx^2}u\Big)\Big\}
$$

in the distribution sense.

Then

$$\left(-\frac{d^2}{dx^2}+\widetilde{V}+\lambda^2\right)|u|\leq \mathrm{Re}\left\{\frac{\overline{u}}{|u|}\left(-\frac{d^2}{dx^2}+\widetilde{V}+\lambda^2\right)u\right\}.$$

Set $\widetilde{T}_\lambda u=f$, then

$$\begin{aligned}&T_\lambda|\widetilde{T}_\lambda^{-1}f|\leq\\&\widetilde{T}|\widetilde{T}_\lambda^{-1}f|\leq \mathrm{Re}\left(\frac{\overline{u}}{|u|}f\right)\\&\qquad\leq|f|.\end{aligned}$$

Let $u\in H^2$, then $f\in L^2$. By (13) the integral kernel of operator T_λ is positive, then

$$|\widetilde{T}_\lambda^{-1}f|\leq T_\lambda^{-1}|f|.$$

For any f, g which satisfy the condition of Lemma 4.2,

$$\|g\,\widetilde{T}_\lambda^{-1}f\|\leq\|g\,T_\lambda^{-1}f\|.$$

So we can complete the proof.

Lemma 4.4. *Let H be the same operator as in Theorem 2.2 and a,b satisfy the same conditions as in Lemma 4.1 and for any α, β, $\gamma\in\mathbf{R}$ with $\alpha<\beta$ and $\gamma>0$, $\psi_{\alpha\beta\gamma}\in C^2$ have the properties that*

(*i*) $0\leq\psi_{\alpha\beta\gamma}\leq 1$
(*ii*) $\mathrm{supp}\ \psi_{\alpha\beta\gamma}=[\alpha-\gamma, \beta+\gamma]$
(*iii*) $\psi_{\alpha\beta\gamma}=1$ *on* $[\alpha, \beta]$
(*iv*) *for some* $c'>0$, $|\psi_{\alpha\beta\gamma}'|\leq\frac{c'}{\gamma}$ $|\psi_{\alpha\beta\gamma}''|\leq\frac{c'}{\gamma^2}$

Then

$$\left\|\left(-\frac{d^2}{dx^2}+1\right)\psi_{\alpha\beta\gamma}P_{(a,b)}(H)\right\|_0\leq C\|\chi_{[\alpha-\gamma,\beta+\gamma]}P_{(a,b)}(H)\|_0$$

where C depends only on v, γ and c'.

Proof. Set $\psi=\psi_{\alpha\beta\gamma}$, $P=P_{(a,b)}(H)$ and $\chi=\chi_{[\alpha-\gamma,\beta+\gamma]}$. Then

$$\left(-\frac{d^2}{dx^2}\right)\psi P=-\psi''P-2\psi'\frac{d}{dx}P+\psi\left(-\frac{d^2}{dx^2}\right)P. \tag{16}$$

For all $f\in L^2$

$$\|-\psi''Pf\|\leq\sup|\psi'|\ \|\chi Pf\|\leq\frac{c'}{\gamma^2}\ \|\chi P\|_0\|f\|.$$

So

(17) $$\|-\psi''P\|_0 \leq \frac{c'}{\gamma^2}\|\chi P\|_0.$$

By the spectral theorem, for all $f \in L^2$

$$\|HPf\|^2 = (HPf, HPf) = (H^2Pf, Pf) = (PH^2Pf, f) = (H^2Pf, f)$$
$$= \int_a^b \lambda^2 dP_\lambda \leq (\min\{-a, v\})^2\|f\|^2 \leq v^2\|f\|^2.$$

Then

$$\left\|\psi\left(-\frac{d^2}{dx^2}\right)Pf\right\| = \|\psi(H-V)Pf\| \leq \|\psi HPf\| + \|\psi VPf\|$$
$$\leq \|\chi PHPf\| + v\|\chi Pf\| \leq 2v\|\chi P\|_0\|f\|.$$

So

(18) $$\left\|\psi\left(-\frac{d^2}{dx^2}\right)P\right\|_0 \leq 2v\|\chi P\|_0.$$

And for $f \in \mathrm{Ran}\, P$,

$$\|\psi' f'\|^2 = (\psi' f', \psi' f') = (f', (\psi')^2 f') = -(f, 2\psi'\psi'' f' + (\psi')^2 f'')$$
$$= -2(\psi'' f, \psi' f') - (\psi' f, \psi' f'') \leq 2\|\psi'' f\|\,\|\psi' f'\| + \|\psi' f\|\,\|\psi' f''\|$$
$$\leq 2\|\psi'' f\|^2 + \frac{1}{2}\|\psi' f'\|^2 + \|\psi' f\|\,\|\psi' f''\|$$

Transposing the second term, multiplying both sides by two and applying (18), we get

$$\|\psi f'\|^2 \leq 4\|\psi'' f\|^2 + 2\|\psi' f\|\,\|\psi' f''\| \leq \frac{4c'^2}{\gamma^4}(1+v\gamma^2)\|\chi f\|^2.$$

So

(19) $$\left\|2\psi\frac{d}{dx}P\right\|_0 \leq \frac{2c'}{\gamma^2}(1+v\gamma^2)^{\frac{1}{2}}\|\chi P\|_0.$$

Combining (16), (17), (18) and (19), the proof is completed.

Lemma 4.5. *Let* $B_{\alpha,\beta} = -\dfrac{d^2}{dx^2} + 1$ *be a self-adjoint operator on* $L^2([\alpha, \beta])$ *with Dirichlet boundary conditions. Then*

$$\|B_{\alpha,\beta}^{-1}\|_{1,L^2([\alpha,\beta])} \leq \frac{\beta-\alpha}{2}$$

Proof. Since the operator $B_{\alpha,\beta}$ is positive and its eigenvalues are

$$\lambda_n=\frac{n^2\pi^2}{(\beta-\alpha)^2}+1 \quad (n=1, 2,\cdots),$$

the eigenvalues of operator $B_{\alpha,\beta}{}^{-1}$ are $\frac{1}{\lambda_n}$ and then

$$\begin{aligned}\|B_{\alpha,\beta}{}^{-1}\|_{1,L^2([\alpha,\beta])}&=\sum_{n=1}^{\infty}\sum\frac{1}{\lambda_n}\\&\leq\sum_{n=1}^{\infty}\frac{(\beta-\alpha)^2}{n^2\pi^2+(\beta-\alpha)^2}\\&\leq(\beta-\alpha)^2\cdot\int_0^{\infty}\frac{dx}{\pi^2x^2+(\beta-\alpha)^2}\\&=\frac{\beta-\alpha}{2}\end{aligned}$$

Lemma 4.6. *Let H be the same operator as in Theorem 2.2 and a,b satisfy the same conditions as in Lemma 4.1 Then for any α, β, $\gamma\in\mathbf{R}$ with $\alpha<\beta$ and $\gamma>0$,*

$$\|\chi_{[\alpha,\beta]}P_{(a,b)}(H)\|_1\leq C(\beta-\alpha+2\gamma)\|\chi_{[\alpha-\gamma,\beta+\gamma]}P_{(a,b)}(H)\|_0$$

where C depends only on v, γ and c'.

Proof. Let $\psi_{\alpha\beta\gamma}$ be the same as in Lemma 4.4. Then

$$\begin{aligned}\chi_{[\alpha,\beta]}P&=\chi_{[\alpha,\beta]}\psi_{\alpha\beta\gamma}P\\&=\chi_{[\alpha,\beta]}*B_{\alpha-\gamma,\beta+\gamma}{}^{-1}B_{\alpha-\gamma,\beta+\gamma}*\psi_{\alpha\beta\gamma}P\end{aligned}$$

where the first $*$ represents embedding from $L^2([\alpha-\gamma, \beta+\gamma])$ to L^2 and second $*$ represents identifying a function which is in L^2 and whose support is a subset of $[\alpha-\gamma, \beta+\gamma]$ as a function in $L^2([\alpha-\gamma, \beta+\gamma])$.

Then by Lemma 4.4 and Lemma 4.5

$$\begin{aligned}\|\chi_{[\alpha,\beta]}P\|_1&=\|\chi_{[\alpha,\beta]}*B_{\alpha-\gamma,\beta+\gamma}{}^{-1}B_{\alpha-\gamma,\beta+\gamma}*\psi_{\alpha\beta\gamma}P\|_1\\&=\|\chi_{[\alpha,\beta]}*\|_{\mathbf{B}(L^2(\alpha-\gamma,\beta+\gamma),L^2)}\|B_{\alpha-\gamma,\beta+\gamma}{}^{-1}\|_{\mathbf{B}(L^2(\alpha-\gamma,\beta+\gamma),L^2(\alpha-\gamma,\beta+\gamma))}\\&\quad\times\|B_{\alpha-\gamma,\beta+\gamma}*\psi_{\alpha\beta\gamma}P\|_{\mathbf{B}(L^2,L^2(\alpha-\gamma,\beta+\gamma))}\\&\leq\frac{(\beta-\alpha+2\gamma)}{2}\left\|\left(-\frac{d^2}{dx^2}+1\right)\psi_{\alpha\beta\gamma}P\right\|_0\\&\leq C(\beta-\alpha+2\gamma)\|\chi_{[\alpha-\gamma,\beta+\gamma]}P\|_0\end{aligned}$$

Proof of Lemma 4.1.

$$\|(H_\lambda{}^{-1}-K)P\|_1=\|\sum_i(H_\lambda{}^{-1}\chi_i-H_{i\lambda}{}^{-1}\chi_i+H_{i\lambda}{}^{-1}\chi_i-\chi_iH_{i\lambda}{}^{-1}\chi_i)P\|_1$$

$$=\|\sum_i \{H_\lambda^{-1}(-\sum_{j\neq i} V_j) H_{i\lambda}^{-1}\chi_i + \sum_{j\neq i}\chi_j H_{i\lambda}^{-1}\chi_i\} P\|_1$$

(20) $$\leq \sum_i (\|H_\lambda^{-1}\|_0 v+1)\|(\sum_{j\neq i,i\pm 1}\chi_j) H_{i\lambda}^{-1}\chi_i\|_0 \|\chi_i P\|_1$$

(21) $$+\sum_i \sum_{j=i\pm 1} \sum_{k=1}^{3} (\|H_\lambda^{-1}\|_0 v+1)\|\chi_j H_{i\lambda}^{-1}\chi_{ik}\|_0 \|\chi_{ik}P\|_1$$

where

$$\chi_{i1}(x)=\chi_{[z_{i-1},z_{i-1}+\frac{d_{i-1}}{8}]}(x)$$

$$\chi_{i2}(x)=\chi_{(z_{i-1}+\frac{d_i}{8})}(x)$$

$$\chi_{i3}(x)=\chi_{[z_i-\frac{d_i}{8},z_i)}(x).$$

First of all in (20) $d(\mathrm{supp}(\sum_{j\neq i,i\pm 1}\chi_j), \mathrm{supp}\,\chi_i)\geq \frac{d_{i*}}{2}$. So by Lemma 4.3

$$\|(\sum_{j\neq i,i\pm 1}\chi_j) H_{i\lambda}^{-1}\chi_i\|_0 \leq C\exp(-\lambda d_{i*}/2)$$

Setting $\gamma=\frac{d}{8}$ and using Lemma 4.6, we get

$$\|\chi_i P\|_1 \leq C\left(w_i+\frac{d_{i-1}+d_i}{2}+\frac{d}{4}\right)\|\chi_{[z_{i-1}-\frac{d}{8},\, z_i+\frac{d}{8}]}P\|_0$$

$$\leq C(w_i+d_i^*).$$

Then

(22) $$\sum_i (\|H_\lambda^{-1}\|_0 v+1)\|(\sum_{j\neq i,i\pm 1} V_j) H_{i\lambda}^{-1}\chi_i\|_0 \|\chi_i P\|_1 \leq C\sum_i (w_i+d_i^{\#})\exp(-\lambda d_{i*}/2)$$

In (21) when $j=i-1$ and $k=2$, 3 or $j=i+1$ and $k=1$, 2, $d(\mathrm{supp}\, V_j, \mathrm{supp}\, \chi_{ik})\geq \frac{d_{i*}}{8}$.

Then

(23) $$\sum_i (\|H_\lambda^{-1}\|_0 v+1)\|\chi_i H_{i\lambda}^{-1}\chi_{ik}\|_0 \|\chi_{ik}P\|_1 \leq C\sum_i (w_1+d_i^*)\exp(-\lambda d_{i*}/8)$$

When $j=i-1$ and $k=1$ or $j=i+1$ and $k=3$,

$$\|\chi_i H_{i\lambda}^{-1}\chi_{ik}\|_0 \leq 1$$

and by Lemma 3.6

$$\|\chi_{ik}P\|_1 \leq C\frac{d_m}{4}\|\chi_{[z_m-\frac{d_m}{4},z_m+\frac{d_m}{4}]}P\|_0 \quad (m=i,\ i+1)$$

$$\leq C\frac{d_m}{4}\exp(-\sqrt{-b}d_m/8). \qquad (m=i,\ i+1)$$

$$\leq C(w_i+d_i^*)\exp(-\sqrt{-b}d_{i*}/8).$$

Then

$$\sum_i (\|H_\lambda^{-1}\|_0 v+1)\|\chi_j H_{i\lambda}^{-1}\chi_{ik}\|_0\|\chi_{ik}P\|_1 \leq C\sum_i (w_i+d_i^*)\exp(-\sqrt{-b}d_{i*}/8). \tag{24}$$

Combining from (22) to (24), we get

$$\|(H_\lambda^{-1}-K)P\|_1 \leq C\sum_i (w_i+d_i^*)\exp\left(-\min\left\{\frac{\sqrt{-b}}{8},\ \frac{\lambda}{8}\right\}d_{i*}\right)<\infty$$

where the last inequality follows from the assumption (A.2).

5. Proof of Main Theorem

In this section we will apply Pearson's theorem to prove Theorem 2.2.

Lemma 5.1. ([6] p.24). *Let A and B be self-adjoint operators and let J be a bounded operator. Suppose that there is a trace class operator C so that $C=AJ-JB$ in the sense that for all $\varphi\in D(A)$ and $\psi\in D(B)$,*

$$(C\psi,\ \varphi)=(J\psi,\ A\varphi)-(JB\psi,\ \varphi)$$

then

$$\Omega^{\pm}(A,\ B;\ J)\equiv \mathrm{s}\text{-}\lim_{t\to\pm\infty} e^{iAt}Je^{-iBt}P_{\mathrm{ac}}(B)$$

exist.

Proof of Theorem 2.2. Since H has no spectra in $(-\infty,\ -v)$, we can assume that $c\geq -v$ without loss of generality. Set $A=K$, $B=H_\lambda^{-1}$ and $J=P_{(a,b)}(H)$ in Lemma 5.1. By the spectral theorem, $P_{(a,b)}(H)=P_{\left(\frac{1}{b+\lambda^2},\frac{1}{a+\lambda^2}\right)}(H_\lambda^{-1})$. Since H_λ^{-1} and $P_{\left(\frac{1}{b+\lambda^2},\frac{1}{a+\lambda^2}\right)}(H_\lambda^{-1})$ commute each other, by Lemma 4.1 we can prove that they satisfy the hypotheses of Lemma 5.1 under the conditions of Theorem 2.2. Then

$$\widetilde{\Omega}\equiv s\text{-}\lim_{t\to\pm\infty} e^{iKt}e^{-iH_\lambda^{-1}t}P_{\left(\frac{1}{b+\lambda^2},\frac{1}{a+\lambda^2}\right)}(H_\lambda^{-1})P_{\mathrm{ac}}(H_\lambda^{-1}) \tag{25}$$

exist.

(25) shows that Range $(P_{\left(\frac{1}{b+\lambda^2},\frac{1}{a+\lambda^2}\right)}(H_\lambda^{-1})P_{\mathrm{ac}}(H_\lambda^{-1}))$ is unitary equivalent to the subspace of Range $(P_{\left(\frac{1}{b+\lambda^2},\frac{1}{a+\lambda^2}\right)}(K)P_{\mathrm{ac}}(K))$. So let's consider the operator K.

For any $i\in\mathbf{Z}$ we can regard the operator $\chi_i H_{i\lambda}^{-1}$ as the one from $L^2(I_i)$ to

$H^2(I_i)$. Since $H^2(I_i)$ is the compact subspace of $L^2(I_i)$, $\chi_i H_{i\lambda}^{-1}$ is compact. So by Riesz-Schauder's theorem its spectra are discrete except for the origin, and the set of its eigenfunctions constitutes a C.O.N.S. of $L^2(I_i)$.

The function which is gained by embedding the eigenfunction of $\chi_i H_{i\lambda}^{-1}$ into $L^2(\mathbf{R})$ is an eigenfunction of K. The set of all such functions for all i consists a C.O.N.S. of L^2.
Then the absolutely continuous subspace of K is $\{0\}$. So we can conclude that

$$\mathrm{Range}\,(P_{(\frac{1}{b+\lambda^2},\frac{1}{a+\lambda^2})}(K)P_{\mathrm{ac}}(K))=\{0\}.$$

Then

$$\mathrm{Range}\,(P_{(\frac{1}{b+\lambda^2},\frac{1}{a+\lambda^2})}(H_\lambda^{-1})P_{\mathrm{ac}}(H_\lambda^{-1}))=\{0\}.$$

By the spectral theorem

$$\mathrm{Range}\,(P_{(a,b)}(H)P_{\mathrm{ac}}(H))=\{0\}.$$

Since a, b are arbitrary with $a<b<0$ and $a, b\notin\sigma_{\mathrm{PP}}(H)$,

$$\sigma_{\mathrm{ac}}(H)\cap(-\infty,0)=\emptyset.$$

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