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Compactness in Boltzmann's equation via Fourier integral operators and applications. II

dedicated to the memory of Ron DiPerna

By

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I. Introduction

This work is the continuation of Part I [51] where a general introduction to Boltzmann's equations (and kinetic models) can be found together with the main goals of this series. In particular, we keep the same notations than in [51] and we recall briefly the Boltzmann's equation

$$\frac{\partial f}{\partial t}+v\cdot\nabla_x f=Q(f,f),\ x\in\mathbf{R}^N,\ v\in\mathbf{R}^N,\ t>0 \tag{1}$$

where $N\geq 2$ and the solutions $f=f(x,v,t)$ are always assumed to be nonnegative on $\mathbf{R}^{2N}_{x,v}\times[0,\infty)$ and the so-called collision term $Q(f,f)$ introduced by L. Boltzmann [11] and J. C. Maxwell [58], [59] is given by

$$Q(f,f)=Q^+(f,f)-Q^-(f,f), \tag{2}$$

$$Q^+(f,f)=\int_{\mathbf{R}^N}dv_*\int_{S^{N-1}}d\omega\, B(v-v_*,\omega)f'f'_*\,, \tag{3}$$

$$\begin{cases} Q^-(f,f)=\int_{\mathbf{R}^N}dv_*\int_{S^{N-1}}d\omega\, B(v-v_*,\omega)ff_*=fL(f)\,,\\ L(f)=A\overset{*}{_v}f\,, \end{cases} \tag{4}$$

and $A(z)=\int_{S^{N-1}}B(z,\omega)d\omega\,(z\in\mathbf{R}^N)$, $f_*=f(x,v_*,t)$, $f'=f(x,v_*,t)$, $f'_*=f(x,v'_*,t)$, $v'=v-(v-v_*,\omega)\omega$, $v'_*=v_*+(v-v_*,\omega)\omega$. Here and everywhere below, we denote indifferently by $a\cdot b$ or (a,b) the usual scalar product of $a,b\in\mathbf{R}^N$.

The so-called collision kernel B that enters the bilinear operator Q is a given function on $\mathbf{R}^N\times S^{N-1}$. We shall always assume (at least) that B satisfies

$$B\in L^1(K\times S^{N-1})\text{ for any compact set } K \text{ of } \mathbf{R}^N,\ B\geq 0, \tag{5}$$

$$B(z,\omega)\text{ depends only on } |z| \text{ and } |(z,\omega)|, \tag{6}$$

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$$(7)\qquad \begin{cases} (1+|z|^2)^{-1}\left(\int_{|v-z|\le R} A(v)\,dv\right) \to 0 \\ \text{as}\,|z|\to+\infty, \text{ for all } R\in(0,\infty). \end{cases}$$

And we will not recall these assumptions in all that follows.

A classical example is given by the so-called hard-spheres model where

$$(8)\qquad B(z,\omega)=|(z,\omega)|\ .$$

Another physically interesting example - that corresponds to soft forces with an angular cut-off - is given by $b(\theta)|z|^{\gamma}$ with $\cos\theta=|(z,\omega)|\,|z|^{-1}$, $\gamma=1-2(N-1)(s-1)^{-1}$ and s is the exponent of the intermolecular potential $(s>1)$. The above assumptions are satisfied as soon as $b(\theta)(\cos\theta)^{N-1}\in L^1(0,\pi/2)$ and $s>1+2\dfrac{N-1}{N+2}$.

We next recall the notion of renormalized solutions of (1) as introduced in R. J. DiPerna and P. L. Lions [25], [26]. First, we complement (1) with an initial condition

$$(9)\qquad f|_{t=0}=f_0 \text{ in } \mathbf{R}^{2N}_{x,v}$$

where $f_0(\ge 0)$ is given on $\mathbf{R}^{2N}_{x,v}$ and satisfies

$$(10)\qquad \iint_{\mathbf{R}^{2N}} dx\,dv\,f(t)\,(1+\omega(x)+|v|^2+|\log f(t)|)<\infty\ .$$

Here and everywhere below, ω is a (weight) function that satisfies

$$(11)\qquad \omega\ge 0,\ (1+\omega)^{1/2} \text{ is Lipschitz on } \mathbf{R}^N,\ e^{-\omega}\in L^1(\mathbf{R}^N).$$

(It was shown in [51] that (11) implies that $\omega\to+\infty$ as $|x|\to+\infty$) .

We then say that $f\in C([0,\infty);L^1(\mathbf{R}^{2N}_{x,v}))$ is a renormalized solution of Boltzmann's equation (1) corresponding to the initial condition (9) if f satisfies

$$(12)\qquad \sup_{t\in[0,T]}\iint_{\mathbf{R}^{2N}} dx\,dv\,f(t)\,(1+\omega(x)+|v|^2+|\log f(t)|)<+\infty,$$

$$(13)\qquad \begin{cases} \dfrac{Q^-(f,f)}{1+f}\in L^\infty(0,\infty;L^1(\mathbf{R}^N_x\times K)); \\[2ex] \dfrac{Q^+(f,f)}{1+f}\in L^1(0,T;L^1(\mathbf{R}^N_x\times K)), \end{cases}$$

for all $T\in(0,\infty)$ and for any compact set K of $\mathbf{R}^N_v$. In addition, we request that

$$(14)\qquad \left(\frac{\partial}{\partial t}+v\cdot\nabla_x\right)\beta(b)=\beta'(f)Q(f,f)\ \text{ in } \mathscr{D}'(\mathbf{R}^{2N}_{x,v}\times(0,\infty))$$

for all $\beta\in C^1([0,\infty);\mathbf{R})$ such that $\beta'(t)(1+t)$ is bounded on $[0,\infty)$. The final property we also require is

$$(15)\qquad \int_0^T dt\int_{\mathbf{R}^N}dx\iint_{\mathbf{R}^{2N}}dv dv_*\int_{S^{N-1}}d\omega\, B(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*}<+\infty$$

for all $T\in(0,\infty)$.

It was shown in [25] that there always exists at least one renormalized solution of (1) with the initial condition (9). Additional properties are available such as conservation of mass (locally) and momentum (globally) and an entropy inequality (shown in R.J. DiPerna and P.L. Lions [26]). And, in fact, we shall derive in section IV more properties satisfied by the solutions we can build. As recalled in the Introduction of Part I [51], this existence result is essentially a consequence of the stability of renormalized solutions under weak L^1 convergence.

We have shown in Part I [51] that for arbitrary sequences of renormalized solutions of (1) with uniform natural bounds (see (12) and (15)), the nonlinear operator Q^+ is always relatively compact (for the convergence in measure). And we applied this compactness (and in fact regularity) result to a new proof of the convergence in L^1 (strongly) to a pure Maxwellian equilibrium in the case of a periodic box.

Here, we use this result to establish the following fact: if f^n is a sequence of renormalized solutions of (1) with uniform natural bounds and if the corresponding initial conditions f_0^n converge strongly in $L^1(\mathbf{R}^{2N}_{x,v})$ to some f_0 then, extracting a subsequence if necessary, f^n converges in $C([0,T];L^1(\mathbf{R}^{2N}_{x,v}))$ $(\forall\, T\in(0,\infty))$ to a renormalized solution of (1) corresponding to the initial condition f_0. The precise statement is given in section II and is proven in section III. For obvious reasons, we say that this result shows that there is propagation of the (strong) L^1 convergence in Boltzmann's equation.

Such a result is then applied in section IV to the derivation of new properties of the solutions of Boltzmann's equation. More precisely, we obtain new differential inequalities satisfied by all smooth solutions of Boltzmann's equation and the propagation of the strong L^1 convergence allows us to check that the weak solutions we build also satisfy these inequalities. In fact, we do not show that any renormalized solution satisfies them but that the approximation procedure used in [25] not only yields the defining properties of renormalized solutions but also these differential inequalities.

Finally, we deduce from these new properties a uniqueness statement: we show that if there exists a strong solution of (1) (say a bounded solution) satisfying (9) then any weak solution (i.e. renormalized solution satisfying the differential inequalities shown in section IV) coincides with the strong one.

Further applications of the compactness results shown in Part I [51] and here will be given in Part III: details can already be found in the Introduction of Part I.

Let us now conclude this Introduction by making a few comments on the results shown in this paper. First of all, the propagation of the strong convergence in $L^1(\mathbf{R}^{2N}_{x,v})$ shows that if initially there are no oscillations, then no oscillations can appear spontaneously during the evolution. This type of questions has become a standard issue for nonlinear evolution problems (of hyperbolic type...) beginning with the works by L. Tartar [65], [66] on compensated compactness. However, in many examples such as nonlinear (hyperbolic) conservation laws, the existing results are slightly different since they show that no oscillations can appear or even persist even if oscillations are present initially - see L. Tartar [66], R. J. DiPerna [22], [23], G. Q. Chen [18], P. L. Lions, B. Perthame and E. Tadmor [56], [57], P. L. Lions, B. Perthame and P. E. Souganidis [56]... And it was shown in P.L. Lions [52] that this phenomenon of immediate damping (for $t>0$) of oscillations is not true for Boltzmann's equation - while it holds for other collision models such as the Landau's model [52]. This is why our result is in fact closer to the results shown on Vlasov-Poisson systems in R. J. DiPerna and P. L. Lions [31] - see also [51] - and of course to the much simpler case of L^1 contractive evolution equations such as, for instance, scalar conservation laws.

We also want to point out that the uniqueness "weak solution = strong solution" shown in section V is a classical substitute to a true uniqueness statement. And one can observe a striking analogy with the state of the art on three dimensional incompressible Navier-Stokes equations. Indeed, the global existence result of weak solutions shown in [25] can be seen as the analogue for Boltzmann's equation of the pioneering work on Navier-Stokes equations by J. Leray [47], [48], [49]. And the uniqueness of "Leray solutions" is not known except for some results which show that weak solutions are equal to a strong one (in a sense to be made precise) whenever the latter exists: examples of such results can be found in R. Temam [67]. Let us also mention that results of a similar type for hyperbolic systems of nonlinear conservation laws can be found in R. J. DiPerna [24].

II. Propagation of strong L^1 convergence in Boltzmann's equation

As explained in the Introduction, we consider a sequence $(f^n)_{n\geq 1}$ of (nonnegative) renormalized solutions of (1) corresponding to a sequence of (nonnegative) initial conditions $(f_0^n)_{n\geq 1}$. We assume uniform natural bounds on f_0^n i.e.

(16) $$\sup_{n\geq 1}\iint_{\mathbf{R}^{2N}} f_0^n\{1+\omega(x)+|v|^2+|\log f_0^n|\}\ dx\ dv<\infty\ .$$

We also assume similar bounds on f^n

(17) $$\sup_{n\geq 1}\ \sup_{t\in[0,T]}\iint_{\mathbf{R}^{2N}} f^n(t)+\omega(x)+|v|^2+|\log f^n(t)|\}\ dxdv<\infty\ ,$$

and

(18) $$\sup_{n\geq 1}\int_0^T dt\int_{\mathbf{R}^N}dx\iint_{\mathbf{R}^{2N}}dvdv_*\int_{S^{N-1}}d\omega\cdot B\left(f^{n'}f_*^{n'}-f^nf_*^n\right)\log\frac{f^{n'}f_*^{n'}}{f^nf_*^n}<\infty$$

for all $T\in(0,\infty)$.

Let us recall (once more) that the existence of such a sequence f^n given a sequence f_0^n satisfying (16) was shown in [25], [26]. In fact, only the case $\omega(x)=|x|^2$ was considered in [25], [26] in which case one can even take $T=+\infty$ in (17) and (18), while the easy modifications of the arguments of [25], [26] needed to allow a general ω satisfying (11) are explained in Part I [51].

Next, we observe that the bounds (16) and (17) imply, extracting subsequences if necessary, that f_0^n (resp. f^n) converges weakly in $L^1(\mathbf{R}^{2N})$ (resp. $L^1(\mathbf{R}^{2N}_{x,v}\times(0,T))$ for all $T\in(0,\infty)$) to some $f_0\geq 0$ (resp. $f\geq 0$) which satisfies (10) (resp. (12)).

And we recall that it was shown in [25], [26] that f is a renormalized solution of (1) corresponding to the initial condition f_0 (i.e. f satisfies (9)). We may now state our main result

Theorem II.1. *If f_0^n converges in $L^1(\mathbf{R}^{2N})$ to f_0, then f^n converges to f in $C([0,T];L^1(\mathbf{R}^{2N}_{x,v}))$ for all $T\in(0,\infty)$.*

Remarks. i) The same result holds for the Boltzmann's equation in a periodic box i.e. when f_0^n, f^n (and thus f_0, f) are assumed to be periodic in each x_i $(1\leq i\leq N)$ with a fixed period $T_i\in(0,\infty)$. In that case, the weight ω is no more necessary and all integrations in x in the assumption are restricted to $\Pi^N=\Pi_{i=1}^N[0,T_i]$.

ii) Let us recall that it was shown in P. L. Lions [52] that if $f^n\xrightarrow[n]{}f$ in $L^1_{loc}(\mathbf{R}^{2N}_{x,v}\times(0,\infty))$ then necessarily $f_0^n\xrightarrow[n]{}f_0$ in $L^1(\mathbf{R}^{2N})$. This fact combined with the above result shows that the strong convergence in $L^1(\mathbf{R}^{2N}_{x,v})$ propagates both forward and backward in time. One might suspect that such a reversible propagation of L^1 convergence is related to the angular cut-off assumption we are making on the collision operator Q and that without angular cut-off the strong L^1 convergence is automatic for $t>0$ (without any further assumption on f_0^n like the strong convergence assumption made in the above result) as it is the case for the so-called Landau model (see P. L. Lions [52]). Of course, this remains highly speculative since very little is known on the

Boltzmann's equation without angular cut-off.

Theorem II.1 is proved in the next section by, in fact, two slightly different arguments. We conclude this section by recalling a few known facts on such sequences of solutions $(f_n)_{n>1}$. First of all, it was shown in [25] that we have for all R, $T \in (0, \infty)$

$$(19) \qquad \begin{cases} Q^{\pm}(f^n, f^n)\,(1+f^n)^{-1} \text{ is weakly relatively compact in} \\ L^1(\mathbf{R}^N_x \times (|v|<R) \times (0, T)), \end{cases}$$

$$(20) \qquad \begin{cases} L(f^n) = A \underset{v}{*} f^n \left(= \int\int_{\mathbf{R}^N \times S^{N-1}} f^n(x, v_*, t)\, B(v-v_*, \omega)\, dv_* d\omega \right) \\ \overset{n}{\longrightarrow} L(f) \text{ in } L^1(\mathbf{R}^N_x \times (|v|<R) \times (0, T)) . \end{cases}$$

And the main compactness result (Theorem II.1) in Part I [51] is

$$(21) \qquad Q^+(f^n, f^n) \underset{n}{\rightarrow} Q^+(f, f) \text{ in measure on } (|x|<R) \times |v|<R) \times (0, T)$$

for all R, $T \in (0, \infty)$.

The assertions (19) and (21) then imply that, in order to prove Theorem II.1, we may assume without loss of generality that we have

$$(22) \qquad Q^+(f^n, f^n) \underset{n}{\rightarrow} Q^+(f, f) \text{ a.e. } \mathbf{R}^{2N} \times (0, \infty)$$

and for all $\delta \in (0, 1]$

$$(23) \qquad Q^{\pm}(f^n, f^n)\,(1+\delta f^n)^{-1} \underset{n}{\rightarrow} \overline{RQ^{\pm}_{\delta}} \text{ weakly in } L^1(\mathbf{R}^N_x \times (|v|<R) \times (0, T))$$

for all R, $T \in (0, \infty)$, where $\overline{RQ^{\pm}_{\delta}}$ are two nonnegative measurable functions in $L^1(\mathbf{R}^N_x \times (|v|<R) \times (0, T))$ for all R, $T \in (0, \infty)$. And we set $\overline{RQ^{\pm}} = \overline{RQ^{\pm}_1}$. Also, $\frac{f^n}{1+f^n}$, $\frac{1}{1+f^n}$ are obviously bounded measurable functions on $\mathbf{R}^{2N} \times (0, \infty)$ $\left(\text{and } \frac{f^n}{1+f^n} \text{ also inherits of the } L^1 \text{ bounds satisfied by } f^n \text{ since } 0 \le \frac{f^n}{1+f^n} \le f^n ...\right)$. Therefore, in order to prove Theorem II.1, we may assume without loss of generality that we have

$$(24) \qquad \frac{f^n}{1+f^n} \underset{n}{\rightarrow} \bar{\gamma} \text{ weakly in } L^\infty(\mathbf{R}^{2N}_{x,v} \times (0, \infty)) \text{ (weak}*)$$

and

$$(25) \qquad \frac{1}{1+f^n} \underset{n}{\rightarrow} \overline{\beta'} \text{ weakly in } L^\infty(\mathbf{R}^{2N}_{x,v} \times (0, \infty)) \text{ (weak}*).$$

We use the awkward notation $\overline{\beta'}$ because $\beta'(t) = \frac{1}{1+t}$ if $\beta(t) = \log(1+t)$.

Next, we remark that $\beta(f^n)=\log(1+f^n)$ is bounded in $L^p(\mathbf{R}^{2N}_{x,v}\times(0,T))$ for all $1\leq p<\infty$, $T\in(0,\infty)$ and that $0\leq\beta(f^n)\leq f^n$. Therefore, once more without loss of generality, we may assume that we have

$$(26)\qquad \beta(f^n)\underset{n}{\rightharpoonup}\bar{\beta}\text{ weakly in }L^n(\mathbf{R}^{2N}_{x,v}\times(0,T))$$

for all $1\leq p<\infty$, $T\in(0,\infty)$. In addition, since f_0^n converges in $L^1(\mathbf{R}^{2N})$ to f_0, we deduce easily that $\beta(f_0^n)$ also converges in $L^1(\mathbf{R}^{2N})$ to $\beta(f_0)$.

Furthermore, the functions $\left(t\rightarrow\frac{t}{1+t}\right)$ and $(t\rightarrow\log(1+t))$ - resp. $\left(t\rightarrow\frac{1}{1+t}\right)$ - are concave on $[0,\infty)$ - resp. is convex on $[0,\infty)$ -. Hence, we deduce from standard functional analysis facts the following inequalities:

$$(27)\qquad \overline{\beta'}\leq\frac{1}{1+f}=\beta'(f)$$

and

$$(28)\qquad \bar{\gamma}\leq\frac{f}{1+f}=\gamma(f),\ \bar{\beta}\leq\log(1+f)=\beta(f).$$

III. Proofs

We shall give two proofs of Theorem II.1. The first one is slightly simpler but the second one shows a bit more clearly how the "calculus and notion" of renormalized solutions allows to deduce Theorem II.1 from the main compactness result shown in Part I [51] namely (21). In fact the first proof will use the fact that we already know from [25] that f is a renormalized solution of (1) while the second one will show directly that f_n converges in L^1.

Proof 1. From the definition of renormalized solutions, we have

$$(29)\qquad \left(\frac{\partial}{\partial t}+v\cdot\nabla_x\right)\beta(f^n)=(1+f^n)^{-1}Q^+(f^n,f^n)-(1+f^n)^{-1}Q^-(f^n,f^n)\quad\text{in }\mathscr{D}'(\mathbf{R}^{2N}_{x,v}\times(0,\infty)).$$

Therefore, if we let n go to ∞ and we use (26) and (23), we deduce

$$(30)\qquad \left(\frac{\partial}{\partial t}+v\cdot\nabla_x\right)\bar{\beta}=\overline{RQ^+}-\overline{RQ^-}\quad\text{in }\mathscr{D}'(\mathbf{R}^{2N}_{x,v}\times(0,\infty)).$$

Next, we observe that $(1+f^n)^{-1}Q^-(f^n,f^n)=\left(\frac{f^n}{1+f^n}\right)L(f^n)$. This observation combined with (20), (23) and (24) implies that we have

$$(31)\qquad \overline{RQ^-}=\bar{\gamma}L(f).$$

Indeed, we have

$$\frac{f^n}{1+f^n}L(f^n) = \frac{f^n}{1+f^n}L(f)+\frac{f^n}{1+f^n}\{L(f^n)-L(f)\} ,$$

$$\left|\frac{f^n}{1+f^n}\{L(f^n)-L(f)\}\right|\leq|L(f^n)-L(f)| .$$

Hence $\frac{f^n}{1+f^n}\{L(f^n)-L(f)\}\underset{n}{\to}0$ in $L^1(\mathbf{R}^N_x\times(|v|<R)\times(0,T))$ for all $R,T\in(0,\infty)$, while, for all $\psi\in L^\infty(\mathbf{R}^N_x\times(|v|<R)\times(0,T)$ (extended by 0 to $\mathbf{R}^{2N}_{x,v}\times(0,\infty))$, $\psi L(f)\in L^1(\mathbf{R}^{2N}_{x,v}\times(0,\infty))$ and thus in view of (24)

$$\int_{\mathbf{R}^{2N}\times(0,\infty)}\frac{f^n}{1+f^n}L(f)\,\psi\,dxdvdt\underset{n}{\to}\int_{\mathbf{R}^{2N}\times(0,\infty)}\bar{\gamma}L(f)\,\psi\,dxdvdt .$$

And this completes the proof of (31).

Next, we consider $(1+f^n)^{-1}Q^+(f^n,f^n)$ and we claim that (22), (23) and (25) imply

$$\overline{RQ^+}=\bar{\beta}'Q^+(f,f) . \tag{32}$$

Indeed, it is of course enough to show (32) on $D_R\times(0,T)$ for any fixed $R,T\in(0,\infty)$ where $D_R=\{(x,v)\mid|x|,|v|<R\}$. We then use Egorov theorem to deduce that for each $\epsilon>0$, there exists a measurable set $E\subset D_R\times(0,T)$ such that $meas_{x,v,t}(E)\leq\epsilon$ and $Q^+(f^n,f^n)$ converges uniformly to $Q^+(f,f)$ on $(D_R\times(0,T))\cap E^c$. In addition, since $\frac{Q^+(f,f)}{1+f}\in L^1(D_R\times(0,T))$ (f is a renormalized solution of (1)), we may assume without loss of generality that $Q^+(f,f)\in L^1((D_R\times(0,T))\cap E^c)$. Hence, in particular,

$$\begin{aligned}\int_{D_R\times(0,T)}&(1+f^n)^{-1}Q^+(f^n,f^n)\,dx\,dv\,dt\\ &\geq\int_{D_R\times(0,T)\cap E^c}(1+f^n)^{-1}Q^+(f^n,f^n)\,dx\,dv\,dt\\ &\geq\int_{D_R\times(0,T)\cap E^c}(1+f^n)^{-1}Q^+(f,f)-\epsilon_n\end{aligned}$$

where $\epsilon_n\underset{n}{\to}0$, and we deduce from (25)

$$\int_{D_R\times(0,T)\cap E^c}\bar{\beta}'Q^+(f,f)\,dxdvdt\leq\lim_n\int_{D_R\times(0,T)}(1+f^n)^{-1}Q^+(f^n,f^n)\,dx\,dv\,dt .$$

In particular letting ϵ go to 0, we deduce that $\bar{\beta}'Q^+(f,f)\in L^1(D_R\times(0,T))$. Next, we have for all $\psi\in L^\infty(\mathbf{R}^{2N}_{x,v}\times(0,\infty))$ supported in $D_R\times(0,T)$

$$\left|\int\psi\{(1+f^n)^{-1}Q^+(f^n,f^n)-\bar{\beta}'Q^+(f,f)\}\,dxdvdt\right|$$

$$\leq \|\psi\|_{L^\infty}\int_E (1+f^n)^{-1}Q^+(f^n,f^n)-\overline{\beta}'Q^+(f,f)\ dxdvdt$$
$$+\left|\int_{E^c}\psi(1+f^n)^{-1}|Q^+(f^n,f^n)-Q^+(f,f)|\ dxdvdt\right|$$
$$+\left|\int_{E^c}\psi Q^+(f,f)\ \{(1+f^n)^{-1}-\overline{\beta}'\}\ dxdvdt\right|.$$

The third integral goes to 0 as n goes to $+\infty$ for each fixed $\epsilon>0$ in view of (25) (recall that $Q^+(f,f)\in L^1(E^c\cap(D_R\times(0,T)))$ and so does the second integral in view of the uniform convergence of $Q^+(f^n,f^n)$ to $Q^+(f,f)$ on $E^c\cap \mathrm{Supp}(\psi)$. Finally, the first integral can be made, uniformly in n, arbitrarily small as ϵ goes to 0 because of (23) and of the integrability of $\overline{\beta}'Q^+(f,f)$ on $\mathrm{Supp}(\psi)$. And the claim (32) is shown.

Next, we combine (30), (31), (32) with the inequalities (27) and (28) to deduce

$$\left\{\begin{aligned}&\left(\frac{\partial}{\partial t}+v\cdot\nabla_x\right)\overline{\beta}\geq\overline{\beta}'(f)Q^+(f,f)-\frac{f}{1+f}L(f)\\&=\beta'(f)Q(f,f)\ \text{in}\ \mathcal{D}'(\mathbf{R}^{2N}_{x,v}\times(0,\infty)).\end{aligned}\right.\tag{33}$$

On the other hand, we already know from [25] that $\beta(f)$ satisfies

$$\left(\frac{\partial}{\partial t}+v\cdot\nabla_x\right)\beta(f)=\beta'(f)Q(f,f)\ \text{in}\ \mathcal{D}'(\mathbf{R}^{2N}_{x,v}\times(0,\infty)).\tag{34}$$

This is really where we use the knowledge that f is also a renormalized solution of (1) (and the weak L^1 stability result shown in [25]). Comparing (33) and (34) we deduce

$$\left(\frac{\partial}{\partial t}+v\cdot\nabla_x\right)[\overline{\beta}-\beta(f)]\geq 0\ \text{in}\ \mathcal{D}'(\mathbf{R}^{2N}_{x,v}\times(0,\infty)).\tag{35}$$

In fact, we know a bit more since (30) implies that $\frac{\partial\overline{\beta}}{\partial t}+v\cdot\nabla_x\overline{\beta}\in L^1(\mathbf{R}^N_x\times(|v|<R)\times(0,T))$ for all $R,T\in(0,\infty)$. In addition, the bound (17) implies that $\sup_{0\leq t\leq T}\iint_{R^{2N}}\overline{\beta}(1+\omega(x)+|v|^2)\ dxdv<\infty$ for all $T\in(0,\infty)$. These two facts allow to check by easy arguments that $\overline{\beta}\in C([0,\infty);L^1(\mathbf{R}^{2N}_{x,v}))$. We next remark that $\overline{\beta}|_{t=0}=\beta(f_0)$ on $\mathbf{R}^{2N}$. Indeed, $(\frac{\partial}{\partial t}+v\cdot\nabla_x)\beta(f_n)$ is bounded in $L^1(\mathbf{R}^N_x\times(|v|<R)\times(0,T))$ (for all $R,T\in(0,\infty)$) and this is enough to ensure that $\beta(f^n)$ converges uniformly on $[0,T]$ in a "negative enough" local Sobolev space (in fact $W^{-s,1}_{\mathrm{loc}}(\mathbf{R}^{2N})$ for $s>1$). Of course, the limit is $\overline{\beta}$ and thus $\overline{\beta}|_{t=0}$ is the limit of $\beta(f^n_0)$ which converges in L^1 to $\beta(f_0)$. Our claim is then shown. Furthermore, $\beta(f)$ also belongs to $C([0,\infty);L^1(\mathbf{R}^{2N}))$ and

$\beta(f)|_{t=0}=\beta(f_0)$ on $\mathbf{R}^{2N}$. In other words, $\bar{\beta}-\beta(f)\in C([0,\infty);L^1(\mathbf{R}^{2N}))$ and $(\bar{\beta}-\beta(f))|_{t=0}=0$ on $\mathbf{R}^{2N}$. This combined with (35) yields easily

$$\bar{\beta}\geq\beta(f) \text{ a.e. on } \mathbf{R}^{2N}\times[0,\infty) \ . \tag{36}$$

Comparing with (28), we deduce that $\bar{\beta}\equiv\beta(f)$ or in other words

$$\log(1+f_n)\underset{n}{\rightarrow}\log(1+f) \text{ weakly in } L^p(\mathbf{R}^{2N}\times(0,T)) \tag{37}$$

for all $1\leq p<\infty$, $T\in(0,\infty)$, while $f_n\underset{n}{\rightarrow}f$ weakly in $L^1(\mathbf{R}^{2N}\times(0,T))$ for all $T\in(0,\infty)$.

We can now conclude by a more or less standard argument observing that $(t\rightarrow\log(1+t))$ is strictly concave on $[0,\infty)$. Indeed, we have on one hand for all $T\in(0,\infty)$ fixed

$$\overline{\lim_n}\int_{\mathbf{R}^{2N}\times(0,T)}\log\left(1+\frac{f_n+f}{2}\right)dxdvdt\leq\int_{\mathbf{R}^{2N}\times(0,T)}\log(1+f)\,dxdvdt \ . \tag{38}$$

On the other hand, for all $M\in(0,\infty)$, we can find $\nu=\nu(M)>0$ such that

$$\log\left(1+\frac{f_n+f}{2}\right)\geq\frac{1}{2}\log(1+f_n)+\frac{1}{2}\log(1+f)+\nu|f_n-f|1_{(f_n\leq M)}1_{(f\leq M)} \ .$$

Hence,

$$\begin{aligned}
&\int_{\mathbf{R}^{2N}\times(0,T)}|f_n-f|\,dxdvdt\\
&\leq\int_{\mathbf{R}^{2N}\times(0,T)}(f_n+f)(1_{(f_n>M)}+1_{(f>M)})\,dxdvdt+\\
&+\frac{1}{\nu}\int_{\mathbf{R}^{2N}\times(0,T)}\left\{\log\left(1+\frac{f_n+f}{2}\right)-\frac{1}{2}\log(1+f_n)-\frac{1}{2}\log(1+f)\right\}dxdvdt \ .
\end{aligned}$$

Therefore, because of (37) and (38),

$$\begin{aligned}
&\overline{\lim_n}\int_{\mathbf{R}^{2N}\times(0,T)}|f_n-f|\,dxdvdt\\
&\leq\overline{\lim_n}\int_{\mathbf{R}^{2N}\times(0,T)}(f_n+f)(1_{(f_n>M)}+1_{(f>M)})\,dxdvdt\\
&\leq C(\log M)^{-1}
\end{aligned}$$

for some $C>0$ independent of $M>0$, $n\geq1$. This last inequality is a consequence of the bound (17).

In conclusion, we have shown for all $T\in(0,\infty)$

$$f_n\underset{n}{\rightarrow}f \text{ in } L^1(\mathbf{R}^{2N}\times(0,T)) \ . \tag{39}$$

There only remains to show that the convergence is uniform in $t\in[0,T]$ for all $T \in (0, \infty)$. In order to do so, we first observe that we may assume without loss of generality that f_n and thus $\frac{1}{1+f_n}$, $\frac{f_n}{1+f_n}$ converge a.e. on $\mathbf{R}^{2N} \times [0, \infty)$ respectively to f, $\frac{1}{1+f}$, $\frac{f}{1+f}$. Hence, $\overline{\beta'} = \frac{1}{1+f}$, $\overline{\gamma} = \frac{f}{1+f}$ and thus for all $\delta \in (0, 1]$, $(1+\delta f^n)^{-1} Q^{\pm}(f^n, f^n)$ converges a.e. on $\mathbf{R}^{2N} \times [0, \infty)$ to $\overline{(1+\delta f^n)^{-1} Q^{\pm}(f^n, f^n)} \equiv RQ^{\pm}$. This combined with (23) implies that $(1+\delta f^n)^{-1} Q^{\pm}(f^n, f^n)$ converges (strongly) in $L^1(\mathbf{R}^N_x \times (|v|<R) \times (0, T))$ (for all $R, T \in (0, \infty)$) to $(1+\delta f)^{-1} Q^{\pm}(f, f)$. Hence, for all $R, T \in (0, \infty)$, we have

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla_x\right) \{\beta_\delta(f_n) - \beta_\delta(f)\} \underset{n}{\rightarrow} 0 \quad \text{in } L^1(\mathbf{R}^N_x \times (|v|<R) \times (0, T)) \tag{40}$$

while because of (39) and (17)

$$\beta_\delta(f_n) - \beta_\delta(f) \underset{n}{\rightarrow} 0 \text{ in } L^1(\mathbf{R}^{2N}_{x,v} \times (0, T)) \ . \tag{41}$$

$$\sup_{t \in [0,T]} \sup_{n \geq 1} \iint_{\mathbf{R}^{2N}} 1_{|x|+|v| \geq R} \{\beta_\delta(f_n) + \beta_\delta(f)\}\, dxdv \rightarrow 0 \text{ as } R \rightarrow +\infty \ . \tag{42}$$

It is then an easy exercise to check that (40) - (42) imply that $\beta_\delta(f^n) \underset{n}{\rightarrow} \beta_\delta(f)$ in $L^1(\mathbf{R}^{2N})$ uniformly in $t \in [0, T]$ for all $T \in (0, \infty)$ and for all $\delta \in (0, 1]$. We may now conclude since because of (17),

$$\sup_{n \geq 1} \sup_{t \in [0,T]} \iint_{\mathbf{R}^{2N}} |f^n - \beta_\delta(f^n)| + |f - \beta_\delta(f)|\, dxdv \rightarrow 0 \text{ as } \delta \rightarrow 0_+$$

(And we refer to [25], [26] where similar arguments are made).

Proof 2. *General convergence properties of renormalized soiutions.* The second proof we want to make consists in establishing a general fact on renormalized solutions of first-order linear equations. More precisely, we consider a sequence $(g^n)_{n \geq 1}$ of solutions of

$$\frac{\partial g^n}{\partial t} + a(y) \cdot \nabla_y g^n = G^n - a^n g^n \quad \text{in } \mathbf{R}^k \times (0, T) \tag{43}$$

where $T \in (0, \infty)$, $k \geq 1$, a is a given vector field satisfying for example for some $C \geq 0$

$$|a(x) - a(y)| \leq C|x-y| \text{ for all } x, y \in \mathbf{R}^k. \tag{44}$$

We could consider as well vector fields a depending on t or even less regular ones using the theory developed in [29]. We assume in addition that $g^n \in C([0, T]; L^1(\mathbf{R}^k))$, $g^n(0) = g^n_0$, G^n, a^n satisfy

$$\{g^n(t) \,|\, t \in [0, T], n \geq 1\} \text{ is relatively weakly compact in } L^1(\mathbf{R}^k), \tag{45}$$

$$(46)\qquad \left\{\begin{array}{l}\text{for all } R\in(0,\infty),\ G^n 1_{(|g^n|\le R)} \text{ is relatively weakly compact in}\\ L^1((|y|<R)\times(0,T)),\end{array}\right.$$

$$(47)\qquad \left\{\begin{array}{l}G^n \text{ is a nonnegative measurable function and converges a.e. on}\\ \mathbf{R}^k\times(0,T),\end{array}\right.$$

$$(48)\qquad \left\{\begin{array}{l}a^n\in L^1((|y|<R)\times(0,T)),\\ a^n \text{ converges in } L^1((|y|<R)\times(0,T)), \text{ for all } R\in(0,\infty),\end{array}\right.$$

$$(49)\qquad g_0^n \quad \text{converges in } L^1(\mathbf{R}^k).$$

This setting contains clearly the situation we are studying for the Boltzmann's equation: take indeed T arbitrary, $k=2N$, $y=(x,\ \xi)$, $a(y)=(\xi,0)$, $g^n=f^n$, $G^n=Q^+(f^n,f^n)$, $a^n=L(f^n)$. Then (17) implies (45), (19) implies (46), (21) implies (47), (20) implies (48) and (49) is assumed to hold in Theorem II.1 Hence, Theorem II.1 follows from the following result.

Of course, we need to explain the meaning of (43) since G^n and $a^n g^n$ do not necessarily belong to L^1_{loc}. We assume that (43) holds in renormalized sense (see [29]) i.e. that we have for all $\beta\in C^\infty(\mathbf{R},\mathbf{R})$ such that $\beta'\in C_0^\infty$

$$(50)\qquad \left(\frac{\partial}{\partial t}+a(y)\cdot\nabla_y\right)\beta(g^n)=[\beta'(g^n)G^n]-a^n[\beta'(g^n)g^n] \text{ in } \mathcal{D}'(\mathbf{R}^k\times(0,T)).$$

Since $\beta'(g^n)=0$ when g^n is large, $\beta'(g^n)G^n\in L^1_{loc}$ because of (46) while $\beta'(g^n)$. g^n is bounded and thus $a^n((\beta'(g^n)g^n)\in L^1_{loc}$

Theorem III. 2. *Under the assumptions* (45) - (49), g^n *converges in* $C([0,T];L^1(\mathbf{R}^k))$.

Proof. We are going to show that g^n is a Cauchy sequence in $C([0,T];L^1(\mathbf{R}^k))$. In fact, (45) implies that it is enough to show that g^n is a Cauchy sequence in $C([0,T];L^1(|y|<R))$ for all $R\in(0,\infty)$. And because of (44), the equation (43) has finite speed of propagation and we may assume without loss of generality that g^n, g_0^n, a^n, G^n are supported in a fixed set $(|y|\le R_0)\times[0,T]$ for g^n, a^n, G^n) - we can always multiply (43) by a cut-off function $\psi(y,t)$ equal to 1 on $(|y|\le R)\times[0,T]$, vanishing for $|y|$ large uniformly in $t\in[0,T]$ and solution of $\left(\frac{\partial\psi}{\partial t}+a(y)\cdot\nabla_y\psi=0\right)$... Then (46) and (48) are now global in y and all integrals below in y or (y,t) are in fact taken on $(|y|\le R_0)$ or on $(|y|\le R_0)\times[0,T]$.

We first prove Theorem III.2 in the case when $a^n\equiv 0$. In order to do so, we introduce $\beta^1 \in C_0^\infty(\mathbf{R})$, such that $0 \leq (\beta^1)' \leq 1$ on $\mathbf{R}$, $\beta^1(t) \equiv t$ on $[-1, +1]$, $(\beta^1)'(t) \equiv 0$ if $|t| \geq 2$ and we set $\beta^M = M\beta^1\left(\frac{t}{M}\right)$ for $M \geq 1$. We then write for all $n, m \geq 1$

$$\left(\frac{\partial}{\partial t} + a(y) \cdot \nabla_y\right)\{\beta^M(g^n) - \beta^M(g^m)\} = G^n(\beta^M)'(g^n) - G^m(\beta^M)'(g^m). \tag{51}$$

Therefore, we have for some $C \geq 0$ independent of $m, n \geq 1$, $M \geq 1$

$$\begin{cases} \sup\limits_{[0,T]} \| \beta^M(g^n) - \beta^M(g^m) \|_{L^1(\mathbf{R}^k)} \\ \leq C\| g_0^n - g_0^m \|_{L^1(\mathbf{R}^k)} + C\int_0^T dt \int_{\mathbf{R}^k} dy |G^n(\beta^M)'(g^n) - G^m(\beta^M)'(g^m)| . \end{cases} \tag{52}$$

Next, we claim that G^n is bounded in L^1 and thus $G \in L^1$ where G is the pointwise limit of G^n. The second part of the claim follows from the first one since $G^n \geq 0$. The L^1 bound is easy: indeed, we integrate the equation satisfied by $\beta^M(g^n)$ and we find for all $M \geq 1$

$$\int 1_{|g^n| \leq M} \, G^n \, dy \, dt \leq C$$

where $C \geq 0$ is independent of M. The L^1 bound follows upon letting M go to $+\infty$.

We then use Egorov's theorem in order to bound the second part of the right-hand side of (52). For all $\epsilon > 0$, there exists a measurable set $E \subset ((|y| \leq R_0) \times [0, T])$ such that $meas_{(y,t)}(E) \leq \epsilon$ and G^n converges uniformly to G on $E^c \cap ((|y| \leq R_0) \times [0, T])$. Next, because of (46),

$$\begin{cases} \sup\limits_{n \geq 1} \int_0^T dt \int_{\mathbf{R}^k} dy G^n 1_{(|g^n| \leq 2M)} 1_E = \omega_M(\varepsilon) \to 0 \\ \text{as } \epsilon \to 0_+, \text{ for all } M \geq 1. \end{cases} \tag{53}$$

In addition, since $G \in L^1$, we have

$$\sup_{n \geq 1} \int G 1_{(|g^n| \geq M)} \, dy dt = \delta(M) \to 0 \text{ as } M \to +\infty . \tag{54}$$

We now combine (52), (53) and (54) and we deduce

$$\begin{cases} \sup\limits_{t \in [0,T]} \| \beta^M(g^n) - \beta^M(g^m) \|_{L^1(\mathbf{R}^k)} \leq C \| g_0^n - g_0^m \|_{L^1(\mathbf{R}^k)} \\ + C\omega_M(\epsilon) + C\delta(M) + C\int_{E^c} |G^n - G| + |G^m - G| \, dy \, dt . \end{cases} \tag{55}$$

Hence, letting n, m go to $+\infty$ and ϵ go to 0_+, we obtain

$$\lim_{n,m \to \infty} \sup_{t \in [0,T]} \| \beta^M(g^n) - \beta^M(g^n) \|_{L^1(\mathbf{R}^k)} \leq C\delta(M) . \tag{56}$$

Furthermore, because of (45), we have

$$\sup_{n\geq 1}\sup_{t\in[0,T]}\|g^n-\beta^M(g^n)\|_{L^1(\mathbf{R}^k)}\to 0 \text{ as } M\to+\infty. \tag{57}$$

And we reach the desired conclusion combining (56) and (57).

We now turn to the general case $(a^n\not\equiv 0)$. We first introduce $h_n^+, h_n^-\in C([0,T];L^1)$ solutions of

$$\begin{cases}\dfrac{\partial h_n^+}{\partial t}+a\cdot\nabla_y h_n^+=(a^n)^+ \text{ in } \mathcal{D}'(\mathbf{R}^k\times(0,T)),\\ h_n^H|_{t=T}\equiv 0 \text{ on } \mathbf{R}^k\end{cases} \tag{58}$$

and

$$\begin{cases}\dfrac{\partial h_n^-}{\partial t}+a\cdot\nabla_y h_n^-=-(a^n)^- \text{ in } \mathcal{D}'(\mathbf{R}^k\times(0,T)),\\ h_n^-|_{t=0}\equiv 0 \text{ on } \mathbf{R}^k\end{cases} \tag{59}$$

And we set $h^n=h_n^++h_n^-$. Since, obviously, $h_n^+, h_n^-\leq 0$ we also have $h^n\leq 0$ on $\mathbf{R}^k\times[0,T]$ and $h^n\in C([0,T];L^1)$ solves

$$\frac{\partial h^n}{\partial t}+a\cdot\nabla_y h^n=a^n \text{ in } \mathcal{D}'(\mathbf{R}\times(0,T)). \tag{60}$$

Then, because of (48), h_n^+, h_n^- and thus h^n converge in $C([0,T];L^1(\mathbf{R}^k))$. Without loss of generality, we may assume that h^n converges a.e. on $\mathbf{R}^k\times(0,T)$.

We then introduce $\mathscr{g}^n=g^ne^{h^n}$, $\tilde{G}^n=G^ne^{h^n}$ and we remark that, since $0\leq e^{h^n}\leq 1$, (45) still holds with $\mathscr{g}^n$ in place of g^n. In addition, (47) holds for $\tilde{G}^n$ and $\mathscr{g}_0^n=\mathscr{g}^n|_{t=0}$ converges *a.e.* while $|\mathscr{g}_0^n|\leq|g_0^n|$ which converges in L^1 so (49) also holds for $\mathscr{g}_0^n$. We next claim that $\mathscr{g}^n(\in C([0,T];L^1))$ is a renormalized solution of

$$\frac{\partial \mathscr{g}^n}{\partial t}+a\cdot\nabla_y\mathscr{g}^n=\tilde{G}^n \text{ in } \mathbf{R}^k\times(0,T)\ . \tag{61}$$

We now prove the claim (61). Of course, if $G^n\in L^1$ then (61) is an easy exercise on linear equations in distributions sense. In order to show (61), we first observe that $e^{h^n}\in C([0,T];L^1)$ (recall that $h^n\leq 0$) solves (in distribution sense) because of (60)

$$\left(\frac{\partial}{\partial t}+a\cdot\nabla_y\right)e^{h^n}=a^ne^{h^n}\quad \text{in } \mathbf{R}^k\times(0,T)\ .$$

And thus we have for all $M\geq 1$ (in distributions sense) on $\mathbf{R}^k\times(0,T)$.

$$\left(\frac{\partial}{\partial t}+a\cdot\nabla_y\right)\left\{e^{h^n}\beta^M(g^n)\right\}$$

$$= (\beta^M)'(g^n)\tilde{G}^n + e^{h^n}a^n\{\beta^M(g^n) - g^n(\beta^M)'(g^n)\} .$$

Therefore, for all $\beta \in C^\infty(\mathbf{R}, \mathbf{R})$ such that $\beta' \in C_0^\infty$, we have

$$(62) \quad \left\{ \begin{aligned} & \left(\frac{\partial}{\partial t} + a \cdot \nabla_y\right)\left\{\beta(e^{h^n}\beta^M(g^n))\right\} \\ & = \beta'(e^{h^n}\beta^M(g^n))(\beta^M)'(g^n)\tilde{G}^n \beta'(e^{h^n}\beta^M(g^n))a^n\{\beta^M(g^n) - g^n(\beta^M)'(g^n)\}e^{h^n}. \end{aligned} \right.$$

We first study the second term of the right-hand side of (62): since $\beta^M(g^n) - g^n(\beta^M)'(g^n)$ and $\beta'(e^{h^n}\beta^M(g^n))$ are non zero respectively when $|g^n| \geq M$ and $|e^{h^n}\beta^M(g^n)| \leq C_0$ hence $|g^n| \geq M$, $e^{h^n}M \leq C_0$.
Then, on this set, $|\beta^M(g^n) - g^n(\beta^M)'(g^n)| \leq CM$ (for some $C \geq 0$ independent of $n \geq 1$, $M \geq 1$). And thus, in conclusion, we have

$$|\beta'(e^{h^n}\beta^M(g^n))a^n\{\beta^M(g^n) - g^n(\beta^M)'(g^n)\}e^{h^n}| \leq C|a^n|1_{|g^n| \geq M} ,$$

and in particular

$$(63) \qquad \sup_{n \geq 1}\|\beta'(e^{h^n}\beta^M(g^n))a^n\{\beta^M(g^n) - g^n(\beta^M)'(g^n)\}e^{h^n}\|_{L^1} \to 0 \text{ as } M \to +\infty.$$

Using (63), we are going to prove that we have for all $C \in (0, \infty)$

$$(64) \qquad \sup_{n \geq 1}\int dydt\ 1_{(|e^{h^n}g^n| \leq C)}\tilde{G}^n < +\infty,$$

and in particular that $1_{(|e^{h^n}g^n| \leq C)}\tilde{G}^n \in L^1$. Indeed, we go back to (62) choosing β in such a way that $\beta' \geq 0$, $\beta' \equiv 1$ on $[-C, +C]$. We then integrate (62) with respect to y and t and we use (63) to find

$$\int dydt\ 1_{(|e^{h^n}g^n| \leq C)}1_{(|g^n| \leq M)}\tilde{G}^n \int dydt\ \beta'(e^{h^n}\beta^M(g^n))(\beta^M)'(g^n)\tilde{G}^n \leq C:$$

And (64) follows upon letting M go to $+\infty$.

We may now complete the proof of (61). Indeed, we let M go to $+\infty$ in (62). Clearly, $\beta(e^{h^n}\beta^M(g^n)) \to \beta(e^{h^n}g^n) = \beta(\tilde{g}^n)$ as $M \to +\infty$. The second term of the right-hand side goes to 0 (see (63)). Finally, (61) holds because

$$\begin{aligned} & |\beta'(e^{h^n}\beta^M(g^n))(\beta^M)'(g^n)\tilde{G}^n - \beta'(e^{h^n}g^n)\tilde{G}^n| \\ & \leq C\left\{1_{(|e^{h^n}g^n| \leq 2C_0)}1_{(M \leq |g^n| \leq 2M)} + 1_{(|e^{h^n}g^n| \leq C_0)}1_{(|g^n| \geq M)}\right\}\tilde{G}^n \to 0 \text{ as } M \to +\infty \end{aligned}$$

in view of (64).

Once (61) is established, we wish to deduce from the proof we already made in the case when $a^n \equiv 0$ the convergence in $C([0, T]; L^1)$ of $\tilde{g}^n$. In order to do so, there only remains to show that (46) holds with $\tilde{G}^n$, $\tilde{g}^n$ replacing respectively G^n, g^n. This will be achieved by showing that $G^n(1+|g^n|)^{-1} \in L^1$ and is in fact weakly relatively compact in L^1. If it is the case, then (46) holds since we have for all $R \in (0, \infty)$

$$0 \leq \tilde{G}_n 1_{|\tilde{g}_n| \leq R} \leq (1+R)\frac{\tilde{G}_n}{1+|\tilde{g}_n|} = (1+R)\frac{e^{h_n}G_n}{1+e^{h_n}|g_n|} \leq (1+R)\frac{G_n}{1+|g_n|} .$$

Next, we are going to deduce our claims on $G^n(1+|g^n|)^{-1}$ using (50) with convenient choices of β. First of all, we consider for $M \geq 1$, $\gamma_M(t) = \int_0^t \frac{1}{1+|s|}\psi_M(s)\,ds$ where $\psi_M \in C_0^\infty(\mathbf{R})$ is even, $0 \leq \psi_M \leq 1$ on $\mathbf{R}$, $\psi_M \equiv 1$ on $[-M, +M]$. We then apply (50) with $\beta = \gamma_M$ and we find integrating over y and t

$$\int \frac{G^n}{1+|g^n|} 1_{|g^n| \leq M}\, dydt \leq \int |a^n| \frac{1}{1+|g^n|}|g^n|\, dydt + C \sup_{t \in [0,T]} \|\gamma_M(g^n(t))\|_{L^1} \leq C$$

since $|\gamma_M(t)| \leq \log(1+|t|) \leq |t|$. Letting M go to $+\infty$, we see that $\frac{G^n}{1+|g^n|} \in L^1$.

Next, we show that we have

$$\text{(65)} \qquad \sup_{n \geq 1} \int 1_{(|g^n| \geq M)} \frac{G^n}{1+|g^n|}\, dy\, dt \to 0 \text{ as } M \to +\infty .$$

Indeed, we now choose in (50) β given by $\gamma_R^M(t) = \int_0^t \frac{1}{1+|s|}\psi^M(s) \cdot \psi_R(s)\, ds$ where $\psi^M \in C_0^\infty(\mathbf{R})$ is even, $\psi^M(t) \equiv 0$ if $|t| \leq M-1$, $\psi^M(t) \equiv 1$ if $M \geq |t|$ and we always take $R > M$. We then obtain exactly as we did before

$$\int 1_{(M \leq |g^n| \leq R)} \frac{G^n}{1+|g^n|} dy\, dt \leq \int |a^n| 1_{M-1 \leq |g^n|}\, dydt C \sup_{t \in [0,T]} \|\gamma_R^M(g^n)\|_{L^1} .$$

We then let R go to $+\infty$ and we obtain easily, setting $\gamma^M(s) = \int_0^t \frac{1}{1+|s|}\psi^M(s)ds$,

$$\sup_{n \geq 1} \int 1_{(|g^n| \geq M)} \frac{G^n}{1+|g^n|}\, dydt \leq \sup_{n \geq 1} \int_{(|g^n| \geq M-1)} |a^n|\, dydt + C \sup_{n \geq 1} \sup_{t \in [0,T]} \|\gamma^M(g^n)\|_{L^1}.$$

This bound proves (65) since, on one hand, the first term goes to 0 as M goes to $+\infty$ because of (48) and (45), while, on the other hand, the second term goes to 0 because of (45) and the fact that $|\gamma^M(t)| \leq 1_{(|t| \geq M-1)} \log(1+|t|) \leq |t| 1_{(|t| \geq M-1)}$.

Finally, (65) and (46) imply that $G^n(1+|g^n|)^{-1}$ is relatively weakly compact in L^1 since we have obsviously for all $M \in (0, \infty)$

$$0 \leq G^n(1+|g^n|)^{-1} \leq (1+M)G^n 1_{(|g^n|\leq M)} + G^n 1_{(|g^n|\geq M)}(1+|g^n|)^{-1} .$$

In conclusion, we have shown that $\tilde{g^n} = g^n e^{h_n}$ converges in $C([0, T]; L^1)$. And we are going to use this information to complete the proof of Theorem III.2. We recall that e^{h^n} (≤ 1) converges in $C([0, T]; L^1)$. And we write for all $K \in (0, \infty)$

$$|g^n - g^m| \leq [|g^n| + |g^m|][1_{(|h^n|\geq K)} + 1_{(|h^m|\geq K)}] + e^K|\tilde{g^n} - \tilde{g^m}| + K|e^{h_n} - e^{h_m}| + 2|g^n|1_{(|g^n|\geq K)} .$$

Hence, for all $K \in (0, \infty)$

$$\overline{\lim_{n,m}} \sup_{t\in[0,T]} \|g^n - g^m\|_{L^1} \leq 2\sup_{n,m} \sup_{t\in[0,T]} \left(\int_{(|h^m|\geq K)} |g^n| dydt\right) + $$
$$+ 2\sup_{n} \sup_{t\in[0,T]} \left(\int_{(|g^n|\geq K)} |g^n|\, dydt\right) .$$

And we conclude letting K go to $+\infty$ in view of (45).

IV. Dissipation inequalities

In this section, we wish to explain how the strong L^1 convergence shown in the preceding sections allows to derive further informations on the global weak solutions built in [25], [26]. These properties take the form of specific differential inequalities that seem to be new even for smooth solutions of Boltzmann's equations. This is why we begin by considering a model example of such inequalities and derive it formally. Then, we justify this example for strong solutions by introducing the general class of differential inequalities we can obtain. And finally, we state and prove the existence of global renormalized solutions satisfying all these inequalities.

The model case of the differential inequalities we wish to obtain is derived formally as follows. Let f be a "nice" solution of Boltzmann's equation (1): by "nice", we mean a bounded solution in $C([0, \infty); L^1(\mathbf{R}^{2N}))$ decaying fast enough as $|(x, v)|$ goes to $+\infty$. We then consider test functions g that, temporarily, we may assume to be smooth in $(x, v, t) \in \mathbf{R}^{2N} \times [0, \infty)$ with compact support in $\mathbf{R}^{2N} \times [0,T]$ where T is arbitrary in $(0, \infty)$. Later on, we shall consider much more general test functions. We then set on $\mathbf{R}^{2N} \times [0, \infty)$

$$E(g) = \left\{\frac{\partial g}{\partial t} + v \cdot \nabla_x g - Q(g, g)\right\}. \tag{66}$$

Obviously, $E(g)$ is bounded, with compact support on $\mathbf{R}^{2N} \times [0, T]$ for all $T \in (0, \infty)$ - notice indeed that $v', v'_* \in Supp\ g\ (t)$ $(t \in [0, T])$ imply $|v|, |v'| \leq C(T)$.

Next, we use (1) and (66) to write

$$\left\{\frac{\partial}{\partial t}+v\cdot\nabla_x\right\}(f-g)=Q(f,f)-Q(g,g)-E(g)$$

and thus

$$\left\{\frac{\partial}{\partial t}+v\cdot\nabla_x\right\}(f-g)=Q(f-g,f)+Q(g,f-g)-E(g) \tag{67}$$

where for all functions φ, ψ, $Q(\varphi,\psi)$ is defined by

$$Q(\varphi,\psi)=\int_{\mathbf{R}^N}dv_*\int_{S^{N-1}}d\omega\, B(\varphi'\psi'_*-\varphi\psi_*). \tag{68}$$

Then, we wish to write an equation for $|f-g|$: in doing so, we use (67) and thus appears a term equal to sign $(f-g)$ at least when $f-g$ does not vanish. We neglect the ambiguous definition of sign $(f-g)$ on the set where $(f-g)$ vanishes and we shall see later on how to justify this decision (assume for the moment that $(f-g)$ has zero measure for instance). Then, formally, (67) implies

$$\left\{\begin{aligned} &\left\{\frac{\partial}{\partial t}+v\cdot\nabla_x\right\}|f-g|=\operatorname{sign}(f-g)\,Q(f-g,f)\\ &\qquad\qquad\qquad+\operatorname{sign}(f-g)\,\{Q(g,f-g)-E(g)\}\ .\end{aligned}\right. \tag{69}$$

We want to integrate this equation with respect to v and we observe that we have

$$\left\{\begin{aligned} &\int_{\mathbf{R}^N}dv\ sign(f-g)\,Q(f-g,f)\\ &=\iint_{\mathbf{R}^N\times\mathbf{R}^N}dvdv_*\int_{S^{N-1}}d\omega\, B\left[sign(f-g)\,(f'-g')f'_*-|f-g|f_*\right]\\ &\leq\iint_{\mathbf{R}^N\times\mathbf{R}^N}dvdv_*\int_{S^{N-1}}d\omega\, B\left\{|f'-g'|f'_*-|f-g|f_*\right\}=0.\end{aligned}\right. \tag{70}$$

Here, we use the change of variables $((v, v_*)\to(v', v'_*)\,(v', v'_*))$ which for each $\omega\in S^{N-1}$ is an isometry. Combining (69) and (70), we obtain the following (macroscopic) differential inequality on $\mathbf{R}^N\times(0,\infty)$

$$\left\{\begin{aligned} &\frac{\partial}{\partial t}\left(\int_{\mathbf{R}^N}|f-g|dv\right)+div_x\left(\int_{\mathbf{R}^N}v|f-g|dv\right)\\ &\leq\int_{\mathbf{R}^N}dv\ sign(f-g)\,\{Q(g,f-g)-E(\mathrm{g})\}.\end{aligned}\right. \tag{71}$$

Let us observe at this stage that f enters at most linearly the inequality (71) and thus makes sense if $f\in C([0,+\infty);L^1(\mathbf{R}^{2N}_{x,v}))$ and is then well suited for the global weak solutions built in [25], [26]. But before we discuss in more details this point, we first want to justify (71) by a slightly more rigorous argument that will, in particular, take care of the discontinuity of the sign function.

We then introduce a class denoted by $\boldsymbol{B}$ of smooth "absolute-value like" functions namely $\boldsymbol{B}=\{\alpha\in C^1(\mathbf{R};\mathbf{R}),\ \alpha(0)=0,\ |\alpha'(t)|\leq 1$ on $\mathbf{R}$, $\alpha'(t)=\text{sign}(t)$ for $|t|$ large$\}$ and we need to incorporate $\alpha(t)=|t|$ in this class. In order to do so, we extend $\boldsymbol{B}$ as follows - this is only one possible choice among many technical ones - : $\boldsymbol{B}=\{\alpha\in C(\mathbf{R};\mathbf{R})$, α is Lipschitz, C^1 except at t finite number of points, α has at each $t\in\mathbf{R}$ left and right derivatives, $\alpha(0)=0$, $|\alpha'(t)|\leq 1$ a.e. on $\mathbf{R}$, $\alpha'(t)=\text{sign}(t)$ for $|t|$ large$\}$. If $\alpha\in\boldsymbol{B}$, we define $\alpha'(t)$ at each of the discontinuity points t_i by imposing an arbitrary value in $[\alpha'(t_{i-}), \alpha'(t_{i+})]$: in this way, we define in the case when $\alpha(t)=|t|$, sign functions (i.e. α') where sign (0) is a fixed but arbitrary value in $[-1,+1]$. And we shall show that (71) holds for all these sign functions. We begin by considering α in $\boldsymbol{B}$ and we follow the derivation of (69) - (71) replacing $|t|$ by $p(v)^{-1}\alpha(tp(v))$ where p is a positive weight-function whose properties will be determined later on. Then, we find instead of (69), (70), (71)

$$
(72)\qquad \begin{cases} \left\{\dfrac{\partial}{\partial t}+v\cdot\nabla_x\right\}[p(v)^{-1}\alpha(p(v)(f-g))] \\ =\alpha'(p(v)(f-g))Q(f-g,f) \\ \qquad +\alpha'(p(v)(f-g))\{Q(g,f-g)-E(g)\}. \end{cases}
$$

$$
\begin{aligned}
&\int_{\mathbf{R}^N} dv\ \alpha'(p(v)(f-g))Q(f-g,f) \\
&\leq\iint_{\mathbf{R}^{2N}} dvdv_*\int_{S^{N-1}} B\,d\omega\,(\alpha')^+(f'-g')^+f'_* \\
&\quad+\iint_{\mathbf{R}^{2N}} dvdv_*\int_{S^{N-1}} B\,d\omega\,(\alpha')^-(f'-g')^-f'_*+ \\
&\quad+\iint_{\mathbf{R}^{2N}} dvdv\int_{S^{N-1}} B\,d\omega\ \alpha'(f-g)f_*.
\end{aligned}
$$

Therehore, we have (setting $p'=p(v')$)

$$
(73)\quad \begin{cases} \displaystyle\int_{\mathbf{R}^N} dv\ \alpha'\ (p(v)(f-g))Q(f-g,f) \\ \displaystyle\leq\iint_{\mathbf{R}^{2N}} dvdv\int_{S^{N-1}} B\,d\omega\ (f-g)^+f_*[\alpha'(p'(f'-g'))^+-\alpha'(p(f-g))] \\ \displaystyle+\iint_{\mathbf{R}^{2N}} dvdv_*\int_{S^{N-1}} B\,d\omega\ (f-g)^-f_*[\alpha'(p'(f'-g'))^-+\alpha'(p(f-g))]. \end{cases}
$$

In particular, since $|\alpha'|\leq 1$ on $\mathbf{R}$, we deduce

$$
(74)\qquad \begin{cases} \displaystyle\int_{\mathbf{R}^N} dv\ \alpha'(p(v)(f-g))Q(f-g,f) \\ \displaystyle\leq\iint_{\mathbf{R}^{2N}} dvdv_*\int_{S^{N-1}} B\,d\omega\ (f-g)^+f_*[1-\alpha'(p(f-g))]+ \\ \displaystyle+\iint_{\mathbf{R}^{2N}} dvdv_*\int_{S^{N-1}} B\,d\omega\ (f-g)^-f_*[1+\alpha'(p(f-g))]. \end{cases}
$$

We denote by R_+, R_- respectively the two terms in the right-hand side of (74). Let us observe that $1-\alpha'(p(f-g))\geq 0$, $1+\alpha'(p(f-g))\geq 0$ and that by assumption $1-\alpha'(p(f-g))]=0$ if $(f-g)p>0$ is large enough while $1+\alpha'(p(f-g))=0$ if $(f-g)p<0$ is large enough. In other words, the integrals defining R_+ and R_- are taken on a set given by $\{|f-g|\leq t_0\, p(v)^{-1}\}$ for some $t_0 \in (0, \infty)$ which depends only on α. And we deduce from (72)-(74)

$$(75)\qquad \begin{cases} \left\{\dfrac{\partial}{\partial t}+v\cdot\nabla_x\right\}[p(v)^{-1}\alpha(p(v)(f-g))] \\ \leq \alpha'(p(v)(f-g))\{Q(g,f-g)-E(g)\}+R_++R_-. \end{cases}$$

We may now choose p satisfying the following requirements: $p\in C(\mathbf{R}^N)$, $p(v)>0$ in $\mathbf{R}^N$, and p satisfies

$$(76)\qquad A*\frac{1}{p}\in C(\mathbf{R}^N) \text{ and } \left[A*\frac{1}{p}\right](1+|v|^2)^{-1}\to 0 \text{ as } |v|\to+\infty.$$

Recall that $A(z)=\int_{S^{N-1}} B(z,\omega)\,d\omega$ and that we assume (7). And (7) implies the existence of such functions p: indeed, let $\chi_n\in C_0^\infty(\mathbf{R}^N)$, $0\leq\chi_n\leq 1$, $\chi_n\equiv 1$ on $\{|v|\leq n\}$. Then, because of (7), $(A*\chi_n)(1+|v|^2)^{-1}\in C_0(\mathbf{R}^N)$. We then set $M_n=\max_{\mathbf{R}^N}\{|A*\chi_n|(1+|v|^2)^{-1}\}$ and $\Phi=\Sigma_{n\geq 1}\chi_n(1+M_n)^{-1}2^{-n}$. We have clearly $\Phi\in C_0(\mathbf{R}^N)$, $(A*\Phi)(1+|v|^2)^{-1}\in C_0(\mathbf{R}^N)$, $\Phi>0$ on $\mathbf{R}^N$. Therefore, $p=\Phi^{-1}$ satisfies the conditions mentioned above and (76) in particular. Let us also mention a few examples: first of all, in the hard spheres model i.e. $B(z,\omega)=|(z,\omega)|$, we find $A(z)=c_0|z|$ for some $c_0>0$ and we may take $p=\Phi^{-1}$ where $\Phi>0$ on $\mathbf{R}^N$, $\Phi\in C(\mathbf{R}^N)$, $\int_{\mathbf{R}^N}(|v|+1)\,\Phi\,dv<\infty$. Also if $A\in L^1(\mathbf{R}^N)$ then we can take $p=\Phi^{-1}$ where $\Phi>0$ on $\mathbf{R}^N$, $\Phi\in C(\mathbf{R}^N)$, Φ is bounded on $\mathbf{R}^N$.

We next explain how (75) yields (71) with an arbitrary (in $[-1,+1]$) normalization for sign (0). Indeed, we can find $\alpha_k\in \boldsymbol{B}$ such that $\alpha'_k(0)=s_0$ fixed in $[-0,+0]$ and $\alpha_k(t)\equiv \operatorname{sign}(t)$ if $|t|\geq 1/k$ for all $k\geq 1$. Let us observe that $\alpha'_k(t)\xrightarrow[k]{}\operatorname{sign}(t)$ for all $t\neq 0$ while $\alpha'_k(0)=s_0$. In addition, we see that the intergrals R_+, R_- are then defined on a set $\{|f-g|\leq\frac{1}{k}p(v)^{-1}\}$ and that $|\{|f-g|\leq p(v)^{-1}\alpha_k\{p(v)(f-g)\}|\leq\frac{2}{k}$. We may then let k go to $+\infty$ in (75) and we recover (71) provided of course we show that $R_+=R_+^k$, $R_-=R_-^k$ go to 0. But this is the case since we have in view of (76)

$$|R_+^k|,\ |R_-^k|\leq\iint_{\mathbf{R}^{2N}} dv\,dv_*\,A(v-v_*)f_*\frac{1}{k}p(v)^{-1}\xrightarrow[k]{} 0.$$

A similar procedure shows that (75) holds in fact for all $\alpha\in\overline{\boldsymbol{B}}$.

All these considerations show that "nice" solutions (i.e. bounded with a fast enough decary at infinity) of Boltzmann's equations satisfy (75) (and thus (71) for all $\alpha \in \overline{\boldsymbol{B}}$, p satisfying (76) and for all test functions g. In fact, in order to be totally rigorous, we need now to explain and detail the class of test functions. We thus introduce a class $\boldsymbol{A}$ of admissible functions g

$$(77)\qquad \begin{cases} \boldsymbol{A}=\Big\{g\in C([0,\infty);L^1(\mathbf{R}^{2N}_{x,v})),\ g|v|^2\in L^\infty(0,T;L^1(\mathbf{R}^{2N}_{x,v})), \\ \dfrac{A*|g|}{1+|v|^2}\in L^1(0,T;L^\infty(\mathbf{R}^{2N}_{x,v})), \\ \dfrac{\partial g}{\partial t}+v\cdot\nabla_x g\in L^1(\mathbf{R}^{2N}_{x,v}\times(0,T)) \text{ for all } T\in(0,\infty)\Big\}\ . \end{cases}$$

Observe in particular that, if $g\in\boldsymbol{A}$, we have for all $T\in(0,\infty)$

$$|Q^-(g,g)|\le|g|(A*|g|)=\{|g|(|v|^2+1)\}\{(1+|v|^2)^{-1}(A*|g|)\}\in L^1(\mathbf{R}^{2N}_{x,v}\times(0,T))$$

$$\int_{\mathbf{R}^N}|Q^+(g,g)|dv\le\int_{\mathbf{R}^N}Q^+(|g|,|g|)dv=\int_{\mathbf{R}^N}Q^-(|g|,|g|)\,dv$$

$$=\int_{\mathbf{R}^N}|g|(A*|g|)\,dv\ \in L^1(\mathbf{R}^N_x\times(0,T)).$$

Therefore, (75) makes sense as soon as $f\in C([0,\infty);L^1(\mathbf{R}^{2N}_{x,v}))$, $f|v|^2\in L^\infty(0,T;L^1(\mathbf{R}^{2N}_{x,v}))$ $(\forall\, T\in(0,\infty))$. In all that follows, we simply say that f satisfies (75) if (75) holds for all $\alpha\in\boldsymbol{B}$, $g\in\boldsymbol{A}$, $p(>0)\in C(\mathbf{R}^N)$ satisfying (76). And we prove the

Theorem IV.1. *Let $f_0(\ge 0)$ satisfy (10), then there exists a renormalized soiution $f\in C([0,\infty);L^1(\mathbf{R}^{2N}_{x'v}))$ of Boltzmann's equation (1) corresponding to the initial condition (9) which satisfies (75).*

Sketch of Proof. We only need to follow the construction made in [25], [26] of a renormalized solution. Indeed, in [25], we approximated (1) by solving

$$(78)\qquad \frac{\partial f^n}{\partial t}+v\cdot\nabla_x f^n=Q_n(f^n,f^n)\quad\text{in } \mathbf{R}^{2N}_{x,v}\times(0,\infty),\ f^n|_{t=0}=f^n_0$$

where

$$f^n_0=\Big\{\min\Big(f_0,\ ne^{-\frac{1}{2}(x^2+|\xi|^2)}\Big)*(2\pi n^{-1})^{-N/2}e^{-\frac{1}{2}(x^2+|\xi|^2)}\Big\}+\frac{1}{n}e^{-\frac{|x|^2+|\xi|^2}{2}}$$

(for example...),

$$Q_n(\varphi,\varphi)=\Big(\int_{\mathbf{R}^N}dv_*\int_{S^{N-1}}d\omega\ B_n(v-v_*,\omega)\,(\varphi'\varphi'_*-\varphi\varphi_*)\Big).$$

$(1+\frac{1}{n}\int_{\mathbf{R}^N}|\varphi|dv)^{-1}$ and $B_n\in C^\infty(\mathbf{R}^N\times S^{N-1})$, $B_n\ge 0$ depends only on $|z|$ and $|(z,\omega)|$, vanishes for $|z|\le 1/n$, for $|z|\ge n$, $|z\cdot\omega|\le\frac{1}{n}|z|$, on $|z\cdot\omega|\ge(1-\frac{1}{n})|z|$ and (7) holds uniformly in n, $B_n\xrightarrow[n]{}B$ in $L^1((|z|\le R)\times S^{N-1})$ $(\forall\, R\in(0,\infty))$.

As shown in [25], (78) is uniquely solvable and admits smooth solutions (with fast decay at infinity). Furthermore, the structure of this truncated equation is essentially the same as the one of (1). In particular, f^n satisfies (75) with Q, E, R_+, R_- replaced respectively by

$$Q_n(g, f^n-g)=\left(1+\frac{1}{n}\int_{\mathbf{R}^N} f^n dv\right)^{-1}\cdot\left\{\int_{\mathbf{R}^N} dv_* \int_{S^{N-1}} d\omega B_n[g'(f^n-g)'_*-g(f^n-g)_*]\right\}$$

$$E_n(g)=\left(\frac{\partial g}{\partial t}+v\cdot\nabla_x g\right)-\left(1+\frac{1}{n}\int_{\mathbf{R}^N} f^n dv\right)^{-1}\cdot\left\{\int_{\mathbf{R}^N} dv_* \int_{S^{N-1}} d\omega\, B_n[g'g'_*-gg_*]\right\}$$

$$R_+^n=\left[\iint_{\mathbf{R}^{2N}} dv\, dv_* \int_{S^{N-1}} B_n d\omega (f^n-g)^+\cdot f_*^n[1-\alpha'(p(f^n-g))]\right]\left(1+\frac{1}{n}\int_{\mathbf{R}^N} f^n dv\right)^{-1}$$

$$R_-^n=\left[\iint_{\mathbf{R}^{2N}} dv\, dv_* \int_{S^{N-1}} B_n d\omega (f^n-g)^-\cdot f_*^n[1-\alpha'(p(f^n-g))]\right]\left(1-\frac{1}{n}\int_{\mathbf{R}^N} f^n dv\right)^{-1}$$

Next, the proofs made in Part I [51] and in section III apply and in particular (17), (18) hold and, extracting a subsequence (still denoted by f^n) if necessary, f^n converges in $C([0,T];L^1(\mathbf{R}^{2N}_{x,v}))$ $(\forall T\in(0,\infty))$ to some f which is a renormalized solution of (1) corresponding to (9). We may then pass to the limit in (75) choosing α in $\boldsymbol{B}$. Once (75) holds for all $\alpha\in\boldsymbol{B}$, one can then check that it also holds for all $\alpha\in\boldsymbol{B}$ by an approximation argument that we detailed above (in the case when $\alpha(t)=sign(t)$).

We have thus shown how certain differential inequalities like (75) or (71) can be obtained. We now wish to explain how differential *equalities* can be obtained for more general classes of functions and how it is possible to combine this idea with the idea of renormalized solutions. Let us also emphasize the fact that the formulations we shall obtain use in fact the entropy bounds and more importantly the entropy dissipation bound (15).

All the equalities we shall derive are of course obtained from (1) by multiplying it by appropriate quantities exactly like an equation is formulated in the sense of distributions by multiplying it by C_0^∞ functions. We thus have to define a convenient class of multipliers that we denote by: m belongs to if

$$m=\varphi(x,t)\alpha'(p(v)\{f(x,v,t)-g(x,v,t)\})+\beta'(f(x,v,t)-g(x,v,t))\psi(x,v,t)$$

where $\varphi\in C_0^\infty(\mathbf{R}^N_x\times[0,\infty))$, $\psi\in C_0^\infty(\mathbf{R}^{2N}_{x,v}\times[0,\infty))$, $g\in\boldsymbol{A}$, $p\in C(\mathbf{R}^N)>0$ satisfies (76), α, $\beta\in C^1$ (or more generally admit left and right derivatives on $\mathbf{R}$, are differentiable except at a finite number of points where α', β' are defined by an arbitrary value between the left and the right derivatives), $\alpha(0)=\beta(0)=0$, α' and β' are bounded, $\alpha'(t)$ is constant for $t>0$ large and finally β' satisfies for some $C\geq 0$

(79) $$|\beta'(t)|\leq C(1+t)^{-1/2} \quad \text{for} \quad t\geq 0.$$

In fact, we could allow more general multipliers m by requiring less regularity on φ, ψ, α, β and by considering functions β that can depend on x, v, t.

These extensions however are rather technical and we skip them since it is not clear that they really add new informations.

First of all, we have to define $<\frac{\partial f}{\partial t}+v\cdot\nabla_x f, m>$. This is easy and we simply set

$$<\frac{\partial f}{\partial t}+v\cdot\nabla_x f, m>=<\left(\frac{\partial}{\partial t}+v\cdot\nabla_x\right)\left[\frac{1}{p}\alpha(p(f-g))\right], \varphi>$$
$$+<\left(\frac{\partial}{\partial t}+v\cdot\nabla_x\right)\beta(f-g), \psi>+<\frac{\partial g}{\partial t}+v\cdot\nabla_x g, \alpha'(p(f-g))\varphi+\beta'(f-g)\psi>$$

or in other words

$$(80)\quad\begin{cases} <\frac{\partial f}{\partial t}+v\cdot\nabla_x f, m> \\ =\int_0^{+\infty}dt\int_{\mathbf{R}^{2N}}dx\,dv\left\{-\frac{1}{p}\alpha[p(f-g)]\left(\frac{\partial\varphi}{\partial t}+v\cdot\nabla_x\varphi\right)\right. \\ -\beta(f-g)\left(\frac{\partial\varphi}{\partial t}+v\cdot\nabla_x\psi\right)+\frac{\partial g}{\partial t}+v\cdot\nabla_x g\Big)[\alpha'(p(f-g))\varphi \\ \left.+\beta'(f-g)\psi]-\int_{\mathbf{R}^{2N}}dx\,dv\left\{\frac{1}{p}\alpha(p(f_0-g_0))\varphi+\beta(f_0-g_0)\psi\right\},\right. \end{cases}$$

where $f_0=f|_{t=0}$, $g_0=g|_{t=0}$ on $\mathbf{R}^{2N}$. Recall that $f, g\in C([0,\infty); L^1(\mathbf{R}^{2N}_{x,v}))$. In view of the properties satisfied by φ, ψ, g, α, β, all the integrals written in (80) make sense: let us only observe that $\left|\frac{1}{p}\alpha(p(f-g))\right|\leq C|f-g|$, $|\beta(f-g)\leq C|f-g|$ for some $C\geq 0$ and let us recall that we assume that (12), (13) and (15) holds while $g|v|^2\in L^\infty(0, T; L^1(\mathbf{R}^{2N}_{x,v}))$ $(\forall T\in(0,\infty))$.

We then have to define $<Q(f,f), m>$. Using the simple change of variables $[(v, v_*)\rightarrow(v', v'_*)]$ we set

$$(81)\quad\begin{cases} <Q(f,f), \varphi\alpha'(p(f-g))>=\int_0^\infty dt\int_{\mathbf{R}^N}dx\,(\varphi(x,t) \\ \cdot\left\{\int_{\mathbf{R}^N}dv\int_{\mathbf{R}^N}dv_*\int_{S^{N-1}}d\omega\; Bff_*\{\alpha'(p'(f'-g'))-\alpha'(p(f-f))\right\}. \end{cases}$$

And we claim that $Bff_*(\alpha'(p'(f'-g'))-\alpha'(p(f-g)))\in L^1(\mathbf{R}^N_x\times\mathbf{R}\times\mathbf{R}^{2N}_{x,v}\times S^{N-1}_\omega\times(0, T))$ for all $T\in(0,\infty)$. Indeed, since α' is constant say for $t\geq t_0>0$, we have

$$Bff_*|\alpha'(p'(f'-g'))-\alpha'(p(f-g)))|\leq Bff_*\{1_{(f'-g'\leq t_0/p')}+1_{(f-g\leq t_0/p)}\}.$$

Then, on the one hand

$$Bff_*1_{(f-g\leq t_0/p)}\leq B|g|f_*+t_0B\frac{1}{p}f_*$$

and for all $T\in(0,\infty)$

$$\int_0^T dt \int_{\mathbf{R}^N} dx \iint_{\mathbf{R}^{2N}} dv dv_* \int_{S^{N-1}} d\omega\, B\Big(|g| f_* + \frac{1}{p} f_*\Big)$$
$$= \int_0^T dt \int_{\mathbf{R}^{2N}} dx dv\, \Big(A * |g| + A * \Big(\frac{1}{p}\Big)\Big) f \le C \int_0^T dt \int_{\mathbf{R}^{2N}} dx dv\, (1+|v|^2) f < +\infty.$$

On the other hand, we use the entropy dissipation bound (15) to deduce

$$Bff_*1_{(f'-g'\le t_0/p')} \le 2Bf'f'_*1_{(f'-g'\le t_0/p')} + \frac{1}{\log 2} D$$

where $D = B(f'f'_* - ff_*)\log \frac{f'f'_*}{ff_*} \in L^1(\mathbf{R}^N_x \times \mathbf{R}^{2N}_{x,v} \times S^{N-1}_\omega \times (0, T))$ for all $T \in (0, \infty)$. Next, we have for all $T \in (0, \infty)$

$$\int_0^T dt \int_{\mathbf{R}^N} dx \iint_{\mathbf{R}^{2N}} dv dv_* \int_{S^{N-1}} d\omega\, Bf'f'_*1_{(f'-g'\le t_0/p')} =$$
$$\int_0^T dt \int_{\mathbf{R}^N} dx \iint_{\mathbf{R}^{2N}} dv dv_* \int_{S^{N-1}} d\omega\, Bff_*1_{(f-g\le t_0/p)}$$

where we used once more the standard change of variables $[(v, v_*) \to (v', v'_*)]$. Our claim is then shown since the last integral is finite as we already proved.

There remains to define $<Q(f, f), \beta'(f-g)\phi>$. We simply write

$$(82)\qquad \begin{cases} <Q(f, f), \beta'(f-g)\phi> \\ = \displaystyle\int_0^T dt \int_{\mathbf{R}^N} dx \int_{\mathbf{R}^{2N}} dv\, dv_* \int_{S^{N-1}} B\, d\omega\phi\, \{f'f'_* - ff_*\}\, \beta'(f-g). \end{cases}$$

And we claim that $B1_{(|v|\le R)}\{f'f'_* - ff_*\}\beta'(f-g) \in L^1(\mathbf{R}^N_x \times \mathbf{R}^{2N}_{x,v} \times S^{N-1}_\omega \times (0, T))$ for all $R, T \in (0, \infty)$. We first remark that we have for all $K > 1$

$$(82)\qquad \begin{cases} f'f'_* \le Kff_* + \dfrac{1}{\log K}(f'f'_* - ff_*)\log \dfrac{f'f'_*}{ff_*} \\ f'f'_* \ge K^{-1}ff_* - \dfrac{1}{\log K}(f'f'_* - ff_*)\log \dfrac{f'f'_*}{ff_*} \end{cases}$$

hence

$$(83)\qquad |f'f'_* - ff_*| \le (K-1)ff_* + \frac{1}{\log K}(f'f'_* + ff_*)\log \frac{f'f'_*}{ff_*}.$$

We then choose $K = e^{|\beta'(f-g)|}$ and we deduce from (83)

$$B1_{(|v|\le R)}|f'f'_* - ff_*||\beta'(f-g)|$$
$$\le B1_{(|v|\le R)}ff_*\beta'(f-g)|\,[e^{|\beta'(f-g)|} - 1] + D$$

where D, because of (15), belongs to $L^1(\mathbf{R}^N_x \times \mathbf{R}^{2N}_{x,v} \times S^{N-1}_\omega \times (0, T))$ for all $T \in (0, \infty)$. Finally, we have since β' is bounded and satisfies (79)

$$B1_{(|v|\le R)}ff_*|\beta'(f-g)|\, \{e^{|\beta'(f-g)|} - 1\}$$

$$\leq CB1_{(|v|\leq R)}ff_*|\beta'(f-g)|^2$$
$$\leq CB1_{(f\leq 2g)}ff_*+CB1_{(|v|\leq R)}(1+(f-g))^{-1}ff_*1_{(f\leq 2g)}$$
$$\leq CB1_{(f\leq 2g)}ff_*+CB1_{(|v|\leq R)}(1+f/2))^{-1}ff_*$$
$$\leq CBgf_*+CB1_{(|v|\leq R)}f_*$$

where C denotes various positive constants.

We already showed before that $B\,gf_*\in L^1(\mathbf{R}_x^N\times\mathbf{R}_{x'v}^{2N}\times S_\omega^{N-1}\times(0,T))$ ($\forall T\in(0,\infty)$) and we conclude the proof of our claim observing that we have because of (7) for some $C\geq 0$

$$\int\int_{\mathbf{R}^{2N}}\int_{S^{N-1}}B1_{(|v|\leq R)}f_*d\omega\,dvdv_*=\int_{\mathbf{R}^N}(A*1_{(|v|\leq R)})f_*dv_*$$
$$\leq C\int_{\mathbf{R}^N}(1+|v|^2)f\,dv.$$

We have thus define $<Q(f,f),m>$ for all multipliers $m\in\boldsymbol{M}$. And we simply say that the Boltzmann's equation (1) holds for all $m\in\boldsymbol{M}$ is we have for all $m\in\boldsymbol{M}$

$$<\frac{\partial f}{\partial t}+v\cdot\nabla_x f,m>=<Q(f,f),m>. \tag{84}$$

Exactly as we proved Theorem IV.1, we can prove the

Theorem IV.2. *Let $f_0\geq 0$ satisfy (10), then there exists $f\in C([0,\infty);L^1(\mathbf{R}_{x,v}^{2N}))$ satisfying (9), (12), (13), (15) and (1) for all $m\in\boldsymbol{M}$.*

Remarks, i) In addition, f satisfies the following properties (see [25], [26])

$$\int_{\mathbf{R}^{2N}}dxdv\,v_k f \text{ is independent of } t\geq 0, \text{ for all } 1\leq k\leq N, \tag{85}$$

$$\left\{\begin{aligned}&\int_{\mathbf{R}^{2N}}dxdv\,f\log f(t)+\frac{1}{4}\int_0^T ds\int_{\mathbf{R}^N}dx\int_{\mathbf{R}^{2N}}dvdv_*\int_{S^{N-1}}d\omega\\&\cdot B(f'f'_*-ff_*)\log\frac{f'f'_*}{ff_*}\leq\int_{\mathbf{R}^{2N}}dxdv\,f_0\log f_0.\end{aligned}\right. \tag{86}$$

ii) If we take $\alpha'\equiv 1$, $g\equiv 0$, $\beta\neq 0$ and thus $m=\varphi(x,t)$, the equality (84) implies

$$\frac{\partial}{\partial t}\left(\int_{\mathbf{R}^{2N}}f\,dx\,dv\right)+\mathrm{div}_x\left(\int_{\mathbf{R}^{2N}}fv\,dx\,dv\right)=0 \tag{87}$$

i.e. the local convervation of mass.

iii) We can check easily that if we take $\beta\equiv 0$ and $\alpha\in\boldsymbol{A}$ (or $\overline{\boldsymbol{A}}$) then (84) can be rewritten in such a way that it yields (75) and thus (84) yields (71) as a very particular case. Let us detail this particular example: we thus take $\beta\equiv 0$, $\alpha(t)\equiv|t|$ and (84) implies

$$
(88)\qquad \begin{cases} \frac{\partial}{\partial t}\Big(\int_{\mathbf{R}^{2N}} f\,dx\,dv\Big)+div_x\Big(\int_{\mathbf{R}^{2N}} f\,v\,dxdv\Big) \\ =-\int_{\mathbf{R}^{2N}}\Big(\frac{\partial g}{\partial t}+v\cdot\nabla_x g\Big)\operatorname{sign}(f-g)\,dv+\int\int_{\mathbf{R}^{2N}} dv\,dv_*\,\cdot \\ \cdot\int_{S^{N-1}} B\,d\omega f f_*\,\{\operatorname{sign}(f'-g')-\operatorname{sign}(f-g)\}\,. \end{cases}
$$

Then we write (and this is allowed in view of the integrabilities shown above)

$$
\begin{aligned}
&\int\int_{\mathbf{R}^{2N}} dv dv_* \int_{S^{N-1}} B\,d\omega f f_*\,\{\operatorname{sign}(f'-g')-\operatorname{sign}(f-g)\} = \\
&\int\int_{\mathbf{R}^{2N}} dv dv_* \int_{S^{N-1}} B\,d\omega[(f-g)f_*+g f_*]\,\{\operatorname{sign}(f'-g')-\operatorname{sign}(f-g)\} \\
&=\int\int_{\mathbf{R}^{2N}} dv dv_* \int_{S^{N-1}} B\,d\omega\, f_*\,\{(f-g)\operatorname{sign}(f'-g')-|f-g|\} + \\
&+\int\int_{\mathbf{R}^{2N}} dv dv_* \int_{S^{N-1}} B\,d\omega\,\operatorname{sign}(f-g)\,[g'f'_*-g f_*].
\end{aligned}
$$

Observing that $(f-g)\ \operatorname{sign}(f'-g')-(f-g)\leq 0$, we then see that (80) yields (71).

iv) In view of the "uniqueness" result shown in the next section, it is quite clear that (71) (or (75), or (88)...) can be used as a *definition* of weak solutions of Boltzmann's equation. This notion of solutious-which could be called dissipative solutious-can be used in many contexts like Fluid Mechanics models and we shall come back to this issue in future publications.

v) If we restrict the class $\boldsymbol{M}$ of multipliers to the case $\alpha\equiv 0$, $g\equiv 0$ so that $m=\psi(x,v,t)\ \beta'(f)$, we see that (84) implies that f is a renormalized solution. In fact, using the entropy dissipation bound (15), we are able to allow a more general decay on β' namely (79) instead of ($|\beta'(t)|\leq C(1+t)^{-1}$).

V. Weak solution=strong solution (when it exists)

We begin with a general uniqueness result:

Theorem V.1 *Let* $T>0, f\in C([0,T];L^1(\mathbf{R}^{2N}_{x,v}))$ *with* $f|v|\in L^1(\mathbf{R}^{2N}_{x,v}\times(0,T))$ *satisfy* (71) *on* $\mathbf{R}^N_x\times(0,T)$ *for all* $g\in\boldsymbol{A}$. *Let* $\bar f\in C([0,T];L^1(\mathbf{R}^{2N}_{x,v}))$ *satisfy*

$$
(89)\qquad A*|\bar f|\in L^1(0,T;L^\infty(\mathbf{R}^{2N}_{x,v})),\ |v|\bar f\in L^1(\mathbf{R}^{2N}_{x,v}\times(0,T)),
$$

and

$$
(90)\qquad \frac{\partial\bar f}{\partial t}+v\cdot\nabla_x\bar f=Q(\bar f,\bar f)\ \text{a.e. on}\ \mathbf{R}^{2N}_{x,v}\times(0,T)\,.
$$

Then, if $f|_{t=0}=\bar f|_{t=0}$ *a.e. on* $\mathbf{R}^{2N}_{x,v}$, $f\equiv\bar f$ *on* $\mathbf{R}^{2N}_{x,v}\times[0,T]$.

Remarks. i) Notice that $Q^-(|\bar{f}|, |\bar{f}|) = (A * |\bar{f}|)|\bar{f}| \in L^1(\mathbf{R}^{2N}_{x,v} \times (0, T))$ and thus we have also $Q^+(|\bar{f}|, |\bar{f}|)$. In particular, (90) implies that $\left(\frac{\partial \bar{f}}{\partial t} + v \cdot \nabla_x \bar{f}\right) \in L^1(\mathbf{R}^{2N}_{x,v} \times (0, T))$.

ii) In practice, the condition (89) essentially means that $A \in L^1(\mathbf{R}^N)$ and $\bar{f} \in L^1(0, T; L^\infty(\mathbf{R}^{2N}_{x,v}))$. More precisely, if these two conditions are satisfied then $A * |\bar{f}| \in L^1(0, T; L^\infty(\mathbf{R}^{2N}_{x,v}))$.

Proof of Theorem V.1. We claim that we can take $g \equiv \bar{f}$ in (71). Let us observe that this is not automatic since the conditions imposed on $\bar{f}$ do not quite imply that $\bar{f} \in \boldsymbol{A}$. Once this claim is proven, Theorem V.1 follows easily. Indeed, using (71), we find

$$\frac{d}{dt}\int_{\mathbf{R}^N} |f-\bar{f}|dv + \mathrm{div}_x\left(\int_{\mathbf{R}^N} v|f-\bar{f}|dx\right) \leq \int_{\mathbf{R}^N} dv \ \mathrm{sign}(f-\bar{f})Q(\bar{f}, f-\bar{f}). \tag{91}$$

Indeed, $E(\bar{f}) \equiv 0$. Next, we observe that we have

$$\begin{aligned}\int_{\mathbf{R}^N} dv \ \mathrm{sign}\ (f-\bar{f})Q(\bar{f}, f-\bar{f}) &\leq \int_{\mathbf{R}^N} dv\ \{Q^+(|\bar{f}|, |f-\bar{f}|) + Q^-(|\bar{f}|, |f-\bar{f}|)\} \\ &= 2\int_{\mathbf{R}^N} dv\ Q^-(|\bar{f}|, |f-\bar{f}|) = 2\int_{\mathbf{R}^N} dv\,(A*|\bar{f}|)|f-\bar{f}| \leq \|A*|\bar{f}|\|_{L^\infty(\mathbf{R}^{2N}{}_{x,v})}\int_{\mathbf{R}^N}|f-\bar{f}|dv.\end{aligned}$$

Inserting this estimate in (91), we find

$$\frac{d}{dt}\int_{\mathbf{R}^N} |f-\bar{f}|dv + \mathrm{div}_x\left(\int_{\mathbf{R}^N} v|f-\bar{f}|dx\right) \leq a(t)\int_{\mathbf{R}^N}|f-\bar{f}|dv \quad \text{in } \mathscr{D}'(\mathbf{R}^N_x \times (0, T))$$

where $a(t) = \|A*|\bar{f}|\|_{L^\infty(\mathbf{R}^{2N}{}_{x,v})} \in L^1(0, T)$.

Since $|v||f-\bar{f}| \in L^1(\mathbf{R}^{2N}_{x,v} \times (0, T))$, we deduce easily from this differential inequality the following inequality

$$\frac{\partial}{\partial t}\int\int_{\mathbf{R}^{2N}} |f-\bar{f}|dx\ dv \leq a(t)\int\int_{\mathbf{R}^{2N}}|f-\bar{f}|dx\ dv \ \text{ in } \mathscr{D}'(0, T).$$

And we conclude easily using Gronwall's lemma since $f - \bar{f} \in C([0, T]; L^1(\mathbf{R}^{2N}_{x,v}))$ and $f|_{t=0} = \bar{f}|_{t=0}$ a.e. on $\mathbf{R}^{2N}_{x,v}$.

Therefore, we only have to show that (71) holds with $g \equiv \bar{f}$. To this end, we introduce $g_n = \varphi_n(v)\bar{f}$ where $\varphi_n \in C_0^\infty(\mathbf{R}^N)$, $\varphi_n \equiv 1$ on $(|v| \leq n)$, $0 \leq \varphi_n \leq 1$. Then, it is easy to check that $g_n \in \boldsymbol{A}$, $g_n \xrightarrow[n]{} \bar{f}$ in $C([0, T]; L^1(\mathbf{R}^{2N}_{x,v}))$. Furthermore, we have in view of (90),

$$E(g_n) = Q(\varphi_n\bar{f}, \varphi_n\bar{f}) - Q(\bar{f}, \bar{f}) \text{ on } \mathbf{R}^{2N}_{x,v} \times (0, T).$$

Hence, we have easily

$$\begin{aligned}\int_{\mathbf{R}^N}|E(g_n)|dv &\leq 2\iint_{\mathbf{R}^{2N}} dv\,dv_* A(v-v_*)\left|(\psi_n\bar{f})(v_*)(\psi_n\bar{f})(v)-\bar{f}(v_*)\bar{f}(v)\right| \\ &\leq 2\int_{\mathbf{R}^N}\{A*[(1-\psi_n)|\bar{f}|]\}\bar{f}\,dv+2\int_{\mathbf{R}^N}(A*|\bar{f}|)(1-\psi_n)|\bar{f}|dv \\ &\leq 2\iint_{\mathbf{R}^{2N}} A(v-v_*)(1-\psi_n)(v_*)|\bar{f}(v_*)||\bar{f}(v)|dv\,dv_*+ \\ &\quad +2\|A*|\bar{f}|\|_{L^\infty(\mathbf{R}^{2N}_{x,v})}\left(\int_{\mathbf{R}^N}(1-\psi_n)|\bar{f}|dv\right).\end{aligned}$$

The second term goes to 0 as n goes to $+\infty$ in $C([0,T];L^1(\mathbf{R}^N_x))$. The first term also goes to 0 as n goes to $+\infty$ (in $L^1(\mathbf{R}^N_x\times(0,T))$) by the dominated convergence theorem since $(1-\psi_n)(v_*)A\underset{n}{\to}0$ and $A(v-v_*)|\bar{f}(v_*)||\bar{f}(v)|\in L^1(\mathbf{R}^N_x\times\mathbf{R}\times(0,T))$. In conclusion, we may apply (71) with $g=g_n$ and we let n go to $+\infty$. Then $E(g_n)\underset{n}{\to}0$ in $L^1(\mathbf{R}^{2N}_{x,v}\times(0,T))$, g_n converges to $\bar{f}$ in $C([0,T];L^1(\mathbf{R}^{2N}_{x,v}))$ and our claim is shown provided we pass to the limit in the two terms that remain i.e. $\int_{\mathbf{R}^N}v|f-g_n|\,dv$ and $\operatorname{sign}(f-g_n)Q(g_n,f-g_n)$. For the first term, this is easy since $v|f-g_n|\underset{n}{\to}v|f-g|$ *a.e.* on $\mathbf{R}_{x,v}\times(0,T)$ while $\Big|v|f-g_n|\Big|\leq|v|(|f|+|\bar{f}|)\in L^1(\mathbf{R}^{2N}_{x,v}\times(0,T))$. The second term is a bit more delicate because of the discontinuity of the sign functions. In fact, in view of the uniqueness proof made above, we do not really need to obtain (71) and it is enough to show that $Q^\pm(|g_n|,|f-g_n|)$ converges, as n goes to $+\infty$, to $Q^\pm(|\bar{f}|,|f-\bar{f}|)$ in $L^1(\mathbf{R}^{2N}_{x,v}\times(0,T))$. Indeed, we have

$$\begin{aligned}&\int_{\mathbf{R}^N}|Q^\pm(|g_n|,|f-g_n|)-Q^\pm(|g|,|f-g|)|\,dv \\ &\leq\iint_{\mathbf{R}^{2N}}dv\,dv_*\,A(v-v_*)\Big||g_n(v_*)(f-g_n)(v)|-|\bar{f}(v_*)(f-\bar{f})(v)|\Big|\end{aligned}$$

and we argue as we did before for $E(g_n)$ in order to complete the proof of our claim and thus of Theorem V.1.

We have seen in the preceding section existence results of weak solutions f of (1) that meet the requirements imposed in Theorem V.1. Of course, the regularity informations imposed on $\bar{f}$ are not known in general. Such existence results are known only in very particular regimes: we refer to [13], [14] and [16] (...) for complete lists of references that provide interesting examples of such regimes. Let us mention two examples (on which we present more or less standard proofs) namely existence results locally in t (i.e. for $T>0$ small enough) and global ones for "not too large initial data". We do not claim the results which follow are original but they illustrate the preceding result and the proofs are simple and short enough to be included here.

Proposition V.1 *i) Let $f_0\geq 0$, $f_0\in L^1\cap L^\infty(\mathbf{R}^{2N}_{x,v})$, $f_0|v|^2\in L^1(\mathbf{R}^{2N}_{x,v})$. We assume that $A\in L^1(\mathbf{R}^N)$ and we set $T_*=2\|A\|_{L^1(\mathbf{R}^N)}\|f_0\|_{L^\infty(\mathbf{R}^{2N}_{x,v})}$. Then, if $T\in$*

$(0, T_*)$, there exists $f \in C([0, T]; L^1(\mathbf{R}^{2N}_{x,v})) \cap L^\infty(0, T; L^\infty(\mathbf{R}^{2N}_{x,v}))$ such that $f|_{t=0} = f_0$, $f|v|^2 \in L^\infty(0, T; L^1(\mathbf{R}^{2N}_{x,v}))$ and (90) holds.

ii) Let f_0 satisfy for some $C_0 \geq 0$, $a, b > 0$, $(x_0, v_0) \in \mathbf{R}^{2N}$

$$0 \leq f_0 \leq C_0 \exp\left(-\frac{|x-x_0|^2}{2a} - \frac{|v-v_0|^2}{2b}\right)(2\pi)^{-N}(ab)^{-N/2}. \tag{92}$$

We assume that $A \in L^q(\mathbf{R}^N)$ for some $q \in \left(\frac{N}{N-1}, +\infty\right)$ and that we have

$$\begin{cases} C_0 \|A\|_{L^q(\mathbf{R}^{2N})} (ab)^{-1/2} (b)^{-\frac{N}{2q}} a^{-\frac{N}{2}} \\ \leq \left(\int_0^\infty (1+\sigma^2)^{-\frac{N}{2p}} d\sigma\right)^{-1} p^{\frac{N}{2p}} \text{ where } p = \frac{q}{q-1} \end{cases}. \tag{93}$$

Then, there exists $f \in C([0, \infty); L^1(\mathbf{R}^{2N}_{x,v})) \cap L^\infty(0, T; L^\infty(\mathbf{R}^{2N}_{x,v}))$ for all $T \in (0, \infty)$, such that $f|_{t=0} = f_0$, (90) holds and

(94)

$$0 \leq f \leq C(t) \exp\left(-\frac{|x-vt-x_0|^2}{2a} - \frac{|v-v_0|^2}{2b}\right)(2\pi)^{-N}(ab)^{-N/2}$$

where $C(t)\ (\geq 0) \in C^1([0, \infty))$.

Remark. This result provides examples of solutions $\bar{f}$ of (90) that satisfy the conditions listed in Theorem V.1. In particular, their uniqueness follows from Theorem V.1 which also shows the uniqueness of arbitrary weak solutions (at least on $[0, T]$) satisfying (71).

Sketch of proof of Proposition V.1: We only explain how one can obtain L^∞ a priori estimates in the case of i) and the bound (94) in the case of ii). The existence then follows from tedious and (more or less) standard approximation or fixed points arguments.

In the case i), we just observe that we have for all $t \geq 0$

$$\|f\|_{L^\infty(\mathbf{R}^{2N}_{x,v})}(t) \leq \|f_0\|_{L^\infty(\mathbf{R}^{2N}_{x,v})} + \int_0^t \|Q(f,f)\|_{L^\infty(\mathbf{R}^{2N}_{x,v})} ds$$
$$\leq \|f_0\|_{L^\infty(\mathbf{R}^{2N}_{x,v})} + 2\int_0^t \|f^2\|_{L^\infty(\mathbf{R}^{2N}_{x,v})}(s) \|A\|_{L^\infty(\mathbf{R}^{2N}_{x,v})} ds.$$

Hence, for all $t \in [0, T_*)$

$$\|f\|_{L^\infty(\mathbf{R}^{2N}_{x,v})}(t) \leq (1 - 2\|A\|_{L^1(\mathbf{R}^N)} \|f_0\|_{L^\infty(\mathbf{R}^{2N}_{x,v})} t)^{-1} \|f_0\|_{L^\infty(\mathbf{R}^{2N}_{x,v})}.$$

In the case ii), we set $\hat{f} = C(t)(2\pi)^{-N}(ab)^{-N/2} \exp\left\{-\left(\frac{|x-vt-x_0|^2}{2a} + \frac{|v-v_0|^2}{2b}\right)\right\}$ where $C(t) \geq 0$ will be determined in such a way that we have

(95) $$\dot{C}C^{-1}\hat{f} \geq Q^{+}(\hat{f},\hat{f}) \text{ on } \mathbf{R}^{2N}_{x,v}\times(0,\infty)\ .$$

Indeed, observe that $\frac{\partial \hat{f}}{\partial t}+v\cdot\nabla_x\hat{f}=\dot{C}C^{-1}\hat{f}$ and that, if f solves (1), we have (recall that $f\geq 0$) $\frac{\partial f}{\partial t}+v\cdot\nabla_x f\leq Q^{+}(f,f)$. We may thus expect that the set $\{0\leq f\leq\hat{f}$ on $\mathbf{R}^{2N}_{x,v}\times[0,\infty)\}$ to be "invariant by the solution operator of the Boltzmann's equation" as soon as (95) holds. And because of (92), we expect (94) to hold if (95) holds and $C(0)=C_0$.

We next remark that $\hat{f}$ is, for all $t\geq 0$, a Maxwellian in v and thus (see [13] for instance) $Q^{+}(\hat{f},\hat{f})\equiv Q^{-}(\hat{f},\hat{f})\equiv\hat{f}A*\hat{f}$. Therefore, (95) holds if we have

(96) $$\dot{C}\geq C^2(2\pi)^{-N}(ab)^{-N/2}\sup_{(x,v)\in\mathbf{R}^{2N}}\left\{A\underset{v}{*}\left[\exp-\left(\frac{|x-vt-x_0|^2}{2a}+\frac{|v-v_0|^2}{2b}\right)\right]\right\}.$$

Next, we observe that we have on $\mathbf{R}^{2N}$

$$\begin{aligned}
&A*\exp\left(-\left(\frac{|x-vt-x_0|^2}{2a}+\frac{|v-v_0|^2}{2b}\right)\right)\\
&\leq\|A\|_{L^q(\mathbf{R}^N)}\left(\int_{\mathbf{R}^N}\exp\left\{-p\left(\frac{|x-vt-x_0|^2}{2a}+\frac{|v-v_0|^2}{2b}\right)\right\}dv\right)^{\frac{1}{p}}\\
&\leq\|A\|_{L^q(\mathbf{R}^N)}\left(\frac{2\pi ab}{p}\right)^{\frac{N}{2p}}(a+bt^2)^{-\frac{N}{2p}}\ .
\end{aligned}$$

Therefore, if we choose $C\in C^1([0,\infty))$ satisfying $C(0)=C_0$ and

(97) $$\dot{C}=C^2(2\pi)^{-N}(ab)^{-N/2}\|A\|_{L^q(\mathbf{R}^N)}\left(\frac{2\pi ab}{p}\right)^{\frac{N}{2p}}(a+bt^2)^{-\frac{N}{2p}} \text{ on } [0,\infty)\ ,$$

then (96) holds. We conclude observing that the condition (93) is precisely the condition that ensures that (97) admits a (uniquie) solution on $[0,\infty)$.

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